Mirror Symmetry: What is it for a Relational Space to be Orientable?*

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1 Introduction

I want to take issue with the definition of enantiomorphy that Pooley (2001) gives in his paper. His account goes something like this:

(i) Suppose that the relationist has an account of the dimensionality of space, according to which space is n-dimensional.
(ii) The relations – especially the mulitple relations – between the parts of a body determine whether it is geometrically embeddable in n-dimensional spaces that are either (only) orientable or (only) non-orientable.
(iii) Then "an object is an enantiomorph iff, with respect to every possible abstract [n]-dimensional embedding space, each reflective mapping of the object differs in its outcome from every rigid motion of it." (2001, 8)

This account depends on the truth of (ii). Suppose that a body were embeddable in both orientable and non-orientable spaces of n-dimensions. Then it might fail to be an enantiomorph, not because any of its possible reflections in physical space was identical to a rigid motion of the body, but because in some abstract space a reflection and a rigid motion of its image are identical. Pooley (in footnote 12) makes this point, but claims that the burden of proof falls on the opponent of his account to show that (ii) is false. I wish first – in §2 – to take up this challenge and, if not deliver such a proof, give convincing grounds for throwing the burden of demonstrating (ii) back on Pooley's relationist. I do not however wish to follow Nerlich in arguing that the relationist project founders on these reefs; instead I want to offer a different proposal for how the relationist should understand orientability – and, as I shall explain, topology and geometry more generally.

*The topics of this paper are developed in far greater detail in Huggett (in progress), "Geometry and Topology for Relationists". That paper in turn owes a lot to discussions with Mihai Ganea. This material is based upon work supported by National Science Foundation Grant SES-0004375.

I asked whether (ii) was true, but I might as well have asked for the details of the account of dimension presupposed by (i), since the account depends on that too – indeed, dimension and orientability are both topological properties, and what is really at stake is how such properties are understood by the relationist. Pooley suggests (in footnote 13) that a theory of mechanics might do the job, and indeed – at least in the example he gives – that it might also specify the geometry of space. If this means that the relationist simply takes facts about the topology and geometry of space to be brute, irreducible facts, among the facts about dynamical laws, then I think that it seriously undermines the interest of the relationist project – it seems to be to admit that there are more spatial facts than facts about relations. (It is in fact worth noting that to their detriment, discussions of relationism typically do simply assume that a full Riemannian geometry – typically Euclidean – is given.) If however Pooley is pointing out that a full theory of mechanics requires an account of the geometry of space (and spacetime) and that such a theory is determined by the relations of bodies alone, then I concur. However, in this case the substantivalist will no doubt demand to hear something more concrete about how such a program is work – in the next section we will consider some of the problems facing the program, and then in §3 I will propose a relationist account of space that resolves the difficulties. Interestingly, we can make considerable progress in this direction largely independently of considerations concerning dynamics.

2 The Problems

Giving an account of orientability is the easiest of a series of problems for the relationist: the hardest is to give an account of the geometry of space, which I take to mean explaining how the relations between bodies determine a Riemannian geometry for the space; then easier, since a geometry has a topology but not *vice versa*, is the problem of the topology of a relational space; and then finally, because it is just one topological property, is the issue of (non-)orientability. I want to go through this series showing – or giving reasons to believe – that relations alone do not in general determine the features in question.

It's easy to produce examples to show that geometry doesn't supervene on relations: consider, for example, a system of bodies that is embeddable in a finite region of an n-dimensional Euclidean space. It is also embeddable in a space that Euclidean everywhere except for some finite 'hole' in which a non-Euclidean region is smoothly joined, since the system of bodies can be embedded in such a way that it is entirely away from the hole. This example and its kind are embarrassing for the relationist – the two spaces could lead to physically different situations if, for example, bodies later head towards the hole.

The relationist has no better luck with the 'easier' problem of topology. Consider a sphere with a finite system of bodies scattered over it. If we cut an open hole in this space in such a way that it doesn't cut any geodesics joining two different (non-polar) bodies then we end up with an embedding of the same bodies and relations in a topologically distinct space: one topologically identical to the plane. (If we do cut a geodesic between two bodies then they are no longer multiply related, and so we don't have the same system of relations.)

You can probably see from these examples how the challege to orientability is going to go. Consider again a system of bodies embedded in a finite region of Euclidean space, then cut a hole in a region that they do not occupy and paste a non-orientable space into it – the result is a non-orientable space, with the same bodies with their Euclidean relations embedded in it. Now, it may seem that this very simple example shows that (ii) is false: it's apparently a recipe for finding a system of bodies and their relations that are embeddable into orientable and non-orientable spaces. But not quite: it is after all possible that the operation of pasting a non-orientable region into the hole introduces a new geodesic connecting the bodies and so leads to a system in which the bodies are related in a different way – namely in that two of the bodies have different numbers of distance relations between them. All the same, the hole can be as small, and as flat, and as far away from the system of bodies – and of course still have an effect on them later in their history – as you like; saving (ii) requires that however the non-orientable region is inserted, the result is a space with new geodesics between bodies. It seems to me unlikely that this is so, and I am absolutely certain that the onus is now on Pooley's relationist to prove that it is so.

Assuming that such a proof is not forthcoming, let's consider how the relationist might respond. He might, against my earlier advice, take it that there is some primitive fact, not reducible to the relations between bodies, about the geometry of physical space. And indeed there is a way in which such a position has been advocated – by the tradition of 'modal relationism', which traces back to Leibniz (e.g., 1956, 26, 42). This position takes it that in addition to the occurent facts about the relations between bodies, all the modal facts about what relations are possible in a world are also primitive. Then it is quite reasonable to suppose that the collection of all facts about whether systems of bodies and relations are possible or not is logically equivalent to the statement that a system is possible iff it is embeddable in some specified Riemannian geometry – the primitive geometry of relational space. As I already suggested, taking this step drastically weakens the relationist position (especially his empiricist credentials), but even so it is not enough to solve all his problems.

To see why, consider that for the substantivalist there are not just facts about geometry of space, but also concerning the location of matter in space (if the geometry is inhomogeneous). For instance, suppose that space had the (flat) geometry of $R^2 \times S^1$ – planar in two dimensions but rolled up in the third, with circumference, say, 10m – except for a hole containing a non-Euclidean region – understood appropriately by the substantivalist and modal relationist. Suppose further that the only bodies are a spaceship and a probe that it can send out with a relative velocity of 10m/s for 1s after which it heads back to the ship at 10m/s. And finally suppose that at first the ship is moving at a constant velocity and if the probe is fired at 10s intervals it always takes on non-Euclidean relations to the ship when it is 1m away, but that later after the ship accelerates by 10m/s the same thing happens when the interval is only 5s.[1] For the substantivalist the facts are that the the ship is moving around the rolled up dimension first at 10m/s and then at 5m/s and is firing at just the right moments to shoot the probe into the inhomogeneity every time it goes past. According to her the ship is travelling around the curled dimension firing the probe at just the right times to reach the inhomogeneity – according to him, whether or not a probe will hit the inhomogeneity is determined by (among other things) facts about the position of the ship in space when the probe is fired. Now, consider things from the point of view of the modal relationist.

At any moment while the probe is on the ship, according to the relationist the spatial facts are exactly the same: that the ship and probe stand in such-and-such actual Euclidean relations, and that they could stand in any relations embeddable in the given space. But there are two (and more) very different relative motions that are embeddable in that space: those that actually take place, in which the probe passes through the non-Euclidean region, and those in which it is fired at the 'wrong' time and always misses – in the latter but not the former case the relations are always Euclidean. Hence the facts which the relationist allows at the moment a probe is fired do not suffice to determine what spatial facts will hold later. This situation is of course absurd: imagine the rocket moving constantly and firing the probe 999 times once every 10s, each time leading to non-Euclidean relations. 10s later, at the start of the 1000th run the relational facts are just as they always are when the probe is on board, and so do not determine what will happen to the probe. Yet we would know by induction exactly what would result – non-Euclidean relations. But how could we possibly know something about the future if it were indeterminate? Clearly we can know something about the spatial facts beyond what relations are actual, and which are possible – something like where in the space of possible relations things are.

Modal relationists do indeed introduce such extra facts, though not necessarily for this reason. For instance, one of the best worked-out accounts was developed by Ken Manders (1982). His scheme has in its primitive vocabulary not only predicates for the relative positions of bodies but also to the effect that two bodies are (or are not) in the 'same place', regardless of whether they are in the same relative positions or not. But if the modal relationist adopts this position then he is only a hair's breadth from substantivalism: for every point (up to isometries) that the substantivalist acknowledges the relationist says that there is the possibility of a body being at a different place – effectively that places are 'permanant possibilities of location'. This is of course a position that one could adopt, but it seems to me to be rather far from the original intent of the relationist, and so I reject it.

[1] I'm assuming that Newtonian mechanics holds in the space, but similar examples could be cooked up for any kind of mechanics.

3 The Answer

So modal relationism is a dead end, even though it offered an account of geometry, topology and (non-)orientability. However, I want to propose what I believe to be a novel account of how these things should be understood by the relationist. This is a back-to-roots relationism: the only primitive spatial facts concern the actual relations between bodies, not their possible relations (though it respects the intuition that there are facts about what relations are and aren't possible, and gives an analysis of such facts). We know that the embeddability of actual relations won't pick out the properties of a space that we need, so let's make a fresh start on the problem.

Reconsider the example of a spaceship moving in $R^2 \times S^1$ regularly firing its probe so that the system takes on non-Euclidean relations every time. As we noted, after a number of repetitions the scientists on the ship are in a position to predict what will happen the next time they perform the experiment. Not only that, they could say what would happen if they were to perform the experiment, whether or not they actually do so. On what basis is this possible? On the basis of induction from past regularities in the way relations that evolved to possible or actual future events – namely that they will also fall under the regularities. Leaving the problem of induction to one side, what this observation indicates is that what is real about space for the relationist is not just the relations of bodies at any particular time, but *the way that relations evolve in regular ways over time*.

Thinking about things in this way connects the discussion up to the topic of laws of nature, since they also concern the regular ways in which events occur. The analogy can be filled out in a number of ways, but here I just want to draw on an account of the nature of laws to help develop my relationism.

A fairly literal minded view of laws takes them to be statements of primitive natural necessities: for example, 'necessarily the gravitational force between any two bodies is proportional to their masses and inversely proportional to the square of their separation'. According to this view the fact that the actual force between two bodies is such-and-such holds because it must as a matter of necessity. A problem for this account though arises if – as is commonly supposed – there are incompatible sets of general statements compatible with any body of specific facts (as we saw many geometries compatible with a set of relations): if they are all true descriptions of the facts, how can one decide which is also *necessarily* true?

There is an account of laws that avoids this problem – and is appealing to empiricists (construed broadly), since it avoids any primitive necessities – namely the 'Mill-Ramsey-Lewis' (MRL) account.[2] According to this view, one does not take the laws as primitive, necessitating the regular ways in which events actually unfold, but instead takes the events and regularities as fundamental, determining the laws. Of course we've just said that if logical compatibility were the only constraint then the laws would be underdetermined by the events, so

[2] See Earman, 1984, for a very clear exposition and references.

according to the MRL theory, a law of nature is, by definition, any member of the *strongest and simplest* set of true statements. [3] And that is all a law is, not a statement that is in addition 'necessarily true' in some primitive, independent sense – how could we hope to know that? As Earman (1986, 99) puts it 'A world W is a world of non-modal facts' (and so our modal talk is unpacked as talk about which truths are simplest and strongest).

I believe we should take a similar approach to space. The only spatial reality of a world concerns the relations between bodies and the regular ways in which they evolve. Any talk about the geometry of space does not concern a substantival space, but instead is talk about those relations and regularities – and in particular the geometry of space is, by definition, the *simplest* one in which the entire history of relations is continuously embeddable.[4] That is, we adopt the MRL insight that law-talk is just the most convenient way to describe the regularities, and say that space talk is just the most convenient way to talk about the regular ways in which relations evolve. According to this view the spatial part of a world is made up only of relations between bodies and the regular ways that those relations evolve, not of any further facts about geometry. Then when we make predictions or counterfactual statements about what will or could happen all we are really talking about is the simplest geometry compatible with the history of relations – what we believe about what will or could happen is grounded in our inferences from the part of the history that we have seen to those that will be.

For obvious reasons I will call this the 'regularity account' of relational space, and we can give it some more bite by considering just what 'simplicity' means. I propose three criteria for simplicity in decreasing order of importance, though I suspect that the list is as yet incomplete. First, of all the simplest space is the one with the lowest possible number of dimensions in which the relational history is embeddable.[5] Next, of two spaces of the same dimension, the simpler is the one in which the geometry varies in a regular way – if a relational history is embeddable in a space in which the metric is a periodic function, and a space just like it except in some finite region, then the regular space is simpler (as with laws, we value regularities). Finally, if two spaces are otherwise equally simple, then simpler is the one whose curvature varies in the smoothest way.

One might object that given any list of simplicity there will still be an underdetermination problem – many equally simple spaces into which a relational

[3] That the set is strongest means, roughly, that we value the most logically powerful statements describing the regular ways in which events unfold – thus we find that laws are typically universal generalizations. That the set is the simplest is a constraint in the other direction – the strongest set of true statements is the set that describes all actual events individually, but when it comes to laws we seek statements that summarize the facts as concisely as possible.

[4] Not strongest as well, since embeddability already ensures that all actual relations are captured.

[5] Now, there might be other reasons to suppose that the number of dimensions is greater than the relations require – evidence for a Kaluza-Klein theory perhaps. True, but here we are trying to consider how relations, independently of dynamical considerations (as far as possible), determine the geometry of space. I will consider briefly how to integrate dynamical considerations below.

history is embeddable. But if so then this is a problem for the substantivalist not the relationist, since he now takes space to have a definite but unknowable geometry. The relationist should respond to this situation by saying that there are geometric facts which are indeterminate. Specifically, the geometic facts are anything that is true of *all* the simplest geometries; so if, for instance, the simplest geometries disagree on the volume of some region, then the volume is in fact indeterminate. And don't think that this is fishy because the inhabitants of the world could measure the volume as accurately as they like – that is just to say that the relational history could have been different, in which case of course a different space could have been the simplest in which the history was embeddable.

And that is that: the regular ways in which relations evolve determine a Riemannian space with its geometry, topology and, crucially for the topic of mirror symmetry, (non-)orientability – conversely these properties are nothing more than succinct and powerful ways of expressing those regularities. This account also provides an acceptable resolution to the problem for the modal relationist, since it explicitly involves an embedding of the relational history into the space in question, and hence a fact about the embedding at any time – even though this too is just a fact about the history as a whole.

Except of course that is not quite that. The limitation of this account is that it has focussed on merely embedding relations into space, when we know that in fact we also have to consider what geometric structures for space and time and spacetime are required by dynamics: An affine connection? Anisotropy? An orientation field? An arrow of time? Extra dimensions? Clearly I do not have time to address these complexities here, except to point out that the account I have given is amenable to development in such directions – the correct space is not only the simplest in which the relations are embeddable, but also that required by dynamics, itself understood in relational terms (for instance as defended in Huggett, 1999). What we have seen here is one part of the way in which the geometry of relational space gets determined.

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