DOES QUANTUM TIME HAVE A PREFERRED DIRECTION?

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Abstract. This paper states and proves a precise sense in which, if all the measurable properties of an ordinary quantum mechanical system are ultimately derivable from position, then time in quantum mechanics can have no preferred direction. In particular, I show that when the position observable forms a complete set of commuting observables, Galilei invariant quantum mechanics is guaranteed to be time reversal invariant.

1. Introduction

There is some precedent for discussing the extent to which the configuration of bodies in motion is sufficient to characterize the nature of matter. For example, Robert Boyle thought that,

> whereas those other philosophers give only a general and superficial account of the phaenomena of nature... both the Cartesians and the Atomists explicate the same phaenomena by little bodies variously figured and moved (Boyle 1772, p.355).

Although this view has an interesting history in its own right, let me here simply point out the useful resemblance it bears to certain descriptions of modern physics. Namely, all the measurable properties of a substance have at times been described in terms derivable entirely from the *spatial positions* of fundamental particles.

There seems to have been a particularly strong inclination toward this ontology in the early days of quantum theory, before the electron's spin was discovered. The hydrogen atom was at the time characterized entirely by placement of electrons in "orbit" around the nucleus. Indeed, Heisenberg later reported being "psychologically" unprepared for Kronig's proposal that the electron had internal spin not reducible to changes in position, recalling that "I just said, 'That is a very funny idea and very interesting,' but in some way I pushed it away" (AIP 1963).

I would like to point out one consequence of the "minimal ontology" preferred by Heisenberg, which is perhaps unexpected. Namely, there seems to be a sense in which, if the measurable properties of an ordinary quantum system are functions of spatial position alone, then motion cannot develop in a preferred direction in time.

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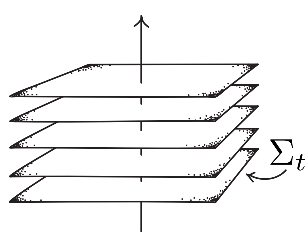
If such a system can develop in time at all, the development can also occur in the reverse temporal order.

In physical language, what I mean by a lack of a preferred temporal direction is that quantum theory is *time reversal invariant*. The "minimal ontology" I have in mind is one in which the position observable forms a *complete set of commuting observables*. The claim that I will argue for, then, is that if position forms a complete set of commuting observables, then ordinary (Galilei invariant) quantum theory must be time reversal invariant.

My endeavor in what follows will be to state and prove a precise expression of this claim. Section 2 sets out the basic quantum structures for the discussion, including position in space, development in time, Galilei invariance, and the "minimal ontology" of a complete set of commuting observables. In Section 3, I discuss the meaning of time reversal invariance, and state the theorem that captures the claim above. Section 4 discusses how the proof of the theorem works in terms of elementary quantum mechanics. Section 5 is the conclusion; a rigorous proof of the theorem is given in the Appendix.

2. Basic Structures

2.1. **Space and time.** We will be discussing non-relativistic space and time, as characterized by a big block. More precisely, the block is a 4-dimensional manifold $\mathbb{R}^4$. To individuate space from time, we slice the block into a family of parallel hypersurfaces $\{\Sigma_t : t \in \mathbb{R}\}$, in such a way that each surface Σ_t represents 3-dimensional Euclidean space at some moment in time t.



The 3-dimensional spaces Σ_t are made up of regions. More precisely, a *spatial region* $\Delta \subseteq \Sigma_t$ for us will be any open set or countable union or intersection of open sets – these are sometimes called the *Borel* sets. They can be assigned three Cartesian coordinate axes, allowing us to label a point p in a spatial region as $p = (x, y, z)$. For convenience of exposition, let us restrict attention to a single one of these axes. That is, let Δ represent a (Borel) set of the real numbers $\mathbb{R}$.

Now we turn to the quantum appropriation of these structures. In quantum theory, the pure states (states of affairs, if one likes) are represented by rays in a

Hilbert space $\mathcal{H}$. This Hilbert space is a vector space, among other things[1], which we take to have a countably infinite basis set. But it is also a considerable abstraction from the spacetime structure set out above. A connection must be made between the two. This can be done in two steps: first, we connect $\mathcal{H}$ to space; then we connect $\mathcal{H}$ to time.

We begin the first step by recognizing that, like a vector space, the Hilbert space $\mathcal{H}$ contains subspaces. A *projection operator* projects each vector in $\mathcal{H}$ onto a subspace of $\mathcal{H}$. The connection to the Euclidean spatial surface Σ_t can then be made as follows: we take each spatial region $\Delta \subseteq \Sigma_t$ and associate it with a projection operator E_Δ onto a subspace of $\mathcal{H}$. Since projection operators have eigenvalues 1 and 0, they have interpretive significance: we follow Mackey (1963) in taking them to represent the true-or-false outcomes of physical experiments. In particular, we take each *spatial projection* E_Δ to represent the experiment, "a macroscopic detection event occurred in the spatial region Δ" (Figure 1).

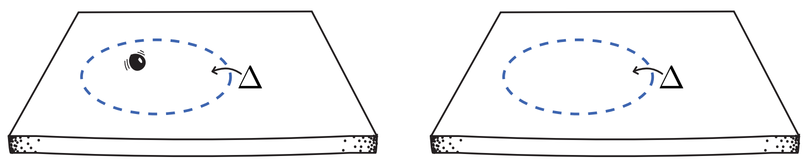


Figure 1. *The projection operator E_Δ has eigenvalue 1 when an experimental detection occurs in the spatial region Δ (left), and eigenvalue 0 when it does not (right).*

This detection event may be a signal from a particle detector in some region of the lab. Philosophers of quantum mechanics are also welcome to read "detection" according to their favorite interpretation, so long as the basic experimental predictions of quantum mechanics are retained. Namely, if the initial state of the system is given by $\psi \in \mathcal{H}$, then the probability of a detection in the region Δ is given by $\langle \psi, E_\Delta \psi \rangle$.

(As a technical aside to those more familiar with Dirac's bra-ket notion: the projection E_Δ is sometimes shirked in favor of Dirac's notation with $|\mathbf{x}\rangle\langle\mathbf{x}|$. I avoid the latter in this treatment, because the ket $|\mathbf{x}\rangle = \delta(\mathbf{x} - \mathbf{x}')$ is not a well-defined vector in the Hilbert space. By dealing with projections E_Δ associated with a *region* Δ instead of a point $\mathbf{x}$, one avoids having to introduce the Dirac delta. This is helpful for the level of mathematical rigor required in the next sections.)

This completes the first step: quantum mechanics can talk about spatial position. In the second step, we need to talk about time.

For this, we recall that by slicing our spacetime into spatial surfaces Σ_t indexed by $t \in \mathbb{R}$, we introduced a time axis. This axis admits a natural notion

[1]A *Hilbert space* $\mathcal{H}$ is a vector space over the complex number field $\mathbb{C}$, equipped with a definite inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, with respect to which it is Cauchy complete. A Hilbert space with a countable basis set is called *separable*.

of "time development" or *translation* forward or backward in time, represented by the group of real numbers under addition $(\mathbb{R}, +)$. Each $t \in \mathbb{R}$ in this group then represents a time development for a duration t.

The connection to Hilbert space $\mathcal{H}$ is now made by assigning each t to an operator $\mathcal{U}_t : \mathcal{H} \to \mathcal{H}$. We do this in a way that is strongly continuous[2], in order to preserve the assumption that systems evolve continuously in time. We also do it in a way that preserves the addition law, $\mathcal{U}_t\mathcal{U}_{t'} = \mathcal{U}_{t+t'}$. The assignment $t \mapsto \mathcal{U}_t$ is then called a *strongly continuous representation* of the group of time translations on $\mathcal{H}$. We interpret it to represent the translation of a pure state $\psi \in \mathcal{H}$ forward or backward in time by a duration t (Figure 2).

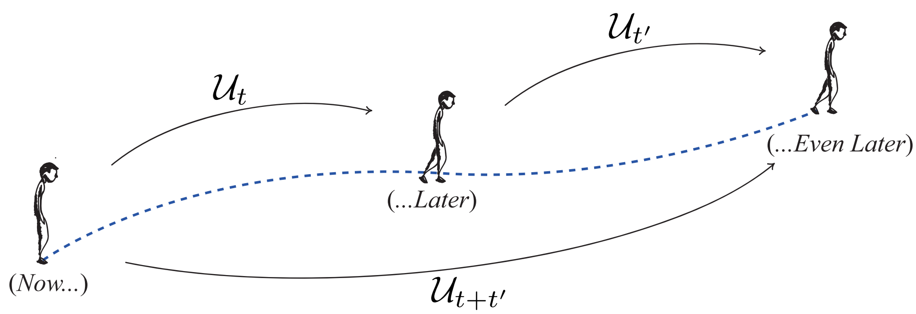


Figure 2. *The operators $\mathcal{U}_t$ represent time translation by a duration t.*

Finally, we take each $\mathcal{U}_t$ in the representation to be a *unitary operator*, in that $\langle \mathcal{U}_t\psi, \mathcal{U}_t\phi \rangle = \langle \psi, \phi \rangle$ for all $\psi, \phi \in \mathcal{H}$. This is grounded in the assumption that the practice of setting up a quantum experiment and predicting a probabilistic outcome is *time translation invariant*; experimental practice does not recognize any preferred moment in time.

In summary, let $\mathcal{H}$ be a Hilbert space with a countably infinite basis set, whose rays represent the pure quantum states of a system; let $\Delta \mapsto E_\Delta$ be a projection valued measure, from regions of space to the lattice of projections on $\mathcal{H}$; and let $t \mapsto U_t$ be a strongly continuous unitary representation of the time translation group $(\mathbb{R}, +)$. The triple $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ contains the basic elements of a quantum description of space and time. It will be the basic object of our analysis.

2.2. **Position and velocity observables.** The projection valued measure $\Delta \mapsto E_\Delta$ on regions of space uniquely defines[3] a self-adjoint operator Q, which I will refer to as the position observable associated with E_Δ, or simply the *position observable*. Those familiar with the formalism will recognize this as the object standing in the

[2]A group of operators is strongly continuous if it is continuous in the Hilbert space norm, for every vector in $\mathcal{H}$. See (Blank et al. 2008, §3.1).

[3]In particular $Q = \int_{\mathbb{R}} \lambda dE_\lambda$, where E_λ is the projection associated with the set $(-\infty, \lambda)$, and the $\int$ is the Lebesgue-Stieltjes integral. See (Jauch 1968, esp. §4.3) for an introduction.

canonical commutation relation,

$$[Q, P]\psi := (QP - PQ)\psi = i\psi, \tag{1}$$

for all ψ in the common dense domain of Q and P. However, note that the triple $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ does not *presume* the existence of a self-adjoint operator P satisfying the canonical commutation relation. Nevertheless, we will now see that this triple does allow us to *construct* an operator $\dot{Q}$, which we may interpret as "velocity," using the given notions of position and time translation. In the next section, we will then introduce assumptions that allow us to *prove* that there exists a non-zero real number μ such that Q and $\mu\dot{Q}$ satisfy the canonical commutation relation (1).

To construct a velocity operator, we make use of our representation of time translation $t \mapsto \mathcal{U}_t$. In the Heisenberg picture, $\mathcal{U}_t$ determines how each operator, such as the position observable Q defined above, changes over time. In particular, at an arbitrary time t, position changes over time as

$$Q(t) = \mathcal{U}_t Q \mathcal{U}_t^{-1},$$

where we note that a unitary operator $\mathcal{U}$ has the property that $\mathcal{U}^* = \mathcal{U}^{-1}$. We will think of "velocity" as the rate of change of this position observable with respect to time, or $\frac{d}{dt}Q(t)$. Without loss of generality, we may consider this quantity at a fixed moment in time $t = 0$; the velocity at an arbitrary time will then be given by the time translation group $\mathcal{U}_t$, just as the position observable was above. We thus define the operator $\dot{Q} := \frac{d}{dt}Q(t)|_{t=0}$, to refer to the rate of change of the position observable in time, evaluated at time $t = 0$.

To get a fix on the particular form of $\dot{Q}$, recall now that $\mathcal{U}_t$ can always be written $\mathcal{U}_t = e^{itH}$, for a unique self-adjoint operator H called the *Hamiltonian*[4]. Then, since $Q = Q(0)$ is independent of time, we have that

$$\begin{aligned}
\dot{Q} :&= \frac{d}{dt}Q(t)|_{t=0} = \frac{d}{dt}\left(\mathcal{U}_t Q \mathcal{U}_t^{-1}\right)|_{t=0} \\
&= \left((\frac{d}{dt}e^{itH})(Qe^{-itH}) + (e^{itH})Q(\frac{d}{dt}e^{-itH}) \right)|_{t=0} \\
&= i(He^{itH}Q - e^{itH}QH)e^{-itH}|_{t=0} \\
&= i[H, Q].
\end{aligned}$$

where the second equality makes use of the chain rule, the third a formal derivative, and the final one evaluates at $t = 0$. This $\dot{Q}$ is a self-adjoint operator, which we may check by observing that $i^* = -i$ and $[H, Q]^* = -[H, Q]$, and hence that $\dot{Q}^* = \dot{Q}$. We thus say that $\dot{Q} = i[H, Q]$ is the velocity observable at time $t = 0$, or more simply the *velocity observable*. It can be constructed whenever the triple $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathcal{U}_t)$ is available to us.

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[4]This fact follows from Stone's theorem (Blank et al. 2008, Thm. 5.9.2).

2.3. **Galilei Invariance.** Ordinary, low-relative-velocity quantum mechanics is Galilei invariant. For us, the experimental significance of this will be the following. Suppose two particle physics experiments are performed in different laboratories, and that the only difference between them is that they were set up in different spatial locations. One may assume that these two experiments will produce the same result; this expresses the fact that particle physics is invariant under *spatial translations*. Similarly, suppose the two experiments are performed at different constant velocities, but are otherwise identical. For example, one experiment might take place on a boat traveling with uniform speed, while the other takes place on shore. These two experiments will again produce the same results. This expresses the fact that particle physics is invariant under *velocity boosts*. In particular, in the present case of ordinary quantum mechanics, these are the Galilei boosts. The two assumptions of invariance under spatial translations and Galilei boosts will be referred to collectively as *Galilei invariance*.

Let us formulate this condition precisely for one of our spatial surfaces Σ_t. To simplify calculations, we choose the $t = 0$ slice Σ_0, since the particular surface in question is irrelevant. Begin by using the triple $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ to define a position observable Q and a velocity observable $\dot{Q}$ as above. We may now take spatial translation by a length $a \in \mathbb{R}$ to have the effect of "translating" the position observable $Q \mapsto Q + aI$, while leaving velocity fixed $\dot{Q} \mapsto \dot{Q}$. Although these "translations" are in fact acting on self-adjoint operators, the terminology is justified by the fact that the mapping translates the spectrum (and thus the measurable values) corresponding to the position observable Q. Indeed, one can check[5] that the mapping $Q \mapsto Q + aI$ is in fact implemented by the mapping $E_\Delta \mapsto E_{\Delta - a}$ on the spatial projections as defined in Section 2.1; that is, the projection assigned to a spatial region Δ is mapped to the projection assigned to the spatial region $\Delta - a$, which is literally a translation of the first by a vector a.

Similarly, we can take a change in velocity by $b \in \mathbb{R}$ to have the effect of boosting the velocity observable $\dot{Q} \mapsto \dot{Q} + bI$, while fixing position[6] $Q \mapsto Q$. We thus arrive at the following.

Definition 1 (Galilei invariance)**.** The structure $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ is *Galilei invariant* only if there exist two strongly continuous one-parameter unitary representations S_a (translations) and R_b (boosts) of the additive group of real numbers,

[5]To verify, consider how $E_\Delta \mapsto E_{\Delta - a}$ effects the position observable Q, defined as $Q := \int_{\mathbb{R}} \lambda dE_\lambda$. By the functional calculus, $\int_{\mathbb{R}} f(\lambda) dE_\lambda = f(Q)$ for any Borel function f. So, $E_\Delta \mapsto E_{\Delta - a}$ has the effect of mapping $Q \mapsto \int_{\mathbb{R}} \lambda dE_{\lambda - a} = \int_{\mathbb{R}} (\lambda + a) dE_\lambda = Q + aI$, where the first equality substitutes $\lambda + a$ for λ, and the second follows from the functional calculus.

[6]Since we normally think of a Galilei boost as transforming a displacement x to $x + vt$, it is clear that boosts can in general have a similar effect on the position observable $Q(t)$; however, we will simplify calculations without loss of generality by choosing to consider only the Σ_0 time slice, on which boosts have no effect on position.

such that

$$S_a Q S_a^{-1} = Q + aI \qquad S_a \dot{Q} S_a^{-1} = \dot{Q}$$
$$R_b Q R_b^{-1} = Q \qquad R_b \dot{Q} R_b^{-1} = \dot{Q} + bI$$

for all $a, b \in \mathbb{R}$.

It may be worth highlighting the significance of taking S_a and R_b to be unitary. Roughly speaking, this captures the assumption that, when an observer in a lab predicts the outcome of an experiment, that prediction will be *independent* of both the position of the lab in space and its velocity.

Here is a way to make this reasoning concrete. Suppose we set up an experimental apparatus, which measures a physical quantity represented by the self-adjoint operator A. Suppose the initial state of the experiment is described by the vector ψ. Then the expectation value for the experiment is given by the inner product, $\langle \psi, A\psi \rangle$. Now, suppose another researcher sets up the same experimental apparatus, but in a lab at a different spatial location. That researcher's apparatus will measure the translated observable[7] $S_a A S_a^{-1}$, and the initial state will be described by the translated vector $S_a \psi$. The expectation value for the translated experiment is thus

$$\langle S_a \psi, (S_a A S_a^{-1}) S_a \psi \rangle = \langle S_a \psi, S_a A \psi \rangle = \langle (S_a^* S_a) \psi, A\psi \rangle,$$

where the first equality follows because $S_a^{-1} S_a = I$, and the second from the definition of the adjoint operation $(^*)$. Similarly, given a researcher performing the experiment at a different uniform velocity, the expectation value will be

$$\langle R_b \psi, (R_b A R_b^{-1}) R_b \psi \rangle = \langle R_b \psi, R_b A \psi \rangle = \langle (R_b^* R_b) \psi, A\psi \rangle.$$

Galilei invariance is meant to capture the assumption that all three of these predictions will agree. This will occur just in case $(S_a^* S_a) = (R_b^* R_b) = I$ is the identity operator, which holds just in case S_a and R_b are unitary[8]. So, we take these operators to be unitary in the definition of Galilei invariance.

The reader familiar with quantum theory may recognize that in fact, Definition 1 expresses that for any $a, b \in \mathbb{R}$, the pairs of operators $(Q, \dot{Q})$, $(Q + aI, \dot{Q})$ and $(Q, \dot{Q} + bI)$ are all *unitarily equivalent*, meaning that each is related to the other by a single unitary transformation. However, it should be emphasized that this definition of Galilei invariance does *not* imply the stronger condition that the Hamiltonian H commute with translations S_a and boosts R_b.

[7]If this seems strange, consider for example the experiment that checks if a particle is in a certain spatial region Δ. The operator representing this experiment is the spatial projection E_Δ. To characterize the same experiment in a different spatial region, we do not use the same projection E_Δ, but rather the translated one $S_a E_\Delta S_a^{-1}$, which corresponds to a particle detection in the translated spatial region $\Delta - a$.

[8]This makes use of the fact that if $\langle \psi, A\psi \rangle = \langle \psi, B\psi \rangle$ for all $\psi \in \mathcal{H}$, then $A = B$ (Messiah 1999, Thm. 1, §XV.2). Note also that both unitary and antiunitary operators have the property that $\mathcal{U}^* \mathcal{U} = I$. But since $t \mapsto \mathcal{U}_t$ is taken to be a strongly continuous representation of $(\mathbb{R}, +)$, the only possibility is then that each $\mathcal{U}_t$ be unitary (Blank et al. 2008, p.354).

2.4. **A minimal ontology.** Roughly speaking, we would now like to characterize the condition that in a given spatial slice Σ_t, all the observables under consideration are "derived entirely from" the position observable. For example, one might notice that composing the position observable with itself gives rise to a new self-adjoint operator, which maps a vector ψ in the domain of Q to $Q^2\psi = (Q \circ Q)\psi$. This operator is different than the position observable. Nevertheless, it is in an obvious sense a "derived entirely" from it. A similar status holds of any polynomial in Q, such as

$$Q^2 + Q + 41.$$

In fact, we need not even restrict ourselves to polynomials; any continuous function[9] of Q is a self-adjoint operator, and the set of all such operators forms an algebra, which we denote $\mathfrak{A}_Q$. It is called the *algebra generated by* Q. In an important sense, the algebra $\mathfrak{A}_Q$ captures the class of operators that are "derived entirely from" the position observable.

This algebra can now be used to describe the central restriction of interest to us: that *all* measurable properties of an ordinary quantum system are derivable entirely from position. Those measurable properties can in general be completely characterized by a set of "simultaneously measurable observables," meaning a set of self-adjoint linear operators on $\mathcal{H}$ that all commute[10]. Suppose we presume that the position observable Q is in that set, and so will commute with all the other simultaneously measurable observables. In order to now express that all the other simultaneously measurable observables can be "derived entirely from position," we need only assert that everything that commutes with Q is in the algebra $\mathfrak{A}_Q$. This property commonly goes under the following title.

Definition 2 (Complete Set of Commuting Observables)**.** A self-adjoint operator Q forms a *complete set of commuting observables* if for every (closed[11]) linear operator A, if $AQ = QA$, then A is in the algebra $\mathfrak{A}_Q$ of functions of Q.

As a technical note relevant for the proof at the end of this paper, let me briefly mention an important consequence of Q forming a complete set of commuting observables. First, let $\{Q\}'$ be the *commutant* of Q, meaning the set of bounded linear operators that commute with Q. Let $\{Q\}''$ be the *extended bicommutant*, meaning the set of closed linear operators such that commute with $\{Q\}'$ on their common domain. Then a generalization of von Neumann's famous bicommutant

[9]Let $f : \mathbb{R} \to \mathbb{C}$ be a Borel function that is defined almost everywhere on the spectral measure $\Delta \mapsto E_\Delta$. Then f defines a function of $Q = \int_{\mathbb{R}} \lambda dE_\lambda$, given by $f(Q) := \int_{\mathbb{R}} f(\lambda) dE_\lambda$; see (Blank et al. 2008, §5.5).

[10]Earman (2008, §5) points out that the existence of such a set can be viewed as a *sine qua non* in the description of a quantum system.

[11]Closure is a technical requirement guaranteeing A will be sufficiently well-behaved. An operator A is *closed* if, whenever a sequence ψ_i in the domain of A is such that $\psi_i \to \psi$ and $A\psi_i \to \phi$, then it follows that ψ is in the domain of A, and $A\psi = \phi$. Closed operators are continuous on their domain, but need not be bounded.

theorem states that $\{Q\}'' = \mathfrak{A}_Q$ (Blank et al. 2008, Theorem 5.5.6); that is, the Borel functions of Q are precisely the elements of the bicommutant of Q. So, one consequence of Q being a complete set of commuting observables is that $\{Q\}' \subseteq \{Q\}''$. Another way of saying this is that, since $\mathfrak{A}_Q$ is commutative, $\mathfrak{A}_Q$ is not a sub-algebra of any closed commutative algebra; that is, $\mathfrak{A}_Q$ is *maximal abelian*.

3. Time reversal invariance

We now have a handle on a what a "minimal ontology" means in ordinary, Galilei invariant quantum mechanics: it is an ontology in which the position observable Q forms a complete set of commuting observables. The main claim of this paper is now, in rough terms, that such quantum systems always admit an operator T such that T-reversal invariance holds – and, in addition, that this T can be reasonably interpreted as 'time reversal.'

Here is how this rough terminology is made precise. First, for T-reversal invariance, we adopt the standard definition:

(i) For any bijection $T : \mathcal{H} \to \mathcal{H}$, we say that the structure $(\mathcal{H}, t \mapsto \mathcal{U}_t)$ is *T-reversal invariant* just in case $T\mathcal{U}_t T^{-1} = \mathcal{U}_{-t}$, for all $t \in \mathbb{R}$.

As one might expect, a system is T-reversal invariant if the group of time translations "reverses sign" under the operation T.

In the discussion of the next section, we will make use of a few other statements of time reversal invariance, which are equivalent whenever T is *antiunitary*, meaning that $\langle T\psi, T\phi\rangle = \langle \psi, \phi\rangle^*$ for all $\psi, \phi \in \mathcal{H}$. In particular, if the bijection $T : \mathcal{H} \to \mathcal{H}$ is antiunitary, then the following statements can be shown[12] to be equivalent:

- $(\mathcal{H}, t \mapsto \mathcal{U}_t)$ is T-reversal invariant in the sense of (i).

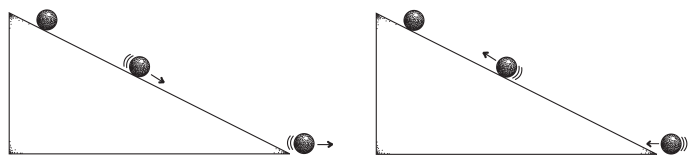
[12] The equivalence of the second and third points was pointed out by Earman (2002, p.248). The equivalence of the first and the third is established as follows. Write $T\mathcal{U}_t T^{-1} = Te^{itH}T^{-1}$ in its Taylor expansion,

$$\begin{aligned} Te^{itH}T^{-1} &= T(I + (itH) + (1/2!)(itH)^2 + \cdots)T^{-1} \\ &= I + T(itH)T^{-1} + (1/2!)T(itH)^2T^{-1} + \cdots \\ &= I + T(itH)T^{-1} + (1/2!)(T(itH)T^{-1})^2 + \cdots \\ &= e^{T(itH)T^{-1}}, \end{aligned}$$

where the penultimate equality follows from the fact that $T^{-1}T = I$. We assumed T is antiunitary, and all antiunitary operators are antilinear, meaning that they conjugate complex numbers. So, $Te^{itH}T^{-1} = e^{T(itH)T^{-1}} = e^{-itTHT^{-1}}$. Thus, if the third point holds and $THT^{-1} = H$, then $Te^{itH}T^{-1} = e^{-itH}$ and we have the first point. Conversely, if the first point holds and $Te^{itH}T^{-1} = e^{-itH}$, then $e^{-itH} = Te^{itH}T^{-1} = e^{-itTHT^{-1}}$. But the since both H and THT^{-1} are self-adjoint, and Stone's theorem guarantees $\mathcal{U}_{-t}$ has a *unique* self-adjoint generator, it follows that $THT^{-1} = H$, and we have the third point.

- If $\psi(t) = \mathcal{U}_t\psi$ (where $\mathcal{U}_t = e^{itH}$ and $\psi \in \mathcal{H}$) is a solution to the Schrödinger equation $i(d/dt)\psi(t) = H\psi(t)$, then $T\psi(-t)$ is also a solution to the Schrödinger equation with the same Hamiltonian H.
- $[T, H]\psi = 0$, for all ψ in the domain of H (and where $\mathcal{U}_t = e^{itH}$).

We have a definition of time reversal invariance. Now, characterizing when an operator T can be reasonably interpreted as "time reversal" is more subtle. Note that we have at our disposal little more than a spatial position observable Q, together with the velocity observable $\dot{Q}$, defined as the rate of change $\dot{Q} := \frac{d}{dt}Q(t)|_{t=0}$. Time reversal in this context is standardly taken to be an antiunitary operator, which preserves states of affairs when applied twice, and which reverses velocities while preserving positions.



This latter claim is made somewhat plausible by our intuitions about films running in reverse. For example, reversing a film of a ball rolling down an inclined plane leads to a depiction in which the directions of velocities are reversed, while all the same positions occur, although they occur in the reverse order. However, a more rigorous justification of the standard definition of time reversal is also possible, as I have shown elsewhere, although I do not have space to discuss that justification here[13]. Instead, I will simply summarize the standard requirements on an adequate "time reversal" operator T as follows.

(ii) T is *faithful* in that it preserves positions and reverses velocities; $TQT^{-1}\psi = Q\psi$ for all $\psi \in \mathcal{D}_Q$, and $T\dot{Q}T^{-1}\phi = -\dot{Q}\phi$ for all $\phi \in \mathcal{D}_{\dot{Q}}$.
(iii) $T : \mathcal{H} \to \mathcal{H}$ is *antiunitary* in that $\langle T\psi, T\phi\rangle = \langle\psi, \phi\rangle^*$, for all $\psi, \phi \in \mathcal{H}$.
(iv) T is an *involution* in that $T^2 = cI$ for some $c \in \mathbb{C}_{unit}$. In other words, T is a 'reversal,' in that applying it twice brings us back to where we started (up to an arbitrary phase factor).

A system is said to be *time reversal invariant* if there exists a bijection $T : \mathcal{H} \to \mathcal{H}$ satisfying T-reversal invariance (i), in addition to the adequacy conditions (ii)-(iv).

With this characterization of the meaning of time reversal invariance in place, we may now turn to our central result.

[13]In particular, we will show in the course of proving the T-theorem that, given our assumptions of Galilei invariance and that Q is a CSCO, Q and $\mu\dot{Q}$ satisfy the canonical commutation relation. The argument development in my "Three Myths about Time Reversal in Quantum Mechanics" may then be used to argue that time reversal has the effects $Q \mapsto Q$ and $\dot{Q} \mapsto -\dot{Q}$, as it is standardly viewed. Note that this is not the case on the "non-standard" view of time reversal discussed by Albert (2000, p.11), Callender (2000, §V), and Maudlin (2007, §4.2). Advocates of the non-standard view may substitute their preferred name for what I am calling "time reversal"; for example, Callender calls it "Wigner time reversal."

Theorem (T Theorem)**.** *Suppose* $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ *is Galilei invariant, and that the self-adjoint operator* Q *associated with* $\Delta \mapsto E_\Delta$ *forms a complete set of commuting observables. Then there exists a bijection* $T : \mathcal{H} \to \mathcal{H}$ *such that the following hold.*

(i) *(T-reversal invariance)* $T\mathcal{U}_t T^{-1} = \mathcal{U}_{-t}$*;*
(ii) *(faithfulness)* $TQT^{-1} = Q$ *and* $T\dot{Q}T^{-1} = -\dot{Q}$*;*
(iii) *(antiunitarity)* T *is antiunitary;*
(iv) *(involution)* $T^2 = cI$ *for some* $c \in \mathbb{C}_{unit}$*;*

Moreover, this T *is unique up to an arbitrary constant.*

The force of the result is that in Galilei invariant quantum theory, if all measurable properties are derivable from position, then time reversal invariance is guaranteed. Equivalently, if one believes that there is a preferred direction of time the level of fundamental quantum interactions, and hence that time reversal invariance fails, then there must be measurable properties that are not functions of the spatial position observable Q.

4. Discussion of the Proof

At first glance, the T Theorem may seem mysterious. How is the connection established between a claim about which measurable properties are available, and a claim about time reversal invariance? In this section, I would like to discuss the central factors that establish this connection.

In the next subsections, I will briefly comment on how some T-violating systems can be prohibited through the requirement of Galilei invariance premise, and then how more can be prohibited by the requirement that Q forms a complete set of commuting observables. I will then sketch an overview of the argument underpinning the T Theorem; a complete statement of the proof is found in the Appendix.

4.1. The significance of Galilei invariance.

Without Galilei invariance, nothing prevents the existence of a Hamiltonian of the form,

$$H = \dot{Q}.$$

With this Hamiltonian, T-reversal invariance will always fail if T satisfies the adequacy conditions set out above. In particular, a time reversal operator T satisfying the faithfulness condition $T\dot{Q}T^{-1} = -\dot{Q}$ cannot be such that $THT^{-1} = H$, so long as $H = \dot{Q}$. As we noted in the previous section, the failure of $THT^{-1} = H$ is equivalent to the failure of time reversal invariance $T\mathcal{U}_t T^{-1} = \mathcal{U}_{-t}$.

However, note that invariance under Galilei boosts also fails when $H = \dot{Q}$. To verify, recall our definition $\dot{Q} := i[H, Q]$ from the first section. Substituting $H = \dot{Q}$ then allows one to write,

$$\dot{Q} = i[\dot{Q}, Q]. \tag{2}$$

Given this equation, there cannot exist a group R_b that "boosts velocity while fixing position," in that $R_b\dot{Q}R_b^{-1} = \dot{Q}+b$ and $R_bQR_b^{-1} = Q$. This can be seen immediately by surrounding both sides of this equation with R_b and R_b^{-1}, and noticing that this fixes the operator on the right hand side but not on the left[14]. Such a group thus cannot exist here, because it would contradict Equation (2).

The important point now is that if we *require* Galilei invariance of our quantum systems, then the Hamiltonian $H = \dot{Q}$ is not allowed. We will of course need a more general argument than this. However, the example gives an idea of how Galilei invariance prohibits *some* of the Hamiltonians for which time reversal invariance fails.

4.2. **The significance of Q forming a CSCO.** Let S be a set of commuting self-adjoint operators that completely describes the measurable properties of a quantum system, and let one of those operators be Q. If Q did not form a complete set of commuting observables, then there could be another observable σ in S that is not a function of Q. Suppose such an operator σ changes sign under time reversal $T\sigma T^{-1} = -\sigma$ (as for example the "intrinsic angular momentum" or "spin" observable does). This is possible independently of how T transforms Q, because σ is by assumption not a function of Q.

This operator σ can now enter into a Hamiltonian in a way that violates time reversal invariance. Consider the Hamiltonian,

$$H = \frac{\mu}{2}\dot{Q}^2 + \sigma.$$

Then time reversal invariance fails, since $THT^{-1} = (\mu/2)\dot{Q}^2 - \sigma \neq H$. This illustrates how self-adjoint operators that commute with Q, but are not functions of Q, can lead to violations of T-reversal invariance[15]. Requiring that Q be a complete set of commuting observables prohibits the existence of this example.

These examples are not intended to provide a general result, but only to give some insight into how the premises of the T Theorem are relevant. To develop an idea of how the general result is obtained, let us now turn the proof itself.

4.3. **Overview of the proof.** The strategy of the T Theorem is to first establish how its two premises, that Galilei invariance holds, and that Q is a complete set of commuting observables, turn out to *severely* restrict the form that the Hamiltonian H can take. In fact, these premises imply that H can only take the form,

$$H = \frac{\mu}{2}\dot{Q}^2 + v(Q),$$

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[14]The explicit calculation: $R_b i[\dot{Q}, Q]R_b^{-1} = i[R_b\dot{Q}R_b^{-1}, Q] = i[\dot{Q} + bI, Q] = i[\dot{Q}, Q]$, so the right hand side is fixed. On the other hand, $R_b\dot{Q}R_b^{-1} = \dot{Q} + bI$, so the left hand side is not.

[15]Indeed, suppose we assume not only that σ commutes with Q, but also that σ is unaffected by spatial translations (S_a) or velocity boosts (R_b): $S_a\sigma S_a^{-1} = R_b\sigma R_b^{-1} = \sigma$. Then the example seems to be perfectly compatible with Galilei invariance, in spite of failing to be time reversal invariant.

where μ is a non-zero real number, and $v(Q)$ is a function of Q alone. This Hamiltonian is in effect the "standard" one, which is the sum of a kinetic energy term $\frac{\mu}{2}\dot{Q}^2$ and a potential function $v(Q)$. Having shown that H must be of this form, the proof of the T theorem then proceeds to choose T to be the usual time reversal operator found in textbooks on this topic, and show that it uniqely satisfies conditions (i)-(iv).

The first step of the proof, which restricts the form of the Hamiltonian H, is a slight adaptation of a lemma due to Jauch (1968, §13-4). Our statement of the lemma is not as strong as Jauch's original claim, but is sufficient for our purposes[16].

Lemma (Jauch)**.** *Suppose $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ is Galilei invariant, and that the self-adjoint operator Q associated with $\Delta \mapsto E_\Delta$ forms a complete set of commuting observables. Then $(\mathcal{H}, e^{ibQ}, e^{ia\mu\dot{Q}})$ is an irreducible unitary representation of the canonical commutation relations in Weyl form, and $\mathcal{U}_t = e^{itH}$, where*

$$H\psi = \frac{\mu}{2}\dot{Q}^2\psi + v(Q)\psi$$

for some non-zero real number μ, some Borel function v, and for all ψ in the domain of H.

A proof of Jauch's lemma is given in the Appendix. The central result of interest is the restricted form of the Hamiltonian mentioned before, as the sum of a kinetic energy term $\frac{\mu}{2}\dot{Q}^2$ and a potential term $v(Q)$ in position alone.

For readers not familiar with the Weyl form of the canonical commutation relation, it is here the statement that

$$e^{ia\mu\dot{Q}}e^{ibQ} = e^{iab}e^{ibQ}e^{ia\mu\dot{Q}}. \tag{3}$$

This equation is appealing to mathematicians, in dealing only with bounded operators. It also turns out to imply[17] the "standard" commutation relation,

$$[Q, \mu\dot{Q}]\psi = i\psi. \tag{4}$$

The main work in proving the lemma goes into showing that these commutation relations follow from our assumptions. This requires a bit of representation theory, which is difficult to describe in simple terms; the result is given in full rigor in the proof at the end of this paper. However, once Equation (4) is in hand, it is instructive to see how the restricted form of the Hamiltonian H falls out almost immediately. Here is how that remaining part runs.

[16]Jauch over-optimistically sought to prove that a related result guarantees minimal electromagnetic coupling. This part of the argument has been criticized by Kraus (1980) and by Brown and Holland (1999), and is unnecessary in the present context.

[17]This follows quickly from the fact that $Q\psi = \lim_{\alpha\to 0}\frac{1}{i\alpha}(e^{iaQ} - I)\psi$ given any ψ for which the limit exists, and similarly for $\dot{Q}\psi$; see (Jauch 1968, p.199).

First, as a direct consequence of Equation (4), one may notice[18] that

$$i[\frac{\mu}{2}\dot{Q}^2, Q]\psi = \dot{Q}\psi. \tag{5}$$

This commutation relation differs only in the first term from the one appearing in the definition of $\dot{Q}$:

$$i[H, Q]\psi = \dot{Q}\psi. \tag{6}$$

So, subtracting Equation (5) from Equation (6), and using the linearity of the commutator bracket, we find that

$$[H - \frac{\mu}{2}\dot{Q}^2, Q]\psi = 0. \tag{7}$$

Now comes the real significance of the assumption that Q forms a complete set of commuting observables. By our definition, this means that whenever an operator commutes with Q, that operator is in fact function of Q (Definition 2). Precisely because of this fact, Equation (7) implies that $H - (\mu/2)\dot{Q}^2$ is a function of Q. Call that function $v(Q)$. Then we have our result: $H - (\mu/2)\dot{Q}^2 = v(Q)$, or

$$H = \frac{\mu}{2}\dot{Q}^2 + v(Q).$$

On the other hand, if Q *did not* form a complete commuting set, then the function v might include some self-adjoint operators other than Q, and time reversal invariance would not be guaranteed.

Completing the proof of the T Theorem is now a simple matter. Roughly speaking, we may choose the "standard" time reversal operator T with respect to the representation of the commutation relations we constructed with Q and $\mu\dot{Q}$. Namely, we take T to be an antiunitary conjugation operator[19] K in this representation. This operator conjugation operator satisfies the adequacy conditions (ii)-(iv), and in particular reverses the sign of $\dot{Q}$, while acting identically on Q. The former implies that $T\dot{Q}^2T^{-1} = (T\dot{Q}T^{-1})^2 = (-\dot{Q})^2 = \dot{Q}^2$, while the latter implies that $Tv(Q)T^{-1} = v(Q)$. It follows that $THT^{-1} = H$, which establishes time reversal invariance.

I hope this helps the reader have some (admittedly rough) idea as to why the T Theorem holds. The proof may be found in full detail in the Appendix.

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[18] This is less mysterious if one has in mind that whenever $[Q, P] = i$, the commutator bracket behaves like a derivative operator. For example, if $P^n = P \circ P \circ \cdots \circ P$ (n times), then one can check that $[Q, P^n] = i(n-1)P^{(n-1)}$. Thus when $(1/2)P^2$, one has that $[Q, (1/2)P^2] = iP$. Now, suppose we have that $[Q, \mu\dot{Q}] = i$ as in Equation (4). Then $[Q, (1/2)(\mu\dot{Q})^2] = i(\mu\dot{Q})$. Multiplying both sides by $-i/\mu$, we now have the stated result that $i[(\mu/2)\dot{Q}^2, Q] = \dot{Q}$.

[19] Messiah (1999, §XV.5) provides an overview of conjugation operators in a representation.

5. Conclusion

Those whose basic ontological commitments include only position have a new commitment to contend with: if the ontology of Galilean quantum theory is derived entirely from the position observable, then time reversal invariance is guaranteed. As it turns out, this kind of result is not unique to quantum theory: one can prove a related theorem in classical Hamiltonian mechanics (Roberts 2011). This suggests a fairly robust sense in which a minimal ontology prohibits time asymmetric phenomena. The result may also raise new issues in the interpretation of quantum mechanics. For example, one may wonder how the de Broglie-Bohm interpretation of quantum mechanics can account for time asymmetric phenomena, given that this interpretation takes spatial position to be the only measurable property that a particle can have. It seems that if time asymmetric phenomena are to be accounted for, then this interpretation cannot identify "position" in its ontology with the spectrum of the position observable Q. Otherwise, to say that all measurable properties are functions of position would prohibit the possibility of time asymmetry.

I do not wish to take a position on the interpretation of quantum mechanics in light of this result. What I would like to commit to is the claim that in Galilei invariant quantum theory, it is no accident that time asymmetric systems admit measurable properties that are not a function of position. Such properties are absolutely essential to the phenomenon of time asymmetry.

6. Appendix

Lemma (Jauch)**.** *Suppose* $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ *is Galilei invariant, and that the self-adjoint operator* Q *associated with* $\Delta \mapsto E_\Delta$ *forms a complete set of commuting observables. Then* $(\mathcal{H}, e^{ibQ}, e^{ia\mu\dot{Q}})$ *is an irreducible unitary representation of the canonical commutation relations in Weyl form, and* $\mathcal{U}_t = e^{itH}$*, where*

$$H\psi = \frac{\mu}{2}\dot{Q}^2\psi + v(Q)\psi$$

for some non-zero real number μ*, some Borel function* v*, and for all* ψ *in the domain of* H*.*

Proof. Define Q and $\dot{Q}$ as above, and recall our definition of Galilei invariance:

$$S_a Q S_a^{-1} = Q + aI \qquad S_a \dot{Q} S_a^{-1} = \dot{Q}$$
$$R_b Q R_b^{-1} = Q \qquad R_b \dot{Q} R_b^{-1} = \dot{Q} + bI$$

where we note that the unbounded operators Q and $\dot{Q}$ act only on their respective domains of definition in $\mathcal{H}$. The essential observation of this proof is that Galilei invariance implies the existence of three representations of the Weyl form of the canonical commutation relations:

(i) $S_a e^{ibQ} S_a^* = e^{iab} e^{ibQ}$
(ii) $R_b e^{ia\dot{Q}} R_b^* = e^{iab} e^{ia\dot{Q}}$

(iii) $S_a R_b^* S_a^* = e^{i\mu ab} R_b^*$

where μ is a fixed non-zero real number. Representations (i) and (ii) follow immediately from our expression of Galilei invariance; we simply exponentiate Q and $\dot{Q}$, and recognize that the unitary operators acting on them may be pulled up into the exponential.

Representation (iii) is somewhat more subtle, being constructed from the fact that the unitary representations S_a and R_b are each defined to be a representation of $\mathbb{R}$, and thus together provide a projective representation of the real plane $\mathbb{R} \times \mathbb{R}$. In particular, it is part of the definition and S_a and R_b that

$$S_a S_b = S_{a+b}, \quad R_a^* R_b^* = R_{a+b}^*. \tag{8}$$

Hence, the mapping $(a, b) \mapsto (S_a, R_b^*)$ is a homomorphism from the vectors $v = (a, b)$ of $\mathbb{R} \times \mathbb{R}$ into the pairs of unitary operators. This means that the group (S_a, R_b^*) is a projective representation of the plane $(\mathbb{R} \times \mathbb{R}, +)$ under vector addition. Since the latter group is abelian and hence satisfies $(a, 0) + (0, b) - (a, 0) = (0, b)$, it follows that the same group properties will hold in the projective representation up to a phase factor $e^{if(a,b)}$:

$$S_a R_b^* S_a^* = e^{if(a,b)} R_b^*$$

for some $f(a, b) \in \mathbb{R}$. Moreover, it is known that the representation may always be chosen such that $f(a, b) = \mu ab$ (Blank et al. 2008, §10 Problem 15). Thus we have the representation expressed in (iii): $S_a R_b^* S_a^* = e^{i\mu ab} R_b^*$.

With a bit of work, (i), (ii) and (iii) can now be shown to imply that Q and $\mu\dot{Q}$ satisfy the canonical commutation relation. Showing this involves three steps.

First, we note that the representation expressed in (ii) implies that $e^{i\mu ab} R_b^* = e^{ia\mu\dot{Q}} R_b^* e^{-ia\mu\dot{Q}}$. Plugging this result into (iii), we find that $e^{ia\mu\dot{Q}} R_b^* e^{-ia\mu\dot{Q}} = S_a R_b^* S_a^*$, and hence that

$$R_{b/\mu}^*(S_a^* e^{ia\mu\dot{Q}}) = (S_a^* e^{ia\mu\dot{Q}}) R_{b/\mu}^*. \tag{9}$$

where we have substituted b/μ in for b, recognizing that the equation holds for any real value of b.

Keeping Equation (9) in mind, we now proceed to the second step, which is to show that $R_{b/\mu}$ is a constant multiple of e^{iaQ}. This draws on the fact that representation (i) contains a term e^{iaQ} for which Q forms a complete set of commuting observables, which implies that the representation is irreducible (Blank et al. 2008, Ex. 6.7.2e). We apply the irreducibility of representation (i) as follows. Rewrite (iii) as

$$S_a R_{b/\mu} S_a^* = e^{-iab} R_{b/\mu}.$$

We know from (i) that $S_a e^{ibQ} S_a^* = e^{iab} e^{ibQ}$. So, multiplying the left sides of these two equations as well as the right sides, we see that

$$(S_a R_{b/\mu} S_a^*)(S_a e^{ibQ} S_a^*) = e^{-iab} e^{iab} (R_{b/\mu} e^{ibQ}),$$

and so since $S_a^* S_a = e^{-iab}e^{iab} = I$, we have that

$$S_a(R_{b/\mu}e^{ibQ})S_a^* = (R_{b/\mu}e^{ibQ})$$

This says that the operator $R_{b/\mu}e^{ibQ}$ commutes with S_a. But the same operator also commutes with e^{ibQ}, since

$$R_{b/\mu}e^{ibQ}R_{b/\mu}^* = e^{ib(R_{b/\mu}QR_{b/\mu}^*)} = e^{ibQ}.$$

Schur's lemma (Blank et al. 2008, Thm. 6.7.1) establishes that multiples of the identity are the only operators that commute with both terms in an irreducible representation. So, since $R_{b/\mu}e^{ibQ}$ commutes with both S_a and e^{iaQ}, we may write $R_{b/\mu}e^{ibQ} = cI$, which implies that

$$R_{b/\mu}^* = \frac{1}{c}e^{ibQ}$$

as claimed.

The third step now substitutes this $R_{b/\mu}^*$ into Equation (9), to get that

$$\frac{1}{c}e^{ibQ}(S_a^* e^{ia\mu\dot{Q}}) = (S_a^* e^{ia\mu\dot{Q}})\frac{1}{c}e^{ibQ},$$

or equivalently,

$$e^{ia\mu\dot{Q}}e^{iaQ} = (S_a e^{iaQ} S_a^*)e^{ia\mu\dot{Q}}.$$

Applying (i) to the right-hand side of this equation, we finally see that $e^{ia\mu\dot{Q}}e^{iaQ} = e^{iab}e^{ibQ}e^{ia\mu\dot{Q}}$, which is the desired representation of the commutation relations in Weyl form. It is irreducible for the same reason that representation (i) is, namely because Q forms a complete set of commuting observables (Blank et al. 2008, Ex. 6.7.2e).

It is now straightforward to determine the form of $\mathcal{U}_t$ in this representation. First, we note that $\mathcal{U}_t = e^{itH}$ for a unique self-adjoint H by Stone's theorem. Moreover, the canonical commutation relation in Weyl form implies the "standard" commutation relation $[Q, \mu\dot{Q}]\psi = i\psi$, for all ψ in the common dense domain of Q and $\dot{Q}$. This in turn implies that $[Q, (1/2)(\mu\dot{Q})^2] = i(\mu\dot{Q})$, which we multiply through by $-i/\mu$ to find that

$$i[(\mu/2)\dot{Q}^2, Q] = \dot{Q}.$$

But by definition, $\dot{Q} = i[H, Q]$, where H is the self-adjoint generator of $\mathcal{U}_t$. We may thus equate $i[(\mu/2)\dot{Q}^2, Q]$ and $i[H, Q]$, which implies that

$$[(H - \frac{\mu}{2}\dot{Q}^2), Q] = 0.$$

Since Q forms a complete set of commuting observables, all operators that commute with Q are Borel functions of it. So the fact that $H - \frac{1}{2}\mu\dot{Q}^2$ commutes with Q

implies that it is in fact a function of Q. Call that function $v(Q)$. Then we have the desired result,

$$H = \frac{\mu}{2}\dot{Q}^2 + v(Q),$$

which proves the lemma. □

Theorem (T Theorem)**.** *Suppose $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto \mathcal{U}_t)$ is Galilei invariant, and that the self-adjoint operator Q associated with $\Delta \mapsto E_\Delta$ forms a complete set of commuting observables. Then there exists a bijection $T : \mathcal{H} \to \mathcal{H}$ such that the following hold.*

(i) *(T-reversal invariance) $T\mathcal{U}_tT^{-1} = \mathcal{U}_{-t}$;*
(ii) *(faithfulness) $TQT^{-1} = Q$ and $T\dot{Q}T^{-1} = -\dot{Q}$;*
(iii) *(antiunitarity) T is antiunitary;*
(iv) *(involution) $T^2 = cI$ for some $c \in \mathbb{C}_{unit}$;*

Moreover, this T is unique up to an arbitrary constant.

Proof. Let $(\mathcal{H}, e^{ia\mu\dot{Q}}, e^{ibQ})$ be the representation guaranteed by Jauch's lemma. We begin by constructing a distinct (Schrödinger) representation, and define a "conjugation operator" K_Q with respect to Q in that representation. This K_Q can then be used to construct an operator T satisfying conditions (i)-(iv).

Following a well-known procedure (Jauch 1968, §12.3), the operator Q can be used to construct a Hilbert space $\mathcal{H}_Q$ of square-integrable functions in which Q is the multiplication operator: $Q\psi(x) = x\psi(x)$. In this representation, define the operator P on the differentiable functions in $\mathcal{H}_Q$ by $P\psi(x) := i(d/dx)\psi(x)$. The operators Q and P together form an irreducible unitary representation of the canonical commutation relations, called the Schrödinger representation. In Weyl form, we may denote this representation, $(\mathcal{H}, e^{iaP}, e^{ibQ})$.

Next, we define the operator $K_Q : \mathcal{H}_Q \to \mathcal{H}_Q$ to be the operator that takes each square integrable function in the Schrödinger representation to its complex conjugate: $K_Q\psi(x) = \psi^*(x)$. We follow Messiah (1999, §XV.5) in noting three relevant properties of K_Q. First, it is antiunitary, since $\langle K_Q\psi(x), K_Q\phi(x)\rangle = \langle\psi^*(x), \phi^*(x)\rangle = \langle\psi(x), \phi(x)\rangle^*$. Second, it is an involution, since $K_Q^2\psi(x) = \psi^{**}(x) = \psi(x)$. Finally, since Q is pure real and P is pure imaginary in this representation, K_Q has the property that $K_QQK_Q = Q$ and $K_QPK_Q = -P$.

We will use this K_Q to construct our desired time reversal operator. The Stone von Neumann theorem (Blank et al. 2008, Theorem 8.2.4) guarantees that the Jauch representation of the Weyl commutation relations is unitarily equivalent to the Schrödinger representation. In particular, there must exist a unitary bijection from the Schrödinger Hilbert space to the Jauch Hilbert space $W : \mathcal{H} \to \mathcal{H}_Q$ such that $WQW^* = Q$ and $WPW^* = \mu\dot{Q}$. We now define our time reversal operator to be the image of K_Q under this mapping:

$$T := WK_QW^*.$$

One may quickly verify that this T inherits properties (ii)-(iv) from K_Q. In particular, it is an involution:

$$T^2 = (WK_QW^*)(WK_QW^*) = WK_Q^2W^* = I.$$

It is the composition of two unitaries W and W^* with one antiunitary K_Q and is therefore antiunitary. And, it has the desired effect on Q and $\mu\dot{Q}$:

$$\begin{aligned} TQT^{-1} &= (WK_QW^*)(WQW^*)(WK_QW^*) \\ &= W(K_QQK_Q)W^* = WQW^* = Q \\ T\mu\dot{Q}T^{-1} &= (WK_QW^*)(WPW^*)(WK_QW^*) \\ &= W(K_QPK_Q)W^* = -(WPW^*) = -\mu\dot{Q}. \end{aligned}$$

The remaining point to prove is (i). Note that since $H = (\mu/2)\dot{Q}^2 + v(Q)$ by Jauch's lemma, and since $\mu\dot{Q}$ and H are both self-adjoint, we know that $v(Q)$ is self-adjoint as well. So, applying T to both sides of H, we get

$$THT^{-1} = \frac{\mu}{2}(T\dot{Q}T^{-1})^2 + Tv(Q)T^{-1} = \frac{\mu}{2}\dot{Q}^2 + Tv(Q)T^{-1}, \tag{10}$$

where by the functional calculus, $v(Q) = \int_{\mathbb{R}} v(\lambda)dE_\lambda$. But $v(Q)$ is known to be self-adjoint. The $v(\lambda)$ are thus real, and

$$Tv(Q)T^{-1} = \int_{\mathbb{R}} v(\lambda)dTE_\lambda T^{-1} = \int_{\mathbb{R}} v(\lambda)dE_\lambda = v(Q).$$

From this, together with (10), it follows that $THT^{-1} = H$, which establishes time reversal invariance.

We finally show that this T is unique up to a constant. Suppose that both T and $\tilde{T}$ satisfy conditions (i)-(iv). In particular, suppose that they are both antilinear involutions satisfying

$$\begin{aligned} TQT^{-1} &= Q & \qquad \tilde{T}Q\tilde{T}^{-1} &= Q \\ T\dot{Q}T^{-1} &= -\dot{Q} & \qquad \tilde{T}\dot{Q}\tilde{T}^{-1} &= -\dot{Q}. \end{aligned}$$

Then $T\tilde{T}$ is a linear operator that commutes with both Q and $\mu\dot{Q}$, since

$$\begin{aligned} (T\tilde{T})Q(\tilde{T}^{-1}T^{-1}) &= TQT^{-1} = Q \\ (T\tilde{T})\mu\dot{Q}(\tilde{T}^{-1}T^{-1}) &= T(-\mu\dot{Q})T^{-1} = \mu\dot{Q}. \end{aligned}$$

But the representation $(Q, \mu\dot{Q})$ provided by Jauch's lemma is irreducible. By Schur's lemma, this implies that the only linear operators commuting with both Q and $\mu\dot{Q}$ are constant multiples of the identity. So, for some $k \in \mathbb{C}$,

$$kI = T\tilde{T} = T\tilde{T}^{-1},$$

where we have used the fact that $\tilde{T}$ is an involution in the second equality. Therefore, $T = k\tilde{T}$ as claimed. □

References

AIP (1963). Interview of Werner Heisenberg by T. S. Kuhn on February 15, 1963, Session V, Niels Bohr Library & Archives, American Institute of Physics, College Park, MD USA.
URL: *http://www.aip.org/history/ohilist/4661_5.html*

Albert, D. Z. (2000). *Time and chance*, Harvard University Press.

Blank, J., Exner, P. and Havlíček, M. (2008). *Hilbert Space Operators in Quantum Physics*, 2nd edn, Springer Science and Business Media B.V.

Boyle, R. (1772). *The Works of the Honourable Robert Boyle*, Vol. 1, London: J. and F. Rivington.

Brown, H. R. and Holland, P. (1999). The Galilean covariance of quantum mechanics in the case of external fields, *American Journal of Physics* **67**(3): 204–214.

Callender, C. (2000). Is Time 'Handed' in a Quantum World?, *Proceedings of the Aristotelian Society*, Aristotelian Society, pp. 247–269.

Earman, J. (2002). What time reversal is and why it matters, *International Studies in the Philosophy of Science* **16**(3): 245–264.

Earman, J. (2008). Superselection Rules for Philosophers, *Erkenntnis* **69**(3): 377–414.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley Series in Advanced Physics, Addison-Wesley Publishing Company, Inc.

Kraus, K. (1980). Galilei Covariance Does Not Imply Minimal Electromagnetic Coupling, *Annalen der Physik* **492**(2): 81–101.

Mackey, G. W. (1963). *Mathematical foundations of quantum mechanics*, New York: W. A. Benjamin, Inc.

Maudlin, T. (2007). *The Metaphysics Within Physics*, New York: Oxford University Press.

Messiah, A. (1999). *Quantum Mechanics, Two Volumes Bound as One*, Dover.

Roberts, B. W. (2011). An earnest account of why classical mechanics is time reveral invariant. Unpublished manuscript, draft available upon request.