STRANGE COUPLINGS AND SPACE-TIME STRUCTURE

STEVEN WEINSTEIN†‡

Northwestern University

General relativity is commonly thought to imply the existence of a unique metric structure for space-time. A simple example is presented of a general relativistic theory with ambiguous metric structure. Brans-Dicke theory is then presented as a further example of a space-time theory in which the metric structure is ambiguous. Other examples of theories with ambiguous metrical structure are mentioned. Finally, it is suggested that several new and interesting philosophical questions arise from the sorts of theories discussed.

In Robert Wald's textbook, *General Relativity,* we find the following (1984, 72–73):

> The entire content of general relativity may be summarized as follows: *Spacetime is a manifold M on which there is defined a Lorentz metric g_{ab}. The curvature of g_{ab} is related to the matter distribution in spacetime by Einstein's equation*
>
> $$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}.$$

This neatly encapsulates the modern view of what is meant by "general relativity." It would seem to suggest that it makes sense, in general relativity, to talk about *the* geometry of space-time. After all, space-time is a manifold equipped with a metric, g_{ab}, which tells us what the shape of space-time *is.* However, depending on how matter couples to gravity, one may encounter situations in which there is more than one geometrical object which could be counted as "the metric." In this paper, I would like to discuss some contexts in which the question, "What is the metric?" may admit more than one reasonable answer.

1. Minimal and Nonminimal Coupling. General relativity admits a Lagrangian formulation, and it is this formulation which is particularly useful for the examples I will be discussing. Briefly, the Lagrangian formulation of a field theory involves the construction of a Lagrangian density (or simply "Lagrangian") L, and from that an action $S = \int L \, d^4x$, and utilizes the principle of least action to find the equations of motion of the fields described in the Lagrangian.[1] For example, the vacuum Einstein equation may be obtained from the Lagrangian

$$L_G = \sqrt{-g}R \tag{1}$$

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‡Department of Philosophy, Northwestern University, 1818 Hinmen Avenue, Evanston, IL 60208-1315.

[1]Fuller discussions of the Lagrangian formulation of general relativity may be found in Appendix E of Wald 1984, Chapter 12 of Weinberg 1972, and Chapter 3 of Hawking & Ellis, 1973.

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(where g is the determinant of g_{ab}, and R is the scalar curvature). Varying the resulting action with respect to the (inverse) metric g^{ab} and setting the result equal to zero (in accord with the principle of least action) yields

$$\frac{\delta S}{\delta g^{ab}} = R_{ab} - \frac{1}{2} R g_{ab} = 0, \tag{2}$$

which is the desired vacuum Einstein equation. The tensor $G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}$ is known as the Einstein tensor. Since G_{ab} is a function of the metric g_{ab}, we will understand g_{ab} to be the metric associated with G_{ab}.

It is straightforward to incorporate matter into the Lagrangian formulation. For example, let the Lagrangian for the matter (be it a scalar field, an electromagnetic field, dust, or what have you) be L_M. Then the total Lagrangian (in units where c = 1) is simply

$$L = (16\pi G)^{-1} L_G + L_M, \tag{3}$$

where G is the gravitational coupling constant, which measures the amount of space-time curvature induced by a given matter distribution. (Note that G, the constant, is unrelated to G_{ab}, the Einstein tensor.) Variation of the associated action with respect to g_{ab} yields

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}, \tag{4}$$

where

$$T_{ab} \equiv \frac{-1}{8\pi G\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}. \tag{5}$$

Here we have recovered the Einstein equation from an action principle. So long as the matter portion L_M of the total Lagrangian L is a scalar, the stress tensor of the system is guaranteed to be locally conserved; i.e., $\nabla^a T_{ab} = 0$ (see Weinberg 1972, 363). If it is locally conserved, one can sensibly regard it as representing the total stress-energy-momentum of the system, and consequently regard the Einstein equation as an equation correlating the spatio-temporal distribution of matter T_{ab} with the curvature of space-time G_{ab}.

A simple instance of (3) is

$$L = \sqrt{-g}[(16\pi G)^{-1} R + g^{ab}\nabla_a\psi\nabla_b\psi + g^{ab}\nabla_a\phi\nabla_b\phi]. \tag{6}$$

This is a theory in which gravity is coupled to two massless scalar fields ψ and ϕ. We can abbreviate this as

$$L = (16\pi G)^{-1} L_G + L_\psi + L_\phi, \tag{7}$$

or equivalently, since multiplication by a constant does not affect the equations obtained by varying the action $S = \int L\, d^4x$, as

$$L = L_G + 16\pi G(L_\psi + L_\phi). \tag{8}$$

Note that the leading term of both $L\psi$ and $L\phi$ is $\sqrt{-g}g^{ab}$ (or $16\pi G\sqrt{-g}g^{ab}$), indicating that the fields ψ and ϕ are "directly coupled" to the metric g_{ab}. The fact that there are no other variables multiplying the metric means that the coupling is "minimal;" minimal coupling Lagrangians are constructed by taking the flat space-time Lagrangian and substituting the curved (Lorentz) metric g_{ab} for the flat (Minkowski) metric.[2] (If the meaning of "minimal coupling" is not yet obvious, it will be once we discuss some non-minimal couplings below!)

In any case, variation of the action $S = \int L\, d^4x$ yields the Einstein equation (4) above, with stress tensor

$$T_{ab} = -\frac{1}{8\pi G\sqrt{-g}}\left(\frac{\delta S_\psi}{\delta g^{ab}} + \frac{\delta S_\phi}{\delta g^{ab}}\right). \tag{9}$$

There is no ambiguity regarding the geometry of space-time; since the metric associated with the l.h.s. of the Einstein equation is g_{ab}, and since the equations of motion (e.g., $g^{ab}\nabla_a\nabla_b\psi = 0$) of the matter on the r.h.s. invoke the same metric, we would likely say that, at least in this case, general relativity associates a unique metric g_{ab} with space-time.

Now consider the Lagrangian

$$L = (16\pi G)^{-1} L_G + L_\phi + e^{2\kappa\phi} L_M, \tag{10}$$

where L_M by itself is a minimally coupled Lagrangian for some matter M of interest. Here, the factor $e^{2\kappa\phi}$ describes a "conformal" coupling of matter to gravity via the scalar field ϕ; the matter is said to be "directly coupled" to the field (in contrast to the "indirect coupling" to be discussed in the next section). "Conformal coupling" means coupling to a term of the form $\Omega^2 g^{ab}$, where Ω is some smooth function. (Ω^{-2} is called a "conformal factor.") As we shall see, this sort of coupling suggests an alternative candidate for "the metric." (Conformal couplings of a similar sort are found in the low-energy effective Lagrangians of various string theories. See for instance Garfinkle et al. 1991 or Horowitz 1990.)

One way of determining the metric would be to look at the equation obtained by varying the action with respect to g^{ab}. Doing this gives an Einstein equation of the standard form (4), where the total stress tensor $T_{ab}(\phi, M)$ is a function of both ϕ and M and satisfies the conservation equation $\nabla^a T_{ab}(\phi, M) = 0$ (where ∇^a is the covariant derivative operator associated with the metric g_{ab}). However, there is an important sense in which one might regard $\tilde{g}_{ab} \equiv e^{2\kappa\phi} g_{ab}$ as an equally good or better candidate for "the metric," since this is the metric to which the matter M is coupled. If M is a pressureless dust, for example, the particles of the dust will follow geodesics of $\tilde{g}_{ab}$, not g_{ab}. In general, we have $\tilde{\nabla}^a T_{ab}(M) = 0$, but not $\nabla^a T_{ab}(M) = 0$, where $T_{ab}(M)$ is the stress tensor resulting from the variation of the matter action $S_M = \int L_M\, d^4x$ with respect to g^{ab}.[3] (See Horowitz 1990 for a discussion of this point.)

Thus there is an ambiguity with regard to the identification of the metric in the theory represented by (10). Though the theory may be put into general relativistic form, the metric associated with the Einstein tensor G_{ab} in the Einstein equation $G_{ab} = 8\pi G T_{ab}$ is not the metric with respect to which the matter M propagates. This speaks against the claim that g_{ab} is the "true" metric.

In short, the conformal coupling in the Lagrangian (10) gives rise to an ambiguity as to which geometrical object, g_{ab} or $\tilde{g}_{ab}$, should properly be called the true metric. Three positions suggest themselves with respect to this example:

1. The true metric is g_{ab}, because that is the metric associated with the general relativistic formulation of the theory, and because the scalar field ϕ is coupled to that metric.

[2] Of course, the Minkowski metric is a special case of a Lorentz metric.

[3] Note, however, that both $\tilde{\nabla}^a T_{ab}(M) = 0$ and $\nabla^a T_{ab}(M) = 0$ hold if the matter field M is conformally invariant. Thus conformally invariant matter does not provide a ready way to distinguish the "true" metric.

2. The true metric is $\tilde{g}_{ab}$, because the matter M is directly coupled to that metric, and the equations of motion for M are thus given by the conservation equation $\tilde{\nabla}^a T_{ab}(M) = 0$. In the case where M is a pressureless dust, the dust particles will traverse geodesics of $\tilde{g}_{ab}$.
3. There is no "true" metric; both g_{ab} and $\tilde{g}_{ab}$ encode relevant physical information.

Rather than attempt to adjudicate between these positions, I will use the space remaining to discuss a theory in which there is a scalar field which is directly coupled to the metric but *in*directly coupled to matter. This is the Brans-Dicke theory of gravitation.

2. Brans-Dicke Theory. The Brans-Dicke theory was developed with an eye toward incorporating various ideas of Berkeley and Mach on the nature of space and the origin of inertia.[4] In particular, the theory is an attempt to realize the idea that gravitational effects are generated by distant matter, rather than simply constrained by matter as in general relativity. In Brans-Dicke theory, the metric is undefined for universes without matter.

Like general relativity, Brans-Dicke theory may be derived from a Lagrangian. The Brans-Dicke Lagrangian is

$$L = \sqrt{-g}(\phi R - \frac{\omega}{\phi} g^{ab}\nabla_a\phi\nabla_b\phi + 16\pi L_M) \tag{11}$$

(Brans & Dicke 1961) where ω is the (dimensionless) Brans-Dicke coupling constant, ϕ is a massless scalar field, and L_M is a normal, minimally coupled (to g_{ab}) Lagrangian for the matter. Here we have "indirect coupling" of the field ϕ to matter; it is called "indirect" because it interacts with matter only indirectly, by shaping the geometry of the space-time in which the matter moves. However, ϕ is *directly* coupled to gravity in the first term (ϕR). Comparing (11) with (7) and (8), one can see that ϕ^{-1} has replaced the gravitational coupling constant G, making the effective gravitational coupling a dynamical variable in the theory.

Varying the action associated with the Brans-Dicke Lagrangian (11) with respect to g^{ab} yields the Brans-Dicke equation

$$R_{ab} - \frac{1}{2} g_{ab}R = 8\pi\theta_{ab} \tag{12}$$

where

$$\theta_{ab} = \frac{8\pi}{\phi} T_{ab} + \frac{\omega}{\phi^2}(\nabla_a\phi\nabla_b\phi - \frac{1}{2} g_{ab}\nabla^c\phi\nabla_c\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$

and T_{ab} is the usual stress tensor resulting from the variation of the matter portion

[4]The original paper of Brans & Dicke (1961) contains a brief but interesting discussion of the philosophical motivations behind their theory, as well as references to some of the relevant papers of Berkeley, Mach, and others. This paper, along with other relevant papers, may be found Dicke 1964. (Thanks to Bill Harper for bringing this collection to my attention.)

It is of some interest to note that Dicke's views on the incompatibility of textbook general relativity and Machian ideas are not universally shared. See Ciufolini & Wheeler 1995, especially chapter 7, for an exposition of the viewpoint that general relativity does indeed realize Mach's views.

of the Lagrangian. (Note that the coefficient of T_{ab} is $8\pi\phi^{-1}$ in place of the typical general relativistic $8\pi G$.) The wave equation for ϕ,

$$\Box\phi \equiv \nabla^a\nabla_a\phi = \frac{8\pi}{2\omega + 3} T^a{}_a,$$

is obtained by varying the action with respect to ϕ, and demonstrates how the matter distribution (specifically, the trace of the stress tensor) serves as a source for ϕ.

Though the l.h.s. of (12) is the Einstein tensor associated with g_{ab}, the r.h.s., θ_{ab}, is not the usual stress tensor, which is to say it is not simply the stress tensor T_{ab} associated with the matter M in the presence of g_{ab}. However, what remains is in some sense an Einstein equation, since $\nabla^a\theta_{ab} = 0$ (from (12) and the contracted Bianchi identity). What we have, then, is arguably a form of general relativity!

Now, Brans-Dicke theory is generally regarded as an *alternative* to general relativity, at least in the form given in (12). There are a couple of reasons for this. One, the various "extra" terms in the tensor θ_{ab} (i.e., the terms other than $8\pi\phi^{-1}T_{ab}$) are regarded as representing an additional degree of freedom of the gravitational field, whereas in general relativity it is typically felt that the gravitational field should be fully described by the metric. Two, θ_{ab} does not satisfy the weak energy condition (Magnano 1995), and many would impose this as an additional requirement on whatever appears on the r.h.s. of an Einstein equation. (See sec. 4.3 of Hawking & Ellis 1973 for further discussion of energy conditions.) Let us just say, then, that it is a matter of some debate as to whether (12) counts as an Einstein equation.

In any case, as Dicke (1962) himself recognized, it *is* possible to rewrite the Brans-Dicke Lagrangian (11) in terms of new variables so as to make it look like a typical general relativistic Lagrangian, and to derive an Einstein equation from the new Lagrangian. This is done by effecting a conformal transformation $\tilde{g}_{ab} = \phi g_{ab}$ and varying the action with respect to the new metric. The form of the new Lagrangian is

$$\tilde{L} = \sqrt{-\tilde{g}}\tilde{R} + 16\pi G(\tilde{L}_\psi + \tilde{L}_M), \tag{13}$$

where $\tilde{L}_\psi$ is the expected Lagrangian for a massless scalar field $\psi = f(\phi)$ and $\tilde{L}_M$ is the matter Lagrangian, which is now a function of both M and ψ, reflecting the fact that in this formulation, the two are directly coupled. This coupling effectively induces a position-dependent change in the rest masses of the matter M. (See Dicke 1964 for discussion of the effects of scalar-matter interaction.)

Variation of the action yields

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{R} = 8\pi G\tilde{\theta}_{ab} \tag{14}$$

where

$$\tilde{\theta}_{ab} = \tilde{T}_{ab} + \tilde{\Lambda}_{ab}.$$

$\tilde{T}_{ab}$ is the usual stress tensor of the matter M, reflecting the mass effect due to the coupling with ψ. $\tilde{\Lambda}_{ab}$ is the stress tensor associated with ψ. The tensor $\tilde{\theta}_{ab}$ is conserved with respect to the connection associated with the new metric and it obeys relevant energy conditions (Magnano and Sokolowski 1994). As a consequence it is plausible to regard this change of variables as effecting a transformation of Brans-Dicke theory into legitimate general relativistic form.

In an important sense, the difference between Brans-Dicke theory in the normal frame (sometimes known as the "Jordan frame") and in the Einstein (general relativistic) frame is simply non-existent, for the two theories are observationally indistinguishable. In another sense, they appear to have different ontologies. Dicke (1962) states that in the Jordan frame, "the gravitational field is described by a metric tensor and a scalar, but . . . the equations of motion of matter in a given field are identical with those of General Relativity, not being explicitly dependent upon the scalar field" (1962, 2163). The indirect dependence of the motion of matter on the scalar arises in the scalar's role as the variable gravitational coupling parameter. (This is also the point of view of Magnano 1995.) In the Einstein frame, the transformed scalar field $\psi = f(\phi)$ essentially appears as an additional *matter* field which couples directly to the matter M, the effect of which is to alter the rest masses of the particles of M. From this perspective, the fact that particles of M do not follow geodesics of the Einstein metric $\tilde{g}_{ab}$ may be seen as resulting from the interaction between M and ϕ, analogous to the deviation from geodesic motion of a charged particle in an electromagnetic field. In short, the Jordan frame formulation of Brans-Dicke theory suggests a metric g_{ab} to which matter is directly coupled, with variable gravitational coupling parameter ϕ^{-1}, whereas the Einstein frame formulation suggests a different metric $\tilde{g}_{ab}$ with matter M directly coupled to a further matter field ψ.

3. Other Theories. The metrical ambiguities discussed in this paper do not typically arise in generic, textbook general relativity, resulting as they do from either non-minimal coupling of matter to geometry or from (apparent) modifications of general relativity itself. However, as noted, they do find their origin in actual, physically motivated models. Motivated by Berkeleyan and Machian ideas, Brans-Dicke theory was developed for use in cosmology, while the sorts of non-minimal couplings discussed in the earlier section have their roots in string theory.

Another recent application of non-standard matter couplings has been to the problem of dark matter in cosmology. Generalizing the Brans-Dicke model, Damour et al. (1990) conjecture that dark matter and visible matter are coupled differently to space-time. (See also Casas et al. 1992 and Garay & García-Bellido 1993.)

The complications involved in coupling fermions to space-time have, in combination with a preference for minimal coupling, given rise to various "metric-affine" theories of gravity, which are not torsion-free. These may well be equivalent to non-affine (torsion-free) theories with non-minimal coupling and a different metric or connection. (See Hehl et al. 1976 for a useful review; see also Anandan & Brown 1995, 355.)

In short, the examples offered here are not simply philosophical thought-experiments, but streamlined versions of actual conjectures as to the proper physical description of space-time and matter for our universe.

4. Philosophical Investigations. There are many interesting philosophical questions raised by the two examples I have discussed. I have taken a somewhat conventionalist attitude toward the issue of distinguishing the metric in the theories I have considered. However, there is surely much room for interesting philosophical debate. Among the questions one could ask are the following:

- Are there reasonable criteria for distinguishing the metric in a space-time theory? If so, under what circumstances (if not all) do these criteria guarantee uniqueness of the metric? If such criteria fail to distinguish a unique metric, is there a fact of the matter nonetheless as to which metric is the true one? Or is this a bad question?

- Is it possible (or desirable) to define "general relativity" in a way which would rule out the existence of theories which admit two metrically inequivalent "general relativistic" formulations?
- Does the sort of ambiguity in metric structure evinced in these examples speak against the idea that space-time is an ontologically distinct entity, that it in some sense exists "absolutely"? (See Earman 1989 and Friedman 1983 for discussion of this idea.)
- Does the ambiguity raise problems for *relationist* views of space-time? This may well be the case, since relationist views typically view space as a relation between objects, and the objects in these theories are arguably as ambiguous as the metric. (Again, see Earman 1989 and Friedman 1983.)
- Is Brans-Dicke theory in the Jordan frame a different theory than in the Einstein frame, or are they just two different interpretations or formulations of the same theory? This question has a close cousin in the question of the relation between the Copenhagen and Bohmian versions of quantum mechanics. (See Cushing et al. 1996 for a collection of recent papers and further references on Bohmian mechanics.)

These are only a few of the areas of investigation suggested by the simple examples discussed in this paper. No doubt there are many shapes future discussion might take!

REFERENCES

Anandan, J. & Brown, H. (1995), "On the reality of space-time geometry and the wavefunction", *Foundations of Physics* 25, 349–360.

Brans, C. & Dicke, R. (1961), "Mach's principle and a relativistic theory of gravitation", *Physical Review* 124, 925–935.

Casas, J., García-Bellido, J. & Quirós, M. (1992), "Scalar-tensor theories of gravity with ϕ-dependent masses", *Classical and Quantum Gravity* 9, 1371–1384.

Ciufolini, I. & Wheeler, J. (1995), *Gravitation and Inertia.* Princeton: Princeton University Press.

Cushing, J., Fine, A. & Goldstein, S. (1996), *Bohmian Mechanics and Quantum Theory: An Appraisal.* Dordrecht: Kluwer.

Damour, T., Gibbons, G. & Gundlach, C. (1990), "Dark matter, time-varying *g*, and a dilaton field", *Physical Review Letters* 64, 123–126.

Dicke, R. (1962), "Mach's principle and invariance under transformation of units", *Physical Review* 125, 2163–2167.

———. (1964), *The Theoretical Significance of Experimental Relativity.* New York: Gordon and Breach.

Earman, J. (1989), *World Enough and Space-Time.* Cambridge, MA: MIT Press.

Friedman, M. (1983), *Foundations of Space-Time Theories.* Princeton: Princeton University Press.

Garay, L. & García-Bellido, J. (1993), "Jordan-Brans-Dicke quantum worm-holes and Coleman's mechanism", *Nuclear Physics* B400, 416–434.

Garfinkle, D., Horowitz, G. & Strominger, A. (1991), "Charged black holes in string theory", *Physical Review* D43, 3140–3143.

Hawking, S. & Ellis, G. (1973), *The Large Scale Structure of Space-Time,* Cambridge: Cambridge University Press.

Hehl, F., von der Heyde, P., Kerlick, G. & Nester, J. (1976), "General relativity with spin and torsion: Foundations and prospects", *Review of Modern Physics* 48, 393–416.

Horowitz, G. (1990), "String theory as a quantum theory of gravity", *in* N. Ashby, D. Bartlett & W. Wyss (eds.), *General Relativity and Gravitation, 1989: Proceedings of the 12th International Conference on General Relativity and Gravitation.* Cambridge: Cambridge University Press, pp. 419–439.

Magnano, G. (1995), "Are there metric theories of gravity other than general relativity?", gr-qc/9511027.

Magnano, G. & Sokolowski, L. (1994), "Physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field", *Physical Review* D50, 5039–5059.

Wald, R. (1984), *General Relativity.* Chicago: University of Chicago Press.

Weinberg, S. (1972), *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: John Wiley and Sons.