

On Tensorial Concomitants and the Non-Existence of a Gravitational Stress-Energy Tensor[†]

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ABSTRACT

Based on an analysis of what it may mean for one tensor to depend in the proper way on another, I prove that, under certain natural conditions, there can be no tensor whose interpretation could be that it represents gravitational stress-energy in general relativity. It follows that gravitational energy, such as it is in general relativity, is necessarily non-local. Along the way, I prove a result of some interest in own right about the structure of the associated jet bundles of the bundle of Lorentz metrics over spacetime.

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1 Gravitational Energy in General Relativity

There seems to be in general relativity no satisfactory, localized representation of a quantity whose natural interpretation would be “gravitational (stress-)energy”. The only physically unquestionable

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expressions of energetic quantities associated solely with the “gravitational field” we know of in general relativity are quantities derived by integration over non-trivial volumes in spacetimes satisfying any of a number of special conditions;¹ These quantities, moreover, tend to be non-tensorial in character. In other words, these are strictly non-local quantities, in the precise sense that they are not represented by fields defined at individual spacetime points (such as tensors or scalars).²

This puzzle about the character and status of gravitational energy emerged simultaneously with the discovery of the theory itself.³ The problems raised by the seeming non-localizability of gravitational energy had a profound, immediate effect on subsequent research. For example, it was directly responsible for Hilbert’s request to Noether that she investigate conservation laws in a quite general setting, the work that led to her famous results relating symmetries and conservation laws.⁴

Almost all discussions of gravitational energy in general relativity, however, dating back to those earliest debates, have been plagued by vagueness and lack of precision. The main result of this paper addresses the issue head-on in a precise and rigorous way. Based on an analysis of what it may mean for one tensor to depend in the proper way on another, I prove that, under certain natural conditions, there can be no tensor whose interpretation could be that it represents gravitational stress-energy in general relativity. It follows that gravitational energy, such as it is in general relativity, is necessarily non-local. Along the way, I prove a result of some interest in own right about the structure of the associated first two jet bundles of the bundle of Lorentz metrics over spacetime.

2 The Principle of Equivalence: A Bad Argument

The most popular heuristic argument used to attempt to show that gravitational energy, such as it is, either does not exist at all or does exist but cannot be localized invokes the “Principle of Equivalence”. Choquet-Bruhat (1983, p. 399), for example, puts the argument like this,

This ‘non local’ character of gravitational energy is in fact obvious from a formulation of the equivalence principle which says that the gravitational field appears as non-existent to one observer in free fall. It is, mathematically, a consequence of the fact that the pseudo-riemannian connexion which represents the gravitational field can always be made to vanish along a given curve by a change of coordinates.

Goldberg (1980, pp. 469-70) makes almost exactly the same argument, though he draws the conclusion in a slightly more explicit fashion:⁵

¹Weyl (1921, pp. 271–272) was perhaps the first to grasp this point with real clarity. See also Dirac (1962).

²Schrödinger (1988, pp. 104–105) gives a particularly clear, concise statement of the relation between the fact that the known energetic, gravitational quantities are non-tensorial and the fact that integration over them can be expected to yield integral conservation laws only under restricted conditions.

³The first pseudo-tensorial entity proposed to represent gravitational stress-energy dates back to Einstein (1915), the paper in which he first proposed the final form of the theory.

⁴See, *e.g.*, Brading (2005).

⁵Goldberg’s formulation of the argument exhibits a feature common in the many instances of it I have found in the literature, the conclusion that a local gravitational energy *scalar* density does not exist and not that a gravitational

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[I]n Minkowski space any meaningful energy density should be zero. But a general space-time can be made to appear Minkowskian along an arbitrary geodesic. As a result, any nontensorial ‘energy density’ can be made to be zero along an arbitrary geodesic and, therefore, has no invariant meaning.

Trautman (1976, pp. 135-6) has also made essentially the same argument. In fact, the making of this argument seems to be something of a shared mannerism among physicists who discuss energy in general relativity; it is difficult to find an article on the topic in which it is not at least alluded to.⁶

There are two fundamental flaws with arguments of this sort. First, the Principle of Equivalence does not deserve its definite article—there are almost as many formulations of it as there are people who invoke it, and many of those formulations are so vague as to leave it unclear whether or not the proposition holds in general relativity.⁷ The second is that it assumes that, if there is such a thing as localized gravitational energy or stress-energy, it can depend only on “first derivatives of the metric”. But that seems wrong on the face of it. If there is such a thing as a localized gravitational energetic quantity, then surely it depends on the curvature of spacetime and not on the affine connection (or, more precisely, it depends on the affine connection at least in so far as it depends on the curvature), for any energy one can envision transferring from the “gravitational field” to another type of system in a different form (*e.g.*, as heat or a spray of fundamental particles) in general relativity must at bottom be based on geodesic deviation,⁸ and so must be determined by the value of the Riemann tensor at a point, not by the value of the affine connection at a single point or even along a curve. There is no solution to the Einstein field-equation that corresponds in any natural way to the intuitive Newtonian idea of a constant gravitational field, *i.e.*, one without geodesic deviation; that, however, would be the only sort of field that one could envision even being tempted to ascribe gravitational energy to in the absence of geodesic deviation, and that attribution is problematic even in Newtonian theory.

An obvious criticism of my response to the standard line, related to another popular argument given for the non-existence or non-locality of gravitational energetic quantities, is that it would make gravitational stress-energy depend on second-order partial derivatives of the field potential (the metric, so comprehended by analogy with the potential in Newtonian theory), whereas all other known forms of stress-energy depend only on terms quadratic in the first partial derivatives of the field potential. To be more precise, the argument almost always runs like this:

One can make precise the sense in which Newtonian gravitational theory is the ‘weak-

stress-energy tensor does not exist. Perhaps one could imagine having a well-defined scalar energy density of a field in the absence of a well-defined stress-energy tensor for that field, though I cannot myself see any way to represent such an idea in general relativity. (Note that if one could, this would appear to be a violation of the thermodynamic principle that all energy is equivalent in character, in the sense that any one form can always in principle be transformed into any other form).

⁶Bondi (1962) and Geroch (1973) are notable exceptions. I take their discussions as models of how one should discuss energetic phenomena in the presence of gravitational fields.

⁷See, for instance, Dicke (1962) and Norton (1985) for discussion of these points.

⁸Penrose (1966) and Ashtekar and Penrose (1990) implicitly rely on the same idea to very fruitful effect.

field' limit of general relativity.⁹ In this limit, it is clear that the metric field plays roughly the role in general relativity that the scalar potential ϕ does in Newtonian theory. In Newtonian theory, bracketing certain technical questions about boundary conditions, there is a more or less well-defined energy density of the gravitational field, proportional to $(\nabla\phi)^2$. One might expect, therefore, based on some sort of continuity argument, or just on the strength of the analogy itself, that any local representation of gravitational energy in general relativity ought to be a “quadratic function of the first partials of the metric”.¹⁰ No invariant quantity at a point can be constructed using only the first partials of the metric, however, so there can be no scalar or tensorial representation of gravitational energy in general relativity.

(No writer I know makes the argument exactly in this form; it is just the clearest, most concise version I can come up with myself.) As Pauli (1921, p. 178) forcefully argued, however, there can be *no* physical argument against the possibility that gravitational energy depends on second derivatives of the metric. Just because the energy of all other known fields have the same form in no way implies that a localized gravitational energy in general relativity, if there is such a thing, ought to have that form as well. Gravity is too different a field from others for such a bare assertion to carry any weight. As I show in §6, moreover, a proper understanding of tensorial concomitants reveals that an expression linear in second partial derivatives is in the event equivalent in the relevant sense to one quadratic in first order partials. This illustrates how misleading the analogy with Newtonian gravity can be.

3 Geometric Fiber Bundles and Concomitants

The introduction of a coordinate system to geometry is an act of violence.

Hermann Weyl

Philosophy of Mathematics and Natural Science

I have argued that, if there is an object that deserves to be thought of as the representation of gravitational stress-energy of the gravitational field in general relativity, then it ought to depend on the Riemann curvature tensor in an appropriate sense. Since there is no obvious mathematical sense in which a general mathematical structure can “depend” on a tensor, the first task is to say what exactly this could mean. I will call a mathematical structure on a manifold that depends in the appropriate fashion on another structure on the manifold, or set of others, a *concomitant* of it (or them).

As near as I can make out, this terminology is due to Schouten.¹¹ The definition Schouten proposed—the only one I know of in the literature—is expressed in terms of coordinates: depending

⁹See, *e.g.*, Malament (1986).

¹⁰In this light, it is interesting to note that gravitational energy pseudo-tensors tend to be quadratic in the first-order partials of the metric.

¹¹See Schouten (1954, p. 15), though of course he used the German *Komitant*. The idea of proving the uniqueness of a tensor that “depends” on another tensor, and satisfies a few collateral conditions, dates back at least to Weyl

on what sort of concomitant one was dealing with, the components of the object had to satisfy various conditions of covariance under certain classes of coordinate transformations. This makes it not only unwieldy in practice and inelegant, but, more important, it makes it difficult to discern what of real significance is encoded in the relation of being a concomitant in particular cases. Schouten's invariance conditions translate into a set of partial differential equations in a particular coordinate system, which even in fairly trivial cases turn out to be forbiddingly complicated.¹² It is almost impossible to determine anything of the general properties of the set of a particular kind of concomitant of a particular object by looking at these equations. I suspect that it is because these conditions are so complex, difficult and opaque that use is very rarely made of concomitants in arguments about spacetime structure in general relativity. This is a shame, for the idea is, I think, potentially rich, and so calls out for an invariant formulation.

The reason I am inquiring into the possibility of a concomitant in the first place, when the question is the possible existence of a representation of gravitational stress-energy tensor, is a simple one. What is wanted is an expression for gravitational energy that does not depend for its formulation on the particulars of the spacetime, just as the expression for the kinetic energy of a particle in classical physics does not depend on the particular interactions one imagines the particle to be experiencing with its environment, and just as the stress-energy tensor for a Maxwell field can be calculated in any spacetime in which there is a Maxwell field, irrespective of the particulars of the spacetime, in contradistinction to the definitions of all known expressions for gravitational energy in general relativity now do (*e.g.*, the ADM mass, which can be defined only in asymptotically flat spacetimes). If there is a well-formed expression for gravitational stress-energy, then one should be able in principle to calculate it whenever there are gravitational phenomena, which is to say,

(1921, pp. 315-18) and Cartan (1922). In fact, Weyl proved that, in any spacetime, the only two-index symmetric covariant tensors one can construct at a point, using only algebraic combinations of the components of the metric and its first two partial derivatives in a coordinate system at that point, that are at most linear in the second derivatives of the metric, are linear combinations of the Ricci curvature tensor, the scalar curvature times the metric and the metric itself. In particular, the only such divergence-free tensors one can construct at a point are linear combinations of the Einstein tensor and the metric with constant coefficients. Using Schouten's definition of a concomitant, Lovelock (1972) proved the following theorem:

Let (\mathcal{M}, g_{ab}) be a spacetime. In a coordinate neighborhood of a point $p \in \mathcal{M}$, let $\Theta_{\alpha\beta}$ be the components of a tensor concomitant of $\{g_{\lambda\mu}; g_{\lambda\mu,\nu}; g_{\lambda\mu,\nu\rho}\}$ such that

$$\nabla^n \Theta_{nb} = 0.$$

Then

$$\Theta_{ab} = rG_{ab} + qg_{ab},$$

where G_{ab} is the Einstein tensor and q and r are constants.

This is a much stronger result in several ways than Weyl and Cartan had been able to attain: one has a more generalized notion of concomitant than algebraic combination of coordinate components; one does not demand that Θ_{ab} be symmetric; and most strikingly, one does not demand that Θ_{ab} be at most linear in the second-order partial derivatives of the metric components.

¹²For a good example of just how hairy these conditions can be, see du Plessis (1969, p. 350) for a complete set written out explicitly in the case of two covariant-index tensorial second-order differential concomitants of a spacetime metric.

in any spacetime whatsoever—it should be a *function* of the curvature in that spacetime, in some appropriately generalized sense of ‘function’. This idea is what a concomitant is supposed to capture.

The machinery of fiber bundles is used to characterize the idea of a concomitant. I give a (brief) explicit formulation of the machinery, because the one I rely on is non-standard. (We assume from hereon that all relevant structures, mappings, *etc.*, are smooth. Nothing is lost by the assumption and it will simplify exposition. All constructions and proofs can easily be generalized to the case of topological spaces and continuous structures.)

Definition 3.1 *A fiber bundle \mathfrak{B} is an ordered triplet, $\mathfrak{B} \equiv (\mathcal{B}, \mathcal{M}, \pi)$, such that:*

FB1. *\mathcal{B} is a differential manifold*

FB2. *\mathcal{M} is a differential manifold*

FB3. *$\pi : \mathcal{B} \rightarrow \mathcal{M}$ is smooth and onto*

FB4. *For every $q, p \in \mathcal{M}$, $\pi^{-1}(q)$ is diffeomorphic to $\pi^{-1}(p)$ (as submanifolds of \mathcal{B})*

FB5. *\mathcal{B} has a locally trivial product structure, in the sense that for each $q \in \mathcal{M}$ there is a neighborhood $U \ni q$ and a diffeomorphism $\zeta : \pi^{-1}[U] \rightarrow U \times \pi^{-1}(q)$ such that the action of π commutes with the action of ζ followed by projection on the first factor.*

\mathcal{B} is the *bundle space*, \mathcal{M} the *base space*, π the *projection* and $\pi^{-1}(q)$ the *fiber* over q . By a convenient, conventional abuse of terminology, I will sometimes call \mathcal{B} itself ‘the fiber bundle’ (or ‘the bundle’ for short). A *cross-section* κ is a continuous map from \mathcal{M} into \mathcal{B} such that $\pi(\kappa(q)) = q$, for all q in \mathcal{M} .

In particular, this definition of a fiber bundle is non-standard in so far as no group action on the fibers is fixed from the start; this implies in particular that no correlation between diffeomorphisms of the base space and diffeomorphisms of the bundle space is fixed.¹³ On the view I advocate, the geometric character of the objects represented by the bundle arises not from the group action directly, but only after one fixes an explicitly defined correlation between diffeomorphisms on the base space with those on the bundle space. For example, depending on how decides that a diffeomorphism on the base space ought to induce a diffeomorphism on the bundle over it whose fibers consist of 1-dimensional vector spaces, one will ascribe to the objects of the bundle the character either of ordinary scalars or of n -forms (where n is the dimension of the base space). The idea is that the diffeomorphisms induced on the bundle space then implicitly define the group action on the fibers appropriate for the required sort of object.¹⁴

I call an appropriate mapping of diffeomorphisms on the base space to those on the bundle space an *induction*. (I give a precise definition in a moment.) In this scheme, therefore, the induction

¹³In this I follow Geroch (1996). See, *e.g.*, Steenrod (1951) for the traditional definition and the way that a fixed group action on the fibers induces a correlation between diffeomorphisms on the bundle space and those on the base space.

¹⁴I will not work out the details of how this comes about here, as they are not needed for the arguments of the paper; see Curiel (2009).

comes first conceptually, and the relation between diffeomorphisms on the base space and those they induce on the bundle serves to fix the fibers as spaces of *geometric objects*, viz., those whose transformative properties are tied directly and intimately to those of the ambient base space. This way of thinking of fiber bundles is perhaps not well suited to the traditional task of classifying bundles, but it turns out to be just the thing on which to base a perspicuous and useful definition of concomitant. Although a diffeomorphism on a base space will naturally induce a unique one on certain types of fiber bundles over it, such as tensor bundles, in general it will not. There is not known, for instance, any natural way to single out a map of diffeomorphisms of the base space into those of a bundle over it whose fibers consist only of spinorial objects.¹⁵

I turn now to making this intuitive discussion more precise. A diffeomorphism ϕ^\sharp of a bundle space \mathcal{B} is *consistent* with ϕ , a diffeomorphism of the base space \mathcal{M} , if, for all $u \in \mathcal{B}$,

$$\pi(\phi^\sharp(u)) = \phi(\pi(u))$$

For a general bundle, there will be scads of diffeomorphisms consistent with a given diffeomorphism on the base space. A way is needed to fix a unique ϕ^\sharp consistent with a ϕ so that a few obvious conditions are met. For example, the identity diffeomorphism on \mathcal{M} ought to pick out the identity diffeomorphism on \mathcal{B} . More generally, if ϕ is a diffeomorphism on \mathcal{M} that is the identity inside an open set $O \subset \mathcal{M}$ and differs from the identity outside O , it ought to be the case that the mapping picks out a ϕ^\sharp that is the identity on $\pi^{-1}[O]$. If this holds, we say that that ϕ^\sharp is *strongly consistent* with ϕ .

Let $\mathfrak{D}_{\mathcal{M}}$ and $\mathfrak{D}_{\mathcal{B}}$ be, respectively, the groups of diffeomorphisms on \mathcal{M} and \mathcal{B} to themselves, respectively. Define the set

$$\mathfrak{D}_{\mathcal{B}}^\sharp = \{\phi^\sharp \in \mathfrak{D}_{\mathcal{B}} : \exists \phi \in \mathfrak{D}_{\mathcal{M}} \text{ such that } \phi^\sharp \text{ is strongly consistent with } \phi\}$$

It is simple to show that $\mathfrak{D}_{\mathcal{B}}^\sharp$ forms a subgroup of $\mathfrak{D}_{\mathcal{B}}$. This suggests

Definition 3.2 An induction is an injective homomorphism $\iota : \mathfrak{D}_{\mathcal{M}} \rightarrow \mathfrak{D}_{\mathcal{B}}^\sharp$.

ϕ will be said to *induce* ϕ^\sharp (under ι) if $\iota(\phi) = \phi^\sharp$.¹⁶

Definition 3.3 A geometric fiber bundle is an ordered quadruplet $(\mathcal{B}, \mathcal{M}, \pi, \iota)$ satisfying

GFB1. $(\mathcal{B}, \mathcal{M}, \pi)$ satisfies FB1-FB5

GFB2. ι is an induction

Geometric fiber bundles are the appropriate spaces to serve as the domains and ranges of concomitant mappings.

Most of the fiber bundles one works with in physics are geometric fiber bundles. A tensor bundle \mathcal{B} , for example, is a fiber bundle over a manifold \mathcal{M} each of whose fibers is diffeomorphic to the

¹⁵See, e.g., Penrose and Rindler (1984).

¹⁶In a more thorough treatment, one would characterize the way that the induction fixes a group action on the fibers, but we do not need to go into that for our purposes. Again, see Curiel (2009).

vector space of tensors of a particular index structure over any point of the manifold; a basis for an atlas is provided by the charts on \mathcal{B} naturally induced from those on \mathcal{M} by the representation of tensors on \mathcal{M} as collections of components in \mathcal{M} 's coordinate systems. A cross-section is then the representation of a particular tensor field of that index structure on \mathcal{M} . There is a natural induction in this case, $\iota : \mathcal{D}_{\mathcal{M}} \rightarrow \mathcal{D}_{\mathcal{B}}^{\#}$, fixed by the pull-back action of a diffeomorphism ϕ of tensors on \mathcal{M} . It is straightforward to show that ι so defined is in fact an induction. Spinor bundles provide interesting examples of physically important bundles that cannot naturally be turned into geometric bundles.

We are finally in a position to define concomitants. Let $(\mathcal{B}_1, \mathcal{M}, \pi_1, \iota_1)$ and $(\mathcal{B}_2, \mathcal{M}, \pi_2, \iota_2)$ be two bundles with the same base space.¹⁷

Definition 3.4 *A mapping $\chi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a concomitant if*

$$\chi(\iota_1[\phi](u_1)) = \iota_2[\phi](\chi(u_1))$$

for all $u_1 \in \mathcal{B}_1$ and all $\phi \in \mathcal{D}_{\mathcal{M}}$.

In intuitive terms, a concomitant is a mapping between bundles that commutes with the action of the induced diffeomorphisms that lend the objects of the bundles their respective geometric characters. It is easy to see that χ must be fiber-preserving, in the sense that it maps fibers of \mathcal{B}_1 to fibers of \mathcal{B}_2 . This captures the idea that the dependence of the one type of object on the other is strictly local, as the respecting of the actions of diffeomorphisms captures the idea that the mapping encodes an invariant relation.

4 Jet Bundles and Higher-Order Concomitants

Just as with ordinary functions from one Euclidean space to another, it seems plausible that the dependence encoded in a concomitant from one geometric bundle to another may take into account not only the value of the first geometrical structure at a point of the base space, but also “how that value is changing” in a neighborhood of that point, something like a generalized derivative of a geometrical structure on a manifold. The following construction is meant to capture in a precise sense the idea of such a generalized derivative in such a way so as to make it easy to generalize the idea of a concomitant to account for it.

Fix a geometric fibre bundle $(\mathcal{B}, \mathcal{M}, \pi, \iota)$, and the space of sections $\Gamma[\mathcal{B}]$. Two sections $\gamma, \eta : \mathcal{M} \rightarrow \mathcal{B}$ *osculate to first-order* at $p \in \mathcal{M}$ if $T\gamma$ and $T\eta$ agree in their action on $T_p\mathcal{M}$.¹⁸ If (x^i, v^α) are coordinates at the point $\gamma(q)$ adapted to the bundle structure (as defined by the induction), then a coordinate representation of this relation is:

$$\left. \frac{\partial(v^\alpha \circ \gamma)}{\partial x^i} \right|_q = \left. \frac{\partial(v^\alpha \circ \eta)}{\partial x^i} \right|_q$$

¹⁷One can generalize the definition of concomitants to cover the case of bundles over different base spaces, but we do not need this here.

¹⁸They osculate to zeroth-order at p if they map p to the same point in the domain.

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for all $i \leq \dim(\mathcal{M})$ and $\alpha \leq \dim(\pi^{-1}[q])$. This constitutes an equivalence relation on $\Gamma[\mathcal{B}]$. A *1-jet* with source q and target $\gamma(q)$, written ' $j_q^1[\gamma]$ ', is such an equivalence class. The set of all 1-jets,

$$J^1\mathcal{B} \equiv \bigcup_{q \in \mathcal{M}, \gamma \in \Gamma[\mathcal{B}]} j_q^1[\gamma]$$

naturally inherits the structure of a differentiable manifold. Let (ϕ, U) be an adapted coordinate chart of \mathcal{B} around $\gamma(q)$, with the coordinate functions (x^i, v^α) . Then the induced coordinate chart on $J^1\mathcal{B}$ is (ϕ^1, U^1) where

$$U^1 \equiv \{j_q^1[\gamma] \mid \gamma(q) \in U\} \tag{4.1}$$

and the coordinate functions associated with ϕ^1 are $(x^i, v^\alpha, v_i^\alpha)$, where

$$\begin{aligned} x^i(j_q^1[\gamma]) &\equiv x^i(q) \\ v^\alpha(j_q^1[\gamma]) &\equiv v^\alpha(\gamma(q)) \\ v_i^\alpha(j_q^1[\gamma]) &\equiv \left. \frac{\partial(v^\alpha \circ \gamma)}{\partial x^i} \right|_q \end{aligned} \tag{4.2}$$

where γ is any member of $j_q^1[\gamma]$; this is well defined since all members of $j_q^1[\gamma]$ agree on $\gamma(q)$ and $\left. \frac{\partial(v^\alpha \circ \gamma)}{\partial x^i} \right|_q$ by definition.

One can naturally fibre $J^1\mathcal{B}$ over \mathcal{M} . The *source projection* $\sigma^1 : J^1\mathcal{B} \rightarrow \mathcal{M}$, defined by

$$\sigma^1(j_q^1[\gamma]) = q$$

gives $J^1\mathcal{B}$ the structure of a bundle space over the base space \mathcal{M} , and in this case we write the bundle $(J^1\mathcal{B}, \mathcal{M}, \sigma^1)$. A section γ of \mathcal{B} naturally gives rise to a section $j^1[\gamma]$ of $J^1\mathcal{B}$, the *first-order prolongation* of that section:

$$j^1[\gamma] : \mathcal{M} \rightarrow \bigcup_{q \in \mathcal{M}} j_q^1[\gamma]$$

such that $\sigma_1(j^1[\gamma](q)) = q$. (We assume for the sake of simplicity that global cross-sections exist; the emendations required to treat local cross-sections are trivial.)

The points of $J^1\mathcal{B}$ may be thought of as coordinate-free representations of first-order Taylor expansions of sections of \mathcal{B} . To see this, consider the example of the trivial bundle $(\mathcal{B}, \mathbb{R}^2, \pi)$ where $\mathcal{B} \equiv \mathbb{R}^2 \times \mathbb{R}$ and π is projection onto the first factor. Fix global coordinates (x^1, x^2, v^1) on \mathcal{B} , so that the induced (global) coordinates on $J^1\mathcal{B}$ are $(x^1, x^2, v^1, v_1^1, v_2^1)$. Then for any 1-jet $j_q^1[\gamma]$, define the inhomogenous linear function $\hat{\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\hat{\gamma}(p) = v^1(\gamma(p)) + v_1^1(j_q^1[\gamma])(p_1 - q_1) + v_2^1(j_q^1[\gamma])(p_2 - q_2)$$

where $\gamma \in j_q^1[\gamma]$, and $p, q \in \mathbb{R}^2$ with respective components (p_1, p_2) and (q_1, q_2) . Clearly $\hat{\gamma}$ defines a cross-section of $J^1\mathcal{B}$ first-order osculant to γ at p and so is a member of $j_q^1[\gamma]$; indeed, it is the unique globally defined, linear inhomogeneous map with this property.

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A 2 -jet is defined similarly, as an equivalence class of sections under the relation of having the same first and second partial-derivatives at a point. More precisely, $\gamma, \eta \in \Gamma[\mathcal{B}]$ *osculate to second order* at $q \in \mathcal{M}$ if $\gamma(q) = \eta(q)$ and

$$\begin{aligned} \left. \frac{\partial(v^\alpha \circ \gamma)}{\partial x^i} \right|_q &= \left. \frac{\partial(v^\alpha \circ \eta)}{\partial x^i} \right|_q \\ \left. \frac{\partial^2(v^\alpha \circ \gamma)}{\partial x^i \partial x^j} \right|_q &= \left. \frac{\partial^2(v^\alpha \circ \eta)}{\partial x^i \partial x^j} \right|_q \end{aligned} \quad (4.3)$$

One then defines $J^2\mathcal{B}$, *et al.*, in the analogous ways. There is a natural projection from $J^2\mathcal{B}$ to $J^1\mathcal{B}$, the *truncation* $\theta^{2,1}$, characterized by “dropping the second-order terms in the Taylor expansion”. In general, one has the natural truncation $\theta^{n,m} : J^n\mathcal{B} \rightarrow J^m\mathcal{B}$ for all $0 < m < n$.

An important fact for the present goal of defining concomitants is that the jet bundles of a geometric bundle are themselves naturally geometric bundles. Fix a geometric bundle $(\mathcal{B}, \mathcal{M}, \pi, \iota)$ and a diffeomorphism ϕ on \mathcal{M} . Then $\iota[\phi]$ not only defines an action on points of \mathcal{B} , but, as a diffeomorphism itself on \mathcal{B} , it naturally defines an action on the cross-sections of \mathcal{B} and thus on the 1-jets. The action can be characterized by the appropriate coordinate transformations of the formulæ (4.2) and (4.3). It is easy to show that the mapping ι^1 so specified from $\mathfrak{D}_{\mathcal{M}}$ to $\mathfrak{D}_{J^1\mathcal{B}}$ is an injective homomorphism and thus itself an induction; therefore, $(J^1\mathcal{B}, \mathcal{M}, \sigma^1, \iota^1)$ is a geometric fiber bundle. One defines inductions for higher-order jet bundles in the same way.

We can now generalize our definition of concomitants. Let $(\mathcal{B}_1, \mathcal{M}, \pi_1, \iota)$ and $(\mathcal{B}_2, \mathcal{M}, \pi_2, j)$ be two geometric fiber bundles over the manifold \mathcal{M} .

Definition 4.1 *A zeroth-order concomitant from \mathcal{B}_1 to \mathcal{B}_2 is a smooth mapping $\chi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ such that*

$$(\forall p \in \mathcal{B}_1)(\forall \phi \in \mathfrak{A}_{\mathcal{M}})\{j(\phi)(\chi(p)) = \chi(\iota(\phi)(p))\}$$

Definition 4.2 *An n^{th} -order concomitant ($n > 0$) from \mathcal{B}_1 to \mathcal{B}_2 is a smooth mapping $\chi : J^n\mathcal{B}_1 \rightarrow \mathcal{B}_2$ such that*

1. $(\forall u \in J^n\mathcal{B}_1)(\forall \phi \in \mathfrak{A}_{\mathcal{M}}) j(\phi)(\chi(u)) = \chi(\iota^n(\phi)(u))$
2. *there is no $(n-1)^{\text{th}}$ concomitant $\chi' : J^{n-1}\mathcal{B}_1 \rightarrow \mathcal{B}_2$ satisfying*

$$(\forall u \in J^n\mathcal{B}_1) \chi(u) = \chi'(\theta^{n,n-1}(u))$$

I will sometimes refer to a zeroth-order concomitant simply as a *concomitant* for the sake of brevity. By another convenient abuse of terminology, I will often refer to the range of the concomitant mapping itself as ‘the concomitant’ of the domain. It will be of physical interest in §6 to consider the way that concomitants interact with multiplication by a scalar field. (Since we consider in this paper only concomitants of linear and affine objects, multiplication of the object by a scalar field is always defined.) In particular, let us say that a concomitant is *homogeneous of weight w* if for any constant scalar field σ

$$\chi(\iota_1[\phi](\sigma u_1)) = \sigma^w \iota_2[\phi](\chi(u_1))$$

An important property of concomitants is that, in a limited sense, they are transitive.

Proposition 4.3 *If $\chi_1 : J^n \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is an n^{th} -order concomitant and $\chi_2 : \mathcal{B}_2 \rightarrow \mathcal{B}_3$ is a smooth mapping, where \mathcal{B}_1 , \mathcal{B}_2 and \mathcal{B}_3 are geometric bundles over the same base space, then $\chi_2 \circ \chi_1$ is an n^{th} -order concomitant if and only if χ_2 is a zeroth-order concomitant.*

This follows easily from the fact that inductions are injective homomorphisms and concomitants respect the fibers.

5 Concomitants of the Metric

As a specific example that will be of use in what follows, consider the geometric fiber bundle $(\mathcal{B}_g, \mathcal{M}, \pi_g, \iota_g)$, with \mathcal{M} a 4-dimensional, paracompact, connected, smooth manifold (*i.e.*, a candidate spacetime manifold), the fibers of \mathcal{B}_g diffeomorphic to the set of Lorentz metrics at each point of \mathcal{M} , all of the same signature $(+, -, -, -)$, and ι_g the induction defined by the natural pull-back. Since the set of Lorentz metrics in the tangent plane over a point of a 4-dimensional manifold, all of the same signature, is a 10-dimensional topological space,¹⁹ the bundle space \mathcal{B}_g is a 14-dimensional manifold. A cross-section of this bundle is equivalent to a Lorentz metric field on the manifold.

The following proposition precisely captures the statement one sometimes hears that there is no scalar or tensorial quantity one can form depending only on the metric and its first-order partial derivatives at a point of a manifold.

Proposition 5.1 *There is no first-order concomitant from \mathcal{B}_g to any tensor bundle over \mathcal{M} .*

To prove this, it suffices to remark that, given any spacetime (\mathcal{M}, g_{ab}) and any two points $p, p' \in \mathcal{M}$, there are coordinate neighborhoods U of p and U' of p' and a diffeomorphism $\phi : \mathcal{M} \rightarrow \mathcal{M}$, such that $\phi(p) = p'$, $\phi^\#(g'_{ab}) = g_{ab}$ at p , and $\phi^\#(\partial'_a g_{bc}) = \partial_a g_{bc}$ at p , where ∂_a (∂'_a) is the ordinary derivative operator associated with the coordinate system on U (U'), and $\phi^\#$ is the map naturally induced by the pull-back action of ϕ .

This is not to say, however, that no information of interest is contained in $J^1 \mathcal{B}_g$. Indeed, two metrics g_{ab} and h_{ab} are first-order osculant at a point if and only if they have the same associated derivative operator at that point. To see this, first note that, if they osculate to first order at that point, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators. Thus, for the derivative operator ∇_a associated with, say, g_{ab} , $\nabla_a(g_{bc} - h_{bc}) = 0$, but $\nabla_a g_{bc} = 0$, so $\nabla_a h_{bc} = 0$ at that point as well. Similarly, if the two metrics are equal and share the same associated derivative operator ∇_a at a point, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators, since their difference will be identically annihilated by ∇_a , and $g_{ab} = h_{ab}$ at the point by assumption. Thus they are first-order osculant at that point and so in the same 1-jet. This proves that all and only geometrically relevant information contained in the 1-jets of Lorentz metrics on \mathcal{M} is encoded in the fiber bundle over spacetime the values of the fibers of which are ordered pairs consisting of a metric and the metric's associated derivative operator at a spacetime point.

¹⁹In fact, it is homeomorphic to a connected, convex, open subset—an open cone with vertex at the origin—in \mathbb{R}^{10} .

The second jet bundle over \mathcal{B}_g has a similarly interesting structure. Clearly, if two metrics are in the same 2-jet, then they have the same Riemann tensor at the point associated with the 2-jet, since they have the same partial-derivatives up to second order at the point. Assume now that two metrics are in the same 1-jet and have the same Riemann tensor at the associated spacetime point. It follows that they are in the same 2-jet, then essentially all and only geometrically relevant information contained in the 2-jets of Lorentz metrics on \mathcal{M} is encoded in the fiber bundle over spacetime the points of the fibers of which are ordered triplets consisting of a metric, the metric's associated derivative operator and the metric's Riemann tensor at a spacetime point. To demonstrate this, it suffices to show that if two Levi-Civita connections agree on their respective Riemann tensors at a point, then the two associated derivative operators are in the same 1-jet of the bundle whose base-space is \mathcal{M} and whose fibers consist of the affine spaces of derivative operators at the points of \mathcal{M} (because they will then agree on the values of first-partial derivatives of their Christoffel symbols at that point in any coordinate system as well as agreeing in the values of the Christoffel symbols themselves, and thus will be in the 2-jet of the same metric at that point).

Assume that, at a point p of spacetime, $g_{ab} = \tilde{g}_{ab}$, $\nabla_a = \tilde{\nabla}_a$ (the respective derivative operators), and $R^a{}_{bcd} = \tilde{R}^a{}_{bcd}$ (the respective Riemann tensors). Let $C^a{}_{bc}$ be the symmetric difference-tensor between ∇_a and $\tilde{\nabla}_a$, which is itself 0 at p by assumption. Then by definition $\nabla_{[b}\nabla_{c]}\xi^a = R^a{}_{bcn}\xi^n$ for any vector ξ^a , and so at p by assumption

$$\begin{aligned} R^c{}_{abn}\xi^n &= \nabla_{[a}\tilde{\nabla}_{b]}\xi^c \\ &= \nabla_a(\nabla_b\xi^c + C^c{}_{bn}\xi^n) - \tilde{\nabla}_b\nabla_a\xi^c \\ &= \nabla_a\nabla_b\xi^c + \nabla_a(C^c{}_{bn}\xi^n) - \nabla_b\nabla_a\xi^c - C^c{}_{bn}\nabla_a\xi^n + C^n{}_{ba}\nabla_n\xi^c \end{aligned}$$

but $\nabla_a\nabla_b\xi^c - \nabla_b\nabla_a\xi^c = R^a{}_{bcn}\xi^n$ and $C^a{}_{bc} = 0$, so expanding the only remaining term gives

$$\xi^n\nabla_a C^c{}_{bn} = 0$$

for arbitrary ξ^a and thus $\nabla_a C^b{}_{cd} = 0$ at p ; by the analogous computation, $\tilde{\nabla}_a C^b{}_{cd} = 0$ as well. It follows immediately that ∇_a and $\tilde{\nabla}_a$ are in the same 1-jet of affine bundles over p . We have proved

Theorem 5.2 *$J^1\mathcal{B}_g$ is naturally diffeomorphic to the geometric fiber bundle over \mathcal{M} whose fibers consist of pairs (g_{ab}, ∇_a) , where g_{ab} is the value of a Lorentz metric field at a point of \mathcal{M} , and ∇_a is the value of the covariant derivative operator associated with g_{ab} at that point, the induction being defined by the natural pull-back. $J^2\mathcal{B}_g$ is naturally diffeomorphic to the geometric fiber bundle over \mathcal{M} whose fibers consist of triplets $(g_{ab}, \nabla_a, R_{abc}{}^d)$, where g_{ab} is the value of a Lorentz metric field at a point of \mathcal{M} , and ∇_a and $R_{abc}{}^d$ are respectively the covariant derivative operator and the Riemann tensor associated with g_{ab} at that point, the induction being defined by the natural pull-back.*

It follows immediately that there is a first-order concomitant from \mathcal{B}_g to the geometric bundle $(\mathcal{B}_\nabla, \mathcal{M}, \pi_\nabla, \iota_\nabla)$ of derivative operators, *viz.*, the mapping that takes each Lorentz metric to its associated derivative operator; likewise, there is a second-order concomitant from \mathcal{B}_g to the geometric bundle $(\mathcal{B}_{\text{Riem}}, \mathcal{M}, \pi_{\text{Riem}}, \iota_{\text{Riem}})$ of tensors with the same index structure and symmetries as the

Riemann tensor, *viz.*, the mapping that takes each Lorentz metric to its associated Riemann tensor. This is the precise sense in which the Riemann tensor associated with a given Lorentz metric is “a function of the metric and its partial derivatives up to second order”.

6 No Gravitational Stress-Energy Tensor Exists

We are finally in a position to prove the main result of the paper, the nonexistence of a gravitational stress-energy tensor. In order to formulate and prove a result having that proposition as its natural interpretation, one must first lay down some natural conditions on the proposed object, to show that no such object exists satisfying the conditions. In general relativity, the invariant representation of energetic quantities is always in the form of a stress-energy tensor, *viz.*, a two-index, symmetric, divergence-free tensor. Not just any such tensor will do, however, for that gives only the baldest of formal characterizations of it. From a physical point of view, at a minimum the object must have the dimension of stress-energy for it to count as a stress-energy tensor. That it have the dimension of stress-energy is what allows one to add two of them together in a physically meaningful way to derive the physical sum of total stress-energy from two different sources. In classical mechanics, for instance, both velocity and spatial position have the form of a three-dimensional vector, and so their formal sum is well defined, but it makes no physical sense to add a velocity to a position because the one has dimension of `length/time` and the other the dimension of `length`.

An essential, defining characteristic of energy in classical physics is its obeying some formulation of the First Law of Thermodynamics. The formulation of the First Law I rely on is somewhat unorthodox: that all forms of stress-energy are in principle ultimately fungible—any form of energy can in principle be transformed into any other form—not necessarily that there is some absolute measure of the total energy contained in a system or set of systems that is constant over time. In more precise terms, this means that all forms of energy must be represented by mathematical structures that allow one to define appropriate operations of addition and subtraction among them, which the canonical form of the stress-energy does allow for.²⁰ I prefer this formulation of the First Law in general relativity because there will not be in a general cosmological context any well-defined global energetic quantity that one can try to formulate a conservation principle for.²¹ In so far as one wants to hold on to some principle like the classical First Law in a relativistic context, therefore, I see no other way of doing it besides formulating it in terms of fungibility. (If one likes, one can take the fungibility condition as a necessary criterion for any more traditional conservation law.) This idea is what the demand that *all* stress-energy tensors, no matter the source, have the same physical dimension is intended to capture.²²

²⁰Note that this is a requirement even if one takes a more traditional view of the First Law as making a statement about global conservation of an absolute magnitude measuring a physical quantity.

²¹It is interesting to note that, for similar reasons, there will in general be no available global statement of the relativistic analogue of the Second Law either.

²²For what it’s worth, this conception has strong historical warrant—Einstein (implicitly) used a very similar formulation in one of his first papers laying out and justifying the general theory (Einstein 1916, p. 149):

It must be admitted that this introduction of the energy-tensor of matter is not justified by the relativity

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To sum up, the stress-energy tensor encodes in general relativity all there is to know of ponderable energetic phenomena at a spacetime point:

1. it has 10 independent components representing with respect to a fixed pseudo-orthonormal frame, say, the 6 components of the classical stress-tensor, the 3 components of linear momentum and the scalar energy density of the ponderable field at that point
2. that it has two covariant indices represents the fact that it defines a linear mapping from timelike vectors at the point (“worldline of an observer”) to covectors at that point (“4-momentum covector of the field as measured by that observer”), and so defines a bi-linear mapping from pairs of timelike vectors to a scalar density at that point (“scalar energy density of the field as measured by that observer”), because energetic phenomena, crudely speaking, are marked by the fact that they are quadratic in velocity and momental phenomena by being linear in velocity
3. that it is symmetric represents, “in the limit of the infinitesimal”, the classical principle of the conservation of angular momentum and the special relativistic equivalence of momentum-density flux and scalar energy density
4. that it is covariantly divergence-free represents the fact that, “in the limit of the infinitesimal”, the classical principles of energy and linear momentum conservation are obeyed
5. the localization of ponderable stress-energy is embodied in the fact that the object representing it is a *tensor*, a multi-linear map acting only on the tangent plane of the point it is associated with
6. finally, the thermodynamic fungibility of energetic phenomena is represented by the fact that the set of stress-energy tensors forms a vector space—the sum and difference of any two is itself a possible stress-energy tensor—all having the same physical dimension

Consequently, the appropriate mathematical representation of localized gravitational stress-energy, if there is one, is a two covariant-index, symmetric, covariantly divergence-free tensor having the physical dimension of stress-energy.

In order to make precise the idea of having the physical dimension of stress-energy, recall that in general relativity all the fundamental units one uses to define stress-energy, namely time, length and mass, can themselves be defined using only the unit of time; these are so-called geometrized units. For time, this is trivially true: stipulate, say, that a time-unit is the time it takes a certain kind of atom to vibrate a certain number of times under certain conditions. A unit of length is then defined as that in which light travels *in vacuo* in one time-unit. A unit of mass is defined as that of

postulate alone. For this reason we have here deduced it from the requirement that the energy of the gravitational field shall act gravitatively in the same way as any other kind of energy.

Moller (1962) also stresses the fact that the formulation of integral conservation laws in general relativity based on pseudo-tensorial quantities depends crucially on the assumption that gravitational energy, such as it is, shares as many properties as possible with the energy of ponderable matter.

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which two, placed one length-unit apart, will induce in each other by dint of their mutual gravitation alone an acceleration towards each other of one length-unit per time-unit per time-unit.²³ These definitions of the units of mass and length guarantee that they scale in precisely the same manner as the time-unit when new units of time are chosen by multiplying the time-unit by some fixed real number $\lambda^{-\frac{1}{2}}$. (The reason for the inverse square-root will become clear in a moment). Thus, a duration of t time-units would become $t\lambda^{-\frac{1}{2}}$ of the new units; an interval of d units of length would likewise become $d\lambda^{-\frac{1}{2}}$ in the new units, and m units of mass would become $m\lambda^{-\frac{1}{2}}$ of the new units. This justifies treating all three of these units as “the same”, and so expressing acceleration, say, in inverse time-units. To multiply the length of all timelike vectors representing an interval of time by $\lambda^{-\frac{1}{2}}$, however, is equivalent to multiplying the metric by λ (and so the inverse metric by λ^{-1}), and indeed such a multiplication is the standard way one represents a change of units in general relativity. This makes physical sense as the way to capture the idea of physical dimension: all physical units, the ones composing the dimension of any physical quantity, are geometrized in general relativity, in the most natural formulation, and so depend only on the scale of the metric itself.

Now, the proper dimension of a stress-energy tensor can be determined by the demand that the Einstein field equation, $G_{ab} = \gamma T_{ab}$, where γ is Newton’s gravitational constant, remain satisfied when one rescales the metric by a constant factor. γ has dimension $\frac{(\text{length})^3}{(\text{mass})(\text{time})^2}$, and so in geometrized units does not change under a constant rescaling of the metric. Thus T_{ab} ought to transform exactly as G_{ab} under a constant rescaling of the metric. A simple calculation shows that $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ remains unchanged under such a rescaling. Thus, a necessary condition for a tensor to represent stress-energy is that it remain unchanged under a constant rescaling of the metric. It follows that the concomitant at issue must be homogeneous of degree 0 in the metric.

It is clear that a gravitational stress-energy tensor, if there is such a thing, ought to be a concomitant of the metric, but the order of the required concomitant is not *a priori* obvious. In fact, the way a homogeneous concomitant of the metric transforms when the metric is multiplied by a constant factor suffices to fix the differential order of that concomitant.²⁴ This can be seen as follows, as exemplified by the case of a two covariant-index, homogeneous concomitant S_{ab} of the metric. A simple calculation based on definition 4.2 and on the fact that the concomitant must be homogeneous shows that the value at a point $p \in \mathcal{M}$ of an n^{th} -order concomitant S_{ab} can be written

²³This definition may appear circular, in that it would seem to require a unit of mass in the first place before one could say that bodies were of the *same* mass. I think the circularity can be mitigated by using two bodies for which there are strong prior grounds for positing that they are of equal mass, *e.g.*, two fundamental particles of the same type. It also suffers from a fundamental lack of rigor that the definition of length does not suffer from. In order to make the definition rigorous, one would have to show, *e.g.*, that there exists a solution of the Einstein Field-Equation (approximately) representing two particles in otherwise empty space (as defined by the form of T_{ab})—*viz.*, two timelike geodesics—such that, if on a spacelike hypersurface at which they both intersect 1 unit of length apart (as defined on the hypersurface with respect to either) they accelerate towards each other (as defined by relative acceleration of the geodesics) one unit length per unit time squared, then the product of the masses of the particles is 1. I will just assume, for the purposes of this paper, that such solutions exist.

²⁴I thank Robert Geroch for pointing this out to me.

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in the general form

$$S_{ab} = \sum_{\alpha} k_{\alpha} g^{q_x} \dots g^{x_r} \left(\tilde{\nabla}_x^{(n_1)} g_{q_x} \right) \dots \left(\tilde{\nabla}_x^{(n_i)} g_{x_r} \right) \quad (6.1)$$

where: $\tilde{\nabla}_a$ is any derivative operator at p other than the one naturally associated with g_{ab} ; the x 's are dummy abstract indices; $\tilde{\nabla}_x^{(n_i)}$ stands for n_i iterations of that derivative operator (obviously each with a different abstract index); α takes its values in the set of all permutations of all sets of positive integers $\{n_1, \dots, n_i\}$ that sum to n , so i can range in value from 1 to n ; the exponents of the derivative operators in each summand themselves take their values from α , *i.e.*, they are such that $n_1 + \dots + n_i = n$; there is exactly one summand for which $n_1 = n$ (which makes it an n^{th} -order concomitant); for each α , k_{α} is a constant; and there are just enough of the inverse metrics in each summand to contract all the covariant indices but a and b .

Now, a combinatorial calculation shows the following:

Proposition 6.1 *If, for $n \geq 2$, S_{ab} is an n^{th} -order homogeneous concomitant of g_{ab} , then to rescale the metric by the constant real number λ multiplies S_{ab} by λ^{n-2} .*

In other words, the only such homogeneous n^{th} -order concomitants must be of weight $\lambda - 2$. So if one knew that S_{ab} were multiplied by, say, λ^4 when the metric was rescaled by λ , one would know that it had to be a sixth-order concomitant. In particular, S_{ab} does not rescale when $g_{ab} \rightarrow \lambda g_{ab}$ if and only if it is a second-order concomitant of g_{ab} .²⁵ Thus, such a tensor has the physical dimension of stress-energy if and only if it is a second-order concomitant. In consequence, the only viable candidates for gravitational stress-energy tensors are two covariant-index, symmetric, divergence-free, second-order concomitants of the metric.

We can now state and prove the main result.

Theorem 6.2 *The only two covariant-index, symmetric, divergence-free, second-order concomitants of the metric are constant multiples of the Einstein tensor.*

(Note that homogeneity of the concomitant does not need to be assumed; it was used only to fix the required order of the concomitant at issue.) Before proving the theorem, I remark that it does bear the required natural interpretation, for the Einstein tensor is not an appropriate candidate for the representation of gravitational stress-energy: the Einstein tensor will be zero in a spacetime having vanishing Ricci tensor but non-trivial Weyl tensor (*i.e.*, one that has only non-trivial conformal curvature); such spacetimes, however, can manifest phenomena, *e.g.*, gravitational radiation, that

²⁵Note that the exponent $(n - 2)$ in this result depends crucially on the fact that S_{ab} has only two indices, both covariant. This result generalizes as follows: if $S^{a \dots b \dots}$ is an n^{th} -order concomitant of g_{ab} such that the number of its covariant indices minus the number of its contravariant indices is 2, then, under constant rescaling of the metric by the real number λ , $S^{a \dots b \dots}$ will get multiplied by a factor of λ^{n-2} . Similar propositions, with different exponents resulting for λ , can be proven in the same way for tensor concomitants of the metric of any index structure. A slight variation of the argument, moreover, shows that there does not in general exist in general a homogeneous concomitant of a given order from a tensor of a given index structure to one of another structure—one may not be able to get the number and type of the indices right by contraction and tensor multiplication alone.

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one naturally wants to say possess gravitational energy in some (necessarily non-localized) form or other.²⁶

Now, to prove the theorem, note first that it follows from theorem 5.2 and proposition 4.3 that any candidate gravitational stress-energy tensor must be a zeroth-order concomitant of $\mathcal{B}_{\text{Riem}}$, the geometric bundle of Riemann tensors over spacetime. (One can take this as a precise statement of the fact that any gravitational stress-energy tensor ought to “depend on the curvature”, as I argued in §2.)

One can form from algebraic combinations of the Riemann tensor and the metric 14 independent scalar invariants,²⁷ which manifestly are themselves concomitants of the metric of order at least second. The important point about these scalar invariants for this argument is that the result of taking the variation of any of them with respect to the metric yields a two covariant-index, symmetric, divergence-free tensor that is itself a concomitant of the metric of the same order as the scalar invariant; moreover, all two covariant-index, symmetric, divergence-free concomitants of the metric, of all orders second and higher, can be derived in this fashion.

To make this claim precise, let \mathfrak{S}_g be the space of scalar invariants of $R_{abc}{}^d$ on \mathcal{M} with associated metric g_{ab} , and $g(\lambda)_{ab}$ be a smooth one-parameter family of metrics defined for $\lambda \in (-1, 1)$ such that $g(0)_{ab} = g_{ab}$. I will be interested in the inverse metric g^{ab} and the associated smooth one-parameter family $g(\lambda)^{ab}$; denote $\left. \frac{dg(\lambda)^{ab}}{d\lambda} \right|_{\lambda=0}$ by ‘ δg^{ab} ’. Define the functional $S : \mathfrak{S}_g \rightarrow \mathbb{R}$ by

$$S[\Sigma] = \int_{\mathcal{M}} \Sigma \sqrt{-g} e_{abcd}$$

for $\Sigma \in \mathfrak{S}_g$, where e_{abcd} is a fixed 4-form (not necessarily the one associated with g_{ab}) and $\sqrt{-g}$ is the determinant of g_{ab} in a coordinate system in which the only components of e_{abcd} are 0, 1 or -1 .²⁸ Clearly $\left. \frac{dS}{d\lambda} \right|_{\lambda=0}$ exists for all smooth one-parameter families $g(\lambda)_{ab}$; denote it by ‘ δS ’. Then the *variation of Σ with respect to g^{ab}* is the unique symmetric tensor field Σ_{ab} such that

$$\delta S = \int_{\mathcal{M}} \Sigma_{nm} \delta g^{nm} \sqrt{-g} e_{abcd}$$

for all smooth one-parameter families $g(\lambda)_{ab}$. A straightforward calculation shows that the variation of any scalar curvature invariant has vanishing covariant divergence with respect to the derivative operator associated with the Riemann tensor given metric; furthermore, the variation is a concomitant of the associated metric of the same order as the scalar curvature invariant itself.

The tensor fields resulting from varying all such scalar curvature invariants with respect to the fixed metric form a real vector space over each point of spacetime of dimension 6, the same as that of all two covariant-index, symmetric, tensors divergence-free with respect to the fixed metric. From

²⁶As an historical aside, it is interesting to note that early in the debate on gravitational energy in general relativity Lorentz (1916) and Levi-Civita (1917) proposed that the Einstein tensor be thought of as the gravitational stress-energy tensor. Einstein criticized the proposal on the grounds that this would result in attributing zero total energy to any closed system.

²⁷Cf. Haskins (1902).

²⁸If \mathcal{M} is not compact, or if the integral does not converge for some other reason, then it ought to be taken over some appropriate compact subset of \mathcal{M} , on the boundaries of which δg^{ab} is to be set to zero.

this, in conjunction with the fact that S is a linear functional, it follows that all two covariant-index, symmetric, divergence-free concomitants of any order of the Riemann tensor (and thus of the metric of any order second or higher) can be formed by taking the variation of all scalar curvature invariants of the Riemann tensor with respect to the metric. Thus any candidate gravitational stress-energy tensor must be the variation with respect to the metric of a scalar curvature invariant.

According to the definition of a homogeneous concomitant, a scalar curvature invariant that is, *e.g.*, quadratic in the Riemann tensor, that is to say, quadratic in second derivatives of the metric, is a fourth-order concomitant of the metric. The rule is that the order of the concomitant is the sum of the exponents of the derivative operators when the concomitant is represented in the form of equation (6.1).²⁹ Consequently a scalar curvature invariant is a second-order concomitant of the metric—and so its variation with respect to the metric yields a second-order concomitant of the metric—if and only if it is linear in the Riemann tensor. The only scalar curvature invariants that are linear in the Riemann tensor are constant multiples of the Gaussian scalar curvature itself, R , *viz.*, the result of contracting the two indices of the Ricci tensor. The tensor that results from taking the variation of R with respect to the metric is the Einstein tensor, which proves the theorem.

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²⁹This remark, by the way, obviates the criticism of the claim that gravitational stress-energy depend on the curvature, that this would make gravitational stress-energy depend on second-order partial derivatives of the field potential, whereas all other known forms of stress-energy depend only on terms quadratic in the first partial derivatives of the field potential. It is exactly second-order, homogeneous concomitants that possess terms quadratic in the first partials.

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