The simple failure of Curie's Principle

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1. Introduction

John Earman has suggested that there is a simple formulation of Curie's Principle that is not only deeply intuitive, but "virtually analytic" (Earman 2004, p.173). He is not the only one to take this view[1], but gives one of its clearest statements. Earman formulates Curie's Principle as the claim: *If*,

(CP1) the laws of motion/field equations governing the system are deterministic;
(CP2) the laws of motion/field equations governing the system are invariant under a symmetry transformation; and
(CP3) the initial state of the system is invariant under said symmetry; then
(CP4) the final state of the system is also invariant under said symmetry. (Earman 2004, p.176)

An intuitive way to express the idea is perhaps: *if no asymmetry goes in, then no asymmetry comes out.*

I would like to point out a simple sense in which this formulation of Curie's Principle fails, when the symmetry transformation is time reversal. I will begin by illustrating a very simple counterexample in classical Hamiltonian mechanics, and then show how this counterexample is endemic to quantum mechanics and quantum field theory. I conclude by discussing two revised principles, which avoid the counterexample, but do not appear to adequately capture the formulation of Curie's principle expressed above.

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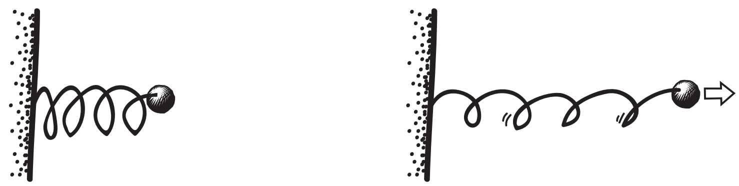
[1]For example, Mittelstaedt and Weingartner (2005, p.231), on the tacit assumption that the laws of physics are deterministic, argue that "from an asymmetric effect and symmetric laws we may conclude asymmetric initial conditions." Ismael (1997, p.170) claims to have *proven* that "all characteristic symmetries of a Curie-cause are also characteristic symmetries of its effect."



(a) Initial state (b) Final state

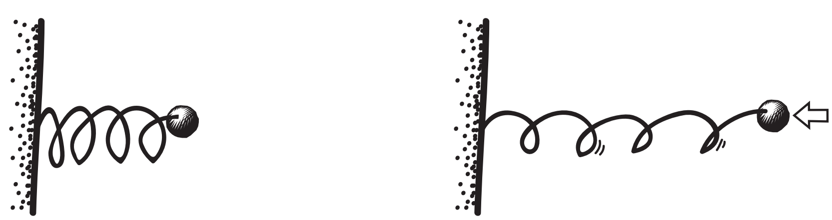
Figure 1. *(a) A harmonic oscillator initially compressed out of equilibrium with zero momentum. (b) A final state for which the system has non-zero momentum.*

2. The simple failure of Curie's Principle

2.1. **In pictures.** Take a harmonic oscillator, such as a bob on a spring. It is manifestly time reversal invariant, meaning that for every possible motion of the bob, there is a "time-reversed motion" that is also possible.

Suppose the system begins its motion at time $t = 0$ with the spring compressed out of equilibrium, and with no initial momentum, as in Figure 1(a). The bob then springs back in the other direction, acquiring some (non-zero) momentum, as in Figure 1(b).

What happens when we time reverse these initial and final states? The time reversal operator leaves the position of a state unchanged, while reversing the direction of the momentum. Our initial state has zero momentum, so it is not changed by the time reversal operator. But the final state has non-zero momentum, which reverses direction under time reversal operator. The result: the laws of motion for the harmonic oscillator are time reversal invariant, and the initial state is preserved by the time reversal operator, but the final state is not. This is a system for which Curie's Principle fails.



(a) Time reversed initial state (b) Time reversed final state

Figure 2. *(a) The initial state has no momentum, and so is preserved by time reversal. (b) A final state has non-zero momentum, and so is not preserved by time reversal.*

2.2. **Mathematical verification.** Let's do the exercise of checking this result in Hamiltonian mechanics. The possible states of the harmonic oscillator are the possible values for the position and momentum (q, p) of the bob in phase space. The laws of motion for the system are Hamilton's equations,

$$\frac{d}{dt}q(t) = \frac{\partial}{\partial p}h(q,p), \quad \frac{d}{dt}p(t) = -\frac{\partial}{\partial q}h(q,p).$$

The Hamiltonian $h(q, p)$ for the harmonic oscillator is $h(q, p) = q^2 + p^2$. The laws of motion are thus manifestly time reversal invariant, in that if $(q(t), p(t))$ is a possible trajectory, then $(q(-t), -p(-t))$ is a possible trajectory as well[2].

We now need to check that there is a trajectory with an initial state that is preserved by time reversal, and a final state that is not. One such trajectory is the following. Note[3] that the following is a solution to the laws of motion above:

$$q(t) = \cos(2t), \quad p(t) = -\sin(2t).$$

At time $t = 0$, this system has zero momentum, since $p(0) = \sin(0) = 0$. But it has non-zero momentum for the subsequent times $0 < t < 2\pi$. The time reversal operator $T : (q, p) \mapsto (q, -p)$ therefore preserves the initial state, but not all later states.

2.3. **Summary.** Here is what we have observed in the example above:

(1) *The harmonic oscillator is time reversal invariant.* This is a simple mathematical fact about the law of motion for the harmonic oscillator.
(2) *The harmonic oscillator has a trajectory for which the initial state is preserved under time reversal.* We choose a trajectory for which the harmonic oscillator is not always in equilibrium, and then choose an initial state with zero momentum.
(3) *Not all later states of the same trajectory are so preserved.* The later states of the harmonic oscillator have non-zero momentum, and so are not preserved by the time reversal operator.

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[2]To verify: Let $(q(t), p(t))$ be a solution to Hamilton's equations. The Hamiltonian $h(q, p) = p^2 + q^2$ has the property that $h(q, p) = h(q, -p)$. So, Hamilton's equations also hold for $h(q, -p)$. But Hamilton's equations hold for all values of t, and therefore under the substitution $t \mapsto -t$. Making this substitution, we thus find that $-(d/dt)q(-t) = \partial h(q, p)/\partial p$ and hence that $(d/dt)q(-t) = \partial h(q, -p)/\partial(-p)$; similarly, $-(d/dt)p(-t) = -\partial h(q, p)/\partial q$, and hence $(d/dt)(-p(-t)) = -\partial h(q, -p)/\partial q$. That is, $(q(-t), -p(-t))$ is also a solution to Hamilton's equations.

[3]Namely, $dq/dt = (d/dt)(\cos(2t)) = -2\sin(2t) = 2p(t) = \partial h/\partial p$, and $dp/dt = (d/dt)(-\sin(2t)) = -2\cos(2t) = -2q(t) = -\partial h/\partial q$.

Curie's Principle thus fails when the symmetry transformation is time reversal.

3. Robust failure in quantum theory

Our example above made use of the way that the classical position and momentum variables (q, p) transform under time reversal. But Curie's Principle fails just as badly in quantum theory, and we need not make any mention of position or momentum to show this[4]. I'll begin by describing the standard definition of time reversal and time reversal invariance in quantum theory, and then show how Curie's Principle fails.

3.1. **Time reversal in quantum theory.** Curie's Principle fails quite generally in both non-relativistic quantum mechanics and in relativistic quantum field theory. To keep the discussion general enough to apply to both, I will characterize the spacetime on which quantum theory takes place as an affine space $\mathcal{M}$, which admits a foliation into spacelike hypersurfaces. This will allow us to think of $\mathcal{M}$ as either a non-relativistic spacetime (such as Newtonian or Galilei spacetime), or a relativistic spacetime (such as Minkowski spacetime).

The vector states of a quantum system will be described by vectors in a Hilbert space $\mathcal{H}$. For any foliation Σ_t of the spacetime $\mathcal{M}$ into spacelike hypersurfaces, we take there to be a continuous one-parameter group of unitary opeartors $\mathcal{U}_t = e^{-itH}$. This group describes the way any initial state $\psi \in \mathcal{H}$ changes over time, by the rule,

$$\psi(t) = e^{-itH}\psi.$$

In differential form, this law becomes the familiar Schrödinger equation $i(d/dt)\psi(t) = H\psi(t)$, which holds for all $\psi(t)$ in the domain of H.

Time reversal in quantum mechanics is a transformation that takes a trajectory $\psi(t)$ to a new trajectory $T\psi(-t)$, where $T : \mathcal{H} \to \mathcal{H}$ is a bijection called the *time reversal operator*. This operator T has the special property of being *antiunitary*. An antiunitary operator satisfies $T^*T = TT^* = I$, but it is antilinear instead of linear, meaning that for any two vectors ψ and ϕ and for any complex constants a and b,

$$T(a\psi + b\phi) = a^*T\psi + b^*T\phi. \tag{1}$$

[4]In fact, there is a similarly robust way to describe this failure in the geometric formulation of classical mechanics (a classic textbook in this formulation is Abraham and Marsden 1978). But that discussion lies outside the scope of the simple point I would like to make here.

Although unusual, antiunitarity is absolutely essential to capturing the meaning of time reversal in quantum theory; Wigner (1931, §20) remains one of the best discussions of this principle.

A quantum system $(\mathcal{H}, e^{-itH})$ is *time reversal invariant* if, whenever $\psi(t)$ is a solution to the law of motion, then so is $T\psi(-t)$. This is equivalent[5] to the statement,

$$THT^{-1} = H, \tag{2}$$

where H is the generator (the "Hamiltonian") appearing in the unitary dynamics $\mathcal{U}_t = e^{-itH}$.

3.2. **Curie's Principle in quantum theory.** Here is how Curie's Principle goes awry in this theory. Let $(\mathcal{H}, e^{-itH})$ be any time reversal invariant quantum system, in that $THT^{-1} = H$. Suppose the initial state ψ is preserved by the time reversal operator, $T\psi = \psi$. Then it is not generally true that $T\psi(t) = \psi(t)$ for all t.

To see why, notice first that since $T\psi = \psi$, we may bring the T over and write $\psi = T^{-1}\psi$, and thus that $Te^{-itH}\psi = Te^{-itH}T^{-1}\psi$. This allows a simple calculation:

$$T\psi(t) = Te^{-itH}T^{-1}\psi = e^{T(-itH)T^{-1}}\psi = e^{itTHT^{-1}}\psi = e^{itH}\psi = \psi(-t).$$

The second equality follows from the functional calculus[6], the third from the antilinearity of T (Equation 1), and the fourth from the assumption of time reversal invariance (Equation 2).

From this it is clear that later states will be preserved by time reversal (that is, $T\psi(t) = \psi(t)$) if and only if $\psi(t) = \psi(-t)$. In other words, the trajectory $\psi(t)$ would have to be symmetric about initial time $t = 0$. This is not generally the case. Even worse: since Curie's Principle is supposed to hold of any initial state, its satisfaction would imply that $\psi(t) = \psi(-t)$ *for all* states, at all times t. This is only possible if the state $\psi(t) = \psi$ is fixed for all of time[7]. So, Curie's

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[5]This was pointed out, for example, in (Earman 2002, p.248)

[6]There is an easy way to see this without the functional calculus, by restricting attention to the so-called "analytic vectors" of H. Such a vector ψ allows the expansion of the exponential as $e^{-itH}\psi = \sum \frac{(-itH)^k}{k!}\psi$. Since $TT^{-1} = I$, we can write $T(-itH)^kT^{-1} = (-TitHT^{-1})^k$. So, applying T to our expansion we see that $Te^{-itH}T^{-1}\psi = \sum \frac{T(-itH)^kT^{-1}}{k!}\psi = \sum \frac{(-TitHT^{-1})^k}{k!}\psi = e^{T(-itH)T^{-1}}\psi$.

[7]Proof: we will show that for any initial state ψ and for all $t \in \mathbb{R}$, $\psi(t) = \psi$. Let $\psi \in \mathcal{H}$ and let $t \in \mathbb{R}$. Define a new initial state $\phi := e^{-i(t/2)H}\psi$, with $\phi(t) := e^{-itH}\phi$. Curie's principle implies that $\phi(t/2) = \phi(-t/2)$. But $\phi(t/2) = e^{-i(t/2)H}\phi = e^{-i(t/2)H}e^{-i(t/2)H}\psi = \psi(t)$, while $\phi(-t/2) = e^{itH}e^{-itH}\psi = \psi$. Therefore, $\psi(t) = \psi$, which proves the claim.

Principle fails for every quantum system that is interesting enough to allow any change whatsoever in time.

In summary: the time reversal invariance of a quantum system $(\mathcal{H}, \mathcal{U}_t)$ implies that $T\mathcal{U}_t\psi = \mathcal{U}_{-t}T\psi$. So, if $T\psi = \psi$, then $T\mathcal{U}_t\psi = \mathcal{U}_{-t}\psi$. This contradicts the conclusion of Curie's Principle, that $T\mathcal{U}_t\psi = \mathcal{U}_t\psi$, in all but the simplest of cases.

4. Revising Curie's Principle

There are at least two ways to revise Curie's Principle to get a true proposition. Neither seems to me to be a satisfactory way to capture the principle. Let me discuss each of them in turn.

4.1. **Argue time reversal is not a symmetry.** One way to go about revising Curie's Principle is to restrict what counts as a "symmetry transformation." By excluding problematic transformations like time reversal, one can produce mathematically correct replacements for Curie's Principle.

Earman himself has formulated one such statement, which he takes to capture Curie's Principle in the algebraic framework for quantum field theory. He begins with a C^* algebra, with an automorphism group α describing the dynamics. His approach is then to characterize a "symmetry" in quantum field theory as (linear) automorphism θ of the C^* algebra. In this framework, Earman writes:

> *Proposition 2 (Curie's Principle).* Suppose that the initial state ω_o is θ-symmetric (i.e. $\widehat{\theta\omega_o} := \omega_o \circ \theta = \omega_o$) and that the dynamics α is also θ-symmetric (i.e. $\theta\alpha\theta^{-1} = \alpha$). Then the evolved state $\omega_1 := \widehat{\alpha\omega_o}$ is θ-symmetric. (Earman 2004, p.198)

This certainly resembles Curie's principle: the dynamics are deterministic (CP1), the initial state is preserved by a symmetry (CP3), the dynamics are preserved by the symmetry (CP2), and we conclude that the final state is preserved by the symmetry (CP4). There is also an easy analogue in non-relativistic quantum mechanics. There, the approach would be to characterize a symmetry θ as (linear) unitary transformation on a Hilbert space $\mathcal{H}$. Then we may write:

> *Non-Relativistic Proposition 2.* Suppose that the initial state $\psi_0 \in \mathcal{H}$ is θ-symmetric (i.e. $\theta\psi_0 = \psi_0$) and that the unitary group e^{-itH} generating the dynamics is also θ-symmetric (i.e. $\theta e^{-itH}\theta^{-1} = e^{-itH}$). Then the evolved state $\psi_1 := e^{-itH}\psi_0$ is θ-symmetric.

Both of these propositions mathematically correct, and their proof is trivial[8]. Time reversal is excluded from the content of the proposition, because the time reversal operator in quantum theory is not linear but *anti*linear; see Section 3.1.

Although Earman's approach saves a Curie-like principle, it is at the expense of the orthodox definition of symmetry transformations in quantum theory. In ordinary quantum theory, symmetry transformations include not only the linear-unitary transformations, but the *anti*linear-*anti*unitary transformations as well. In the algebraic framework in which Earman works, symmetry transformations include both linear-automorphisms and antilinear-anti-automorphisms. So, these revised statements fall short of capturing the original statement of Curie's Principle, in excluding an important class of orthodox symmetries.

A clever response would be to notice that, although Earman's discussion does not mention antilinear operators, the above two propositions actually do hold when θ is antilinear! Unfortunately, this response is a red herring. Taking $\theta = T$ to be the time reversal operator, the premise that $\theta e^{-itH}\theta^{-1} = e^{-itH}$ in the non-relativistic proposition (or $\theta\alpha\theta^{-1} = \alpha$ in the algebraic proposition) does *not* capture statement of time reversal invariance in quantum theory. As we saw in Section 3.1, time reversal invariance is equivalent to the statement that $THT^{-1} = H$. But since T is antilinear, this implies that

$$Te^{-itH}T^{-1} = e^{T(-itH)T^{-1}} = e^{itTHT^{-1}} = e^{itH}.$$

That is, time reversal does not leave the dynamics unchanged, but rather reverses the temporal order. So, although antilinear operators do fall within the scope of the above propositions, one cannot interpret these propositions as capturing the "invariance of the laws under time reversal" required by the premises of Curie's Principle (in particular CP2). The propositions thus fall short of the original claim.

4.2. **Argue that Curie's Principle is about trajectories.** Another response is to retain the orthodox definition of symmetry and invariance, but to modify the kind of object that Curie's Principle is about. The last premise and the conclusion of Curie's Principle (Earman's CP3 and CP4) are about *states*; recall:

(CP3) [if] the initial state of the system is invariant under said symmetry;
then,
(CP4) the final state of the system is also invariant under said symmetry.

[8]Proof of the latter: $\theta\psi_1 = \theta e^{-itH}\psi_0 = e^{-itH}\theta\psi_0 = e^{-itH}\psi_0 = \psi_1$

But the premise about invariance of the laws (Earman's CP2) is a statement about states *in an entire trajectory*; namely, a system is time reversal invariant if, whenever $\psi(t)$ is a possible trajectory, it follows that $T\psi(-t)$ is too. So, we can view the trouble with Curie's principle as one of discord between two objects interest: one premise is about states, while the other is (most generally) about trajectories.

One way to fix the problem is to bring these objects of interest back into agreement, by making *all* the premises of Curie's principle about trajectories. To do this, let us write $\{\psi(t) = e^{-itH}\psi \mid t \in \mathbb{R}\}$ to denote the trajectory with initial state ψ. We begin by distinguishing two senses in which a state $\psi(t)$ in that trajectory can be "symmetric" with respect to a symmetry transformation.

(1) A state $\psi(t)$ at a time t is *S-symmetric in the original order* if $S\psi(t) = \psi(t)$.
(2) A state $\psi(t)$ at a time t is *S-symmetric in the reverse order* if $S\psi(t) = \psi(-t)$.

This is not such an unusual distinction, when one recalls (from the end of the previous subsection) that the standard definition of time reversal invariance is expressed with a similar reversal of sign: $Te^{-itH}T^{-1} = e^{itH}$. We can now express a revision of Curie's Principle: *If*,

(CP1) the laws of motion/field equations governing the system are deterministic;
(CP2) the laws of motion/field equations governing the system are invariant under a symmetry transformation; and
(CP3′) the state of the system at some fixed time t_0 is symmetric under said symmetry (in the original or reverse order);
then,
(CP4′) the state of the system at any time t is symmetric under said symmetry (in the same order).

In the context of ordinary quantum mechanics, this statement corresponds to the following mathematical fact[9].

Fact 1. Suppose a state $\psi(t_0) := e^{-it_0H}\psi$ at a fixed time t_0 is θ-symmetric in the original order (i.e. $\theta\psi(t_0) = \psi(t_0)$), and that the unitary group e^{-itH} generating the dynamics is invariant under θ in the original order (i.e. $\theta e^{-itH}\theta^{-1} = e^{-itH}$). Then for all times t, the state $\psi(t) = e^{-itH}\psi$ is θ-symmetric in the same order.

[9] Fact 1 follows from the non-relativistic version of Proposition 2 in the last subsection. Fact 2 is proved: $T\psi(t) = Te^{-i(t-t_0)H}e^{-it_0H}\psi = Te^{-i(t-t_0)H}\psi(t_0) = e^{i(t-t_0)H}T\psi(t_0) = e^{i(t-t_0)}\psi(-t_0) = e^{i(t-t_0)H}e^{it_0H}\psi = e^{itH}\psi = \psi(-t)$.

Fact 2. Suppose a state $\psi(t_0) := e^{-it_0 H}\psi$ at a fixed time t_0 is θ-symmetric in the reverse order (i.e. $\theta\psi(t_0) = \psi(-t_0)$), and that the unitary group e^{-itH} generating the dynamics is invariant under θ in the reverse order (i.e. $\theta e^{-itH}\theta^{-1} = e^{itH}$). Then for all times t, the state $\psi(t) = e^{-itH}\psi$ is θ-symmetric in the reverse order.

We have again arrived at a correct mathematical statement. Moreover, time reversal is no longer excluded, being captured now by Fact 2. But is this Curie's Principle? Strictly speaking, Curie's Principle stated that if the initial state is preserved by a symmetry transformation, then so is the final state. This is not true of Fact 2 above, where the symmetry transformation "flips" each state about the temporal origin. Facts 1 and 2 perhaps express a more natural principle, in bringing the premises into closer alignment. But it does not capture the original expression of Curie's Principle.

5. Conclusion

Chalmers (1970) has suggested that Curie's Principle is a useful heuristic device. Perhaps this is the most that we can say. We have seen that there are statements *like* Curie's Principle that are mathematically correct. However, the general statement of Curie's Principle is false for an important class of symmetry transformations that includes time reversal. It is false in Hamiltonian mechanics, false in quantum mechanics. This appears to be a dramatic failure indeed.

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