Derivation of the Meaning of the Wave Function

Shan Gao*

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Abstract

We show that the physical meaning of the wave function can be derived based on the established parts of quantum mechanics. It turns out that the wave function represents the state of random discontinuous motion of particles, and its modulus square determines the probability density of the particles appearing in certain positions in space.

The wavefunction gives not the density of stuff, but gives rather (on squaring its modulus) the density of probability. Probability of what exactly? Not of the electron being there, but of the electron being found there, if its position is 'measured'. Why this aversion to 'being' and insistence on 'finding'? The founding fathers were unable to form a clear picture of things on the remote atomic scale. (Bell 1990)

The meaning of the wave function in quantum mechanics is often analyzed in the context of conventional impulse measurements. Although the wave function of a quantum system is in general extended over space, an ideal position measurement will inevitably collapse the wave function and can only detect the system in a random position in space. Thus it seems natural to assume the wave function is only related to the probabilities of these random measurement results as in the standard probability interpretation. However, it has been widely argued that the probability interpretation is not wholly satisfactory because of resorting to the vague concept of measurement (Bell 1990). On the other hand, although the wave function is regarded as a physical entity in some alternative formulations of quantum mechanics such as the de Broglie-Bohm theory and the many-worlds interpretation (de Broglie 1928; Bohm 1952; Everett 1957; De Witt and Graham 1973), it remains unclear what physical entity the wave function really represents. One of the main reasons, as we think, is that conventional impulse measurements

*Unit for HPS and Centre for Time, University of Sydney, NSW 2006, Australia. E-mail: sgao7319@uni.sydney.edu.au.

can hardly provide useful information about the wave function of a single quantum system beyond the probability interpretation[1].

Fortunately, it has been known that the physical state of a single quantum system can be protectively measured (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993; Aharonov, Anandan and Vaidman 1996; Vaidman 2009). A general method is to let the measured system be in a nondegenerate eigenstate of the whole Hamiltonian using a suitable protective interaction (in some situations the protection is provided by the measured system itself), and then make the measurement adiabatically so that the state of the system neither changes nor becomes entangled with the measuring device appreciably. In this way, such protective measurements can measure the expectation values of observables on a single quantum system without disturbing its state, and in particular, the mass and charge density of a quantum system as one part of its physical state, as well as its wave function, can be measured as expectation values of certain observables. Since the principle of protective measurement is irrelevant to the controversial process of wavefunction collapse and only based on the linear Schrödinger evolution and the Born rule, which are two established parts of quantum mechanics, its results as predicted by quantum mechanics can be used to investigate the real meaning of the wave function[2].

According to protective measurement, the charge of a charged quantum system (e.g. the basic charge of an electron) is distributed throughout space, and the charge density in each position is proportional to the modulus square of the wave function of the system there. The key to unveil the meaning of the wave function is to find the origin of the charge distribution. The charge distribution has two possible existent forms: it is either real or effective. The charge distribution is real means that it exists throughout space at the same time, and the charge distribution is effective means that there is only a localized particle with the total charge of the system at every instant, and its ergodic motion forms the effective charge distribution. If the charge distribution is effective, then there will exist no electrostatic self-interaction of the effective charge distribution, as there is only a localized charged particle at every instant. By contrast, if the charge distribution is real, then there will exist electrostatic self-interaction of the real charge distribution, as the charge distribution exist throughout space at the same time[3]. Since the superposition principle of quantum mechanics prohibits the

[1]For a recent interesting result based on conventional impulse measurements see Pusey, Barrett and Rudolph (2011).

[2]It can be expected that protective measurement will be realized in the near future with the rapid development of quantum technologies.

[3]That a real charge distribution has electrostatic self-interaction has been confirmed not only in the classical domain but also in the quantum domain for many-body systems. For example, two charged quantum systems such as two electrons have electrostatic interaction, and thus a real charge distribution containing these two charges has electrostatic self-interaction.

existence of electrostatic self-interaction, and especially, the existence of the electrostatic self-interaction of the charge distribution of an electron already contradicts experimental observations (Gao 2011), the charge distribution of a quantum system cannot be real but must be effective. This means that for a charged quantum system, at every instant there is only a localized particle with the total charge of the system, while during a time interval the time average of the ergodic motion of the particle forms the effective charge distribution, and the charge density in each position is proportional to the modulus square of the wave function of the system there.

The next question is which sort of ergodic motion the particle undergoes. If the ergodic motion of the particle is continuous, then it can only form the effective charge distribution during a finite time interval. However, the charge density of a particle, which is proportional to the modulus square of its wave function, is an instantaneous property of the particle. In other words, the ergodic motion of the particle must form the effective charge density during an infinitesimal time interval (*not* during a finite time interval) at a given instant. We can also reach this conclusion by analyzing a specific example. Consider an electron in a superposition of two energy eigenstates in two boxes. Even if the electron can move with infinite velocity, it cannot *continuously* move from one box to the other due to the restriction of box walls. Therefore, any type of continuous motion cannot generate the effective charge density proportional to the modulus square of the superposition state[4]. In conclusion, the ergodic motion of the particle cannot be continuous but must be discontinuous. Moreover, in order to form the charge distribution predicted by quantum mechanics, the probability density of the particle appearing in each position must be equal to the modulus square of its wave function there.

The above result implies that the wave function of a quantum system describes the state of random discontinuous motion of a localized particle representing the system, and the modulus square of the wave function gives the probability density of the particle appearing in certain position in space. However, there may exist a deeper level of meaning of the wave function. From a logical point of view, for the random discontinuous motion of a particle, there should exist a probabilistic instantaneous condition that determines the probability density of the particle appearing in every position in space; otherwise it would not "know" how frequently it should appear in each position in space. This condition cannot come from otherwhere but must come from the particle itself. In other words, the particle must have an instantaneous property that determines its motion in a probabilistic way. This property is usually called indeterministic disposition or propensity in the literature[5]. Therefore, at a deeper level, the wave function of a quantum

[4]For a more detailed argument for the discontinuity of the ergodic motion see Gao (2011).

[5]The propensity here denotes single case propensity. In addition, it is worth stress-

particle represents the dispositional property of the particle that determines its random discontinuous motion, and its modulus square determines the probability density of the particle appearing in certain position in space. In this sense, we may say that the motion of a particle is "guided" by its wave function in a probabilistic way.

The picture of random discontinuous motion of a single particle can be extended to the motion of many particles. For a many-particle system, its wave function represents the propensity property of the whole system, and the modulus square of the wave function determines the probability density of the particles appearing in certain positions in space. In addition, the picture of random discontinuous motion may exist not only for position but also for other dynamical variables such as momentum and energy, and thus the above interpretation of the wave function in position space may also apply to the wave function in momentum space etc. Due to the randomness of motion for each variable, the probability distributions of all variables for an arbitrary wave function can be consistent with quantum mechanics[6].

To sum up, we have shown that quantum mechanics already spells out the real meaning of the wave function. There are three steps to reach this conclusion. First, protective measurement, which is based on the established parts of quantum mechanics, shows that the charge of a charged quantum system is distributed throughout space, and the charge density in each position is proportional to the modulus square of its wave function there. Next, the superposition principle of quantum mechanics requires that the charge distribution is not real but effective, that is, it is formed by the ergodic motion of a localized particle with the total charge of the system. Lastly, in order to form the charge distribution predicted by quantum mechanics, the ergodic motion of the particle must be discontinuous and random, and the probability density of the particle appearing in every position must be equal to the modulus square of its wave function there. Therefore, according to quantum mechanics, the wave function describes the state of random discontinuous motion of particles, and at a deeper level, it represents the dispositional property of the particles that determines their random discontinuous motion. In particular, the modulus square of the wave function determines the probability density of these particles appearing in certain positions in space.

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ing that these propensities possessed by the particles relate to their objective motion, not to measurements on them (as in the existing propensity interpretations of quantum mechanics).

[6]Note that for random discontinuous motion the properties (e.g. position) of a quantum system in a superposed state are indeterminate in the sense of usual hidden variables, though they do have definite values at every instant. This makes the theorems that restrict hidden variables such as the Kochen-Specker theorem irrelevant.

Appendix: Protective measurement of the charge distribution of a quantum system

Since the existence of the charge distribution of a charged quantum system is the basis of our derivation of the meaning of the wave function, we will briefly illustrate this important result here. For a more detailed analysis see Gao (2011).

Consider the spatial wave function of a single quantum system with negative charge Q:

$$\psi(x,t) = a\psi_1(x,t) + b\psi_2(x,t), \tag{1}$$

where $\psi_1(x,t)$ and $\psi_2(x,t)$ are two normalized wave functions respectively localized in their ground states in two small identical boxes 1 and 2, and $|a|^2 + |b|^2 = 1$. An electron, which initial state is a small localized wave packet, is shot along a straight line near box 1 and perpendicular to the line of separation between the boxes. The electron is detected on a screen after passing by box 1. Suppose the separation between the boxes is large enough so that a charge Q in box 2 has no observable influence on the electron. Then if the system were in box 2, namely $|a|^2 = 0$, the trajectory of the electron wave packet would be a straight line as indicated by position "0" in Fig.1. By contrast, if the system were in box 1, namely $|a|^2 = 1$, the trajectory of the electron wave packet would be deviated by the electric field of the system by a maximum amount as indicated by position "Q" in Fig.1.

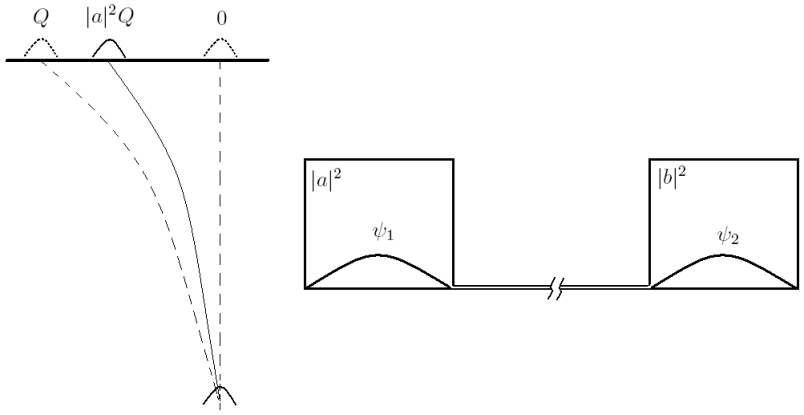

Fig.1 Scheme of a protective measurement of the charge density of a single quantum system

We first suppose that $\psi(x,t)$ is unprotected, then the wave function of the combined system after interaction will be

$$\psi(x,x',t) = a\varphi_1(x',t)\psi_1(x,t) + b\varphi_2(x',t)\psi_2(x,t), \tag{2}$$

where $\varphi_1(x',t)$ and $\varphi_2(x',t)$ are the wave functions of the electron influenced by the electric fields of the system in box 1 and box 2, respectively, the trajectory of $\varphi_1(x',t)$ is deviated by a maximum amount, and the trajectory of $\varphi_2(x',t)$ is not deviated and still a straight line. When the electron is detected on the screen, the above wave function will collapse to $\varphi_1(x',t)\psi_1(x,t)$ or $\varphi_2(x',t)\psi_2(x,t)$. As a result, the detected position of the electron will be either "Q" or "0" on the screen, indicating that the system is in box 1 or 2 *after* the detection. This is a conventional impulse measurement of the projection operator on the spatial region of box 1, denoted by A_1. A_1 has two eigenstates corresponding to the system being in box 1 and 2, respectively, and the corresponding eigenvalues are 1 and 0, respectively. Since the measurement is accomplished through the electrostatic interaction between two charges, the measured observable A_1, when multiplied by the charge Q, is actually the observable for the charge of the system in box 1, and its eigenvalues are Q and 0, corresponding to the charge Q being in box 1 and 2, respectively. Such a measurement cannot tell us the charge distribution of the system in each box *before* the measurement.

Now let's make a protective measurement of A_1. Since $\psi(x,t)$ is degenerate with its orthogonal state $\psi'(x,t) = b^*\psi_1(x,t) - a^*\psi_2(x,t)$, we need an artificial protection procedure to remove the degeneracy, e.g. joining the two boxes with a long tube whose diameter is small compared to the size of the box[7]. By this protection $\psi(x,t)$ will be a nondegenerate energy eigenstate. The adiabaticity condition and the weakly interacting condition, which are required for a protective measurement, can be further satisfied when assuming that (1) the measuring time of the electron is long compared to $\hbar/\Delta E$, where ΔE is the smallest of the energy differences between $\psi(x,t)$ and the other energy eigenstates, and (2) at all times the potential energy of interaction between the electron and the system is small compared to ΔE. Then the measurement of A_1 by means of the electron trajectory is a protective measurement, and the trajectory of the electron is determined by the expectation value of the charge of the system in box 1. In particular, when the size of box 1 can be ignored compared with the separation between it and the electron wave packet, the wave function of the electron will obey the following Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m_e}\nabla^2\psi(\vec{r},t) - k\frac{e\cdot|a|^2Q}{|\vec{r}-\vec{r_1}|}\psi(\vec{r},t), \tag{3}$$

where m_e is the mass of electron, k is the Coulomb constant, $\vec{r_1}$ is the position of the center of box 1, and $|a|^2Q$ is the expectation value of the charge Q in box 1. Correspondingly, the trajectory of the center of the electron wave

[7]It is worth noting that the added protection procedure depends on the measured state, and different states need different protection procedures in general.

packet, $\vec{r_c}(t)$, will satisfy the following equation by Ehrenfest's theorem:

$$m_e \frac{d^2\vec{r_c}}{dt^2} = -k\frac{e\cdot|a|^2Q}{|\vec{r_c}-\vec{r_1}|(\vec{r_c}-\vec{r_1})}. \quad (4)$$

Then the electron wave packet will reach the position "$|a|^2Q$" between "0" and "Q" on the screen as denoted in Fig.1. This shows that the result of the protective measurement is the expectation value of the charge Q in the state $\psi_1(x,t)$ in box 1, namely the integral of the charge density $Q|\psi(x)|^2$ in the region of box 1.

The result of the above protective measurement can tell us the charge distribution of the system in each box *before* the measurement. Suppose we can continuously change the measured state from $|a|^2 = 0$ to $|a|^2 = 1$ (and adjust the protective interaction correspondingly). When $|a|^2 = 0$, the single electron will reach the position "0" of the screen one by one, and it is incontrovertible that no charge is in box 1. When $|a|^2 = 1$, the single electron will reach the position "Q" of the screen one by one, and it is also incontrovertible that there is a charge Q in box 1. Then when $|a|^2$ assumes a numerical value between 0 and 1 and the single electron reaches the position "$|a|^2Q$" between "0" and "Q" on the screen one by one, the result will similarly indicate that there is a charge $|a|^2Q$ in the box by continuity. The point is that the definite deviation of the trajectory of the electron will reflect that there exists a definite amount of charge in box 1[8]. Moreover, the above equation that determines the result of the protective measurement, namely Eq. (4), gives a more direct support for the existence of a charge $|a|^2Q$ in box 1. The r.h.s of Eq. (4) is the formula of the electric force between two charges located in different spatial regions. It is incontrovertible that e is the charge of the electron, and it exists in position $\vec{r}$. Then $|a|^2Q$ should be the other charge that exists in position $\vec{r_1}$. In other words, there exists a charge $|a|^2Q$ in box 1.

In conclusion, protective measurement shows that the charge of a charged quantum system is distributed throughout space, and the charge density in each position is proportional to the modulus square of its wave function there. This conclusion is based on two established parts of quantum mechanics, namely the linear Schrödinger evolution and the Born rule. In the above example, the linear Schrödinger evolution determines the deviation of the electron wave packet, and the Born rule is needed to obtain the information about the center of the electron wave packet detected on the screen.

[8]Any physical measurement is necessarily based on certain interaction between the measured system and the measuring system. One basic form of interaction is the electrostatic interaction between two electric charges as in our example, and the existence of this interaction during a measurement, which is indicated by the deviation of the trajectory of the charged measuring system, means that the measured system also has the charge responsible for the interaction. Note that the arguments against the naive realism about operators and the eigenvalue realism in the quantum context are irrelevant here.

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