Possibilities without possible worlds/histories

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Abstract

The paper puts forward a theory of historical modalities that is framed in terms of possible continuations rather than possible worlds or histories. The proposal is tested as a semantic theory for a language with historical modalities, tenses, and indexicals.

1 Motivations

Possible histories/worlds are philosophically demanding. They are posited to analyze either modal discourse or indeterminism, or both. To qualify for this task they must be in some sense real, but in which sense exactly? A further problem concerns individuals: does an individual as a whole occur in many possible worlds/histories, or does it have parts contained in those worlds/histories, or is it contained in exactly one history? Still further, if our world is indeterministic, the notion of 'actual history' does not seem to be legitimate. A related complaint is that we do not have epistemic access to histories, as these are differentiated by the minutest details, which might be located in a remote future. In contrast, possible continuations of events appear to be innocuous since they are local. After all, our talk of possible continuations seems to be a natural translation of utterances like: "After opening the fridge, I can take out either beer or milk". That is translated as: "There are two possible continuations of my opening the fridge: in the

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first I take out a beer whereas in the second I take out milk".[1] Note that a continuation may involve inconsistent events, as my taking a bottle of milk out of the fridge will be followed by me either drinking it, or not.

There is also a theoretical concern motivating a search for a notion alternative to that of history. In the branching space-times of Belnap (1992) (henceforth BST1992), two possible events belong to one history if and only if they have a common upper bound, where the ordering is defined in terms of light cones. As Belnap explicitly acknowledges, this criterion for being a history will not do for some spacetimes of general relativity. Even simple general relativistic spacetimes (e.g., Schwarzschild's spacetimes) have pairs of events that are not below any event. By applying a BST1992 criterion for history to such a pair, we seem to get a result that the two events do not belong to one history. For this to be possible, histories must split somewhere—this is required by axioms of BST1992. A single spacetime or history does not split from itself, however.[2] The upshot is that BST1992 cannot model a general relativistic spacetime of the mentioned sort.[3] Finally, local objects like continuations and events fit the spirit of branching theories better than structures as large as histories. I take these motivations as sufficiently serious to attempt a refurbishment of BST1992. The paper aims at two goals: to construct a theory of possible branching continuations (BCont) and to show that it can serve as a semantics for a language with indexicals, historical modalities, and tenses. The further task of investigating whether the proposed theory can accommodate the insights of general relativity is left for some later project. The underlying spirit of this paper is a search for small structures for representing indeterminism, shared by Müller (2010) as well.

The essay is organized as follows. The theory of branching continuations is motivated and informally introduced in Section 2.1, while its axioms are presented and discussed in Section 2.2. Section 2.3 investigates how continuations branch and Section 2.4 describes some further developments of the theory. Section 3.1 constructs semantic models for languages with tenses, historical modalities and indexicals. The resulting semantic theory is then tested by some puzzles in Section 3.2.

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[1] It is perhaps a better English to say that a stage, rather than an event, has possible continuations, but this way of talking brings in questions of endurantism, which are not philosophically innocent.

[2] The idea underlying BST1992 is that in "physically interesting" BST models histories are identifiable, or are isomorphic, to spacetimes of physics, e.g., Definition 16 of BST1992 introduces Minkowskian Branching Space-Times as BST models in which every history is Minkowski spacetime.

[3] This issue was clarified to me by T. Müller.

2 The theory of branching continuations (BCont)

2.1 Continuations rather than histories

In this section we will informally introduce the basic notions of the theory of possible branching continuations, leaving the axioms for the next section. We assume here the rudiments of any branching theory: a nonempty set W partially ordered by $\leqslant$, with the usual interpretation. That is, W is the set of possible point events, and $e \leqslant e'$ means that e' lies in a possible future of e.

Our point of reference is BST1992, which is an axiomatic theory combining indeterminism with rudiments of relativity. For a discussion of its axioms we refer the reader to the "Postprint" of Belnap (1992), and here we merely list them in plain language. BST1992 defines histories as maximal upward directed subsets of a base set W. The axioms require that W has no maximal elements, the ordering $\leqslant$ is dense on W, each lower bounded chain in W has an infimum in W, and each upper bounded chain has supremum in each history of which it is a subset. Finally, an axiom called the prior choice principle says that if a lower bounded chain is contained in one history but has no overlap with some other history, then there is a maximal element in the intersection of the two histories that lies below the chain in question.

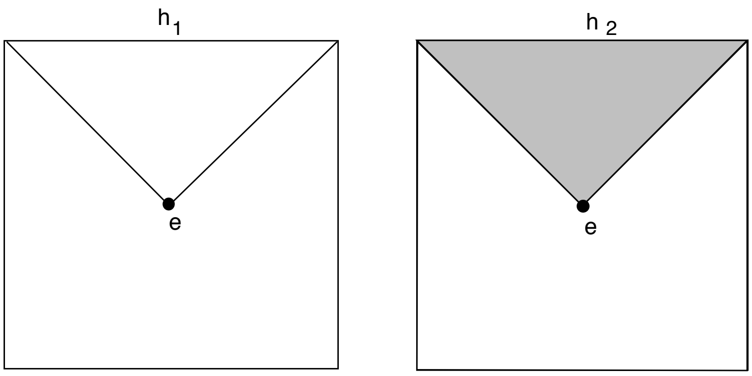

Figure 1: A BST1992 model: two histories h_1 and h_2 with a single choice event e. The shaded area indicates where h_2 diverges from h_1. Note that no events on the rim of future light-cone of e belong to the overlap but e is in the overlap.

A feature of BST992 that we want our theory to preserve is the so called indeterminism without choice—cf. section 9 of Belnap (1992). To assist the explanation, we turn to a particular class of models of BST1992, called Minkowskian Branching Structures (MBS's), in which histories are isomorphic to Minkowski spacetime.[4] Observe that the prior choice principle guarantees that every two histories intersect, and that there is at least one maximal element in their intersection. Such maximal elements are called choice events of the corresponding histories. Another consequence of axioms of BST1992 is that two histories with a single choice event e have separate future light-cones of e, whereas the events that are space-like related to e or are in the past light-cone of e belong to the overlap of the two histories – see Figure 1. Notably, with the exception of e itself, events *on* the future light-cone of e do not belong to the overlap. This pattern brings in a distinction between chanciness and indeterminism without choice. Consider a maximal chain joining an event in the overlap and some event above a choice event. The chain has different topological features, depending on whether or not it contains the choice event. If it does, there is in the chain the "last element" from the overlap, namely e; if it does not, there is not in the chain the "last element" from the overlap, but the first element from the difference of the two histories.

Since diverging futures of a choice event e look like e's continuations, a glimpse at a BST1992 model above might suggest that x and y belong to a same continuation of e provided that there is a V-like link above e, joining x and y. In symbols, $\exists z \in W : \; e < z \wedge z \leqslant x \wedge z \leqslant y$. This is not a good suggestion, however, since events *on* the future light cone of (and above) e do not belong to the overlap of histories that split at e, and accordingly some such events cannot be joined by a V-link above e. They can be joined by some zigzagging line above e, however, which points to a need to generalize V-links to (what we call): snake-links.

Definition 1 (snake-link)
$\langle e_1, e_2, \ldots, e_n \rangle \subseteq W$ $(1 \leqslant n)$ *is a snake-link iff*

$$\forall i : \; 0 < i < n \rightarrow (e_i \leqslant e_{i+1} \vee e_{i+1} \leqslant e_i).$$

A snake-link is above (below) $e \in W$ *if every element of it is strictly above (below)* e.

Note that a single x above e as well as a chain above e constitute snake-links above e.

Definition 2 (snake-linked)
For $x, y \in W$*,* x *and* y *are snake-linked above* e*,* $x \approx_e y$*, iff there is a snake-link* $\langle e_1, e_2, \ldots, e_n \rangle$ *above* e *such that* $x = e_1$ *and* $y = e_n$.

[4]Cf. Müller (2002) and Wroński and Placek (2009).

Analogously one may define the relations of being snake-linked below e, being snake-linked not-above e, being snake-linked not-below e, being snake-linked in a subset of W, etc. For the record, we put down the definition of the last concept:

Definition 3 (snake-linked in a subset of W)
Let $W' \subseteq W$ and $x, y \in W'$. We say that x and y are snake-linked in W' iff there is a snake-link $\langle e_1, e_2, \ldots, e_n \rangle$ such that $x = e_1$ and $y = e_n$ and $e_i \in W'$ for every $0 < i \leqslant n$.

Clearly, being snake-linked above a point event is a special case of being snaked-linked in a subset yet, for expository reasons, we gave both the definitions.

In BST1992, any two events can at worst be connected (ordering-wise) by a path in the shape of an M (cf. Fact 14 of Belnap (1992)), which may raise doubts as to whether we need snake-links in the present approach. First, a model of BCont may fail to extend upward long enough, leaving no room for an M-path above the two events considered. Secondly, in our construction we aim to produce a snake-link joining two events, where typically the snake-link is required to be above some third event e_0; accordingly, although some two events are joined by an M-shaped path, the region above e_0 might be too small to contain the bottom vertex of any such M-shaped path.

Observe now that for any $x, y, z > e$, the following is true about being snake-linked above e:

1. $x \approx_e x$;

2. if $x \approx_e y$, then $y \approx_e x$;

3. if $x \approx_e y$ and $y \approx_e z$, then $x \approx_e z$.

Perhaps the third property requires an argument. Thus, let $x \approx_e y$ and $y \approx_e z$ be true due to the snake-links above e, $\langle e_1, \ldots, e_n \rangle$ and $\langle f_1 \ldots, f_m \rangle$, respectively. Since $e_n = f_1 = y$, the sequence $\langle g_1, \ldots, g_{m+n} \rangle$, such that $g_k = e_k$ if $k \leqslant n$ and $g_k = f_{k-n}$ if $n < k \leqslant m + n$, is a snake-link above e. Also, $g_1 = x$ and $g_{n+m} = z$. Hence $x \approx_e z$.

The above properties mean that $\approx_e$ is an equivalence relation on the set $W_e = \{e' \in W \mid e < e'\}$. Accordingly, $\approx_e$ induces a partition of the set W_e. This has a particular significance for our construction, since we will identify this partition with the set of possible continuations of e.

Definition 4 (continuations)
For $e \in W$ we define Π_e as the partition of $W_e = \{e' \in W \mid e < e'\}$ by the relation $\approx_e$ of being snake-linked above e. The elements of Π_e are called

possible continuations *of* e. *By* $\Pi_e\langle x\rangle$ ($e < x$) *we will mean the unique continuation of* e *to which* x *belongs.*[5]

As a simple consequence of the definition of $\approx_e$, we get that two comparable events located above some third event belong to the same continuation of this event:

Fact 5
$\forall e, e', e_o \in W((e \leqslant e' \vee e' \leqslant e) \wedge e_0 < e \wedge e_0 < e' \rightarrow \exists\, H \in \Pi_{e_0}\, e, e' \in H)$.

Events that have more than one possible continuation are called: choice events.

Definition 6 (set CE of choice events)
For $e \in W$, $e \in CE$ *iff* $card(\Pi_e) > 1$.

We will not impose any limitation on the number of possible continuations of events.

An important concept for any modal framework is that of consistency, which is typically explained in terms of possible worlds/histories. BCont thus poses a challenge of explaining consistency without an appeal to these notions. As a first intuition, one might want to say that two events are consistent in two cases: they are comparable by $\leqslant$ or they are incomparable but do not belong to different continuations of any choice event. This idea will not do, however, unless we exclude cases reminiscent of EPR, or of law-like connections.[6] To illustrate, consider two incomparable choice events, $c_1, c_2 \in CE$, each of which has two continuations, H_1^+, H_1^- and H_2^+, H_2^-, respectively, such that $H_1^+ \cap H_2^+ = \emptyset$ and $H_1^- \cap H_2^- = \emptyset$. Now, an event $e \in H_1^+$ and an event $e' \in H_2^+$ such that $c_1 \not< e'$ and $c_2 \not< e$ belong neither to alternative continuations of c_1 nor to alternative continuations of c_2. But e and e' look like a paragon of inconsistent events, intuitively speaking. A relevant observation is that to snake-link e and e', some element of the snake

[5] Possible continuations of events generalize elementary possibilities open at events of BST1992. An elementary possibility open at event e is some particular subset P of the set $H_{(e)}$ of histories passing through e; since possible continuations of e are naturally thought as occurring above e, it is better to identify continuations with sets of particularly located events than to identify them with sets of histories. In this vein, a continuation C corresponding to elementary possibility P open at e can be defined as $C = \{x \in \bigcup P \mid e < x\}$. Somewhat simpler, in Kowalski and Placek's (1999) framework continuations of e can be identified with the so-called atomic outcomes of e. In both frameworks, however, continuations are definable in terms of histories, and hence will not suffice for the present purpose.

[6] To use BST1992 terminology, these are cases of modal funny business, defined in Belnap (2002), and intensively studied in Müller et al. (2008) and Placek and Wroński (2009).

link will be neither above c_1 nor above c_2. To take care of any pattern of choice events, we say that e and e' are consistent if they are snake-linked in such a way that each element of a snake-link in question is either above all choice events in the past of e or above all choice events in the past of e'. This leads to the following definition:

Definition 7 (consistency)
For $e, e' \in W$, let $W_e := \{x \in W \mid \forall c(c \in CE \wedge c < e \rightarrow c < x)\}$ and $W_{e'} := \{x \in W \mid \forall c(c \in CE \wedge c < e' \rightarrow c < x)\}$.
We say that e and e' are consistent iff they are snake-linked within $W_e \cup W_{e'}$, i.e., they are snake-linked by l such that $l \subseteq W_e \cup W_{e'}$.
$A \subseteq W$ is consistent iff every two elements of A are consistent.
And we say that $A \subseteq W$ is inconsistent iff A is not consistent.

This definition permits that two events are consistent, although they have no common upper bound. Observe also that two comparable events are consistent, and also that singleton $\{e\}$ is a consistent subset of W. Moreover, two events e and e' that belong to different continuations of some third event e_0 are inconsistent since they cannot be snake-linked above e_0. Finally, the present concept of consistency generalizes the BT/BST1992 notion consistency: If e and e' are consistent in the BT/BST1992 sense, they belong to one history, which entails that there is e^* such that $e \leqslant e^*$ and $e' \leqslant e^*$, from which it follows that e and e' are joined by a snake-link with a shape of $\bigwedge$. Each element of this link, i.e., e, e^*, e' is either above all choice events below e or above all choice events below e', so e is consistent with e'.

It remains to be seen that the two events e and e' of our EPR-inspired illustration are indeed inconsistent. For *reductio*, let us suppose that they are consistent. Then e and e' are linked by a snake-link l in $W_e \cup W_{e'}$. Accordingly l must have two consecutive comparable elements, say $t_i, t_{i+1} \in l$ such that e and t_i are snake-linked above c_1 and t_{i+1} and e' are snake-linked above c_2, so (†) $t_{i+1} \in \Pi_{c_2}\langle e' \rangle = H_2^+$. To be specific, let us assume $t_i \leqslant t_{i+1}$. It follows that $c_1 \leqslant t_{i+1}$ and that e and t_{i+1} are snake-linked above c_1. Hence $t_{i+1} \in \Pi_{c_1}\langle e \rangle = H_1^+$. This together with (†) contradicts $H_1^+ \cap H_2^+ = \emptyset$. (The argument for the case $t_{i+1} < t_i$ is analogous.)

We will now define "large events", l-events for short; they are consistent and generalize the notion of point events, i.e., elements of W.

Definition 8 (l-events)
We say that $A \subseteq W$ is an l-event, $A \in$ l-Events, iff $A \neq \emptyset$ and A is consistent.

Clearly every $e \in W$ is an l-event. Since W is assumed to be nonempty, *l-Events* is nonempty.

One might ask at this point what the relation is between l-events and the objects defined in BT/BST1992? There is an affinity between l-events and the initial events of BT/BST1992, the latter being defined as upper bounded subsets of histories. Since an upper bound of initial event I can be used to snake-link every two events $e, e' \in I$ in the region $W_e \cup We'$, every initial event is consistent in the BCont sense, i.e., it is an l-event. The implication in the other direction fails, however, since an l-event might fail to be upper bounded.

The present framework does not appeal to the notion of possible courses of events, and accordingly abstains from identifying histories with maximal upward directed subsets of a base set W. Nevertheless, a model of BCont has maximal upward directed subsets of its base set. Since a model of BCont is a nonempty partially ordered set $\langle W, \leqslant \rangle$, it follows from the Zorn-Kuratowski lemma that every chain in $\langle W, \leqslant \rangle$ can be extended to a maximal chain in W, and every upward directed subset of W can be extended to a maximal upward directed subset of W.[7] Thus each model of the theory of branching continuations (BCont) has maximal chains (ie., BT histories) and maximal upward directed subsets of the base set (i.e., BST1992 histories).

In the introduction, we spelt out our motivations for avoiding the notion of possible courses of events, and its related notion of possible worlds and possible histories. In this spirit, we recommend working with continuations and l-events, and not ascribing any ontological significance to maximal chains or maximal upward directed subsets of base set W. This decline of the role of maximal upward directed subsets is somewhat reminiscent of how the status of maximal chains in a base set changes from BT to BST. In BT it is interpreted as a history, and in BST it has no ontological significance (generally speaking), since in this theory histories are identified with maximal upward directed subsets of W.[8]

2.2 Axioms

We proceeded thus far without paying any attention to axioms of our BCont theory. Our policy is to refurbish any BST1992 axiom that is violated in our construction. Here is a result of this policy.

Definition 9 (model of BCont)
A model of the theory of branching continuations (BCont) is a pair $\mathcal{W} = \langle W, \leqslant \rangle$ that satisfies the following axioms:

[7] For a proof that there are maximal upward directed subsets of W, see Belnap (1992).

[8] We say "generally speaking" since in some BST models, maximal upward directed subsets of W are maximal chains in W.

1. *$\langle W, \leqslant\rangle$ is a nonempty partially ordered set;*

2. *the ordering $\leqslant$ is dense on W;*

3. *W has no maximal elements;*

4. *every lower bounded chain $C \subseteq W$ has an infimum;*

5. *if a chain $C \subseteq W$ is upper bounded and $C \leqslant b$, then there is a unique minimum in $\{e \in W \mid C \leqslant e \wedge e \leqslant b\}$;*[9]

6. *for every $x, y, e \in W$, if $e \nleq x$ and $e \nleq y$, then x and y are snake-linked in the subset $W_{\ngeq e} := \{e' \in W \mid e \nleq e'\}$ of W;*

7. *if $x, y \in W$ and $W_{\leqslant xy} := \{e \in W \mid e \leqslant x \wedge e \leqslant y\} \neq \emptyset$, then $W_{\leqslant xy}$ has a maximal element;*

8. *For every $x, y \in W$, if $\neg\exists c\colon c \in CE \wedge (c < x \vee c < y)$, then x and y are snake-linked in the subset $W_{\ngeq CE} := \{e \in W \mid \forall c \in CE\ c \nleq e\}$ of W.*

Axioms (1)—(4) are exactly like in BST1992, and axiom (5) is reminiscent of a BST1992 axiom of history-relative suprema. Axiom (6) requires that any two events that are not above some third event e, are snake-linked in the region not-above e. This is intended to enforce a distinction between the future and the past of an event. That is, using the relation of being snake-linked in the subset $W_{\ngeq e}$ of W (which is an equivalence relation on $W_{\ngeq e}$), one may define a concept opposite to that of continuations of e, call it *anti*nuations of e. Analogously to continuations, antinuations of e are elements of the partition of $W_{\ngeq e}$ by the relation of being snake-linked not-above e. Against this background, axiom (6) says that for every e, every two events in $W_{\ngeq e}$ are snake-linked in this subset, which means that each event has at most one antinuation. The opposition between (possibly) many continuations and a single antinuation of an event is how the present theory reflects the future–past asymmetry. Note that since the region not-above e includes e, axiom (6) does not prohibit models with minimal elements (we allow for a Big Bang). Axiom (7) is to exclude branching without a choice event; it produces maximal elements of common pasts, which in *appropriate* cases serve as choice events. Note that the partition of possible continuations of an event is defined in a straightforward manner, since being snake-linked is an equivalence relation—cf. Definition 4. This is in contrast to BST1992, which need the axiom of prior choice principle (not assumed here) to ensure

[9] I write $C \leqslant b$ for $\forall x \in C\colon x \leqslant b$. I owe the present form of axiom 5 to T. Müller and N. Belnap.

that a relation used to induce a partition of possibilities open at an event is indeed an equivalence relation. Finally, the role of axiom (8) is similar to the BT/BST1992 axiom that all histories intersect: at the bottom of a model there is a consistent event in the sense of Definition 7. Observe that continuity axioms (4) and (5), and axiom (3) require that the cardinality of a BCont model must be (at least) that of the continuum.

The list of axioms above suggests another way to go, the aim of which is to assure that a resulting theory be a generalization of BST1992 (as we will shortly see, BCont is not a generalization of BST1992).[10] Having a nonempty partially ordered set W and Definition 7 of consistency, we may define *histories* as maximal consistent subsets of W, and assume all axioms of BST1992. In other words, the idea is to change the definition of history, but leave intact the axioms. We do not develop this proposal here, since our main objective is to show how to represent indeterminism without positing possible histories.

It is interesting to learn what are the relations between models of BCont, models of BST1992 in general, and a particular class of BST1992 models, MBS's, in which histories are isomorphic to Minkowski spacetime.

Lemma 10 *Some models of BCont do not satisfy axioms of BST1992, some models of BST1992 do not satisfy axioms of BCont, but every MBS satisfies axioms of BCont.*

Sketch of the proof: For a BCont model that is not a BST1992 model consider a half-plane in $\mathbb{R}^2$ without the upper rim, i.e., $W := \{\langle t, x\rangle \in \mathbb{R}^2 \mid t < 0\}$, with Minkowskian ordering $\leqslant_M$, defined as follows:

$$\langle t_1, x_1\rangle \leqslant_M \langle t_2, x_2\rangle \text{ iff } (x_2 - x_1)^2 \leqslant (t_2 - t_1)^2 \text{ and } t_1 \leqslant t_2, \tag{1}$$

(I assume here that the first coordinate is temporal and the second is spatial and this is stated in units in which the speed of light c equals 1.) In this model, the maximal upward directed subsets of W (i.e., BST histories) do not satisfy the prior choice principle of BST1992.

For a BST1992 model that is not a model of BCont, consider a base set $W := \{\langle t, x\rangle \in \mathbb{R}^2 \,|\, t > 0\} \cup \{\langle 0, 0\rangle\}$ with Minkowskian ordering $\leqslant_M$. This is a one-history BST model. Axiom (6) of BCont fails with respect to $\langle 0, 0\rangle$ and two events that are SLR to, and located, respectively, left and right to $\langle 0, 0\rangle$, for instance $\langle 1/2, -1\rangle$ and $\langle 1/2, 1\rangle$. These events cannot be snaked-linked not-above $\langle 0, 0\rangle$.[11]

[10] This alternative approach was suggested to me by T. Müller.

[11] One might think (incorrectly) that the problem stems from there being a minimal element in this model. But axiom (6) is violated as well in the following model without

Finally, it is relatively easy to check that MBS's satisfy axioms of BCont. Since this task requires one to introduce a full machinery of MBS's, we leave it for a reader (for a construction of MBS's, cf. Wroński and Placek (2009)). □

2.3 Patterns of branching

BST1992 has a specific pattern of branching: every two histories intersect, their intersection has at least one maximal element, called choice point, and, if c is a choice point for two histories in a BST1992 model that permits a notion of light-cones, then the events *on* the future light cone of c (and different from c) do not belong to the intersection of the two histories.

We will argue informally that axioms of BCont enforce a similar pattern of branching. More precisely, we will see that patterns of branching excluded by BST1992 are forbidden by axioms of BCont as well.

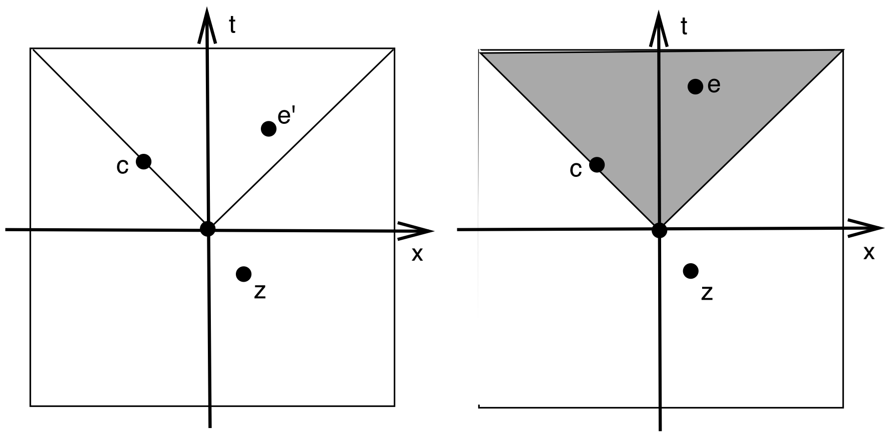

Figure 2: This diagram represents three structures: (1) the vertex $\langle 0,0\rangle$ and events on its future light-cone do not belong to the overlap, (2) as in (1), with the exception of c that belongs to the overlap, and (3) as in (1), with the exception of vertex $\langle 0,0\rangle$.

We will consider three structures produced by pasting together some regions of two copies of $\mathbb{R}^2$, each equipped with the Minkowskian ordering $\leqslant_M$, as defined by Equation 1 above.

minima: $W = W_1 \cup W_2$, where $W_1 := \{x \in \mathbb{R} \,|\, x \leqslant 0\}$ and $W_2 := \{\langle t,x\rangle \in \mathbb{R}^2 \mid 0 < t\}$. The BST ordering $\leqslant\cdot$ coincides with the natural ordering of reals on W_1, and with Minkowskian ordering on W_2, and for the "mixed" case it is this: for $y \in W_1$ and $\langle t,x\rangle \in W_2$: $\langle t,x\rangle \nleqslant \cdot\, y$ and $y \leqslant\cdot \langle t,x\rangle$ iff $\langle 0,0\rangle \leqslant_M \langle t,x\rangle$.

Let the base set W of our first structure be the union of these three sets:

$$\begin{aligned} O_0 &= \{\langle t, x\rangle \in \mathbb{R}^2 \mid \langle 0,0\rangle \not\leqslant_M \langle t, x\rangle\}, \\ O_1 &= \{\langle t, x, 1\rangle \mid \langle t, x\rangle \in \mathbb{R}^2 \wedge \langle 0,0\rangle \leqslant_M \langle t, x\rangle\}, \\ O_2 &= \{\langle t, x, 2\rangle \mid \langle t, x\rangle \in \mathbb{R}^2 \wedge \langle 0,0\rangle \leqslant_M \langle t, x\rangle\}. \end{aligned} \tag{2}$$

The ordering $\leq$ is defined in terms of the Minkowskian ordering $\leqslant_M$:

1. For $\langle t_1, x_1\rangle, \langle t_2, x_2\rangle \in O_0$: $\langle t_1, x_1\rangle \leq \langle t_2, x_2\rangle$ iff $\langle t_1, x_1\rangle \leqslant_M \langle t_2, x_2\rangle$;

2. For $\langle t_1, x_1, i\rangle, \langle t_2, x_2, i\rangle \in O_i$ (where i =1, 2): $\langle t_1, x_1, i\rangle \leq \langle t_2, x_2, i\rangle$ iff $\langle t_1, x_1\rangle \leqslant_M \langle t_2, x_2\rangle$;

3. Every $\langle t_1, x_1, 1\rangle \in O_1$ and every $\langle t_2, x_2, 2\rangle \in O_2$ are incomparable with respect to $\leq$;

4. For $\langle t_1, x_1\rangle \in O_0$ and $\langle t_2, x_2, i\rangle \in O_i$ $(i = 1, 2)$: (1) $\langle t_2, x_2, i\rangle \not\leq \langle t_1, x_1\rangle$ and (2) $\langle t_1, x_1\rangle \leq \langle t_2, x_2, i\rangle$ iff $\langle t_1, x_1\rangle \leqslant_M \langle t_2, x_2\rangle$.

We thus obtained two separate triangles O_1 and O_2, each with "its" separate zero point $\langle 0, 0, i\rangle$ $(i = 1, 2)$ at the vertex—see Figure 2. O_1 and O_2 look like possible continuations, but what are these continuations of? Our structure has no choice event since (strictly) above every event z in O_0 there is another event in O_0 that can be used to snake-link above z every event from O_1 with every event from O_2. Accordingly, each $z \in O_0$ has one possible continuation and hence is not a choice event. Note however that any two events, each from a different triangle, are not snake-linked within the union of the triangles; this fact, however, does not induce a choice event.

Our structure has two maximal upward directed subsets of W, namely $O_0 \cup O_1$ and $O_0 \cup O_2$, but their intersection, O_0 has no maximal element, which means that it is not a BST1992 model. Neither is the structure a model of BCont, as it violates axiom 7. Take two events, $e \in O_1$ and $e' \in O_2$. The common past of e and e', i.e., the set $W_{\leqslant ee'} \subseteq O_0$, is nonempty, but there is no maximal element in$W_{\leqslant ee'}$. A contradiction with axiom 7.

To improve on the first construction, let us consider a seemingly better structure in which some single event from a triangle's side, say $c = \langle 1, -1\rangle$, is assumed to belong to the joint region. That is, we redefine the subsets of base set W': $O'_0 = O_0 \cup \{\langle 1, -1\rangle\}$, $O'_i = O_i/\{\langle 1, -1, i\rangle\}$, where $i = 1, 2$. Clearly, c is a maximal element of O'_0, and O'_0 is the intersection of two maximal upward directed subsets of W'. Since c looks like a BST choice point, it is natural to expect that it has more than one continuation, i.e., there are events above c that cannot be snake-linked *above* c. And this is indeed so: $\{x \in O'_1 \mid c < x\}$ and $\{x \in O'_2 \mid c < x\}$ are two continuations of c, in the sense of Definition 4. Nevertheless the oddity that O'_1 and O'_2 are not continuations is still there. This oddity need not trouble us, however, since the structure is neither a BCont model nor a BST1992 model. It does not satisfy axiom 7 of BCont, for the reasons already explained. And it fails to satisfy the prior choice principle of BST1992.

To turn finally to a "good" structure, let its base set W'' be the union of $O''_0 = O_0 \cup \{\langle 0,0\rangle\}$, $O''_1 = O_1/\{\langle 0,0,1\rangle\}$ and $O''_2 = O_2/\{\langle 0,0,2\rangle\}$. $\langle 0,0\rangle$ is a maximal element in O''_0, which is the intersection of two maximal upward directed subsets of W'': $O''_0 \cup O''_1$ and $O''_0 \cup O''_2$. Thus $\langle 0,0\rangle$ serves as a witness for the prior choice principle and it is easy to see that $\langle W'', \leq\rangle$ satisfies the other axioms of BST1992. From a perspective of BCont, $\langle 0,0\rangle$ is a choice event with two possible continuations, O''_1 and O''_2. It is straightforward to see that this structure satisfies the axioms of BCont.

The point of these constructions is that if one tries to implement branching on structures resulting from pasting copies of Minkowski spacetime, the axioms of the present theory will enforce a pattern of branching assumed in BST1992.

Another specific feature of BT/BST1992 branching is that histories do not branch backwards. We finish this section with an argument that the BCont axioms entail an analogue of "no backward branching". The "no backward branching" condition has a clear sense in the presence of histories: for all events x, y, z, if $y \leqslant x$ and $z \leqslant x$, then there is a history h to which both y and z belong. In BST1992 this immediately follows from the definition of history as a maximal upward directed subset of a base set. Since in BCont we have continuations and l-events rather than histories, backward branching does not have such a clear-cut meaning there. It may mean one of these two conditions:

1. for some x, y, z such that $y < x$ and $z < x$, y and z belong to separate continuations of some event e;

2. some y, z, such that neither is above any choice event, are not snake-linked in region $W_{\ngeq CE} := \{e \in W \mid \forall c \in CE \; c \nless e\}$.

Note that the second case is explicitly forbidden by axiom (8). But the first case cannot occur, either. For *reductio* suppose x, y, z are like in the premise, that is, for some e: $y \in H_1$ and $z \in H_2$, where $H_1, H_2 \in \Pi_e$ and $H_1 \neq H_2$. It follows that there is no snake-link above e joining y and z. But obviously there is a snake-link of this sort as $y < x$, $z < x$, and each x, y, z is above e. Contradiction. Hence, there is no backward branching in the BCont theory.

2.4 Further developments

In this subsection we will sketch how the BCont theory can handle three concepts that have turned out to be particularly fruitful in the applications of BST1992: basic transitions, the relation of being space-like related (SLR), and spatiotemporal locations, s-t locations for short.

Basic transitions. In BST1992 and BT, a basic transition is a pair consisting of an event e and a possibility H open at the event e, where the latter notion is defined as a set of histories containing e and not divided at e. Basic transitions are the building blocks of Belnap's (2005b) theory of causation, are indispensable for causal probability spaces of Weiner and Belnap (2006) and Müller (2005), and play a crucial role in the *stit* account of agency– cf. Belnap (2005a). A BCont analogue of basic transition is immediate to recognize:

Definition 11 (basic transitions in BCont)
Let $\langle W, \leqslant \rangle$ be a model of BCont. A basic transition is a pair $\langle e, H \rangle$, where $e \in W$ and $H \in \Pi_e$ is a continuation of e.

SLR. The BST1992 relation of being space-like related (SLR) generalizes the relation of space-like separation of special relativity to the modal context. Two BST events are said to be SLR if they are incomparable by the BST ordering, but are consistent (in the sense of belonging to one history). BCont gives the same definition, yet, with consistency newly explained by Definition 7 above.

Definition 12 (SLR)
$e, e' \in W$ are SLR iff they are consistent but incomparable.

Spatiotemporal locations and Instants. To link events of branching theories to events of spacetime theories of physics, one needs to be able to assign coordinates to the former events. A first step in this direction is the BT concept of Instants and BST1992 concept of spatiotemporal locations (s-t locations, aka space-time points).[12] The idea underlying Instants is that some events belonging to alternative histories occur at the same instant. Analogously, in BST1992 one envisages events from different histories, yet occurring at the same spatiotemporal location. Clearly, the concepts are defined in BT and BST1992 in terms of histories. Can they be defined without recourse to histories? Here is our proposal:

[12]For the former, cf. Belnap et al. (2001), and for the latter cf. Müller (2005).

Definition 13 (s-t locations)
We say that a model $\langle W, \leqslant\rangle$ of BCont has spatiotemporal locations (s-t locations) iff there is a partition S of W such that

1. *For each l-event A and each $s \in S$, the intersection $A \cap s$ contains at most one element;*

2. *S respects the ordering, that is, for all l-events A, B, and all $s_1, s_2 \in S$, if all the intersections $A \cap s_1, A \cap s_2, B \cap s_1$, and $B \cap s_2$ are nonempty, and $A \cap s_1 = A \cap s_2$, then $B \cap s_1 = B \cap s_2$,*

3. *and analogously for the strict ordering $<$; and*

4. *if $e_1 \leqslant e_2 \leqslant e_3$, then for every l-event A such that $s(e_1) \cap A \neq \emptyset$ and $s(e_3) \cap A \neq \emptyset$, there is an l-event A' such that $A \subseteq A'$ and $s(e_2) \cap A' \neq \emptyset$, where $s(e_i)$ stands for a (unique) $s \in S$ such that $e_i \in s$, and*

5. *if L is a chain of choice events in $\langle W, \leqslant\rangle$ upper bounded by e_0 and such that $\exists s \in S\ \forall x \in L\ \exists e \in W:\ (x < e \wedge s(e) = s)$, then $\exists e^*(e^* \in \bigcap_{x \in L} \Pi_x\langle e_0\rangle \wedge s(e^*) = s)$.*

We call S a set of s-t locations for $\langle W, \leqslant\rangle$ and we will call the triple $\langle W, \leqslant, S\rangle$ a BCont models with set S of s-t locations.

Clause 1 is a weakening of the BT/BST1992 condition that each history and each instant/s-t location intersect at a single event. Clause 2 and 3 weaken the conservativeness conditions assumed in these theories. Clause 4 allows one to extend an event, in rather specific circumstances, by an event at a given s-t location. Finally, clause 5 permits one to transfer a structure of s-t locations from one continuation to another continuation.

Let us next introduce a relation $\preccurlyeq$ on S, a candidate for a partial ordering on S:

Definition 14 (ordering of s-t locations)
For $s_1, s_2 \in S$, let $s_1 \preccurlyeq s_2$ iff $\exists e_1, e_2\ (e_1 \in s_1 \wedge e_2 \in s_2 \wedge e_1 \leqslant e_2)$.

In MBS's, which is a physically motivated class of BST models, the set of s-t locations is partially ordered, and the ordering is similar to the Minkowski ordering. It is thus satisfying to see that $\preccurlyeq$ is a partial ordering, provided a natural condition holds true:

Fact 15
If a BCont model $\langle W, \leqslant, S\rangle$ with set S of s-t locations is downward directed, i.e., $\forall e_1, e_2 \in W\ \ \exists e_0 \in W\ \ (e_0 \leqslant e_1 \wedge e_0 \leqslant e_2)$, then $\preccurlyeq$ is a partial dense ordering on S.

Proof: Reflexivity holds since by the definition of partition, each $s \in S$ is nonempty, and hence $\exists e \in s\ (e \leqslant e)$ is true, which entails $s \preccurlyeq s$. To prove antisymmetry, let $e_1 \in s_1$, $e_2 \in s_2$ be a witness for $s_1 \preccurlyeq s_2$ and $e' \in s_1$, $e'' \in s_2$ be a witness for $s_2 \preccurlyeq s_1$. This means in particular that (†) $e_1 \leqslant e_2$ but ($\star$) $e'' \leqslant e'$. Clearly, $A := \{e_1, e_2\}$ and $B := \{e', e''\}$ are l-events. By clause 3 of Definition 13, we get: $e_1 < e_2$ iff $e' < e''$. Thus, $e_1 < e_2$ leads to a contradiction with ($\star$), so by (†): $e_1 = e_2$. By the definition of partition, $s_1 = s_2$. Turning to transitivity, consider $e_1 \in s_1$, $e_2 \in s_2$ that witness $s_1 \preccurlyeq s_2$ and $e'_2 \in s_2$, $e'_3 \in s_3$ that witness $s_2 \preccurlyeq s_3$. If $\{e_1, e_2, e'_2, e'_3\}$ is consistent, then by clause 1 of Definition 13: $e_2 = e'_2$, and by transitivity of $\leqslant$: $e_1 \leqslant e'_3$, and hence $s_1 \preccurlyeq s_3$. If $\{e_1, e_2, e'_2, e'_3\}$ is inconsistent, then (since $\langle W, \leqslant\rangle$ is assumed to be downward directed), there is e_0 such that $e_0 \leqslant e_1$ and $e_0 \leqslant e'_2$. Now observe that since $e_o \leqslant e_1 \leqslant e_2$ and $s(e_0) \cap \{e_0, e'_2, e'_3\} \neq \emptyset$ and $s(e_2) \cap \{e_0, e'_2, e'_3\} \neq \emptyset$, by clause 4 of Definition 13 we get: there is an l-event A' and $e^* \in W$ such that $e^* \in s(e_1) \cap A'$ and $\{e_0, e'_2, e'_3\} \subseteq A'$. By clauses 2 and 3 of Definition 13 applied to $\{e_1, e_2\}$ and A' we get: $e^* \leqslant e'_2$. Since $e'_2 \leqslant e'_3$ we get $e^* \leqslant e'_3$, which proves $s_1 \preccurlyeq s_3$.
To prove density of $\preccurlyeq$, let $s_1 \preccurlyeq s_3$ and $s_1 \neq s_3$. Then there is $e_1 \in s_1$ and $e_3 \in s_3$ such that $e_1 < e_3$. By density of $<$, there is e_2 such that $e_1 < e_2 < e_3$. By the definition of $\preccurlyeq$, we thus have: $s_1 \preccurlyeq s(e_2) \preccurlyeq s_3$. Since $\{e_1, e_2, e_3\}$ is an l-event, clause 1 of Definition 13 dictates that $s_1 \neq s(e_2)$ and $s(e_2) \neq s_3$. □

A distinctive feature of BT models is that, if they allow for the introduction of *Instants*, the set of *Instants* is linearly ordered. We prove a fact to the effect that if a BCont model is like a BT model, then "its" set S of s-t locations is linearly ordered:

Fact 16
Let $\langle W, \leqslant, S\rangle$ be a model of BCont with set S of s-t locations that is downward directed, satisfies "no backward forks" condition, i.e., $\forall e_1, e_2, e_3 \in W (e_1 \leqslant e_3 \wedge e_2 \leqslant e_3 \rightarrow e_1 \leqslant e_2 \vee e_2 \leqslant e_1)$, and the following condition:
for all $e, e' \in W$, if e and e' are incomparable by $\leqslant$, then there are $H_1, H_2 \in \Pi_m$ such that $H_1 \neq H_2$ and $e \in H_1$, and $e' \in H_2$, where m is a maximal element of $W_{\leqslant ee'} = \{y \in W \mid y \leqslant e \wedge y \leqslant e'\}$.
Then (1) S is linearly ordered by $\preccurlyeq$ and (2) every l-event of $\langle W, \leqslant, S\rangle$ is a chain.

Proof: (1) The existence of m is guaranteed by the assumption of downward directedness and clause 5 of Definition 9. Let us assume for *reductio* that for some $s, s' \in S$: $s \not\preccurlyeq s'$ and $s' \not\preccurlyeq s$, which entails $\forall e \in s \forall e' \in s' : (e \not\leqslant e' \wedge e' \not\leqslant e)$. We will pick some $e \in s$ and $e' \in s'$ and aim to produce a chain of choice events satisfying the assumptions of clause 5 of Definition 13. To begin, since

e and e' are incomparable, by downward directedness and the condition of our Fact, there is a choice event m_0 such that e and e' belong to separate continuations of m_0. In particular, ($\blacktriangle$) $m_0 < e$. No backward forks imply that the events below e are linearly order. Moreover, the existence of m_o guarantees that there is a chain L of choice events, defined as follows:

$$m \in L \text{ iff } m < e \wedge \exists e_1 (m < e_1 \wedge e_1 \notin \Pi_m \langle e \rangle \wedge s(e_1) = s'). \qquad (\star)$$

Applying clause 5 of Definition 13 to L, we get ($\star\star$) $e^* \in \bigcap_{m \in L} \Pi_m \langle e \rangle$ such that ($\dagger$) $s(e^*) = s'$. If e and e^* are comparable, we immediately get that s and s' are comparable. Suppose thus that they are incomparable, from which it follows that there is a choice event m^* such that (\$) $m^* < e$ and (£) $m^* < e^*$, and ($\ddagger$) $e^* \notin \Pi_{m^*} \langle e \rangle$. Due to (\$), (£), ($\ddagger$), and ($\dagger$) we get that m^* satisfies the right-hand side of ($\star$), and hence $m^* \in L$. But then by ($\star\star$) we have $e^* \in \Pi_{m^*} \langle e \rangle$, contradicting ($\ddagger$).

To prove (2), assume for reductio that there is an l-event A that is not a chain, which means that A has incomparable elements e and e'. By the assumption of the fact, e and e' must belong to separate continuations of some m, which is a maximal element of $W_{\leqslant ee'} = \{y \in W \mid y \leqslant e \wedge y \leqslant e'\}$. Since clearly $m \in CE$, e and e' are not consistent, which entails that A is not consistent, so it is not an l-event. Contradiction. $\square$

3 Semantics without histories

3.1 A sketch of BCont semantics

In this section we will test the theory of BCont, asking if it yields a reasonable semantics for a language with indexicals, tenses, and historical modalities. Our point of reference is the branching-time semantics, as suggested in S. Kripke's letter to A. N. Prior (dated September 3, 1958, unpublished), discussed then briefly in Prior (1967), and worked out in Thomason (1970). We take the Kripke/Prior/Thomason semantics for our reference theory, since it is relatively simple and we have some intuitions concerning tenses. We do not have comparable intuitions concerning relativistic notions, and for this reason it will not be revealing to take BST for our reference theory.

We need thus to consider BCont models that are similar to BT models with $Instants$, which we define as follows:

Definition 17 ((BT+$Instants$)-like model)
A model $\langle W, \leqslant, S \rangle$ of BCont with set S of s-t locations is said to be (BT+Instants)-like if it satisfies these conditions:

1. *downward directedness,*

2. *no backward forks, and*

3. *for all $e, e' \in W$, if e and e' are incomparable by $\leqslant$, then there are $H_1, H_2 \in \Pi_m$ such that $H_1 \neq H_2$ and $e \in H_1$, and $e' \in H_2$, where m is a maximal element of $W_{\leqslant ee'} = \{y \in W \mid y \leqslant e \wedge y \leqslant e'\}$.*

In fact, a (BT+$Instants$)-like model of BCont is a BT model with $Instants$, since it has maximal chains (which BT identifies with histories), every two of which overlap, and S of the former model interplays with maximal chains exactly like $Instants$ of BT interplays with BT histories. On a related note, since l-events in a (BT+$Instants$)-like model of BCont are chains, from a BT perspective, they are subsets (also improper subsets) of histories.

Facts 15 and 16 guarantee that the set S of each (BT+$Instants$)-like model $\langle W, \leqslant, S\rangle$ of BCont is dense and linearly ordered by $\preccurlyeq$. Accordingly, for every (BT+$Instants$)-like model $\langle W, \leqslant, S\rangle$ there is a coordinalization, that is, an order-preserving bijection X between $\langle S, \preccurlyeq\rangle$ and a dense subset of some linearly ordered set. What this latter set is, depends on the cardinality of S, which is determined by the cardinality of maximal chains in W. By BCont axioms (2)–(5) the cardinality of S is not smaller than the cardinality of the set $\mathbb{R}$ of reals. Putting these together, if the cardinality of S is not larger than that of the reals, $|S| = |\mathbb{R}|$, S can be mapped on a dense subset of $\mathbb{R}$. In these cases we may use a given coordinalization X to define an interval relation:

$$int(e_1, e_2, t) \text{ iff } X(s(e_2)) - X(s(e_1)) = t,$$

meaning "e_1 is t units before e_2", where t is a real number. In terms of this relation we will state the truth-conditions for metric tense operators.

We will ultimately opt for evaluating sentences at evaluation points that are built out of l-events. Since l-events can be small, it is preferable to work with metric tenses, $F(x)$ and $P(x)$, read as: "in x units, it will be the case that ..." and "x units ago, it was the case that ...", respectively, rather than open-ended tenses "it will be the case" and "it was the case". We assume here that x ranges over the set $\mathbb{R}$ of real numbers. Accordingly, a (BT+$Instants$)-like structure for this language requires a coordinalization of S with $\mathbb{R}$, which means that $|S| = |\mathbb{R}|$. We will call a coordinalization of S with $\mathbb{R}$ "real coordinalization" and we will assume it in the rest of this paper.

To finally characterize the language $\mathcal{L}$ we consider, it is assumed that its atomic formulas are significantly present-tensed, like "It is so-and-so now".

$\mathcal{L}$ has metric tense operators $F(x)$ and $P(x)$ with $x \in \mathbb{R}$, the familiar truth-conditional connectives: $\neg$, $\wedge$, $\vee$, and $\rightarrow$, modal operators $Sett$: ("It is settled that") and $Poss$: ("It is possible that"), and an operator Now: ("it is now the case that").

We define a structure and a model for language $\mathcal{L}$:

Definition 18 (structure and model)
A structure for $\mathcal{L}$ is a pair $\mathfrak{S} = \langle \mathcal{W}, X \rangle$, where $\mathcal{W} = \langle W, \leqslant, S \rangle$ is a (BT+Instants)-like model of BCont such that $|S| = |\mathbb{R}|$, and X is a real coordinalization of S.
A pair $\mathfrak{M} = \langle \mathfrak{S}, \mathfrak{J} \rangle$ is a model for language $\mathcal{L}$, where $\mathfrak{S}$ is a structure for $\mathcal{L}$ and $\mathfrak{J} : Atoms \rightarrow \mathcal{P}(W)$ is an interpretation function and Atoms is the set of atomic formulas of $\mathcal{L}$.[13]

A novelty of Prior's (1967) approach is that formulas are evaluated at event-history pairs where an event is assumed to belong to the history in question. Such pairs are written as e/h; accordingly $e/h \models \varphi$ is read as "formula φ is evaluated as true at evaluation point e/h." What should we use as evaluation points in the present theory? In particular, what should be used in place of history? A natural answer is that an evaluation point is an event/l-event pair $\langle e, A \rangle$, subject to the condition that e is consistent with A, that is $\{e\} \cup A$ is an l-event. We will write such pairs as e/A.

Accordingly, we assume this concept of evaluation points:

Definition 19 (evaluation points)
Let $\mathfrak{S} = \langle \mathcal{W}, X \rangle$ be a structure for language $\mathcal{L}$, where $\mathcal{W} = \langle W, \leqslant, S \rangle$.

Then $\langle e, A \rangle$, written as e/A, is an evaluation point in $\mathfrak{S}$ for formulas of $\mathcal{L}$ iff (1) $\{e\} \cup A \subseteq W$ is an l-event and (2) $A \neq \emptyset$.

Note that we do not require for an evaluation point e/A that $e \in A$.

What does it mean, however, that a sentence is true at an evaluation point so defined, as the point involves an l-event? Perhaps what pops first to someone's mind is a reading in terms of information. On this interpretation $e/A \models \varphi$ means that at event e, given the available information about event A, sentence φ is true. We oppose this reading since informational notions are not wedded to BT/BST1992. The branching theories are naturally interpreted as attempts to capture what our indeterministic world is like rather than what we know, or how we gather information about the world. We thus

[13] Another option is to take for the interpretation function $\mathfrak{J} : Atoms \rightarrow \mathcal{P}(W \times l\text{-}Events)$. The situation exactly mimics a BT choice between an interpretation function sending atomic formulas into subsets of W, or into subsets of $W \times Hist$, where $Hist$ is the set of BT histories – cf. Prior (1967, p. 123).

recommend an unabashedly ontological reading. That is, we say: at event e and a large event A sentence φ is true. We expect that a sentence might be evaluated differently at e/A and e/A', if A and A' lie above (or partly above) e. For instance,

"I will drink the milk from my fridge in two minutes. "

is likely to be evaluated as true at the event e of my present opening the fridge and the l-event A of my taking out a bottle of milk from there in one minute, but as false or indefinite at the same event e and l-event A' of my taking out a beer from the fridge in one minute. It is also natural to require that if φ is true at e and A, it is as well true at events larger than A, that is, at e/A' such that $A \subseteq A'$. This is guaranteed by Lemma 26 below. Finally, which l-event A in the past of e is considered, should not influence whether or not φ is true at e/A, since event e fixes its unique past. That is, if φ is true at e/A and A is in the past e, then φ is true at every e/A' such that A' is in the past of e. This is the content of Lemma 27 below.

Truth, falsity, and indefiniteness at evaluation points is an objective matter, and should not be confused with our assessment, under available information, of whether sentences are true, false, or indefinite at evaluation points. We might be mistaken in our beliefs that something is really possible, and mistakenly take something for an l-event. This kind of mistake is not essentially different from other mistakes leading to incorrect assessment concerning truth, falsity, and indefiniteness. Despite such mistakes, however, since l-events are typically small and may have plenty of holes, we can form a premonition, a hunch, a forecast, or a scientific prediction that there is such-and-such future l-event and, moreover, that at this l-event combined with an event of utterance, the uttered sentence is true/false/indefinite. Such epistemic access to l-events is to be strongly contrasted with our lacking epistemic access to histories. Since histories are maximally large (and there are no holes in them), it is not in our power to form even an intuition of which history will be realized. A semantic theory framed in terms of histories leads thus to a grave tension between ontology and epistemology, which (we believe) is alleviated in our BCont framework.

An intuition underlying our notion of truth says that whether a formula is (definitely) true at evaluation point e/A, depends on how things stand appropriately high above e. That is, for a formula to be definitely true at e/A, it should be fulfilled in every sufficiently long extension of e/A. We thus need a few auxiliary notions before turning to definite truth. We will define the following:

- An evaluation point e/A goes x-units-above e;

- One evaluation point is an x-units-above-e extension of some other evaluation point;
- A set of evaluation points is a fan determined by a given evaluation point;
- An evaluation point fulfills a formula;
- A formula is definitely true at an evaluation point.

Let us then get to work.

Definition 20 (extensions of an evaluation point)
Let $\mathfrak{S} = \langle \mathcal{W}, X \rangle$ be a structure for $\mathcal{L}$, $\mathcal{W} = \langle W, \leqslant, S \rangle$, and e/A be an evaluation point in $\mathfrak{S}$ for $\mathcal{L}$. Then we say:

e/A goes at least x-units-above e ($0 \leqslant x$) iff $\exists e_1 \in W\ \exists e_2 \in A\ (e_1 \leqslant e_2 \ \wedge\ int(e, e_1, x))$;

e/A' is an x-units-above-e extension of e/A ($0 \leqslant x$) iff (1) $A \subseteq A' \subseteq W$ and (2) e/A' goes at least x-units-above e.

Observe that if an evaluation point is an x-units-above-e extension of e/A, it is also an y-units-above-e extension of e/A for any $0 < y \leqslant x$ (by the density of S and since X is an order-preserving bijection). The phrase "at least" implies that an x-units-above-e extension of e/A can also be an $(x+y)$-units-above-e extension of e/A for some $y > 0$. Also an evaluation point can be an x-units-above-e extension of itself.

In a proof later on we will need the following fact concerning extensions:

Fact 21
Let $\mathfrak{S} = \langle \mathcal{W}, X \rangle$ be a structure for $\mathcal{L}$, $\mathcal{W} = \langle W, \leqslant, S \rangle$, and e_2/B' be an x-units-above e_2 extension of e_2/B and $e_1 < e_2$. Then e_1/B' is an x-units-above e_1 extension of e_1/B.

Proof: Let e_2/B' be as in the premise. Then there is e' such that ($\star$) $e' \leqslant e_{B'}$ for some $e_{B'} \in B'$ and ($\star\star$) $X(s(e')) - X(s(e_2)) = x \geqslant 0$. Since $\{e_2\} \cup B'$ is an l-event, it must be a chain, and hence by ($\star$) and no downward forks, e_2 and e' must be comparable. By ($\star\star$) $e_2 \leqslant e'$ and hence $e_1 < e_2 \leqslant e'$. Since S is dense and X is bijective order preserving and onto $\mathbb{R}$, there is $s'' \in S$ such that $s(e_1) < s'' < s(e')$ and $X(s'') - X(s(e_1)) = x$. We will now argue that there is $e'' \in s''$ such that $e_1 < e'' < e'$. If there is $e'' \in s''$ such that $e'' < e'$, then by the condition of no downward forks $e_1 < e'' < e'$. Otherwise we pick some $e \in s''$ and, since it is incomparable with e', by clause 3 of Definition 17, there is $x \in CE$ such that $x < e$ and $x < e'$ and hence (by no downward forks) $e_1 < x$. This event x guarantees that the set l below is nonempty:

$$l := \{x \in CE \mid e_1 < x < e' \land \exists e''(e'' \in s'' \land x < e'')\}.$$

By no downward forks, since all element of l are below e', l is a chain. By clause 5 of Definition 13, there is e'' such that (†) $e'' \in \bigcap_{x \in l} \Pi_x\langle e'\rangle \land s(e'') = s''$. It must be that $e'' < e'$. Otherwise e' and e'' were incomparable, so there would be $y \in CE$ such that $e'' \notin \Pi_y\langle e'\rangle$. But then $y \in l$ and hence by (†) $e'' \in \Pi_y\langle e'\rangle$. Contradiction. Thus, $e'' < e'$ and then by no downward forks $e_1 < e'' < e'$. The event e'' so constructed is a witness that e_1/B' is an x-units-above e_1 extension of e_1/B. □

We now define an auxiliary concept of a fan of evaluation points, which considerably simplifies the truth conditions for sentences with *Sett* as a principal operator.

Definition 22 (fan of evaluation points)
Let $\mathfrak{S} = \langle \mathcal{W}, X\rangle$ be a structure for $\mathcal{L}$, $\mathcal{W} = \langle W, \leqslant, S\rangle$ and e/A be an evaluation point in $\mathfrak{S}$ for $\mathcal{L}$.

Two l-events A_1 and A_2 of $\mathcal{W}$ are isomorphic instant-wise iff $\forall e_1 \in A_1 \exists e_2 \in A_2 : s(e_1) = s(e_2)$ and $\forall e_2 \in A_2 \exists e_1 \in A_1 : s(e_1) = s(e_2)$.

The fan $\mathcal{F}_{e/A}$ of evaluation points determined by evaluation point e/A is a set of evaluation points subject to this condition:
$e/A' \in \mathcal{F}_{e/A}$ iff e/A' is an evaluation point in $\mathfrak{S}$ and A and A' are isomorphic instant-wise.

Note that while producing the fan out of a given evaluation point e/A, we keep e fixed and vary A subject to the condition of instant-wise isomorphism and the condition that a resulting A' is consistent with e. In many cases, the only A' that satisfies this condition is A itself. Thus, many evaluation points permit only trivial fans, i.e., singletons of themselves.

Since the formulas of $\mathcal{L}$ are significantly tensed, a formula can be evaluated differently at different contexts of use. The operators of $\mathcal{L}$ determine that a single relevant aspect of the context of use is the moment of use, symbolized by e_C. In general, a moment of use e_C of a sentence is different from the event-part of evaluation point e/A for this sentence. In this respect, sentences considered as stand-alone are somewhat exceptional, since for such a sentence, the e of evaluation point e/A is identified with moment e_C of use. For embedded sentences e of e/A and e_C may diverge.[14] On a related issue, we believe that a context of use does not determine which future l-event will

[14]For more on this issue, see Belnap et al. (2001) and Belnap (2007).

occur. There is no future l-event of use, for quite similar reasons as there is no history of use in the BT semantics – cf. Belnap et al. (2001). On the other hand, if an l-event A is in the past of moment e_C of use, it does not add anything to the context, since, given the language we consider, all relevant features of contexts of use are coded by moments of use. Thus, to put it bluntly, there is no A_C, though there is e_C. We will accordingly define what it means that a formula is evaluated as true in a model, at a moment of use, and at a point of evaluation. But first we need a semi-technical notion: "in a model a moment of use and an evaluation point fulfills a formula". In symbols: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$.

Definition 23 (point fulfills a formula)
Let $\mathfrak{M} = \langle \mathfrak{S}, \mathcal{J} \rangle$ be a model for $\mathcal{L}$. Then

1. *if $\psi \in Atoms$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff $e \in \mathcal{J}(\psi)$;*
2. *if ψ is $\neg\varphi$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \varphi$;*
3. *if ψ is $\beta \vee \varphi$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \beta$ or $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \varphi$;*
4. *and similarly for $\wedge$ and $\rightarrow$;*
5. *if ψ is $F(x)\varphi$ for $x > 0$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff there are $e' \in W$ and $e^* \in A$ such that $e' \leqslant e^*$ and $int(e, e', x)$, and $\mathfrak{M}, e_C, e'/A \approx\!\!\mid\, \varphi$;*[15]
6. *if ψ is $P(x)\varphi$ for $x > 0$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff there is $e' \in W$ such that $\{e'\} \cup A \in$ l-Events and $int(e', e, x)$, and $\mathfrak{M}, e_C, e'/A \approx\!\!\mid\, \varphi$;*
7. *if ψ is $Sett\colon \varphi$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff for every evaluation point e/A' from fan $\mathcal{F}_{e/A}$ determined by e/A: $\mathfrak{M}, e_C, e/A' \approx\!\!\mid\, \varphi$;*
8. *if ψ is $Now\colon \varphi$: $\mathfrak{M}, e_C, e/A \approx\!\!\mid\, \psi$ iff there is $e' \in s(e_c)$ such that $\{e'\} \cup A$ is consistent and $\mathfrak{M}, e_C, e'/A \approx\!\!\mid\, \varphi$.*

The historical possibility operator ($Poss$:) is defined as usual:

$$Poss\colon \psi \text{ iff } \neg Sett\colon \neg\psi.$$

Finally, we have come to the definite truth: here is the notion of a formula being definitely true in a model, at a moment of use, and at an evaluation point, i.e., $\mathfrak{M}, e_C, e/A \models \psi$.

[15] Note that the condition $e' \leqslant e^*$ ensures that $\{e'\} \cup A$ is an l-event, so e'/A is a legitimate point of evaluation.

Definition 24 (definite truth)
ψ is definitely true at $\mathfrak{M}, e_C, e/A$, in symbols $\mathfrak{M}, e_C, e/A \models \psi$, iff there is an $x \geqslant 0$ such that for every x units-above e extension e/A' of e/A: $\mathfrak{M}, e_C, e/A' \approx\!\!\!\models \psi$;

ψ is indefinite at $\mathfrak{M}, e_C, e/A$, in symbols $\mathfrak{M}, e_C, e/A \mathrel{?\!\!\models} \psi$, iff there is no $x \geqslant 0$ such that for every x-units-above-e extension e/A' of e/A: $\mathfrak{M}, e_C, e/A' \approx\!\!\!\models \psi$ or for every x-units-above-e extension e/A' of e/A: $\mathfrak{M}, e_C, e/A' \approx\!\!\!\models \neg\psi$.

As the first consequence of this definition, consider the following theorem:

Theorem 25
For any formula ψ and any evaluation point e/A, exactly one of the following three options must hold:

$$e/A \models \psi \text{ or } e/A \models \neg\psi \text{ or } e/A \mathrel{?\!\!\models} \psi.$$

Proof: As for the fulfillment of a formula at extensions of e/A, these three cases exhaust all possibilities:

1. there is an $x \geqslant 0$ such that for every x units-above e extension e/A' of e/A: $e/A' \approx\!\!\!\models \psi$, or
2. there is an $x \geqslant 0$ such that for every x units-above e extension e/A' of e/A: $e/A' \not\approx\!\!\!\models \psi$, or
3. there is no $x \geqslant 0$ satisfying (1) or (2) above.

The first case means that $e/A \models \psi$. Given the definition of fulfillment (the clause on negation) and the first clause of definition (24), the second case is equivalent to $e/A \models \neg\psi$. And given the definition of fulfillment (the clause on negation) the third case is equivalent to $e/A \mathrel{?\!\!\models} \psi$. □

We need to check on the stability of the notion of truth. We have the following lemma.

Lemma 26 *If $\mathfrak{M}, e_C, e/A \models \psi$, then for every extension e/A^* of evaluation point e/A: $\mathfrak{M}, e_C, e/A^* \models \psi$.*

Proof: It follows from the premise that there is an $x \geqslant 0$ such that for every x-units-above-e extension e/A' of e/A: $\mathfrak{M}, e_C, e/A' \approx\!\!\!\models \psi$. We need to check that for an arbitrary extension e/A^* of e/A there is $z \geqslant 0$ such that for every z-units-above-e extension e/A'' of e/A^*: $\mathfrak{M}, e_C, e/A'' \approx\!\!\!\models \psi$. Put $z := x$. Since $A \subseteq A^* \subseteq A''$ and e/A'' goes at least x units above e, e/A'' is an x-units-above-e extension of e/A. Accordingly, by the premise: $\mathfrak{M}, e_C, e/A'' \approx\!\!\!\models \psi$.

□

Thus, a sentence definitely true at an evaluation point stays definitely true as the extensions of the initial evaluation point become increasingly longer. A sentence indefinite at some evaluation point may become definitely true or definitely false at an extension of the initial evaluation point, whereby we call a sentence definitely false iff its negation is definitely true. As for the dynamics of definite truth, one would like to know if each sentence becomes definitely true or definitely false at some sufficiently long extension of an initial evaluation point. In general, the answer depends on the details of coordinalization, however. But for (what we called) real coordinalizations, every sentence is definitely true or definitely false at some (sufficiently long) extension of an initial evaluation point.

Next we shall see that l-events in the past of e are irrelevant for the evaluation of sentences.

Lemma 27 *If* $\forall e_A \in A : e_A \leqslant e$, $\forall e_B \in B : e_B \leqslant e$, *and* $e/A \models \varphi$, *then* $e/B \models \varphi$.

Proof: By the premise, there is $y \geqslant 0$ such that for every y-units-above e extension e/A' of e/A we have $e/A' \models\!\approx \varphi$. Consider an y-units-above e extension e/B' of e/B. Since $\{e\} \cup B'$ is an l-event, and hence a chain, and $\forall e_A \in A : e_A \leqslant e$, by no downward forks, $\{e\} \cup B' \cup A$ is an l-event, yielding the evaluation point e/B^*, with $B^* := A \cup B'$. Since e/B^* is an y-units-above e extension of e/A, we get $e/B^* \models\!\approx \varphi$. To finish the proof we should argue by induction with respect to the complexity of formula φ that if $e/B^* \models\!\approx \varphi$, then $e/B' \models\!\approx \varphi$. Since the argument is straightforward, we present only the case with $F(x)$ as the principal operator of φ. That is, we assume that $\varphi := F(x)\psi$ and for every e': if $e'/B^* \models\!\approx \psi$, then $e'/B' \models\!\approx \psi$. $e/B^* \models\!\approx F(x)\psi$ entails that there is e^* such that $int(e, e^*, x)$ and $e^*/B^* \models\!\approx \psi$. It follows from the induction step that $e^*/B' \models\!\approx \psi$, and hence $e/B' \models\!\approx F(x)\psi$. □

As a conclusion to this section, we state some simple observations of how definite truth, definite falsity and indefiniteness mesh together.

1. If ψ is indefinite at a point, so is its negation.

2. if ψ is indefinite at a point, $\psi \vee \varphi$ is either definitely true or indefinite at this point;

3. if ψ is indefinite at a point, $\psi \wedge \varphi$ is either indefinite or definitely false at this point;

4. if ψ is indefinite at a point, $\psi \to \varphi$ is either definitely true or indefinite at this point (same for $\varphi \to \psi$).

5. settled cannot be indefinite: $Sett : \psi$ is definitely true or $\neg Sett : \psi$ is definitely true.

3.2 Puzzles

The branching-time semantics of Kripke/Prior/Thomason has a few distinctive features that make it different and (we believe) superior, to other systems of tense logic. In this chapter we will check if the semantics based on BCont theory that we have sketched above retains these features. Whether or not a semantic theory has some aspects can often be seen by focusing on some puzzles and asking how the theory analyzes them. It is this task to which we will soon turn.

To list the features we alluded to, the Kripke/Prior/Thomason semantics permits a distinction between what will happen and what will necessarily happen, which Peircean approaches fail to draw. Second, it rejects the view that sentences in the past tense, if true, are necessarily true. Third, it allows for a careful analysis of the interplay of tenses, settledness, and indeterminism, as involved in sentences like "Einstein was born a Nobel prize winner", asserted in an indeterministic context. That is, BT semantics analyzes this sentence (assumed to be uttered now, in 2010) to be settled true, while granting that years ago it was possible that Einstein would fail to win a Nobel prize. Finally, the theory could be naturally supplemented by Belnap's (2001) technique of double-time reference, which offers a persuasive model of how a sentence that is not definitely true at one moment, becomes definitely true at some later moment.[16] Our aim now is to see if indeed these four aspects of BT theory are preserved in the BCont semantics.

A Peircean future? No. For a sentence in the future tense, the Peircean approach equates its being true at e with its being true in every possible history to which e belongs. Since the truth-value of a future-tensed sentence is thus defined in terms of the quantification over the set of possible histories, the Peircean approach fails to accommodate a distinction between what will happen and what will necessarily happen. In the present framework, however, we similarly define definite truth by quantifying over possible extensions of an evaluation point. One might thus suspect that we fail to draw the mentioned

[16]Needless to say, these features have either had antecedents, likely dating back to antiquity, or are attempts to solve some issues, known and discussed since antiquity, cf. Øhrstrøm and Hasle (1995).

distinction and thus our account is a version of the Peircean approach. The suspicion is incorrect, however: the present framework sharply distinguishes between the definite true of $F(x)\phi$ and of $Sett\colon F(x)\phi$. To see this, consider a (BT+*Instants*)-like model $\mathfrak{M}$ of BCont, as depicted on Figure 3.

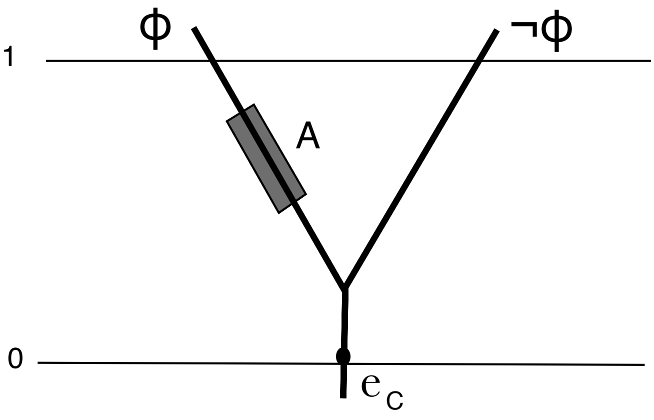

Figure 3: Difference between $Sett\colon F_1\phi$ and $F_1\phi$.

Horizontal lines indicate two elements of S, called 0 and 1, with $s(e_c) = 0$. The shadowed rectangle symbolizes the l-event A. $F(1)\phi$ is fulfilled by every 1-units-above-e_C extension of e_C/A, and hence: $\mathfrak{M}, e_C, e_C/A \models F(1)\phi$. On the other hand, no matter how large $x \geqslant 0$ is, every fan determined by each x-units-above-e_C extension of e_C/A has an element lying on the right branch of the model, where $\neg\phi$ is true at s-t location 1. Accordingly, we have: $\mathfrak{M}, e_C, e_C/A \models \neg Sett\colon F(1)\phi$.

Is the past always settled? There is a powerful intuition that, in contrast with the future, the past is already settled. This intuition, interpreted as a semantical claim:

(†) From $P(x)\varphi$ it follows that $Sett\colon P(x)\varphi$,

was likely a premise of Diodor's Master Argument as well as of some versions of the argument from divine foreknowledge to the necessity of the future.[17] A way of blocking these arguments, which the Ockhamist tradition and BT semantics take, is to reject (†). A modern rationale for this move can be seen in von Kutschera's (1986) distinction between a sentence being about the past and the sentence being in the past tense. The intuition about settledness of

[17]Cf. Øhrstrøm and Hasle (1995).

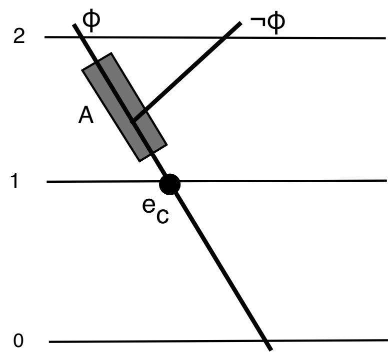

Figure 4: $\mathfrak{M}, e_C, e_C/A \models P(1)F(2)\phi$ but $\mathfrak{M}, e_C, e_C/A \models \neg Sett\colon P(1)F(2)\phi$.

the past is upheld if it concerns sentences about the past, but rejected if it concerns sentences in the past tense but not about the past, e.g., sentences like $P(x)F(y)\varphi$ with $x < y$ and φ an atomic formula. Since a formula in the past tense is a formula with $P(x)$ as the principal operator, the implication (†) is rejected. But is it rejected in the present framework as well? Consider the model of Figure 4, with the symbol conventions explained as above. Let ϕ be an atomic formula and $s(e_C) = 1$. $P(1)F(2)\phi$ is fulfilled by every 1-units-above-e_C extension of e_C/A, and hence: $\mathfrak{M}, e_C, e_C/A \models P(1)F(2)\phi$. On the other hand, for any $x \geqslant 0$, every fan determined by each x-units-above-e_C extension of e_C/A has an element lying on the right branch of the model, where $\neg\phi$ is true at s-t location 2.

Accordingly, we have: $\mathfrak{M}, e_C, e_C/A \models \neg Sett : P(1)F(2)\phi$, which shows that (†) is not valid. On the other hand, it can be proved that if ϕ atomic and $P(x)\phi$ is definitely true, so is $Sett\colon P(1)\phi$. Thus, some sentences in the past tense are settled true, and some are not.

3.2.1 "Einstein was born a Nobel Prize winner"

This conundrum sentence is ascribed to Arthur Prior. Let us suppose it is asserted now (in 2010) and that the assertor grants that Einstein might have failed to receive the Nobel Prize. Despite this indeterminism, the sentence appears to be settled true now. That is, it is settled (now) that it was true one hundred years ago that Einstein would receive the Nobel Prize in 11 years time (in 1921). In symbols, $Sett\colon P(100)F(11)\phi$. This should be compatible with the truth of the sentence that one hundred years ago it was not settled that Einstein would receive the Nobel Prize in 11 years' time, that is, in 1921. We thus want to show that $Sett\colon P(100)F(11)\phi$ does not imply $P(100)\ Sett\colon F(11)\phi$. Consider Figure 5. Every element of every fan

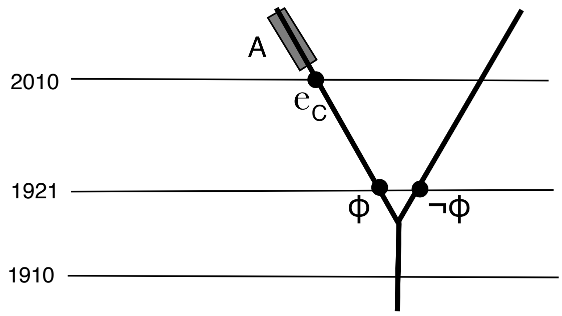

Figure 5: From $Sett: P(100)F(11)\phi$ it does not follow that $P(100)$ $Sett: F(11)\phi$.

determined by any extension of e_C/A lies on the left branch of the model, where ϕ holds. Hence, $\mathfrak{M}, e_C, e_C/A \models Sett: P(100)F(11)\phi$. On the other hand, no matter how large x is, every x-units-above e_C extension e_C/A' of e_C/A fails to fulfill $P(100)$ $Sett$: $F(11)$ ϕ. Namely, in our model there is exactly one e' such that $int(e', e_C, 100)$ (which means that $X(s(e')) = 1910$). No matter how large (or small) the extension e_C/A' of e_C/A is, some element of the fan determined by e'/A' is on the right branch, where $\neg\phi$ holds. Accordingly, $Sett: F(11)\phi$ is not fulfilled at $\mathfrak{M}, e_C, e'/A'$, and hence $P(100)$ $Sett: F(11)\phi$ is not fulfilled at $\mathfrak{M}, e_C, e_C/A'$. As x is arbitrary, we conclude that $\mathfrak{M}, e_C, e_C/A \not\models P(100)$ $Sett: F(11)\phi$.

Double time reference. The double time reference is a BT/BST1992 technique developed by Belnap (2001) to explain how something that was not settled true at one event, becomes settled true at an appropriately later event. I have intentionally phrased this vaguely since a part of the problem is what is an object which turns into settled truth?

In the BT framework, with metric rather than open tenses (to make it more similar to BCont theory) and with $Instants$, the following variation of Belnap's story illustrates the problem. (Note that in the next paragraph $\models$ stands for the BT truth at an event/history pair, with $Ins(e)$ standing for the instant of event e.)

At e_1 Themistocles makes the promise—"In t units of time Themistocles will fight a sea battle"— in symbols: $F(t)\varphi$. It is reasonable to assume that the core of the promise, "$F(t)\varphi$", is not a settled truth at the event of making the promise, and also that the promise is not already vitiated at this event. So:

$$e_1/h \not\models Sett\colon F(t)\varphi \text{ and } e_1/h \models Poss\colon F(t)\,\varphi. \qquad (*)$$

(The reference to history h plays no role above, since the BT truth conditions for *Sett* and for *Poss* require one to quantify over histories.) The event at which the promise is satisfied is one e_2 at which it is settled true that Themistocles fights the battle. As for the timing of event e_2, it may occur during or after the battle as well as before it. What does it mean, generally speaking, that the promise made at e_1 is satisfied at e_2? The double-time reference yields this verdict:

Double-time reference—BT analysis
The promise "$F(t)\varphi$" made by Themistocles at e_1 is satisfied at a later event e_2 iff $\forall h$: $(e_2 \in h \to e_1/h \models F(t)\varphi)$.

This analysis entails that at event e_2 at which the promise made at e_1 is satisfied we have:

$$e_2/h \models \mathit{Sett}\colon P(x)F(t)\varphi, \text{ where } x = \mathit{Ins}(e_2) - \mathit{Ins}(e_1) > 0.^{18} \qquad (\dagger)$$

(Again, the reference to history h is inessential). This consequence is desirable since at the event at which Themistocles's promise is satisfied (say two days after making the promise) Themistocles should truly say:

It is settled that two days ago it was the case that I would fight a battle next day.

We may introduce a similar device in the present framework. (Warning: from now on, we are again using $\models$ and $\not\approx$ of the BCont theory.)

Double-time reference—BCont analysis
The promise "$F(t)\varphi$" made by Themistocles at e_1 is satisfied at a later event e_2 iff

$$\forall A\ (A \cup \{e_2\} \in l\text{-}\mathit{Events} \to e_1/A \models F(t)\varphi). \qquad (**)$$

Since $A \cup \{e_2\}$ is an l-event and $e_1 < e_2$, $A \cup \{e_1\}$ is an l-event as well, so it yields the evaluation point e_1/A.

We need to check if this analysis has a consequence analogous to ($\dagger$):

Fact 28
If the premise "$F(t)\varphi$" made at e_1 is satisfied at e_2 ($e_1 < e_2$), then for every l-event A such that $A \cup \{e_2\} \in l\text{-}\mathit{Events}$

$$e_2/A \models \mathit{Sett} : P(x)F(t)\varphi, \text{ where } x \text{ is such that } \mathit{int}(e_1, e_2, x). \qquad (\ddagger)$$

[18] The condition on x results from allowing that the premise might be satisfied before t units of time and from the left-hand side conjunct of ($\star$).

Proof: Let us assume that the premise "$F(t)\varphi$" made at e_1 is satisfied at e_2 ($e_1 < e_2$) and (\$) $int(e_1, e_2, x)$ is true; since $e_1 < e_2$, $x > 0$.

We will first show that $e_2/\{e_2\} \models Sett: P(x)F(t)\varphi$. Since $e_1 < e_2$, $\{e_1, e_2\} \in$ *l-Events*, so by the assumption and (**), $e_1/\{e_2\} \models F(t)\varphi$, and hence there is $y \geqslant 0$ such that (£) for every y-units-above e_1 extension e_1/A of $e_1/\{e_2\}$: $e_1/A \not\approx F(t)\varphi$.

Consider now an arbitrary y-units-above e_2 extension e_2/B of $e_2/\{e_2\}$. Since $e_1 < e_2$, by Fact 21 we get that e_1/B is an y-units-above e_1 extension of $e_1/\{e_2\}$. By (£) we get $e_1/B \not\approx F(t)\varphi$. By (\$) and clause 6 of Definition 23 we have $e_1/B \not\approx F(t)\varphi$ iff $e_2/B \not\approx P(x)F(t)\varphi$. Accordingly, $e_2/B \not\approx P(x)F(t)\varphi$. Pick now an arbitrary $e_2/B' \in \mathcal{F}_{e_2,B}$. This is an y-units-above e_2 extension of $e_2/\{e_2\}$. By the reasoning above, $e_2/B' \not\approx P(x)F(t)\varphi$. It follows that $e_2/B \not\approx Sett: P(x)F(t)\varphi$ for an arbitrary y-units-above e_2 extension e_2/B of $e_2/\{e_2\}$. Hence $e_2/\{e_2\} \models Sett: P(x)F(t)\varphi$.

By Lemma 26 we get that for every l-event A, if $e_2 \in A$, then $e_2/A \models Sett: P(x)F(t)\varphi$. To arrive at the sought-for conclusion (‡), it remains to show that $e_2/B \models Sett: P(x)F(t)\varphi$ for any B such that $e_2 \notin B$. By the argument above, the sentence is true at e/A, where $A := B \cup \{e_2\}$. Thus, it is enough if we show that generally, for every l-event B and every formula ψ:

if $A = B \cup \{e_2\}$, then (if $e_2/A \models \psi$, then $e_2/B \models \psi$).

Now $e_2/A \models \psi$ means that there is $y \geqslant 0$ such that the family $\{e_2/A'\}$ of y-units-above e_2 extensions of e_2/A is a witness for $e_2/A \models \psi$. Consider now an arbitrary y-units-above e_2 extension e_2/B' of e_2/B. Clearly, $e_2/\{B' \cup \{e_2\}$ is an y-units-above e_2 extensions of e_2/A, so $e_2/(B' \cup \{e_2\}) \not\approx \psi$. The final part of this proof is an inductive argument (with respect to the complexity of ψ) to the effect that if $e_2/(B' \cup \{e_2\}) \not\approx \psi$, then $e_2/B' \not\approx \psi$. The argument is exactly like the last part of the proof of Lemma 27. We conclude that $e_2/B \models \psi$. □

The fact shows that on our analysis the sentence:

It is settled that x days ago it was the case that Themistocles would fight a battle in t days,

is evaluated true at the event at which Themistocles's promise is satisfied and which occurs x days after the promise was made.

4 Conclusions

We put forward a possible-worlds theory, of a branching variety, that works in terms of possible continuations and large events rather than in terms of possible worlds or histories. The theory has similar explanatory virtues as BST1992. Some specific models of the theory, which we called (BT+*Instants*)-like models, can be used to construct semantical models for languages with indexicals, metric tenses, and historical modalities. The resulting semantical theory preserves the distinctive insights of Kripke/Prior/Thomason semantics.

It thus seems that we have here a philosophical paradise on the cheap: an ontological theory for indeterminism (in the sense of an open future) that can be used as a semantics for languages with tenses and modalities, and that is shy about possible histories. The open problem (and a large project) is to produce an ensemble of branching manifolds, i.e., a kind of generalization of an individual manifold of general relativity.

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