When can statistical theories be causally closed?

Balázs Gyenis
Eötvös University, Budapest
BKÁE, Budapest
e-mail: gyepi@hps.elte.hu

Miklós Rédei
Department of History and Philosophy of Science
Eötvös University, Budapest
e-mail: redei@ludens.elte.hu

Abstract

The notion of common cause closedness of a classical, Kolmogorovian probability space with respect to a causal independence relation between the random events is defined, and propositions are presented that characterize common cause closedness for specific probability spaces. It is proved in particular that no probability space with a finite number of random events can contain common causes of all the correlations it predicts; however, it is demonstrated that probability spaces even with a finite number of random events can be common cause closed with respect to a causal independence relation that is stronger than logical independence. Furthermore it is shown that infinite, atomless probability spaces are always common cause closed in the strongest possible sense. Open problems concerning common cause closedness are formulated and the results are interpreted from the perspective of Reichenbach's Common Cause Principle.

1 The problem and informal review of results

Let T be a theory formal part of which contains classical probability theory $(\mathcal{S}, p)$, where $\mathcal{S}$ is a Boolean algebra of sets representing random events (with Boolean operations $\cup, \cap, \perp$) and where p is a probability measure possessing the standard properties. Typically, T predicts correlations between certain elements of $\mathcal{S}$: if $A, B \in \mathcal{S}$, then the quantity

$$Corr(A, B) \doteq p(A \cap B) - p(A)p(B) \tag{1}$$

is called the *correlation* between A and B. Events A and B are said to be positively correlated if $Corr(A, B) > 0$, negatively if $Corr(A, B) < 0$. In what follows we restrict ourselves to positive correlations (see Remark 1 concerning negative correlations).

According to a classical tradition in philosophy of science, articulated especially by H. Reichenbach [16] and more recently by W. Salmon [19], correlations are always results of causal relations. This is the content of what became called Reichenbach's Common Cause Principle (RCCP). The principle asserts that if $Corr(A, B) > 0$ then either the events A, B stand in a direct causal relation responsible for the correlation, or there exists a third event C causally affecting both A and B, and it is this third event, the so-called *(Reichenbachian) common cause*, which brings about the correlation by being related to A, B in a specific way (see Definition 2). So formulated, Reichenbach's Common Cause Principle is a metaphysical claim about the causal structure of the World, its status has been investigated extensively in the literature, especially by Butterfield [2]; Cartwright [3]; Placek [9, 10]; Salmon [17, 18, 19]; Sober [20, 21, 22]; Spohn [23]; Suppes [26]; Uffink [28] and Van Fraassen [29, 30, 31].

Assuming that RCCP is valid, one is led to the question of whether our theories predicting probabilistic correlations can be causally rich enough to contain also the causes of the correlations. The aim of the present paper is to formulate precisely and to investigate this question.

According to RCCP, causal richness of a theory T would manifest in T's being *causally closed* in the sense of being capable to explain the correlations by containing a common cause of every correlation between causally *independent* events A, B. More explicitly, if T is a theory containing probability theory $(\mathcal{S}, p)$ and if $R_{ind}(A, B)$ is a causal independence relation on $\mathcal{S}$, then we call T *common cause closed with respect to* R_{ind}, if for every pair (A, B) of correlated events such that $R_{ind}(A, B)$ holds, there exists a common cause $C \in \mathcal{S}$ of the correlation. Our problem can then be formulated as the question of whether probabilistic theories can be common cause closed with respect to certain causal independence relations.

We shall prove that if the causal independence relation R_{ind} contains *every* pair of correlated events then no theory with a *finite* set $\mathcal{S}$ of random events can be common cause closed with respect to R_{ind} (Proposition 2). This result means that in the case of a finite number of random events, correlations cannot be explained exclusively in terms of common causes. Causal closedness with respect to an R_{ind} that contains *every* pair of correlated events is an unreasonably strong notion, however. For instance, one does not expect a common cause to exist for a correlation between A and B if the correlation is a result of a logical implication relation between A and B, i.e. if $A \subseteq B$ or $B \subseteq A$ holds. Thus R_{ind} should be such that $R_{ind}(A, B)$ implies that A and B belong to two, logically independent Boolean subalgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{S}$ (for the definition of logical independence see Definition 1). We shall show that, with one exception, no $(\mathcal{S}, p)$ with a finite event structure $\mathcal{S}$ can be common cause closed with respect to *every* two, logically independent Boolean subalgebras (see Proposition 3 and Proposition 4 for details). A common cause extendability theorem proved earlier in [8] and recalled here in Proposition 1 entails that any probability space $(\mathcal{S}, p)$ with a *finite* $\mathcal{S}$ and with an arbitrary fixed pair $(\mathcal{L}_1, \mathcal{L}_2)$ of logically independent Boolean subalgebras can be extended in such a way that the extension is common cause closed with respect to $(\mathcal{L}_1, \mathcal{L}_2)$ (Proposition 6).

We also shall demonstrate that for *any* finite $\mathcal{S}$ having more than 4 atoms there exist a probability measure p on $\mathcal{S}$ and two logically independent Boolean subalgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{S}$ such that $(\mathcal{S}, p)$ is common cause closed with respect to $(\mathcal{L}_1, \mathcal{L}_2)$ (Proposition 5).

The problem of common cause closedness of an $(\mathcal{S}, p)$ with respect to a general causal independence relation on an arbitrary, not necessarily finite event structure $\mathcal{S}$ remains an open question. All we can show (Proposition 7) is that atomless probability spaces are always common cause closed with respect to *any* independence relation R_{ind}.

The structure of the paper is the following. Reichenbach's notion of common cause is recalled in Section 2 together with a few other definitions needed in the paper. Section 3 presents the propositions mentioned. The proofs of the propositions are collected in the Appendix. Section 3 also formulates a few open problems.

2 The notion of common cause

Throughout the paper $(\mathcal{S}, p)$ denotes a probability space with Boolean algebra of sets $\mathcal{S}$ and additive map (probability measure) p on $\mathcal{S}$. If $\mathcal{S}$ is a σ-algebra, then p is assumed to be σ-additive. If $\mathcal{S}$ is finite, then it is the power set $\mathcal{P}(X)$ of a set X having n elements denoted by a_i $(i = 1, 2 \ldots n)$, and in this case we write $\mathcal{S}_n$. The atoms in $\mathcal{S}_n$ are the one element sets $\{a_i\}$. The probability space is called *atomless* if for any $A \in \mathcal{S}$, $p(A) \neq 0$ there exists $B \subseteq A$, $B \in \mathcal{S}$ such that $p(B) < p(A)$.

Note that $(\mathcal{S}, p)$ can be atomless with $\mathcal{S}$ having atoms (example: $\mathcal{S}$ = Lebesgue measurable sets of real numbers $\mathbb{R}$ and p = Lebesgue measure on $\mathbb{R}$); atomlessness of $(\mathcal{S}, p)$ and (non)existence of atoms in $\mathcal{S}$ are different concepts.

One can think of the elements of $\mathcal{S}$ in two, equivalent ways: either as representing *random events*, or as representing *propositions* spelling out that the corresponding random event occurs. In the latter case the Boolean operations represent the classical logical connectives; in particular $\subseteq$ represents the implication relation of classical propositional logic. For later purposes we recall the notion of logical independence of Boolean algebras.

Definition 1 Two propositions $A \in \mathcal{S}$ and $B \in \mathcal{S}$ are called *logically independent* if all of the following relations hold

$$A \cap B \neq \emptyset, \quad A^{\perp} \cap B \neq \emptyset, \quad A \cap B^{\perp} \neq \emptyset, \quad A^{\perp} \cap B^{\perp} \neq \emptyset \tag{2}$$

Two Boolean subalgebras $\mathcal{L}_1, \mathcal{L}_2$ of the Boolean algebra $\mathcal{S}$ are called *logically independent* if A and B are logically independent whenever $\emptyset, X \neq A \in \mathcal{L}_1$ and $\emptyset, X \neq B \in \mathcal{L}_2$.

Logical independence of $\mathcal{L}_1$ and $\mathcal{L}_2$ means that *any* two non-selfcontradictory propositions $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$ can be jointly true in some interpretation (namely in the interpretation that makes $(A \cap B)$ true).

Logical independence is hereditary: if $\mathcal{L}'_1$ and $\mathcal{L}'_2$ are Boolean subalgebras of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, then logical independence of $\mathcal{L}_1, \mathcal{L}_2$ entails logical independence of $\mathcal{L}'_1, \mathcal{L}'_2$. The pair $(\mathcal{L}'_1, \mathcal{L}'_2)$ is called a *maximal* logically independent pair, if logical independence of Boolean subalgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ containing respectively $\mathcal{L}'_1$ and $\mathcal{L}'_2$ as Boolean subalgebras implies $\mathcal{L}'_1 = \mathcal{L}_1$ and $\mathcal{L}'_2 = \mathcal{L}_2$. (For a detailed analysis of the notion of logical independence see [14, 15] and Chapter 7 in [12].) For later purposes we also need the following notion: the pair (A, B) is called *logically independent modulo zero probability* if there exist A', B' such that $p(A') = p(B') = 0$ and $(A \setminus A')$ and $(B \setminus B')$ are logically independent.

Recall that a map h from $\mathcal{S}$ into another Boolean algebra $\mathcal{S}'$ is called a *Boolean algebra homomorphism* if it preserves all Boolean operations. A Boolean algebra homomorphism is called *embedding* if $X \neq Y$ implies $h(X) \neq h(Y)$. The probability space $(\mathcal{S}', p')$ is called an *extension* of $(\mathcal{S}, p)$ if there exists an embedding of $\mathcal{S}$ into $\mathcal{S}'$ such that $p'(h(X)) = p(X)$ for every X in $\mathcal{S}$.

Reichenbach's definition of a common cause of a correlation is formulated in terms of classical probability spaces as follows.

Definition 2 *C is a* common cause *of the correlation (1) if the following (independent) conditions hold:*

$$\begin{aligned} p(A \cap B|C) &= p(A|C)p(B|C) && (3)\\ p(A \cap B|C^{\perp}) &= p(A|C^{\perp})p(B|C^{\perp}) && (4)\\ p(A|C) &> p(A|C^{\perp}) && (5)\\ p(B|C) &> p(B|C^{\perp}) && (6) \end{aligned}$$

where $p(X|Y) = p(X \cap Y)/p(Y)$ denotes the conditional probability of X on condition Y, $C^{\perp}$ denotes the complement of C and it is assumed that none of the probabilities $p(X)$, $X = A, B, C, C^{\perp}$ is equal to zero.

The above definition was first given by Reichenbach in [16]. We shall occasionally refer to conditions (3)-(6) as "Reichenbach(ian) conditions". It is customary to express eqs. (3) and (4) by saying that C and $C^{\perp}$ *screen off* the correlation $Corr(A, B) > 0$, and eqs. (3) and (4) are usually called *screening off* conditions. To exclude trivial common causes we call a common cause C *proper* if it differs from both A and B by more than a probability zero event. In what follows "common cause" will always mean a proper common cause.

A correlation will be called *non-degenerate* if A and B differ by more than a probability zero event. It can happen that, in addition to being a probabilistic common cause, the event C logically implies A and B, i.e. $C \subseteq A \cap B$. If this is the case then we call C a *strong* common cause. If C is a common cause such that $C \not\subseteq A$ and $C \not\subseteq B$ modulo measure zero event, then C is called a *genuinely probabilistic* common cause (cf. [13, 12]).

Remark 1: Reichenbach's definition specifies the common cause of a *positive* correlation. Reichenbach himself never gave a definition of negative correlations, which are however as much in need of an explanation as are positive ones. One natural way of defining a common cause of a negative correlation is to utilize the following two facts:

$$Corr(A, B) < 0 \ \text{ implies } \ Corr(A^{\perp}, B) > 0 \ \text{ and } \ Corr(A, B^{\perp}) > 0. \tag{7}$$

$$C \text{ and } C^{\perp} \quad \text{screen off} \quad Corr(A, B) < 0 \tag{8}$$

$$\begin{array}{ccc} & \text{iff} & \\ C \text{ and } C^{\perp} & \text{screen off} & Corr(A^{\perp}, B) > 0 \\ & \text{iff} & \\ C \text{ and } C^{\perp} & \text{screen off} & Corr(A, B^{\perp}) > 0. \end{array}$$

To remain consistent with Definition 2 one can then stipulate that C is a common cause of $Corr(A, B) < 0$ if C is either a common cause of $Corr(A^{\perp}, B) > 0$ or is a common cause of $Corr(A, B^{\perp}) > 0$ (in the sense of Definition 2). The common cause of a negative correlation is thus not unique; note however that Definition 2 does not determine a unique common cause of a positive correlation either, as we will see later. Under this specification of common cause of a negative correlation all the propositions proved in this paper remain valid irrespective of whether correlation means positive or negative correlation, and in all proofs it suffices to consider positive correlations only.

It is easy to see that there exist finite probability spaces that contain a pair of correlated events but do not contain a common cause of that correlation (see [7]). Such probability spaces are called *common cause incomplete* (with respect to that correlation). Existence of common cause incomplete probability spaces are clearly a threat for Reichenbach's Common Cause Principle. But this threat can be met by saying that the two correlated events are in fact causally related and this causal relation is responsible for the correlation. Suppose, however, that a causal independence relation R_{ind} is given on $\mathcal{S}$ and that $\mathcal{S}$ does not contain a common cause C of a correlation between elements A, B that are causally independent. Even this threat for RCCP can be met, however, for one has the following result (for the proof of which see [8]):

Proposition 1 *Any probability* $(\mathcal{S}, p)$ *space which is common cause incomplete with respect to a* finite *set of correlation can be extended in such a manner that the extension* $(\mathcal{S}', p')$ *contain a common cause of each correlations in the finite set.*

Because of conditions (5)–(6) required of the common cause, extending a given $(\mathcal{S}, p)$ by "adding" a common cause to it entails that the extension $(\mathcal{S}', p')$ contains correlations which are not present in the original structure (for instance the correlations between C and A and between C and B). It is therefore a nontrivial matter whether a given common cause incomplete space can be made *common cause closed*, or whether common cause closed spaces exist at all, where "common cause closedness" of a probability space means that the space contains a common cause of *every* correlation in it. Our next definition fixes the notion of common cause closedness with respect to a causal independence relation.

Definition 3 Let $(\mathcal{S}, p)$ be a probability space and R_{ind} be a two-place causal independence relation between elements of $\mathcal{S}$. The $(\mathcal{S}, p)$ is called *common cause closed* with respect to R_{ind}, if for every correlation $Corr(A, B) > 0$ with A and B such that $R_{ind}(A, B)$ holds, there exists a common cause C in $\mathcal{S}$. If there are no elements in $\mathcal{S}$ that are positively correlated, then $(\mathcal{S}, p)$ is called *trivially common cause closed*.

Note that if $\mathcal{S}$ is a Boolean algebra of subsets of a set X and p is concentrated at single point b in X, (i.e. if p is defined by $p(A) = 1$ if $b \in A$ and $p(A) = 0$ if $b \notin A$), then there are no correlations in $(\mathcal{S}, p)$; hence there exist trivially common cause closed probability spaces. We are interested in non-trivial common cause closedness.

3 Propositions on common cause closedness

Proposition 2 *Let* $(\mathcal{S}_n, p)$ *be a finite probability space. If* R_{ind} *contains all the pairs of events* A, B *in* $\mathcal{S}_n$ *that are correlated in* p*, then* $(\mathcal{S}_n, p)$ *is not non-trivially common cause closed with respect to* R_{ind}*.*

This proposition shows that a probability space containing a finite number of random events contains more correlations than it can account for in terms of common causes. But this is not surprising because common cause closedness with respect to a causal independence relation that leaves no room for causal dependence is unreasonably strong. How to strengthen R_{ind} so as to obtain an intuitively more acceptable, weaker notion of common cause closedness with respect to R_{ind}? Intuitively, causal independence of A and B should imply that from the presence or absence of A one should not be able to infer either the occurrence or non-occurrence of B, and conversely: presence or absence of B should not entail occurrence or non-occurrence of A. Taking, as it is common, the partial ordering $\subseteq$ in the Boolean algebra $\mathcal{S}$ as the implication relation between events (equivalently: between propositions that the corresponding events occur), this requirement about R_{ind} can be expressed by the demand that $R_{ind}(A,B)$ should imply all of the following relations

$$\begin{array}{ccccccc} A \not\subseteq B, & A^{\perp} \not\subseteq B & , & A \not\subseteq B^{\perp}, & A^{\perp} \not\subseteq B^{\perp} \\ B \not\subseteq A, & B^{\perp} \not\subseteq A & , & B \not\subseteq A^{\perp}, & B^{\perp} \not\subseteq A^{\perp} \end{array}$$

This requirement can be expressed compactly by saying that $R_{ind}(A,B)$ implies that A and B are logically independent (equivalently, that $\{\emptyset, A, A^{\perp}, X\}$ and $\{\emptyset, B, B^{\perp}, X\}$ are logically independent Boolean subalgebras of $\mathcal{S}$). This motivates the following definition.

Definition 4 $(\mathcal{S}, p)$ is called common cause closed with respect to the pair $(\mathcal{L}_1, \mathcal{L}_2)$ of logically independent Boolean subalgebras of $\mathcal{S}$, if for every $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$ that are correlated in p, there exists a common cause C in $\mathcal{S}$ of the correlation between A and B.

Proposition 3 *Let $(\mathcal{S}_5, p_u)$ be the probability space with the Boolean algebra $\mathcal{S}_5$ generated by 5 atoms and with p_u being the probability measure defined by the uniform distribution on atoms of $\mathcal{S}_5$. Then $(\mathcal{S}_5, p_u)$ is common cause closed with respect to every pair of logically independent Boolean subalgebras $(\mathcal{L}_1, \mathcal{L}_2)$ of $\mathcal{S}_5$.*

Our next proposition shows that the behavior of the probability space $(\mathcal{S}_5, p_u)$ described in Proposition 3 is exceptional.

Proposition 4 *If the probability space $(\mathcal{S}_n, p)$ is not $(\mathcal{S}_5, p_u)$, then it is not non-trivially common cause closed with respect to every pair of logically independent Boolean subalgebras.*

Proposition 5 *For any $n \geq 5$, if $\mathcal{S}_n$ is a finite Boolean algebra generated by n atoms, then there exists a probability measure p on $\mathcal{S}_n$ and there exist two logically independent Boolean subalgebras $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{S}_n$ such that $(\mathcal{S}_n, p)$ is common cause closed with respect to $(\mathcal{L}_1, \mathcal{L}_2)$.*

In view of Propositions 4 and 5 the best one can generally hope for in the case of finite event structures is that *some* are common cause closed with respect to *some* logically independent pairs of Boolean subalgebras. We do not know at this point, how typical or untypical common closedness is in the case of probability spaces with finite event structures: Given an arbitrary probability space $(\mathcal{S}_n, p)$, it is not known how large is the maximal set L of maximal logically independent pairs of Boolean subalgebras of $\mathcal{S}_n$ with respect to which $(\mathcal{S}_n, p)$ is common cause closed. Given a certain "type" of distribution p on $\mathcal{S}_n$ (e.g. the p determined by the uniform probability on atoms, or the Poisson distribution) how does the size of L depend on n? – these are open questions.

Proposition 6 *If $(\mathcal{S}, p)$ with finite $\mathcal{S}$ is not common cause closed with respect to a logically independent pair $(\mathcal{L}_1, \mathcal{L}_2)$, then it can be extended into a $(\mathcal{S}', p')$, with $\mathcal{S}'$ being also finite, in such a manner that $(\mathcal{S}', p')$ is common cause closed with respect to the logically independent pair $(h(\mathcal{L}_1), h(\mathcal{L}_2))$, where $h(\mathcal{L}_i)$ is the homomorph image in $\mathcal{S}'$ of $\mathcal{L}_i$ $(i = 1, 2)$.*

Proposition 6 is a direct consequence of Proposition 1 and the fact that the homomorph image $(h(\mathcal{L}_1), h(\mathcal{L}_2))$ in $\mathcal{S}'$ of the logically independent pair $(\mathcal{L}_1, \mathcal{L}_2)$ is also a logically independent pair of Boolean subalgebras in $\mathcal{S}'$. The logically independent pair $(h(\mathcal{L}_1), h(\mathcal{L}_2))$ will not in general be maximal however, not even if $(\mathcal{L}_1, \mathcal{L}_2)$ is a maximal pair, and it is not known whether extensions always exist that are common cause closed with respect to a *maximal* logically independent pair containing $(h(\mathcal{L}_1), h(\mathcal{L}_2))$. In particular, it is not known whether Proposition 5 remains true if "logically independent" means *maximal* logically independent.

If $\mathcal{S}$ is not finite, and if there are an infinite number of random events in either $\mathcal{L}_1$ or in $\mathcal{L}_2$, where $(\mathcal{L}_1, \mathcal{L}_2)$ is a logically independent pair, then one cannot invoke Proposition 1 to conclude that $(\mathcal{S}, p)$ can be extended into a probability space that is common cause closed with respect to $(h(\mathcal{L}_1), h(\mathcal{L}_2))$, and it is not known if such extensions exist or not. Common cause closedness is not impossible however for probability spaces with infinite $\mathcal{S}$, as the next proposition shows.

Proposition 7 *If $(\mathcal{S}, p)$ is an atomless probability space, then it contains uncountably many proper common causes of every non-degenerate correlation in it. Moreover if A and B are logically independent modulo measure zero event, then $\mathcal{S}$ contains both uncountably many strong and genuinely probabilistic common causes of their correlation.*

4 Concluding comments

The propositions in the preceding section show that demanding common cause closedness of probabilistic theories is not a mathematically impossible requirement, not even if the event structure is finite – provided that the causal independence relation is at least as strong as logical independence; however, the propositions also show that strictly empirical hence necessarily finite theories will not in general be common cause closed. (The strong causal closedness of $(\mathcal{S}_5, p_u)$ is clearly not more than a mathematical coincidence: $(\mathcal{S}_5, p_u)$ just happens to contain precisely the "right amount" of correlations to be common cause closed.) If one considers Reichenbach's Common Cause Principle a valid principle reflecting the true causal structure of our World, then one position one can take in view of the causal incompleteness of finite probabilistic theories is that the search for causes that explain observed correlations is a never ending quest: with every step that enriches the observed world by adding new types of events to it to explain observed correlations new connections and correlations emerge that do not have a causal explanation in the given theory. In this regard the Common Cause Principle serves as a heuristic principle driving research, a research which, in view of Proposition 6 is always locally promising: as long as one considers a finite number of correlations, it is never inconceivable that a large enough (but still finite) theory explains all the correlations in question.

Another position one can take in view of the causal incompleteness of finite probabilistic theories is that the causal independence relation R_{ind} should be strictly stronger than logical independence. The stronger R_{ind}, the weaker the corresponding notion of common cause closedness with respect to R_{ind}, and the more easily a probabilistic theory can be common cause complete. One cannot strengthen R_{ind} in an arbitrary manner, however; especially not, if $(\mathcal{S}, p)$ is part of a larger theory T whose laws and principles must be consistent with R_{ind}. Those laws may even entail a causal independence relation R_{ind} between the random events. This happens in the case of relativistic quantum field theory (QFT), where the event structure $\mathcal{S}$ is given by a lattice of projections $\mathcal{P}(\mathcal{N}(V_1 \cup V_2))$ of the von Neumann algebra $\mathcal{N}(V_1 \cup V_2)$ determined by the observables measurable in the *spacelike separated* spacetime regions V_1, V_2 of the Minkowski spacetime and where a (normal) state ϕ takes the role of p (see [5] for the involved operator algebraic notions and their quantum field theoretic interpretation). (Since $\mathcal{P}(\mathcal{N}(V_1 \cup V_2))$ is not a Boolean algebra, Definition 2 of common cause has to be amended by *requiring* explicitly the common cause to commute with the events in the correlation, see [13] for details.) Since V_1 and V_2 are spacelike (hence causally independent according to the theory of relativity) $R_{ind}^{QFT}(A, B)$ is defined to hold whenever $A \in \mathcal{N}(V_1)$ and $B \in \mathcal{N}(V_2)$. (We remark that $R_{ind}^{QFT}(A, B)$ implies logical independence of A and B, cf. Chapter 11 in [12].) On the other hand, it is a characteristic feature of QFT that there exist projections $A \in \mathcal{N}(V_1)$ and $B \in \mathcal{N}(V_2)$ that are correlated in many states ϕ (see [24, 25, 11] for a review of the relevant results), so the problem of common cause closedness (with respect to the relation R_{ind}^{QFT}) of the non-commutative probability space $(\mathcal{P}(\mathcal{N}(V_1 \cup V_2)), \phi)$ determined by QFT arises naturally. This problem was precisely formulated in [13] (see also Chapter 12 in [12]), and it is still open, only partial results are known (see [11]). These partial results indicate that QFT *might* be common cause closed with respect to R_{ind}^{QFT}.

In contrast to the finite probability spaces, atomless probability spaces are common cause closed in the strongest possible sense: they contain a proper common cause of each and every non-degenerate correlation in them; in fact, as the proof of Proposition 7 shows, they contain uncountably many proper common causes of every non-degenerate correlation. Hence these spaces need not be enlarged to become causally rich. Note that atomless probability spaces are not rare: if $\mathcal{S}$ is the Borel σ-algebra of real numbers, then p given by a density function (with respect to the Lebesgue measure) yields an atomless probability space. Probability spaces of this sort occur frequently in applications. A possible interpretation of existence of a multitude of common causes in atomless probability spaces could be that the Reichenbachian conditions (3)-(6) are just necessary but *not sufficient* conditions for an event to be accepted as a common cause. This interpretation is reinforced by the fact that common causes, if they exist, are generally not unique even in finite probability spaces. If one views the Reichenbachian conditions just necessary, then one is obliged to specify the additional conditions an event must possess to be accepted as a common cause, for otherwise the discussion of causal completeness becomes impossible. Clearly, whatever those additional conditions might be, whether empirical or mathematical, requiring them restricts further the class of theories that can be considered causally closed.

Acknowledgements

Work supported by OTKA, contract numbers: T 015606 and T 032771.

Appendix

First we formulate a couple of conditions that *exclude* an event $C \in \mathcal{S}$ from being a common cause of a correlation. Throughout the Appendix $(\mathcal{S}, p)$ is a probability space, A and B are events, which, if correlated, are assumed to be positively correlated, and recall that a common cause is always assumed to be a *proper* common cause. Also recall that if $\mathcal{S}$ is finite, then it is the power set $\mathcal{P}(X)$ of a set X having n elements and in this case we write $\mathcal{S}_n$. Elements of X will be denoted by letters a_i $(i = 1, 2 \ldots n)$, so the atoms in $\mathcal{S}_n$ are the one element sets $\{a_i\}$.

Lemma 1: Let $(\mathcal{S}, p)$ be arbitrary and assume that A and B are correlated. Then C can not be a proper common cause of the correlation between A and B if any of the following conditions holds.

(i) $C \subseteq (A \cap B)^\perp$

(ii) $A \cap B = C \cap B$ and $A \subseteq C$

(iii) $C = A \cap B$

(iv) $C = A \cup B$

(v) $A \subseteq B \subseteq C$

Proof: Elementary algebraic calculations show that conditions (3) and (4) together with any of (i)-(v) imply either $p(A) = p(C)$ or $p(B) = p(C)$, which contradicts the assumption that C is a proper common cause. We omit the elementary details.

Lemma 2: If $p(A \cup B) = 1$ then A and B are not positively correlated.

Proof: If $p(A \cup B) = 1$, then $p((A \cup B)^\perp) = 0$, so $p(A^\perp \cap B^\perp) = 0$, which implies that $Corr(A^\perp, B^\perp) \leq 0$, but the sign of the correlations of pairs (A, B) and $(A^\perp, B^\perp)$ are the same.

Proof of Proposition 2: The assertion in this proposition is a consequence of Proposition 4 for the following reason. If R_{ind1} and R_{ind2} are two causal independence relations such that R_{ind1} is stronger than R_{ind2}, then if $(\mathcal{S}, p)$ is not common cause closed with respect to R_{ind1}, then $(\mathcal{S}, p)$ is not common cause closed with respect to R_{ind2} either. We nevertheless give an explicit proof of Proposition 2 here because the correlated pairs in $(\mathcal{S}, p)$ that are shown not to possess a common cause in $\mathcal{S}$ is *not* a logically independent pair, and this provides a motivation to make the causal independence relation at least as strong as logical independence.

If $\mathcal{S}$ is finite and is generated by n atoms, then we may assume that $n \geq 3$ because $(\mathcal{S}_1, p)$ is trivially common cause closed and $(\mathcal{S}_2, p)$ is either trivially common cause closed or, if $Corr(\{a_1\},\{a_1\}) > 0$ and/or $Corr(\{a_2\},\{a_2\}) > 0$ (which are the only possible correlations in $(\mathcal{S}_2, p)$), then obviously there is no proper common cause in $\mathcal{S}_2$ of either of these two correlations. If $n \geq 3$ and there are only two atoms in $\mathcal{S}_n$ with non-zero probability, then this is essentially the case of $(\mathcal{S}_2, p)$, so we may assume that there exist three atoms $\{a_1\}, \{a_2\}, \{a_3\}$ in $\mathcal{S}_n$ such that $p(\{a_i\}) \neq 0$ $(i = 1, 2, 3)$. Consider the following two events: $A = \{a_1\}$ and $B = \{a_1, a_2\}$. Then $p(\{a_1\}) + p(\{a_2\}) < 1$ and so

$$p(A \cap B) = p(\{a_1\}) > p(A)p(B) = p(\{a_1\})[p(\{a_1\}) + p(\{a_2\})]$$

which means that A and B are positively correlated. We claim that there exists no common cause C in $\mathcal{S}_n$ of this correlation. Indeed, by (i) in Lemma 1, a common cause C must contain $\{a_1\}$. By (iii) and (iv) in Lemma 1, C cannot be of the form of $C = A \cap B$ and $C = A \cup B$. For any set $M \subseteq \{3, 4, \ldots n\}$ of indices C cannot be of the form $C = \{a_1, a_i \; i \in M\}$ by (ii) in Lemma 1, and C cannot be of the form $C = \{a_1, a_2, a_i \; i \in M\}$ by (v) in Lemma 1. These 4 cases include all possible C's in $\mathcal{S}_n$. So the proposition is proved.

Proof of Proposition 3: We show first (Step I.) that if p is the uniform distribution on $\mathcal{S}_n$ and $(\mathcal{S}_n, p)$ is common cause closed with respect to every logically independent pair (A, B), then $n \leq 5$. Then we will demonstrate (Step II.) that $(\mathcal{S}_5, p)$ is indeed common cause closed with respect to every logically independent pair.

Step I. Assume that p is the uniform distribution on the atoms: $p(a_i) = \frac{1}{n}$ for every i. Let $A = \{a_1, a_2, a_3\}$ and $B = \{a_2, a_3, a_4\}$. Then $Corr(A, B) = \frac{2}{n} - \frac{3}{n}\frac{3}{n}$, which implies that A and B are positively correlated for any $n \geq 5$.

By (i) in Lemma 1, a common cause C of this correlation must contain either a_2 or a_3 or both. By (ii) in Lemma 1, for any set $M \subseteq \{5, 6, \ldots n\}$ of indices C cannot have the form $C = \{a_1, a_2, a_3, a_i \; i \in M\}$. By (iii) in Lemma 1, $C \neq A \cap B$, and by (iv) in Lemma 1, it holds that $C \neq A \cup B$. It follows then that C can only be one of the following forms (where $M \subseteq \{5, 6, \ldots n\}$ is again a set of indices):

$$C = \{a_2, a_i \; i \in M\} \tag{9}$$
$$C = \{a_2, a_3, a_i \; i \in M\} \tag{10}$$
$$C = \{a_1, a_2, a_i \; i \in M\} \tag{11}$$
$$C = \{a_1, a_2, a_4, a_i \; i \in M\} \tag{12}$$
$$C = \{a_1, a_2, a_3, a_4, a_i \; i \in M\} \tag{13}$$

One can verify by explicit calculations that a necessary condition for any of these C's to satisfy the screening off conditions (3) and (4) is that $n \leq 5$. For instance, in the case of a C of the form (11), the screening off condition (3) requires the index set M to be empty, and the screening off condition (4) entails $n = 4$. To verify the remaining cases is left to the reader.

Step II. If $n = 5$, then the event pairs that are positively correlated are of the following form

$$A = \{a_1, a_2\} \quad \text{and} \quad B = \{a_2, a_3\} \tag{14}$$
$$A = \{a_1, a_2, a_3\} \quad \text{and} \quad B = \{a_2, a_3, a_4\} \tag{15}$$

One can verify by explicit calculations that $C = \{a_1, a_2, a_3, a_4\}$ is a common cause of the correlation between A and B of the form (14), and that $C = \{a_2\}$ is a common cause of the correlation between A and B of the form (15).

Proof of Proposition 4: If $(\mathcal{S}_n, p)$ is not $(\mathcal{S}_5, p_u)$, then there are the following three cases to consider:

Case 1. p is the uniform distribution on the atoms of $\mathcal{S}_n$ and $n \geq 2$, $n \neq 5$;
Case 2. p is not the uniform distribution on the atoms of $\mathcal{S}_n$, $n \geq 2$ is arbitrary and none of the atoms has zero probability.
Case 3. (S_n, p) is arbitrary, but some atoms have zero probability.
Case 1.: If p is the measure defined by the uniform distribution on the atoms of $\mathcal{S}_n$, then the argument in Step I. in the proof of Proposition 3 shows that if $n > 5$ then $(\mathcal{S}_n, p)$ is not common cause closed with respect to every logically independent Boolean subalgebras of $\mathcal{S}_n$. For $2 \leq n \leq 4$ the proof of proposition is left to the reader.

Case 2.: Assume that p is not the uniform distribution on the atoms and none of the atoms has zero probability. Order and number the atoms in the decreasing order of the magnitude of their probability: $p(\{a_i\}) \geq p(\{a_j\})$ if $i < j$. Since p is not the uniform distribution on the atoms, we have

$$p(\{a_1\}) > p(\{a_n\}) \tag{16}$$

In consequence of Lemma 2 we can assume that $n \geq 4$, so consider the events $A = \{a_1, a_{n-1}\}$ and $B = \{a_1, a_n\}$. These events are logically independent and we have $p(A \cap B) = p(\{a_1\})$; furthermore, using (16) we can compute

$$\begin{aligned}
p(A)p(B) &= [p(\{a_1\} + p(\{a_{n-1})\})][p(\{a_1\}) + p(\{a_n\})] && (17)\\
&= p(\{a_1\})^2 + p(\{a_1\})p(\{a_{n-1}\}) + p(\{a_1\})p(\{a_n\}) + p(\{a_{n-1}\})p(\{a_n\}) && (18)\\
&< p(\{a_1\})^2 + p(\{a_1\})p(\{a_{n-1}\}) + p(\{a_1\})p(\{a_n\}) + p(\{a_{n-2}\})p(\{a_1\}) && (19)\\
&= p(\{a_1\})[p(\{a_1\}) + p(\{a_{n-2}\}) + p(\{a_{n-1}) + p(\{a_n\})] && (20)
\end{aligned}$$

So A and B are correlated because $[p(\{a_1\}+p(\{a_{n-2}\})+p(\{a_{n-1})+p(\{a_n\})] \leq 1$. A common cause C of this correlation must contain a_1 by (i) in Lemma 1, but it cannot be $\{a_1\}$ by (iii) in Lemma 1. C cannot be $\{a_1, a_n\} = A$ and $\{a_1, a_{n-1}\} = B$ because C must be proper, and C cannot be $\{a_1, a_n, a_{n-1}\}$ by (iv) in Lemma 1. By (ii) in Lemma 1, C cannot be of the form $\{a_1, a_{n-1}, a_i \; i \in M\}$ and $\{a_1, a_n, a_i \; i \in M\}$ with $M \subseteq \{2, 3, \ldots n-2\}$. The remaining possibility is

$$C = \{a_1, a_{n-1}, a_n, a_i \; i \in M\} \quad \emptyset \neq M \subseteq \{2, 3, \ldots n-2\} \tag{21}$$

If the screening off condition (3) for the C of form (21) did hold, then we would have

$$p(\{a_1\})[p(\{a_1\})+p(\{a_{n-1}\})+p(\{a_n\})+\sum_{i\in M} p(\{a_i\})] = [p(\{a_1\})+p(\{a_{n-1}\})][p(\{a_1\})+p(\{a_n\})] \tag{22}$$

which implies

$$p(\{a_1\}) \sum_{i\in M} p(\{a_i\}) = p(\{a_{n-1}\})p(\{a_n\}) \tag{23}$$

Since M is nonempty we have $\sum_{i\in M} p(\{a_i\}) > 0$. Since $p(\{a_i\}) > 0$ for every i, the inequality (16) together with $p(\{a_i\}) \geq p(\{a_j\})$ for $i < j$ entails that the left hand side of (23) is strictly greater than the right hand side, consequently the screening off condition cannot hold for C.
Case 3.: Assume now that the number of atoms in $\mathcal{S}_n$ that have zero probability is equal to k. Re-number the atoms in such a way that $p(\{a_i\}) \neq 0$ if $i = 1, 2, \ldots, n-k$. Let $\mathcal{S}_{n-k}$ be the Boolean algebra generated by the atoms $\{a_i\}$ $(i = 1, 2, \ldots, n-k)$, and let p_{n-k} be defined on $\mathcal{S}_{n-k}$ by

$$p_{n-k}(\cup_{j\in M}\{a_j\}) = p(\cup_{j\in M}\{a_j\}) \tag{24}$$

for every $M \subseteq \{1, 2, \ldots, n-k\}$. Then all correlations of $(\mathcal{S}_{n-k}, p_{n-k})$ appears in $(\mathcal{S}_n, p)$, and it follows that $(\mathcal{S}_n, p)$ is not common cause closed with respect to a certain R_{ind} if $(\mathcal{S}_{n-k}, p_{n-k})$ is not common cause closed with respect to the natural restriction of R_{ind} to

$\mathcal{S}_{n-k}$. Since $(\mathcal{S}_{n-k}, p_{n-k})$ contains no atom with non-zero probability, if it is not equal with $(\mathcal{S}_5, p_u)$, then Case 3 is reduced to Case 1 or to Case 2 and the proof is complete.

Lemma 3: If $A \subset C \subset B$ modulo measure zero sets, then C is a proper common cause of the correlation between A and B.

Proof: Using the fact that $Corr(A, B) > 0$ implies $p(B) < 1$ it is easy to verify that the conditions (5)-(6) hold.

Lemma 4: Assume that none of the logically independent A and B contains the other apart from a measure zero event. Then C is a proper common cause of the correlation between A and B if one of the following two conditions holds.

(i) $C \subseteq (A \cap B)$ and

$$p(C) = \frac{p(A \cap B) - p(A)p(B)}{1 + p(A \cap B) - p(A) - p(B)} \tag{25}$$

(ii) $(A \cup B) \subseteq C$ and

$$p(C) = \frac{p(A)p(B)}{p(A \cap B)} \tag{26}$$

Proof: If $C \subseteq A \cap B$ then the screening off condition (3) holds trivially. Equation (25) is precisely an (algebraically equivalent) expression of the second screening off condition (4) for such a C. If $(A \cup B) \subseteq C$, then the screening off condition (4) holds trivially for C. Equation (26) is precisely an (algebraically equivalent) expression of the first screening off condition (3) for such a C. Since the denominators of (25)-(26) are non-zero by Lemma 2 and by the fact that A and B are positively correlated, it remains to show that $p(C)$ always satisfies $0 < p(C) < p(A \cap B)$ in (25) and $p(A \cap B) < p(C) < 1$ in (26). These conditions are equivalent to $p(A) \neq p(A \cap B)$ and $p(B) \neq p(A \cap B)$, respectively, but these latter ones hold due to the assumption that A and B do not contain each other apart from a measure zero event.

Proof of Proposition 5: If $n = 5$, then, in view of Proposition 3 we can choose any logically independent pair of subalgebras. If $n \geq 6$, then let

$$p(\{a_1\}) = \frac{3}{12}, \quad p(\{a_2\}) = \frac{1}{12}, \quad p(\{a_3\}) = \frac{2}{12}, \quad p(\{a_4\}) = \frac{2}{12}, \quad p(\{a_5\}) = \frac{3}{12} \tag{27}$$

and for $M = \{6, 7, \ldots, n\}$ let $p(\{a_i\}) > 0$, $(i \in M)$ such that $\sum_{i \in M} p(\{a_i\}) = \frac{1}{12}$. Define A and B by

$$A = \{a_1, a_2, a_3\} \quad B = \{a_1, a_2, a_4\} \tag{28}$$

and define the two subalgebras by

$$\mathcal{L}_1 = \{\emptyset, A, A^\perp, X\} \quad \mathcal{L}_2 = \{\emptyset, B, B^\perp, X\} \tag{29}$$

Obviously, $\mathcal{L}_1$ and $\mathcal{L}_2$ are logically independent subalgebras of $\mathcal{S}_n$ with $Corr(A, B) > 0$ and $Corr(A^\perp, B^\perp) > 0$ (no other positive correlations exist between the elements of $\mathcal{L}_1$ and $\mathcal{L}_2$). To complete the proof we just have to show that the two correlations have proper common causes. We claim that $C = \{a_1\}$ and/or $C = \{a_5\}$ are appropriate common causes for the two correlations under consideration. Indeed, by its definition, for the measure p it holds that

$$p(\{a_1\}) + p(\{a_2\}) = p(\{a_5\}) + \sum_{i \in M} p(\{a_i\}) = \frac{4}{12} \tag{30}$$

which implies eq. (25), so C is a common cause by (i) in Lemma 4.

Proof of Proposition 7: Since $(\mathcal{S}, p)$ is an atomless probability space, for every $A \in \mathcal{S}$, $p(A) = \alpha$, $0 < \beta < \alpha$ there exists a $B \in \mathcal{S}$ such that $B \subseteq A$ and $p(B) = \beta$ (see [6][p. 174]).

Call this property of an atomless probability space *denseness*. Let $A, B \in \mathcal{S}$, $p(A) = \alpha$, $p(B) = \beta$ be such that $Corr(A, B) > 0$. Let's suppose first that $A \subset B$ modulo measure zero set. Due to the denseness of the atomless probability spaces, for any γ such that $\alpha < \gamma < \beta$ there exists a $C_\gamma \in \mathcal{S}$ with $A \subset C_\gamma \subset B$ and such that $p(C_\gamma) = \gamma$ and we can invoke Lemma 3 to conclude that this C_γ is a common cause of the correlation $Corr(A, B) > 0$. This argument shows that there exist in fact uncountably many common causes for this type of correlation, since γ can be chosen in uncountably many ways. If $A \not\subseteq B$ and $B \not\subseteq A$ modulo measure zero event, then A and B are logically independent because $Corr(A, B) > 0$, so $A \cap B \neq \emptyset$. Let γ be real number defined by the right hand side of eq. (25). Denseness implies that there exists a $C \subset (A \cap B)$ such that $p(C) = \gamma$, hence by (i) in Lemma 4 this C is a proper common cause, which also is a *strong* common cause. Denseness and (ii) in Lemma 4 entail that there also exists a common cause C such that $(A \cup B) \subset C$, i.e. in the case of atomless probability spaces every correlation has both a *strong* and a *genuinely probabilistic* common cause. To see that in fact there exist uncountably many strong common causes one just has to consider that denseness implies that for every $0 < \delta < min\{p((A \cap B)\backslash C), \gamma\}$ there are $C^1_\delta, C^2_\delta \in \mathcal{S}$, $C^1_\delta \subset C$, $C^2_\delta \subset (A \cap B)\backslash C$ such that $p(C^1_\delta) = p(C^2_\delta) = \delta$, so for $C_\delta \doteq (C\backslash C^1_\delta) \cup C^2_\delta \subset A \cap B$ the equation $p(C_\delta) = \gamma$ needed for C_δ to be a common cause holds. The proof of existence of uncountably many genuinely probabilistic common causes is essentially the same thus omitted.

References

[1] Belnap, N. and Szabó, L.E. [1996]: 'Branching Space-time Analysis of the GHZ Theorem',
Foundations of Physics **26**, pp. 989-1002.

[2] Butterfield, J. [1989]: 'A space-time approach to the Bell inequality', in J. Cushing and E. McMullin (*eds.*), *Philosophical Consequences of Quantum Theory*, Notre Dame: University of Notre Dame Press, pp. 114-144.

[3] Cartwright, N. [1987]: 'How to tell a common cause: Generalization of the conjunctive fork criterion', in J. H. Fetzer (*ed.*), *Probability and Causality*, Boston: Reidel Pub. Co., pp. 181-188.

[4] J.H. Fetzer (ed.) [1988]: *Probability and Causality*, Boston: Reidel Pub. Co.

[5] R. Haag [1992]: *Local Quantum Physics*, Berlin, Heidelberg, New York: Springer Verlag.

[6] P. Halmos [1950]: *Measure Theory*, D. Van Nostrand, New York.

[7] G. Hofer-Szabó, M. Rédei and L.E. Szabó [2000]: 'Reichenbach's Common cause Principle: Recent results and open questions', *Reports on Philosophy* Nr. 20, pp. 85-107.

[8] G. Hofer-Szabó, M. Rédei and L.E. Szabó [1999]: 'On Reichenbach's common cause principle and Reichenbach's notion of common cause', *The British Journal for the Philosophy of Science*, **50**, pp. 377-398.

[9] T. Placek [2000]: 'Stochastic outcomes in branching space-time. An analysis of Bell theorems', *The British Journal for the Philosophy of Science*, **51**, pp. 445-475.

[10] T. Placek [2000]: *Is Nature Deterministic?*, Cracow: Jagellonian University Press.

[11] M. Rédei, S.J. Summers [2002]: 'Local Primitive Causality and he Common Cause Principle in quantum field theory', *Foundations of Physics*, **32**, pp. 335-355.

[12] M. Rédei [1998]: *Quantum Logic in Algebraic Approach*, Dordrecht: Kluwer Academic Publishers.

[13] Rédei, M. [1997]: 'Reichenbach's Common Cause Principle and quantum field theory', *Foundations of Physics*, **27**, pp. 1309-1321.

[14] Rédei, M. [1995a]: 'Logical independence in quantum logic', *Foundations of Physics*, **25**, pp. 411-422.

[15] Rédei, M [1995b]: 'Logically independent von Neumann lattices', *International Journal of Theoretical Physics*, **34**, pp. 1711-1718.

[16] Reichenbach, H. [1956]: *The Direction of Time*, Los Angeles: University of California Press.

[17] Salmon, W.C. [1978]: 'Why ask "Why?"?', in *Proceedings and Addresses of the American Philosophical Association*, **51**, pp. 683-705.

[18] Salmon, W.C. [1980]: 'Probabilistic causality', *Pacific Philosophical Quarterly*, **61**, pp. 50-74.

[19] Salmon, W.C. [1984]: *Scientific Explanation and the Causal Structure of the World*, Princeton: Princeton University Press.

[20] E. Sober [1984]: 'Common cause explanation', *Philosophy of Science*, **51**, pp. 212-241.

[21] E. Sober [1988]: 'The principle of the common cause' in [4] pp. 211-228.

[22] E. Sober [2001]: 'Venetian sea levels, British bread prices, and the principle of common cause', *The British Journal for the Philosophy of Science*, **52**, pp. 331-346.

[23] Spohn, W. [1991]: 'On Reichenbach's Principle of the Common Cause', in W. Salmon and G. Wolters (*eds.*), *Logic, Language and the Structure of Scientific Theories*, Pittsburgh: University of Pittsburgh Press, pp. 211-235.

[24] Summers, S.J. [1990a]: 'Bell's Inequalities and Quantum Field Theory', in *Quantum Probability and Applications V.* Lecture Notes in Mathematics # 1441, Berlin-Heidelberg-New York: Springer, pp. 393-413.

[25] Summers, S.J. [1990b]: 'On the independence of local algebras in quantum field theory', *Reviews in Mathematical Physics*, **2**, pp. 201-247.

[26] Suppes, P. [1970]: *A Probabilistic Theory of Causality*, Amsterdam: North-Holland.

[27] Suppes, P. and Zanotti, M. [1981]: 'When are probabilistic explanations possible?', *Synthese*, **48**, pp. 191-199.

[28] J. Uffink [1999]: 'The principle of the common cause faces the Bernstein Paradox', *Philosophy of Science*, Supplement, **66**, pp. 512-525.

[29] Van Fraassen, B.C. [1977]: 'The pragmatics of explanation', *American Philosophical Quarterly*, **14**, pp. 143-150.

[30] Van Frassen, B.C. [1982]: 'Rational belief and the common cause principle', in R. McLaughlin (*ed.*), *What? Where? When? Why?*, Boston: D. Reidel Pub. Co., pp. 193-209.

[31] Van Fraassen, B.C. [1989]: 'The Charybdis of Realism: Epistemological Implications of Bell's Inequality', in J. Cushing and E. McMullin (*eds.*), *Philosophical Consequences of Quantum Theory*, Notre Dame: University of Notre Dame Press, pp. 97-113.