CIFOL: Case-intensional first order logic (I) Toward a theory of sorts

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1 Introduction

This is Part I of a two-part essay. We present CIFOL, an (1) easy-to-use, (2) uniform, (3) powerful, and (4) useful combination of first order logic with modal logic resulting from philosophical and technical modifications of Bressan 1972. Such an intensional first-order logic, or quantified modal logic, must be able to represent facts about the identity and distinctness of things in different possible circumstances or, as we will say, possible cases.[1] One way or the other, this may bring with it a distinction between the extension of an expression in a case, and its intension as the pattern of extensions across cases.[2] The crucial question is how this distinction can be made productive. There are many systems of quantified modal logic out there, each with its own virtues, but in our view none of them exhibits easiness of use, uniformity, expressive power, and usefulness in the precise way that we will motivate

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[1]We consistently use "thing" not in a wide sense, but rather for a "proper" thing, a resident of our concrete world of which it makes sense to say that it exists in different cases or at different times.

[2]Warning: Although philosophers often use "intension" to connote meaning and "extension" to connote reference, we do not, as we soon explain.

and indeed implement in our proposed system of case-intensional first order logic, CIFOL. As we noted, CIFOL is a modification of Bressan 1972. There Bressan, a theoretical physicist, presents the system ML^{ν}, which in turn is a profound deepening of the "method of extension and intension" of Carnap 1947. His purpose was to devise a quantified modal logic not in the interests of either metaphysics or the philosophy of language, but rather to help with understanding some aspects of scientific theory. His touchstone was Mach's well-known but murky definition of "mass" in terms of *possible* experiments. We think that among Bressan's many innovations, his introduction of *modal separation* (Def. 17) stands out as a testament to his originality: No other work in quantified modal logic even hints at this fundamental concept.

Bressan's system is both ν-sorted and higher order. CIFOL is first order, not because we think the first order is of paramount logical interest, but merely because ascending to higher types is unavoidably complex when everything is spelled out. Just to begin, one has to keep track of four kinds of type entities: the types themselves, typed expressions, typed domains of extensions, and typed domains of intensions. For a summary description of ML^{ν}, see Belnap 2006. It is a scandal that Bressan's brilliant work has been almost universally ignored for forty years.

1.1 Four logical virtues

We begin with the first of the four qualities numbered above. (1) For a first-order intensional logic, by easiness of use we mean (only) that a system relies chiefly on procedures familiar from first order and modal logic. CIFOL makes two large exceptions: (a) To the minimal syntax, CIFOL adds a Frege-inspired singular term, "$*$," to mark nonexistence such as occasioned by failed definite descriptions; and (b) CIFOL subtracts the general replacement property from the usual logic of identity, while retaining the replacement property for necessary identity. CIFOL has nearly the full power of first-order quantification—including unrestricted instantiation and generalization principles and unrestricted λ-conversion and context-free formation of definite descriptions ($\imath$-terms)—combined with **S5** modal logic. This combination of modality and quantification stands in contrast to almost every other extant quantified modal logic. On the other hand, the primitive syntax of CIFOL has no "extra" features that go beyond first order logic and **S5**. It is nevertheless natural to suppose that the quantificational and modal features of CIFOL working in combination go beyond what either can accomplish alone.

From a semantic point of view, of principal significance will be quantification over intensions and non-extensional predication.

(2) By uniformity we intend (only) a requirement on the treatment of terms and predicates, both syntactically and semantically. There are certainly good linguistic, metaphysical or scientific reasons for treating some terms or predicates differently than others (e.g., for having a separate category of proper names, or of sortal or extensional predicates). These reasons seem, however, to be extra-logical: Logic is supposed to be formal in the sense of subject-neutral. Logical regimentation at this juncture means running the risk of artificially narrowing down the range of what can be expressed, and thus, in the end, of constraining empirical and conceptual work. A first-order intensional logic uniform in our somewhat restricted sense has just one syntactical category of terms, and one syntactical category of predicates, with a uniform semantic clause characterizing the application of a predicate to a term (or to several terms, for a many-place predicate). CIFOL lives up to this ideal (except for its treatment of the identity predicate itself) including the treatment of definite descriptions as categorematic terms. Nearly all extant systems of quantified modal logic need to have recourse to special features of some syntactically identified class of terms, for example, by treating proper names and variables as so-called "rigid designators" that alone allow one to trace an individual across cases as a matter of logic, since their intension is forced to be constant across all cases, while other terms are not so constrained. This amounts to a breach of uniformity of the kind we have in mind, and to making the various ways of tracing individuals, which is an empirical, scientific matter, a matter of logic. It is a scientific discovery that a common frog can be traced as one living organism from spawn to tadpole to frog under the sortal term, "common frog" (*rana temporaria*), no matter whether you give the beast a name or not, nor how you identify it in a given case. CIFOL leaves terms and predicates unconstrained by logic in the name of logical uniformity.

Of course our requirement of uniformity runs against another person's felt need of a complex syntax. For example, it would make no sense for Gupta, in his book titled *The logic of common nouns* (1980), to avoid introducing a formal representation of common nouns along side of predicates and singular terms. It is not a question of taste, but of purpose.

(3) CIFOL is expressively powerful in certain ways found only in case-intensional systems with non-extensional predication, a category that in-

cludes, aside from CIFOL, only Bressan's ML^{ν} and Montague's **IL**.[3] Only these systems seem to have the power to define, in the language itself, what it *means* to be able to trace an individual from one case to another. The source of this extra power comes partly through quantification over intensions and partly through allowing and exploiting non-extensional predication. These expressive features allow CIFOL to offer a definitional interface with which to introduce sortal (tracing) predicates—the so-called "absolute" predicates—and "extensional" (qualitative) predicates via non-logical axioms as in §4 below.

(4) Finally and above all, a logic worth its salt ought to be useful, in the sense of being able to represent everyday and (especially) scientific scenarios and of giving some guidance to the perplexed.[4] Consider, for example, the following "horse story": There is a paddock outside; the occupant of the paddock is Andy, a brown stallion (male horse). An hour later, Andy has been moved into the barn, and the occupant of the paddock is now Daisy, a grey mare (female horse). Andy could, however, have been left in the paddock as well, in which case there would be two horses in the paddock. It is perfectly easy to picture what's going on; a child can do it. It is, however, difficult to give a perspicuous formal representation of the situation. The hard bit is to come up with a good account of what the terms in the story, "Andy," "Daisy" and "the occupant of the paddock," stand for and how the predicative expressions, "brown," "grey," "male," "female" and "horse," function to combine with the terms to form propositions that are true or false, as the case may be. Specifically, a major challenge is to explain how certain terms (e.g., "Daisy") allow one to trace an individual (a horse) through the various cases and sustain the ascription of essential properties (e.g., being female), while other terms (e.g., "the occupant of the paddock") are unfit for these purposes. Most quantified modal logics, if they can handle the example at all, fail to illuminate the notion of tracing. (We essay to meet the challenge in §5.) For a second example, consider that only Bressan touches on the

[3] Montague introduced his full system of intensional logic, now commonly referred to as **IL**, in Montague 1973. While he did not exercise the system's power (his application was to a fragment of ordinary English), later work by Gallin 1975 does so, making it clear that **IL** can do anything that ML^{ν} can do. For similarly powerful systems, see also Tichý 1988 and Fitting 2004. Gupta 1980, Ch. 3.4, also considers a first-order version of ML^{ν}.

[4] There are of course other notable uses of modal logic; for instance, clarifying metaphysical doctrines, and analyzing natural-language constructions. Bressan 1972 touches on these matters.

problem of literally "possible measurements," nearly all of which never get made but which must always give the same results if there is to be a stable concept of what is measured. (See §5.5.) We don't know of another case in which a quantified modal logic has been applied to generating rigorous scientific arguments.

1.2 Semantics: preliminary informal discussion

Following Bressan, CIFOL takes the notion of a *possible case* as the basis for modal semantics. Possibility is truth in a case; necessity is truth in all cases. No metaphysical assumptions about the structure of the cases are made beforehand. Specifically, cases are not taken to be "worlds" with an internal temporal structure; the cases themselves can be taken to be temporal if that is useful in an application. Bressan applies with full generality the *method of extension and intension* introduced in a limited way by Carnap 1947: Extension is always extension-in-a-case, and every expression has an extension in each case and an intension, which is the pattern of extensions across cases. *Both extensions and intensions are therefore thoroughly objective*: In contrast to standard philosophical usage, intensions are not taken to be analytic of meanings, nor indeed to be always of a kind that a human mind can grasp. All individual terms, including variables, constants and definite descriptions, are handled uniformly. Semantically, there is a "domain," introduced in §2.3, that harbors the extensions of individual terms, but take care right from the beginning: Members of "the domain" need not be construed as individuals. Rather, in certain typical applications it is the *intensions* of certain individual terms that correspond to concrete individuals. To lose sight of this feature of CIFOL is entirely to misconstrue our enterprise. It follows that it is wrong to think of intension vs. extension as rather like Frege's sense and denotation; even the grammar is unlike, since an extension is always extension-in-a-case. Partly for this reason, we altogether avoid the semantic word "reference" (sticking to extension/intension), but it may help to observe that if we were to use it, reference in CIFOL would be to intensions.

Predication and quantification are intensional: Variables, like all other terms, have an intension, and an extension in each case. Furthermore, in a profoundly significant departure from almost every other quantified modal logic known to us, whether a predicate (other than the identity predicate) applies to a sequence of terms in a case, may depend not just on the extensions of the terms in that case, but on their intensions ("non-extensional predication"). No restrictions are placed on the (intensional) values of the variables, so that instantiation and λ-abstraction are allowed with full generality. Non-extensional predication allows for and is required by the definitional (*logical* rather than syntactical) characterizations of extensional and absolute properties given in §4. We count these definitions as part of CIFOL. In contrast, axioms declaring a particular predicate as extensional or absolute are an extra-logical part of an application. Absolute properties have the intensional features of sortal properties, so they can be employed to trace an object between cases. As such, it is only individual intensions that fall under an absolute property that can reasonably represent concrete individuals. In other words, *in its definition of "absolute,"* CIFOL *gives an account of the logical* nature *of the tracing of individuals between cases*, whereas whether a particular predicate is absolute is assumed to be not a matter of logic, but rather of science and metaphysics. There is in CIFOL no recourse to "rigid designators" nor to any notion of "trans-world identity."[5]

1.3 A brief overview of other systems

We briefly discuss a number of proposed systems of quantified modal logic to indicate how they do not further our aims (that is, in particular, sorting out complicated everyday reasoning and—above all—scientific modal reasoning) or fall short of our ideals. This is meant to provide additional motivation for introducing CIFOL.[6]

The first steps combining first order and modal logic were made purely

[5]Indeed, CIFOL's minimalistic, non-metaphysical outlook on cases gives one an argument against a category of rigid designators over and above our appeal to ease of use and uniformity: Such a syntactical category should (we may want to suppose) work for any set of cases. But surely rigid designation in the sense of "same extension in every case" only makes sense, if it makes sense at all, for cases = worlds; it makes no sense for cases = times, or cases = moment/history pairs as in branching histories. In those applications of the framework, rigidity would amount to freezing the thing in question; it couldn't change (supposing that the change of a thing is reflected by temporal variation in the extension of a term denoting the thing).

[6]Our aim is to be brief rather than comprehensive, both with respect to the selection of systems mentioned and with respect to the discussion of the systems themselves. For more detailed information on many systems of quantified modal logic, see, e.g., Hughes and Cresswell 1996, Garson 2005, and Fitting 2011. It is striking that no mainstream discussion of quantified modal logic with the exception of Parks 1972, Gupta 1980 and Bacon 1980 takes substantial notice of Bressan's ideas; and as we will make clear, even the latter two systems fall short of answering to our purposes.

syntactically in Barcan 1947. Carnap 1947 gives a semantics and introduces his method of extension and intension. Cases in his system are syntactical entities, viz., "state-descriptions" that contain "for every atomic sentence either this sentence or its negation, but not both, and no other sentences" (p. 9). While opening the way to the method of extension and intension, the syntactic approach severely limits both uniformity and usefulness. To name one striking limitation, Carnap does not allow modalities in definite descriptions (p. 184).

Post-Carnapian systems are not based on syntactically individuated cases, but rather on a general notion of cases, like CIFOL, or—more commonly—on possible worlds. In Kripke 1959 and others, the development of quantified modal logic has been an area of tight contact between logic and metaphysics, in that logical features of a system are made subservient to metaphysical views about worlds, objects, individuals, substances, and so on. This approach has led to a proliferation of logical systems. All of them offer means for tracing individuals across cases in the logic itself, which restricts usefulness. As we will indicate, that approach also reduces ease of use and uniformity. A second feature of all non-Bressanian systems is what we think of as "extensionalism," namely, a desire to be as extensional as possible, the traditional ideal being first-order predicate logic. This hankering manifests itself in more than one way, but the overwhelmingly common result of extensionalism is that it supports the view that *all* predication is extensional; that is, the view that the extension (truth value) of the result of applying a predicate to a term depends only on the extension of the term.[7] As one might put it, as a matter of extensionalist logic, all predication is said to be extensional like the negation connective, rather than being non-extensional like the necessity connective. No extensionalist logic has the expressive power to illuminate the idea of tracing an individual from case to case or time to time.

Extensionalism is a natural child of first-orderism. How so? In second-order modal logic it would indeed be quixotic to insist that all predication be extensional. Think of the second-order property of a first-order property such as "white," applying contingently: Whether or not a first-order property, P applies contingently to Socrates clearly must depend on the intension of P (i.e., its extension in many possible cases), not merely on its extension

[7] As an extra-logical matter, CIFOL describes "most" predicates as extensional, while avoiding extensionalism by leaving room for non-extensional predication.

in a single case. Now uniformity kicks in: If predication at higher types needs to be non-extensional, uniformity suggests that the same should hold at lower types. It is therefore not surprising that the higher order system of Montague 1973 allows for predicates of all types, including predicates applying to intensions. It is also unsurprising that all quantified modal logics that ignore higher types demand that all predication be extensional.[8]

The pioneering semantics of Kripke 1959 (for **S5**) and Kripke 1963 (for weaker systems, based on a relational structure of possible worlds), endows each world with its own domain of individuals. Extensionalism comes in twice: (1) Although constants are given intensions, variables are given only extensions, which amounts to non-uniformly taking variables, but not constants, to be rigid designators. (2) Predication is extensional, even when its argument is a constant bearing an intension.

Extensionalism is also a driving force behind the counterpart-theoretic approach to quantified modal logic pioneered by Lewis 1968, who considers the non-extensionality of modal logic "a historical accident" (p. 113) that can be overcome. In Lewis's approach, the need for a tracing principle is denied, as individuals are strictly world-bound. Our conversational practice of tracing individuals across different cases is catered for by a context-dependent counterpart relation between inhabitants of different worlds. The Lewis scheme may work well for conversations; it seems unlikely, however, that our protean conversational practice can undergird a serious use of quantified modal logic in science. Metaphysically, Lewis answers the question, whether a thing can be present in more than one case, in the negative. CIFOL works out the contrary position, that the person who answered the doorbell on Monday is the same person as the one who answered on Tuesday, and that the person who is going to answer a knock in case it's loud is the same person who is going to answer a soft knock. It is inconceivable that the philosophical verdict on such a basic question is going to be unanimous. We will only maintain that all the parts of the CIFOL scheme fit together intuitively, comfortably, and without strain or loose pieces.

Like Bressan's ML^{ν}, the system of Montague 1973 (known as **IL**) is both higher order and also invites concentration on its first order fragment (see note 3). **IL** is properly intensional in that it assigns intensions to all terms, including variables. Nevertheless, **IL** treats variables and constants differently:

[8]Except CIFOL, which, though first order, self-consciously uses ideas from Bressan's higher-order logic.

As in Kripke 1963, constants may have non-constant intensions, but variables are rigid designators. On the other hand, in **IL**, variables are available at all types, so that there are also variables ranging over individual intensions, and we can express intensional predication. This means, however, that the different types have to be converted explicitly in order to maintain well-formedness; there is a primitive intension-of operator and a corresponding primitive extension-of operator. **IL** uses these in order to be able to explain the failure of arguments such as the following: "The temperature is ninety. The temperature rises. Therefore, ninety rises." With the (indirect) availability of non-extensional predication, **IL** has resources sufficient to introduce extra-logical tracing principles; Montague, however, does not take this step.

We repeat: **IL** adds notation beyond the **S5** modalities and quantification, namely, the "intension of" operator and a companion "extension of" operator. These syntactic additions, which seem to arise out of extensionalism, spoil ease of use and make it more difficult to connect **IL** with scientific argumentation. There is, however, no doubt that **IL** is equal in power to ML^{ν} in strictly mathematical terms.

Muskens 2007 develops a higher-order quantified modal logic, **MTT** (modal type theory), which deserves credit for cleaning up **IL**, partly by letting predication be nonextensional in a sense even stronger than that available in ML^{ν}. **MTT** prides itself on two features orthogonal to the concerns of CIFOL: (1) It is what we may call "hyperintensional" in that it is suitable for epistemic and doxastic modalities as well as possibility and necessity. (2) It admits of formulation as a cut free tableau calculus. **MTT** is noticeably more complex than CIFOL without, we think, offering comparable logical insight at the first order. In our opinion, its doctrine of intension and extension, which is not at all case-intensional, is, for that reason, relatively opaque and by so much difficult to apply.

Tichý 1988 introduces "transparent intensional logic" (TIL), a system that bears many similarities to **IL** while at the same time differing in crucial respects. TIL is based on constructions, which allows for hyperintensionality, e.g., in belief contexts, which CIFOL does not.[9] Variables are treated as simple constructions selecting an object from a given sequence of objects (a valuation); thus, again, first-order variables are rigid designators, with the mentioned negative effects. To be fair, Tichý 1988 also offers variables of

[9]We believe it is still a major open question whether hyperintensionality is amenable to useful formal treatment.

different types, e.g., variables for intensions, but this route breaks uniformity in the treatment of terms. Nevertheless, TIL, like **IL**, illuminates many bits and pieces of English.

Williamson 2012 offers a second order modal logic as a context for discussing the "Barcan formula," which we mention in §3.1.6. Based on extensional predication and an extensional treatment of quantifiers, that study concerns problems quite different from those that arise from our first-order adaptation of ML^{ν}.

Thomason 1969 discusses systems of quantified modal logic with the aim of combining "modal logic and metaphysics" (so the title of the paper): "[A]s well as shaping logical work in the light of heuristic philosophical considerations, I have often found myself modifying metaphysical preconceptions in the light of technical considerations" (Thomason 1969, 120). Thomason considers, but rejects a system **Q2** with variables for intensions (but extensional predication); his ultimate choice, in the system **Q3**, is to treat variables as standing for substances and to require constant intensions for them: again, as in so many other systems, making the tracing of individuals a matter of logic. As before, we mention this as a salient difference rather than as a value judgment.

Bacon 1980, building on Thomason 1969, aims at a first-order version of Bressan's framework, although in our judgment he misses the mark. Terms (including variables) have intensions, and an extension in each case. Bacon adds to the grammar: He introduces a single tracing predicate, ▯, which is read as "subsists" or "straight," and is meant to single out "what is rigidly designated" (p. 193, n1). As in other cases, extensional predication restricts the usefulness of the system, and allowing only a single tracing predicate restricts the metaphysics of substance as a logical matter.

In contrast, Parks 1972 anticipates CIFOL by offering a *faithful* first-order version of Bressan 1972, which Parks calls "the Bressan language." Parks applies the Bressan language to the question whether a class such as the Supreme Court can change its members. Later philosophers, sadly, completely ignore this interesting paper.

Gupta 1980 develops a system similar in spirit to Bressan's. He stresses the "important logical and semantic difference between common nouns and predicates" (p. 1) and builds his system to reflect these differences both syntactically and semantically. Quantifiers are of the form $(\forall K, x)$, with K a common noun, so that variables are always variables for things of a certain kind. In this way, a tracing principle, provided by the kind K, is built into the quantificational machinery of the logic. This feature has the advantage of nicely enhancing the fit of the formal system with English syntax and semantics. It comes at the cost of ease of use, as well as syntactic and semantic uniformity. The system is powerful; these features, however, seem to make it less easy to use the system to help with clarifying technical scientific arguments.[10]

Fitting 2004 gives a readable account of certain difficulties of the most common frameworks of quantified modal logic, criticizing both the idea of variables as rigid designators and the counterpart theory of Lewis 1968. Against this background he introduces his first-order intensional logic, **FOIL**. As in other systems, object variables are distinguished from intension variables, thus reducing ease of use—a trade-off that we have encountered before.

Garson 2005 embarks on the laudable enterprise of unifying quantified modal logic by starting from a minimal basis, G. G has some similarities to CIFOL: Variables as well as other terms of G take intensions as values, and identity is case-relative. Garson describes how a great many extensions of G, including "[w]ith one notable exception, the major systems found in the literature,"[11] can be treated uniformly with regard to soundness and completeness. All these systems share extensional predication. Consequently, Garson's unification excludes CIFOL and its case-intensional cousins.

To sum up: With the exception of CIFOL, Bressan 1972, and Parks 1972, quantified modal logic has, for better or worse, been affected by (1) extensionalist tendencies and (2) the felt need to introduce logical means for tracing individuals across cases. This has led to systems whose usefulness for certain applications is restricted since they treat extra-logical matters as fixed by logic, and, except for **IL**, thereby fail to provide expressive resources that are needed for scientific or conceptual work. The systems also pay a price in that they do not fully live up to standards of ease of use and uniformity that

[10]Gupta 1980, Ch. 3.4, discusses the option of adopting Bressan's approach to common nouns, acknowledging its uniformity. For a thoughtful discussion and comparison of Gupta's system with Montague's **IL**, see Van Leeuwen 1991, which also documents an exchange between Montague and Dana Scott in which Montague seems ready to adopt stages, or heaps-of-molecules-at-a-moment, as extensions in a temporal reading of his intensional logic, similar to what we will propose in our examples (§5). Furthermore, Van Leeuwen argues that such an understanding of extensions would be the most "favorable interpretation" (p. 76) of Gupta 1980. Van Leeuwen thereby comes close to the CIFOL view that extensions of terms are not individuals (see §2.3).

[11]The reference is to the expanding domain system LPC+S and its relatives defined in Hughes and Cresswell 1996, Ch. 15.

CIFOL can satisfy. On the other hand, several of the systems provide an easier and more illuminating fit with English. We think of CIFOL as being in this respect rather like that bedrock of logic, first-order predicate logic.

1.4 Structure of the essay

Part I of the essay is structured as follows: We introduce the basic notions of CIFOL, including its grammar and the general machinery of extension and intension, in §2. In §3, we lay out the detailed semantics of CIFOL. We take two shortcuts here: First, we explicitly describe only one-place predicates and operators; second, we omit a lengthy discussion of the proof theory, much of which is a straightforward combination of standard first-order proof theory and the proof theory for **S5** (modulo the CIFOL-specific treatment of identity as extensional; see §3.1.5 and §3.3). For details, see Bressan 1972 or, more gently, Belnap 2006. In §4 we discuss absolute properties and related notions, and their uses in characterizing sortals; we illustrate their use in §5. We summarize in §6. The separate Part II (Belnap and Müller 2012) continues the discussion.

2 Basics

2.1 Grammar of CIFOL

The principal "parts of speech" in CIFOL are terms, sentences, operators, and predicates, all defined by recursion on complexity, and certain connectives. Among the atomic constants there are sentential constants, p, predicate constants, P, individual constants, c, and operator constants, f. Among the atomic terms, there is also a set *Vars* of individual variables, with x, y, z ranging over them, and there is a special individual constant, $*$, to figure as a sign of non-existence. Individual terms, with α, β ranging over them, arise by applying an n-ary operator (either constant or λ-operator), η, to an n-tuple of terms: $\eta(\alpha_1, \ldots, \alpha_n)$.There is a distinguished two-place predicate constant for use in case-dependent identity sentences: $\alpha_1 = \alpha_2$.[12] Using Θ to range over predicates, additional sentences come by applying an n-ary predicate

[12]Even though it is hard to think of identity as *world* dependent, taking it as *case* dependent is natural: "The winner will be (identical to) Ralph in case it rains, but not in case the sun shines."

(either constant or λ-predicate), Θ, to an n-tuple of terms: $\Theta(\alpha_1, \ldots, \alpha_n)$. Sentences arise from these via the usual truth-functional connectives such as negation, conjunction, disjunction, and the conditional and biconditional: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$; the modal connectives $\Box$ and $\Diamond$ for necessity and possibility; and the usual first-order quantifiers, $\exists x$ and $\forall x$, applied to sentences. Φ ranges over sentences.

CIFOL features unrestricted formation of $\imath$-terms (definite descriptions) $\imath x(\Phi)$, λ-predicates $\lambda x(\Phi)$, and, λ-operators $\lambda x(\alpha)$.[13] A definite description is an individual term. Applying a λ-operator, $\lambda x(\alpha)$ to a term, β, issues in a term, $(\lambda x(\alpha))\beta$. Applying a λ-predicate, $\lambda x(\Phi)$, to a term, β, issues in a sentence, $(\lambda x(\Phi))\beta$. A λ-operator [λ-predicate] may occur only in an operator [predicate] position (on pain of ascending past the first order). Finally, an expression, whether open or closed, is either an operator, η, or a predicate, Θ, or an individual term, α, or a sentence, Φ; and in the latter two cases is categorematic. We let ξ range over expressions.

2.2 Cases

In contrast to the exotic practice of quantifying over "worlds," quantification over *cases* is standard English:

- What if it rains? In any such a case, the game will be canceled; but the game will proceed in case it doesn't rain.
- Yes, there are cases in which Sam will agree to wash the car, but also cases in which he won't.
- In no case will I attend the meeting.

Partly because of their ties to conversational English, cases are less pretentious than worlds, so that their use invites substantial increase of flexibility in applications. On the other hand, invoking a set, Γ, of cases to help with modal thinking in CIFOL brings along with it certain interconnected idealizing presuppositions that go beyond the everyday meaning of "case." (1) In any application, Γ is to be the set of *all* possible cases; there are no others. (2) Accordingly, each case $\gamma \in \Gamma$ is elementary, which is to say, there are no

[13] λ-operators and -predicates in CIFOL are all one-place. One may simulate the use of binary λ-operators, for instance, with $\lambda x_1(\lambda x_2(\alpha)\beta_2)\beta_1$.

subcases. (3) Put linguistically, no elementary case harbors a contradiction, and each elementary case decides all disjunctions. Point (3) makes it clear that there is an interaction, be it ever so slight, between postulating a set of cases on the one hand, and the language (grammar and semantics) on the other. (4) At this point, we have no need for "the real case" since even though many examples and modal puzzles call for it, we are not in this essay introducing an "It's actually true that" connective.

The set of cases might be humdrum and finite, and elementary only relative to the conversational context. Let there be a horserace coming up next Saturday, and let Γ be $\{\gamma_1, \gamma_2\}$, where γ_1 is a case in which the track is dry, and γ_2 a case in which the track is muddy. A stable owner might ask her manager if there is a horse that can win the race in any case. Or Γ might be arcane and infinite, as in Bressan's application of ML^ν to Mach's definition of mass in terms of the ratio of consequent relative velocities when the measured mass-point strikes the unit mass-point. In this context, each case, γ, represents a distribution of mass-points, at a certain fixed time, t_0, with their attendant masses and velocities. Given that the to-be-measured mass-point, here-now, does not in fact collide with the unit mass stored in Sèvres (and they won't let you play with it anyway), the problem is to make sense of using, here and now, the result of a merely possible collision.

In an application, Γ might or might not be structured in some way; for example, an application to tense logic would interpret Γ as a linearly ordered set of intervals or moments of time. In another application, Γ might be construed as the set of momentary events in a branching (indeterministic) structure (branching histories). In Part II of this essay (Belnap and Müller 2012), we will develop our approach with a view towards temporal-modal cases in branching histories. In this Part, however, we make no assumptions whatsoever about the structure of the set of cases. At the present level of generality all we need to know about cases is that there are some—indeed, at least two, so as to avoid modal triviality.[14] We always let Γ be the set of all cases, and let γ range over Γ. Truth and extension for closed expressions will be relativized to cases. More fundamentally, in case-intensional semantics, every closed expression has an extension-in-γ relative to each case, γ, and an intension that can always be represented as a function from the set, Γ, of

[14]We can't think of a word that applies idiomatically to both modal alternatives and to times; CIFOL, however, operates at a level of generality that treats times and alternatives alike, which is our reason for sticking to "cases" when we are being most general.

all cases into extensions of an appropriate sort. To help us keep track, given sets X and Y, we use $X \mapsto Y$ as the set of functions that map X into Y. Thus, intensions will uniformly have the form $\Gamma \mapsto Y$, for appropriate Y.

2.3 Extensional domain

Having specified the cases, we need an extensional domain, D, to provide case-dependent extensions of singular terms, and the set of truth values $\{\mathbf{T}, \mathbf{F}\}$, which we abbreviate as $\mathbf{2}$, to provide case-dependent extensions of sentences of CIFOL. In standard quantified modal logic, where the cases are thought of as separate "worlds," such a domain is normally thought of as containing individuals: the inhabitants of the separate worlds. In CIFOL, however, concrete individuals are *not* represented by members of D. More fundamentally, in CIFOL there are no variables that range over D. The only function of D is to enable a case-dependent identity: We will say later that $\alpha_1 = \alpha_2$ is true in a case, γ, when and only when the extensions in D of α_1 in γ and of α_2 in γ are identical. CIFOL puts D to no further use. In particular, there is in CIFOL no facility for comparing the extension of a term in one case with its extension in another, so that the idea of a term being "rigid" by having the same extension in every case turns out to be inexpressible, which is a matter of no consequence, since expressing rigidity would add nothing useful to the expressive powers of CIFOL. You will see that in CIFOL the work of rigidity is accomplished via "absolute properties," as in §4 below. With these means, we will be able to say in CIFOL that the horse that wins the race in case it rains is the same horse as the horse that wins the race in case it doesn't rain—or that they are *not* the same horse—without comparing the extension of "the horse that wins" in the case in which it rains with the extension of "the horse that wins" in the case in which it doesn't rain.

Thus, we assume almost no structure on D, and for simplicity, we assume that the domain is specified independently of the cases. It is essential to observe, however, that CIFOL is not comparable to a "constant domain" logic in the sense of standard modal logic. The profound reason is that the individual variables of CIFOL take values that are intensional, rather than extensional.[15]

[15] As Bressan observed, one obtains exactly the same logic if one allows that extensional domains are case relative, as long as these case-relative or "variable" domains are all of the same cardinality; but CIFOL itself has a single extensional domain. As noted above, this does not render CIFOL a "constant domain" logic.

There is one more thing to say about D: We have introduced $*$ as an individual constant. Its intension must therefore be a function in $\Gamma \mapsto D$. Following Frege, we assume that D contains, apart from at least one "proper" extension, a "throwaway" to be the extension of terms, such as failed definite descriptions, that do not otherwise have an extension in a given case. We call the throwaway "$*$". Since we let $*$ be a term, and since we don't care what its extension is, we may as well think of $*$ as autonymous by giving it the extension $*$ in every case.[16] The sentence "$\alpha \neq *$" then expresses a kind of case-dependent existence, and is to be read "α exists." For convenience, we define a case-dependent existence predicate, E:[17]

Definition 1 (Existence predicate)

$$\Box\forall x[Ex \leftrightarrow_{df} x \neq *].$$

This case-relative existence predicate will play an important role in the discussion of properties of properties in §4.

2.4 Intensions

The defining feature of case-intensional semantics is that every expression of every type shall have both an intension and an extension-in-γ for each $\gamma \in \Gamma$. Starting with categorematic expressions, by an "individual intension" we mean a function in $\Gamma \mapsto D$, and by a "propositional intension" a function in $\Gamma \mapsto \mathbf{2}$. It is then obvious that in our lexicon intensions are neither linguistic nor subjective.[18] We let $\overline{z}$, in our use-language, range over individual intensions, so that $\overline{z}(\gamma) \in D$. Of note is that in case-intensional semantics, for certain predicates, Θ, the truth value in a case γ of a sentence, $\Theta(\alpha)$, may depend not just on the extension in γ of the singular term, α, but on its intension (intensional predication).

You should expect that *concrete individuals are represented in CIFOL by intensions, not extensions*, a thought that we spell out a little in the

[16] As a matter of convenience, we permit use-mention ambiguity for $*$.

[17] We use the style of definition prescribed by Suppes 1957, Ch. 8; so we are thinking of Def. 1 as an "axiom" added to a theory formulated in CIFOL. In contrast many subsequent definitions should be conceived as added to our semantic theory of CIFOL rather than to CIFOL itself.

[18] Carnap 1947 and many later logicians say "individual concept" where we write "individual intension." We make the change to avoid false connotations: In our usage, there is nothing "conceptual" about individual intensions.

next section. It is worth repeating that a CIFOL intension is not meant to give the meaning of an expression: There is nothing either linguistic or mental or social about intensions. This is easiest to see when the cases are taken as times: Certain intensions can represent the entire history of concrete individuals and their possibilities, and regardless of whether or not one endorses the view that an individual is to be metaphysically identified with its history and possibilities, it is good to say that this identification is perfectly reasonable for logic. As for extensions, they may in such a temporal setting be thought of as "stages" of individuals. Extensions are a powerful technical tool; however, like stages, they have no place in the ontology of common sense, nor in the ontology of CIFOL (see also §5.1, esp. note 49).

2.5 Assignments and individuals

As noted, the intensional type of each individual term is, uniformly, $\Gamma \mapsto D$. In addition to cases, Γ, we must consider that expressions may contain free variables that are later to be bound by quantifiers or other devices. We therefore need to work with assignments to the variables; we let δ range over the set of all such assignments, that is, over the set $\Delta = \mathit{Vars} \mapsto (\Gamma \mapsto D)$. Thus, each $\delta \in \Delta$ assigns an individual intension to each individual variable.[19]

In typical applications of CIFOL, individuals of the concrete world, to use Quine's phrase, are represented as certain individual intensions, namely, those that fall under certain absolute properties in the sense of Def. 18 below in §4.3. It is a confusion to identify CIFOL extensions and individuals—a confusion that is made seemingly plausible by thinking of cases as "worlds," and of the extensions of individual terms at a world (the "inhabitants" of these worlds, as one might say), as individuals. Case-intensional logic, which makes no mandatory assumptions about the structure of the cases, helps to avoid this confusion: Why should the extension of an individual term at a case be the individual itself? Certainly individuals are not case-bound in natural language: Socrates, who is running in one case, is the same individual that is not running in another case; and Socrates at 2:00 p.m. is the same man as Socrates at 4:00 p.m. The discussion of §5 will show how to spell all this out without invoking "trans-world identity."

[19]Note that this type is mathematically equivalent to $\Gamma \mapsto (\mathit{Vars} \mapsto D)$, so that an assignment can alternatively be pictured as a family, indexed by Γ, of assignments of extensions to the variables. While mathematically equivalent, this way of thinking about variables can create confusions and is to be avoided.

In accord with Quine's famous comment as guest in Carnap 1947, extensions have "been dropped from among the values of the variables" of CIFOL. Quine took this as a fault; we do not: There is no danger that "the individuals of the concrete world" might disappear, "leaving only their concepts behind them" (p. 197). The reason is that in accord with the still more famous Quine dictum, that to be is to be the value of a variable, CIFOL represents concrete individuals precisely as certain values of its variables: Such values must fall under a natural absolute property.[20] The temporal application makes the point vivid: Socrates is represented as a function in $\Gamma \mapsto D$, which (accurately) represents Socrates as persisting through many moments. Of course Socrates is not eternal; CIFOL will represent this via the truth of "Socrates $= *$" in each case γ in Γ at which Socrates fails to exist.

3 Semantics

3.1 Basic case-intensional semantics

We can now lay out, recursively, case-intensional semantics for CIFOL.[21] Throughout, we silently assume that variables have been chosen so as to avoid collision or confusion. We begin by laying out in one place all the various semantic parameters needed for the recursive account of intension and extension. For ease of exposition, we treat only single-argument predicate and operator constants.

Semantic parameters for CIFOL. Γ is the set of cases, and γ is a member of Γ. D is the extensional domain, d is a member of D. Δ is the set *Vars* $\mapsto (\Gamma \mapsto D)$ of intensional assignments of values to the variables, and δ is a member of Δ. $\mathcal{I}$ is an intensional interpretation of the individual, predicate, and operator constants, each having a form dictated by the grammar. Thus, $\mathcal{I}(c) \in \Gamma \mapsto D$, $\mathcal{I}(p) \in \Gamma \mapsto \mathbf{2}$, $\mathcal{I}(P) \in \Gamma \mapsto ((\Gamma \mapsto D) \mapsto \mathbf{2})$ and $\mathcal{I}(f) \in \Gamma \mapsto ((\Gamma \mapsto D) \mapsto D)$, provided P and f are one place—which is the only case that we treat explicitly.[22] All semantic information required

[20] Other values can represent gerrymandered individuals: "Socrates at 2:00 p.m. and Plato at 4:00 p.m.," which might be the intension of "the philosopher on the corner."

[21] Like all model-theoretic semantics, CIFOL's falls on the "B" rather than on the "A" side of McTaggart's famous dichotomy.

[22] In these terms, the signature of extensional predication (see Def. 13) would lie in the characterization of $\mathcal{I}(P)$: $\mathcal{I}(P) \in \Gamma \mapsto (D \mapsto \mathbf{2})$, or equivalently, $\Gamma \mapsto \wp(D)$, instead of

for understanding closed expressions (i.e., no free variables) is included in a "model," $\mathcal{M} = \langle \Gamma, D, \mathcal{I} \rangle$.

As announced, we will stay general and talk about cases $\gamma \in \Gamma$ only in the abstract here (we give some structure to Γ in some examples). The heart of case-intensional semantics is the double dictum that each meaningful expression, ξ, including those with free variables, has (1) an intension that depends only on the model, $\mathcal{M}$, and an assignment, δ, to the variables, and (2) for each case, γ, an extension that depends on the model, $\mathcal{M}$, the assignment, δ, and the case, γ.[23]

3.1.1 Intension and extension

Expressions will have intensions, relativized to $\mathcal{M}$, and also to δ if the expression may contain free variables. We write

$$int_{\mathcal{M},\delta}(\xi)$$

(omitting δ when irrelevant), which will always have the form $\Gamma \mapsto X$, for X the set of appropriate extensions for expressions having the same type as ξ. Expressions will also have an extension in each $\gamma \in \Gamma$, also relativized to $\mathcal{M}$, and also to δ when there might be free variables. We write

$$ext_{\mathcal{M},\delta,\gamma}(\xi),$$

which will always satisfy the fundamental equations

Definition 2 (Extension from intension)

$$ext_{\mathcal{M},\delta,\gamma}(\xi) =_{df} (int_{\mathcal{M},\delta}(\xi))(\gamma)$$

and

Definition 3 (Intension from extension)

$$int_{\mathcal{M},\delta}(\xi) =_{df} \lambda\gamma[\gamma \in \Gamma](ext_{\mathcal{M},\delta,\gamma}(\xi)).$$

$\Gamma \mapsto ((\Gamma \mapsto D) \mapsto \mathbf{2})$ or $\Gamma \mapsto \wp(\Gamma \mapsto D)$.

[23] In the context of CIFOL, the double dictum must not be shortened to the appalling slogan, "each meaningful expression has an intension and an extension." Note also that δ is irrelevant and may be dropped in speaking of the intension, or extension-at-γ, of expressions that contain no free variables.

Henceforth we omit explicit use of the restricting clause "$[\gamma \in \Gamma]$" that occurs in Def. 3.[24] The core idea is that an intension, far from being something either mental or linguistic, is a pattern of extensions in γ, as γ ranges over the set Γ of cases.

Since extension and intension are correlative, in some contexts we take Def. 3 as semantically prior, in which case Def. 2 serves as a definition, and in other contexts the other way around. Circularity is avoided by our recursive use of extensions and intensions. We go over the clauses for each grammatical type of CIFOL as listed in §2.1.

3.1.2 Generalities: Sentences and terms

CIFOL virtuously treats in strict parallel the semantics of the two categorematic syntactic classes, sentences and terms. To support this parallelism, it has to turn out that each sentence, Φ [each term, α] will take as value, for each assignment, δ, a *sentential* (or *propositional*) [an *individual*] *intension*, which is a mapping from the cases to the set of truth values, $\mathbf{2}$ [to the domain, D] :

$$int_{\mathcal{M},\delta}(\Phi) \in \Gamma \mapsto \mathbf{2} \quad [int_{\mathcal{M},\delta}(\alpha) \in \Gamma \mapsto D].$$

A *sentential* [*individual*] *extension* on assignment δ in case γ is a truth value, that is, a member of $\mathbf{2}$ [a member of D]

$$ext_{\mathcal{M},\delta,\gamma}(\Phi) \in \mathbf{2} \quad [ext_{\mathcal{M},\delta,\gamma}(\alpha) \in D].$$

We obtain part of these requirements by means of parallel constraints on $\mathcal{I}$ and δ that are used in the base case of an inductive account of extension and intension: For each sentential constant, p, [individual constant, c], its intensional interpretation must be a mapping from Γ into $\mathbf{2}$ [into D]:

$$int_{\mathcal{M}}(p) =_{df} \mathcal{I}(p) \text{ (hence } int_{\mathcal{M}}(p) \in \Gamma \mapsto \mathbf{2})$$

$$int_{\mathcal{M}}(c) =_{df} \mathcal{I}(c) \text{ (hence } int_{\mathcal{M}}(c) \in \Gamma \mapsto D)$$

and for each individual variable, x, its intensional value on assignment δ must be of the same type as interpretations of individual constants:

$$int_{\mathcal{M},\delta}(x) =_{df} \delta(x) \text{ (hence } int_{\mathcal{M},\delta}(x) \in \Gamma \mapsto D).$$

[24] Risking the remote possibility of use-mention confusion, we will both mention λ as an element of CIFOL in the context λx (with x an individual variable), and use it in the context $\lambda\gamma$ (with γ ranging over cases).

Then the extensions-in-γ come from Def. 2. It is part of the uniformity of case-intensional semantics that the constant/variable distinction makes no difference to the type of their intensions, a choice that greatly simplifies the use of CIFOL. For example, this uniformity licenses unrestricted use of the rules of universal instantiation and existential generalization. More important, however, is that case-intensional semantics satisfies two critical requirements on any fully adequate logic, namely, "the essential substitution property" and "the vacuous assignment property." To state them succinctly, we need to introduce clear notation both for syntactic subsitution and for assignment-shift.

Definition 4 (Syntactic substitution) *$[\alpha/x](\xi)$ is the result of putting the closed individual term, α, for every free occurrence of the variable x in expression ξ.*

Definition 5 (Assignment-shift) *Where $x, y \in Vars, \delta \in Vars \mapsto (\Gamma \mapsto D), \overline{z} \in (\Gamma \mapsto D)$, and $[\overline{z}/x](\delta) \in Vars \mapsto (\Gamma \mapsto D)$:*

$$([\overline{z}/x](\delta))(y) =_{df} \begin{cases} \overline{z} & \textit{iff } y = x; \\ \delta(y) & \textit{otherwise.} \end{cases}$$

Thus, $[\overline{z}/x](\delta)$ may be read "the assignment that results from shifting the assignment δ by giving x the intensional value $\overline{z}$, and leaving alone the assignment to all other variables."

Definition 6 (Essential substitution property) *If there is no confusion or collision of variables (e.g., if α is closed),*

$$ext_{\mathcal{M},\delta,\gamma}([\alpha/x](\xi)) = ext_{\mathcal{M},[int_{\mathcal{M},\delta}(\alpha)/x](\delta),\gamma}(\xi).$$

At this point, we intend Def. 6 as a desideratum, or as a fact that depends on later developments. It says that you can calculate the extension on assignment δ in case γ of the result of substituting closed α for x in ξ by first calculating the assignment, call it δ', that results by altering δ by giving x as intensional value the intension of α, and leaving alone the assignment to all other variables, and then calculating the extensional value of ξ on δ' in γ. The essential substitution property is, *mutatis mutandis*, fundamental to extensional first order logic, required (among other things) for verifying that universal instantiation and existential generalization preserve truth. No modal logic that breaks uniformity in the sense of treating variables and constants differently (giving constants a more extensive set of values than the set allowed for variables) can be expected to have the "essential substitution property."

Next comes the apparently trivial but surprisingly deep companion to the essential substitution property, which is also required of a fully adequate logic:[25]

Definition 7 (Vacuous assignment property.) *The semantic value of an expression, ξ, shall never depend on assignments to variables that do not occur free in ξ.*

In developing a semantics for CIFOL, we take pains to satisfy the essential substitution property and the vacuous assignment property.

3.1.3 Operator constants, predicate constants, complex terms, predications

CIFOL permits operators and predicates of arbitrary n-arity; however, we give explicit treatment only to one-place operators and predicates, leaving those with many places to *mutatis mutandis*. Because of uniformity, we may treat operators and predicates together. Since the intension of a term [sentence] is always a function in $\Gamma \mapsto D$ [$\Gamma \mapsto \mathbf{2}$], it is hardly surprising that the intension of a one-place operator constant, f, [predicate constant, P] is conceptually equivalent to a function in $(\Gamma \mapsto D) \mapsto (\Gamma \mapsto D)$ [$(\Gamma \mapsto D) \mapsto (\Gamma \mapsto \mathbf{2})$].

In other words, the natural type of a one-place predicate constant would have the type of functions from individual intensions into sentential intensions (propositions), and would underwrite the characterization of the intension of a predicate as a propositional function (*à la* Russell).[26] In a strictly parallel fashion, the natural type of an intensional operator would be the type of functions from individual intensions into individual intensions (intensions in,

[25]See Kishida 2010 for a category-theoretic explanation of this claim.

[26]The overwhelmingly common (but regrettable) choice of standard first-order quantified modal logics is to let the type of the intension of a predicate constant be $\Gamma \mapsto (D \mapsto 2)$, or, equivalently, $\Gamma \mapsto \wp(D)$, thereby forcing all predicates to be extensional as a matter of logic. This choice severely reduces the expressive power of any such quantified modal logic.

intensions out). The official type, however, as promised in §3.1.2, must have the form $\Gamma \mapsto X$, and so we use the transposes instead:

$$int_{\mathcal{M}}(f) =_{df} \mathcal{I}(f) \in \Gamma \mapsto ((\Gamma \mapsto D) \mapsto D);$$

$$int_{\mathcal{M}}(P) =_{df} \mathcal{I}(P) \in \Gamma \mapsto ((\Gamma \mapsto D) \mapsto \mathbf{2}).$$

(Just this once: If f were binary, the type of its intension would need to be $\Gamma \mapsto (((\Gamma \mapsto D) \times (\Gamma \mapsto D)) \mapsto D)$, and so on.) So, by Def. 2,

$$ext_{\mathcal{M},\gamma}(f) \in ((\Gamma \mapsto D) \mapsto D); \quad ext_{\mathcal{M},\gamma}(P) \in ((\Gamma \mapsto D) \mapsto \mathbf{2}).$$

All that is intuitive and good; but note that the types of f $[P]$ and α don't quite match as function and argument, which threatens blockage of a semantic account of complex terms, $f(\alpha)$ [predications, $P(\alpha)$]. For this reason, the intension of a complex term, $int_{\mathcal{M},\delta}(f(\alpha))$ [predication, $int_{\mathcal{M},\delta}(P(\alpha))$], has to be defined in a somewhat roundabout way. First, recursively defining the extension of $f(\alpha)$ $[P(\alpha)]$ on δ relative to γ is straightforward:

$$ext_{\mathcal{M},\delta,\gamma}(f(\alpha)) =_{df} (ext_{\mathcal{M},\gamma}(f))(int_{\mathcal{M},\delta}(\alpha)) \in D;$$

$$ext_{\mathcal{M},\delta,\gamma}(P(\alpha)) =_{df} (ext_{\mathcal{M},\gamma}(P))(int_{\mathcal{M},\delta}(\alpha)) \in \mathbf{2}$$

(the types match). Then, in accord with the uniform definition of $int_{\mathcal{M},\delta}(\xi)$ in terms of $ext_{\mathcal{M},\delta,\gamma}$, as in Def. 3, we may define

$$int_{\mathcal{M},\delta}(f(\alpha)) =_{df} \lambda\gamma((ext_{\mathcal{M},\gamma}(f))(int_{\mathcal{M},\delta}(\alpha)));$$

$$int_{\mathcal{M},\delta}(P(\alpha)) =_{df} \lambda\gamma((ext_{\mathcal{M},\gamma}(P))(int_{\mathcal{M},\delta}(\alpha))).$$

The upshot is that case-intensional semantics treats predicate constants as strictly analogous to operator constants: Just replace the target, D, by $\mathbf{2} = \{\mathbf{T}, \mathbf{F}\}$.

It is occasionally useful to employ the epsilon (element of) notation for predication.

Definition 8 ($\in$) *If P is a predicate (including a λ-predicate as introduced just below) and α is an individual term,*

$$\alpha \in P \leftrightarrow_{df} P(\alpha).$$

The two notations are not intended to mark a logical difference.

3.1.4 Lambda-operators, -terms, -predicates, and -predications

A λ-operator [-predicate] of CIFOL is made by prefixing λx, for x an individual variable, to a term [sentence], for example, $\lambda x(f(x,(g(x))))$ [$\lambda x(x = \alpha)$], presumably a term [a sentence] that contains x free. Because CIFOL is to be first order (no quantification over higher types), we require that a λ-operator [-predicate] can occur only in operator [predicate] position. The semantics of λ-operators and -predicates is best understood as derivative:

$$int_{\mathcal{M},\delta}((\lambda x(\beta))\alpha) =_{df} int_{\mathcal{M},\delta}([\alpha/x]\beta),$$

$$int_{\mathcal{M},\delta}((\lambda x(\Phi))\alpha) =_{df} int_{\mathcal{M},\delta}([\alpha/x]\Phi),$$

as long as collision or confusion of variables is avoided. Then Def. 2 gives the extension-in-γ of of the λ-term $(\lambda x(\beta))\alpha$ [of the λ-predication $(\lambda x(\Phi))\alpha$].

Given that in CIFOL λ-operators $\lambda x(\beta)$ [λ-predicates $\lambda x(\Phi)$] are restricted to operator [predicate] positions, it is obvious that adding these λ-forms is conservative.

3.1.5 Identity

The two-place predicate constant, =, deserves special notice, because it is both an *extensional* predicate and used to help characterize extensional predicates in CIFOL (Def. 13). In each case, γ, the extension in γ of an identity statement depends only on the extension in γ of the terms:

$$ext_{\mathcal{M},\delta,\gamma}(\alpha_1 = \alpha_2) =_{df} \begin{cases} \mathbf{T} & \text{iff } ext_{\mathcal{M},\delta,\gamma}(\alpha_1) = ext_{\mathcal{M},\delta,\gamma}(\alpha_2); \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

As you can see, the fundamental thought is not that identity is "contingent"; rather, the basic feature of identity is that it is *case-dependent*.[27] Case-dependent identity is extensional identity, or identity of extensions. The

[27] Discussions of so-called contingent identity are concerned with examples such as the one given by Gibbard 1975, in which Lumpl (a lump of clay) and Goliath (a statue) exist at the same place at exactly the same times (they come into and go out of existence together), while that was not necessarily so. This story relies on treating constitution as identity (a matter on which CIFOL, as a logic, remains silent), and on an asymmetric handling of temporal cases "in the real world" and modal cases, which leads to the idea of world-bound individuals, Lumpl and Goliath, that have counterparts making true modal properties (such as possibly going out of existence in a different way) that underwrite the contingency of their identity. CIFOL has no use for this complex machinery, and it can represent the relevant aspects of the story in a straightforward manner: Allowing for

definition of the intension of an identity accords with Def. 3:

$$int_{\mathcal{M},\delta}(\alpha_1 = \alpha_2) =_{df} \lambda\gamma(ext_{\mathcal{M},\delta,\gamma}(\alpha_1 = \alpha_2)).$$

Although tedious to prove, it is intuitively reasonable to expect that identity of intensions of terms α and β suffices for replacement of α by β in an arbitrary CIFOL context, as long as one doesn't run up against confusion or collision of bound variables.

3.1.6 Truth-functional, modal, and quantificational connectives

The clauses for the truth-functional connectives are standard: Taking $\neg$ and $\wedge$ to be basic, we have the following for the extensions (the intensions are again defined via Def. 3, being the respective functions from the set of cases Γ):

$$ext_{\mathcal{M},\delta,\gamma}(\neg\Phi) =_{df} \begin{cases} \mathbf{T} & \text{iff } ext_{\mathcal{M},\delta,\gamma}(\Phi) = \mathbf{F}; \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

$$ext_{\mathcal{M},\delta,\gamma}(\Phi_1 \ \wedge \ \Phi_2) =_{df} \begin{cases} \mathbf{T} & \text{iff } ext_{\mathcal{M},\delta,\gamma}(\Phi_1) = ext_{\mathcal{M},\delta,\gamma}(\Phi_2) = \mathbf{T}; \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

Then the semantics of other truth functional connectives such as the conditional, $\rightarrow$, and the biconditional, $\leftrightarrow$, fall right out as expected.

For the alethic modal connectives, we employ the standard **S5** semantics. (We beg leave to doubt the usefulness of alethic modalities that rely on relational semantics.)

$$ext_{\mathcal{M},\delta,\gamma}(\Box\Phi) =_{df} \begin{cases} \mathbf{T} & \text{iff for all } \gamma' \in \Gamma, (ext_{\mathcal{M},\delta,\gamma'}(\Phi) = \mathbf{T}); \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

$$ext_{\mathcal{M},\delta,\gamma}(\Diamond\Phi) =_{df} \begin{cases} \mathbf{T} & \text{iff for some } \gamma' \in \Gamma, (ext_{\mathcal{M},\delta,\gamma'}(\Phi) = \mathbf{T}); \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

temporal-modal cases, there is a whole range of cases (in the example, a full history) in which Lumpl = Goliath holds, and there are cases in which Lumpl $\neq$ Goliath. Using the notion of an absolute property (see §4.3), CIFOL can even allow that Lumpl and Goliath fall under different sortal predicates while sharing their extension in many cases: such an extension is then traced under two different principles of identity, allowing for the intensional difference of Lumpl and Goliath even if both are proper substances. (On the issue of whether this makes metaphysical sense, CIFOL, as a logic, remains silent.)

Again, the intension is derived from the various extensions-in-γ via Def. 3. As usual, the strong modality $\Box$ and the weak modality $\Diamond$ turn out to be duals, so that we can treat $\Box$ as semantically basic and introduce the weak modality as an abbreviation: $\Diamond =_{df} \neg\Box\neg$.

The semantics of quantifiers is, not surprisingly, more complicated, in precisely the same fashion as the semantics of quantifiers in classical logic. Using $\overline{z}$, as before, as ranging over $\Gamma \mapsto D$, we rely on the assignment-shift notation (Def. 5). Then semantic clauses for the quantifiers come naturally by quantifying over individual intensions:[28]

$$ext_{\mathcal{M},\delta,\gamma}(\exists x\Phi) =_{df} \begin{cases} \mathbf{T} & \text{iff } \exists\overline{z}(\overline{z} \in (\Gamma \mapsto D) \text{ and } ext_{\mathcal{M},[\overline{z}/x](\delta),\gamma}(\Phi) = \mathbf{T}); \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

$$ext_{\mathcal{M},\delta,\gamma}(\forall x\Phi) =_{df} \begin{cases} \mathbf{T} & \text{iff } \forall\overline{z}(\text{if } \overline{z} \in (\Gamma \mapsto D) \text{ then } ext_{\mathcal{M},[\overline{z}/x](\delta),\gamma}(\Phi) = \mathbf{T}); \\ \mathbf{F} & \text{otherwise.} \end{cases}$$

Then, by Def. 3,

$$int_{\mathcal{M},\delta}(\exists x\Phi) =_{df} \lambda\gamma(ext_{\mathcal{M},\delta,\gamma}(\exists x\Phi)),$$

and

$$int_{\mathcal{M},\delta}(\forall x\Phi) =_{df} \lambda\gamma(ext_{\mathcal{M},\delta,\gamma}(\forall x\Phi)).$$

Since $\forall x$ and $\exists x$ are interdefinable in the same fashion as in first order logic, we could stick to the former as basic. Finally, we note that as a result of the independence of the choice of case, $\gamma \in \Gamma$, and the choice of intension, $\overline{z} \in (\gamma \mapsto D)$, both the Barcan formula, $\Diamond\exists x\Phi \to \exists x\Diamond\Phi$, and its converse, $\exists x\Diamond\Phi \to \Diamond\exists x\Phi$, turn out valid, as do the equivalent statements formulated with $\Box$ and $\forall$.[29] Both are usually found problematic. Consider the Barcan

[28] As we mentioned in §2.5 above, logicians mesmerized by two familiar Quinean dogmas can hardly avoid thinking that it is somehow a requirement of an honest logic that the values of variables should be extensions, and that to be is to be the value of a variable. To repeat, CIFOL rejects the first dogma, but is happy to accommodate the second dogma in the following sense: To be an individual in the concrete world is to be the value of a variable ranging over individual intensions that fall under some natural sortal. See also §4.3 below.

[29] A valid sentence of CIFOL that to the uninformed is more astonishing than the Barcan formula is this: If Θ is extensional (Def. 13),

$$\mathcal{M} \models \Box\exists x\Theta x \to \exists x\Box\Theta x.$$

It looks as if there is a permutation-of-quantifiers mistake; but it isn't so. (It would appear that the axiom of choice is needed for the proof of validity.)

formula, and take Φ to stand for "x is a Dodo." Let us consider an English sentence that looks like an instance of the Barcan formula scheme:

If it could be that there exists a Dodo, then
there exists something that could be a Dodo. (1)

Now it is true to say that it could be that there exists a Dodo (it wasn't necessary that the Dodo became extinct), but it seems false to say that there exists something that could be a Dodo (after all, only a Dodo could be a Dodo). This, however, does not exhibit a problem with the validity of the Barcan formula in CIFOL. Rather, since the range of CIFOL variables includes the "nonexistent entity," $*$, the CIFOL correspondent to (1) needs the existence predicate (Def. 1) in addition to the existential quantifier:

$$\Diamond\exists x(Ex \ \wedge \ Dx) \rightarrow \exists x(Ex \ \wedge \ \Diamond Dx). \tag{2}$$

And this formula can be falisfied in CIFOL.

For brevity we only give an explicit model for converse Barcan. A typical English counterexample takes Φ to stand for "x doesn't exist":

If there exists a thing that possibly doesn't exist, then
possibly there exists a thing that doesn't exist. (3)

Again, the antecedent is true, but the consequent is even logically false. But again, the correct rendering of that sentence in CIFOL has to make explicit the use of the existence predicate, leading to

$$\exists x(Ex \ \wedge \ \Diamond\neg Ex) \rightarrow \Diamond\exists x(Ex \ \wedge \ \neg Ex). \tag{4}$$

It is easy to show that this formula is not a CIFOL validity. Let $\Gamma = \{\gamma_1, \gamma_2\}$ and $D = \{*, a\}$ with $a \neq *$, so that $\Gamma \mapsto D$ can be represented as the set $\{a*, aa, *a, **\}$.[30] Let $int(\alpha) = a*$. Then since $E(a*)$ is true in γ_1 but false in γ_2, the antecedent of (4) is true in γ_1. But its consequent, being a logical falsehood, must fail in γ_1, thus showing how CIFOL, in which the converse Barcan formula is a validity, correctly counterexamples the English converse Barcan lookalike, (3).

[30]For notation, see the beginning of §5.1.

3.1.7 Unique existence and definite descriptions

In contrast to the standard Russellian approach, the CIFOL semantics of definite descriptions, following Frege, treats them as categorematic expressions having the same semantic type as individual constants and individual variables, namely, $\Gamma \mapsto D$. As a preliminary, we introduce $\exists_1 x\Phi$ as saying that there exists exactly one x such that Φ—extensionally speaking. Then we define extensions and intensions for $\imath$-terms.

Definition 9 (Unique existence)

$$\exists_1 x\Phi \leftrightarrow_{df} \exists x(\Phi \wedge \forall y([y/x]\Phi \rightarrow y = x)).^{31}$$

Let us abbreviate $(\Phi \wedge \forall y([y/x]\Phi \rightarrow y = x))$ by Φ_1. Then, given an assignment δ and a case γ, if $ext_{\mathcal{M},\delta,\gamma}(\exists_1 x\Phi) = \mathbf{T}$, then there is a unique $d \in D$ such that for any $\overline{z} \in \Gamma \mapsto D$, we have:

$$\text{if} \quad ext_{\mathcal{M},[\overline{z}/x](\delta),\gamma}(\Phi_1) = \mathbf{T} \quad \text{then} \quad \overline{z}(\gamma) = d.$$

We call this d *the extensional witness for* $\exists_1 x\Phi$ *at* $\mathcal{M}, \delta, \gamma$. Now we define definite descriptions as certain $\imath$-terms that have an extension in each case:

Definition 10 (Definite descriptions)

$$ext_{\mathcal{M},\delta,\gamma}(\imath x\Phi) =_{df} \begin{cases} * & \textit{iff } ext_{\mathcal{M},\delta,\gamma}(\exists_1 x\Phi) = \mathbf{F}, \\ \textit{the extensional} & \\ \textit{witness for} & \\ \exists_1 x\Phi \textit{ at } \mathcal{M},\delta,\gamma & \textit{iff } ext_{\mathcal{M},\delta,\gamma}(\exists_1 x\Phi) = \mathbf{T}. \end{cases}$$

Note that the definite description in CIFOL is a purely extensional construct. Of course there is an intension for each definite description, defined by Def. 3. The point, however, is that only extensional information is used, so that the $\imath$ construction applied to extensionally equivalent properties gives rise to extensionally equivalent terms.[32]

[31]Recall that by the convention mentioned at the beginning of §3.1, here and in the following, y stands for a variable that does not occur in Φ.

[32]Bressan worked out a useful intensional description construction; because, however, his account is second order, we do not try to reconstruct it in CIFOL.

As with Frege and in contrast to Russell and his followers, the semantics of a definite description is quite independent of the context in which it is embedded.

Because the CIFOL treatment of definite descriptions is transparent but unusual, we note that definite descriptions in CIFOL always work properly for contexts that are extensional (for the notation $(extnl\ x)\Phi$, see Def. 13 below). That is, the following is valid (see Def. 11):

$$(\exists_1 x\Phi \wedge (extnl\ x)\Phi) \rightarrow [\imath x\Phi/x]\Phi.$$

The predicate version of this is perhaps easier to read:

$$(\exists_1 x\Theta x \wedge (extnl\ x)\Theta x) \rightarrow \Theta(\imath x\Theta x).$$

3.1.8 Defined predicates

Part of what makes CIFOL easy to use is its unfettered ability to permit the introduction of new predicates by definition. The following schema is available for arbitrary Φ, provided one satisfies the standard criteria of eliminability and conservativeness, just as in extensional logic—see Suppes 1957, 154.

$$\text{Definition:}\quad \Box\forall x(Px \leftrightarrow_{df} \Phi).$$

At a slightly higher type, a logically equivalent form of definition would employ λ abstracts:

$$\text{Definition:}\quad \Box(P =_{df} \lambda x\Phi).$$

It is understood that identity of properties is logically definable at the first order by, for example, $\forall y(Py \leftrightarrow \lambda x(\Phi)y)$. The powerful upshot is that there is no difference, in CIFOL, between the role of predicate letters, P, and λ-abstracts.

3.2 Truth and validity

The *ext*/*int* notation shows off the ease of use and, above all, the uniformity of CIFOL semantics: The central idea is applied uniformly to every part of speech. Nevertheless it is helpful to recast the extension of sentences also in more familiar terms, leaving intension to be construed by way of Def. 3.

Accordingly, we define a truth predicate, relative to $\mathcal{M}$, δ and γ, in the common way, first in what Carnap calls "the word language," and then in a common symbolic form that we shall use heavily. Then we define validity in the usual way, assuming as usual that $\mathcal{M} = \langle \Gamma, D, \mathcal{I} \rangle$.

Definition 11 (True, valid, $\models$)

$$\Phi \textit{ is true on } \mathcal{M} \textit{ and } \delta \textit{ in } \gamma \leftrightarrow_{df} ext_{\mathcal{M},\delta,\gamma}(\Phi) = \mathbf{T}.$$

$$\mathcal{M}, \delta, \gamma \models \Phi \leftrightarrow_{df} ext_{\mathcal{M},\delta,\gamma}(\Phi) = \mathbf{T}.$$

$$\mathcal{M} \models \Phi \leftrightarrow_{df} \forall \gamma \in \Gamma,\ \delta \in \Delta\ (\mathcal{M}, \delta, \gamma \models \Phi).$$

$$\models \Phi \leftrightarrow_{df} \Phi \textit{ is valid} \leftrightarrow_{df} \mathcal{M} \models \Phi \textit{ for all models } \mathcal{M}.$$

As always, δ can be dropped when irrelevant.

3.3 Proof theory of CIFOL

What is an appropriate proof theory for CIFOL? Bear in mind that CIFOL is intended to be a first-order limited version of Bressan's infinitely typed theory, ML^{ν}. Since ML^{ν} squarely contains elementary arithmetic, it cannot have a sound and complete proof theory. (Bressan 1972 offers a kind of relative completeness; see section N64.) Nevertheless, for routine applications, the following version of CIFOL proof-theory will do. (1) Take any formulation of the first order logic of truth functions and quantifiers. (2) Add postulates characterizing identity as an equivalence relation (thus omitting the usual replacement principles), and a postulate characterizing necessary identity as a sufficient condition for replacement of one closed term by another in all (reasonable) CIFOL contexts. Add what is needed to ensure that Frege-style definite descriptions and first-order-respecting lambda abstraction are available. (3) Add standard **S5** postulates for necessity and possibility. (4) Finally, add use of the definitions of absoluteness, etc.: Defs. 13–18. We do not claim that such a proof theory is complete with respect to the rich CIFOL semantics that we have presented; indeed, it certainly is not.[33] CIFOL remains, however, as a tool that is helpful in the endeavor

[33] As Bressan has shown, ML^{ν} has the resources to formulate an interesting and apt theory of truth-in-a-case. First he introduces a predicate, ElR, such that the intensions falling under ElR are in perfect correspondence with the set, Γ, of cases. It is good to think of ElR as giving an account of "internal cases." For technical reasons, Bressan switches

to separate good arguments from bad ones among those framed in CIFOL's grammar. We illustrate applications of an informally described CIFOL "natural deduction" style proof theory in §4.3 and §5.5.

4 CIFOL qualities and CIFOL sortals

We have finished presenting the core of the logic CIFOL in terms of its primitive grammar, semantics and proof theory: We will add nothing more "creative." On the other hand, in our view, no presentation of CIFOL can be taken to be complete without a family of noncreative definitions characterizing certain logical properties applicable to predicates. We take these first-order-definable properties of properties to be as much an integral part of CIFOL as one takes the definitions of addition and multiplication to be an integral part of Peano arithmetic. Accordingly, we complete our formal account of CIFOL with seven definitions: Def. 13 of extensionality, Def. 15 of extensionalization and existence, Def. 16 of modal constancy, Def. 17 of modal separation, Def. 18 of absoluteness, Def. 14 of CIFOL quality, and Def. 19 of CIFOL sortal. The definitions of modal separation and absoluteness are entirely original with Bressan.[34] Finally, we set down two theses that connect certain informal notions with (formal) CIFOL notions. Namely, we advance the thesis that, up to an idealization, each natural quality satisfies the defining conditions of a CIFOL quality, and each natural sort satisfies the defining conditions of a CIFOL sortal.[35]

We turn now to the key CIFOL definitions of certain properties of properties; later, in §5, we illustrate their usefulness. This means that we are going beyond purely logical motivation: Considerations of metaphysics, and

to a second internal representation, El, of the cases. Letting x be an ML^ν variable ranging over the internal cases, Bressan uses this equipment to show how to define an ML^ν construction that we take to have the sense of "That Φ is true in the internal case, x." Bressan's treatment of this matter appears to go beyond CIFOL as we have presented it. It seems, however, that redefining CIFOL as containing but a single additional first order axiom suffices for defining "true in a case." This theme is developed in Belnap 2013.

[34]The ability to characterize certain predicates as absolute is, we think, the principal virtue of systems allowing non-extensional predication. It is an order of magnitude more important than the ability to theorize wisely about various non-extensional ("opaque") contexts of conversational English on which so much of the literature focuses.

[35]It will be clear that we use "CIFOL quality" and "CIFOL sortal" as technical terms intended as revelatory of the informal notions of "natural quality" and "natural sortal." You will also see from our usage that we are relaxed in our use of "property" and "predicate."

specifically the metaphysics of individuals and sorts, will play a guiding role from now on. We repeat that this will not lead us to tinker with the logic, but rather enable the easy addition of extra-logical concepts, each with its characteristic axioms, to do the metaphysical work. An adequate metaphysical picture of individuals and sorts must be based on explicit considerations of time, modality, and their interaction. These explicit considerations, leading to cases as moment/history pairs, will be the subject of Part II of this essay. In this Part, we stick to an abstract notion of cases. The general idea of tracing an individual across cases via a sortal property can already be spelled out at that level of generality.

As we noted in §2.4, the semantics of applying a predicate to an argument-term in a case depends in general on the intension of the argument-term, not just on its extension in the case. The concept of a property in case-intensional semantics is therefore broad. Many predicates, however, are "extensional" (Def. 13): Whether they apply to closed α in case γ depends only on the extension of α in γ (i.e., on $ext_{\mathcal{M},\gamma}(\alpha)$). Ordinary *natural qualities* are like that: Whether or not *Jack is lean* is true in case γ depends only on whether Jack is lean in that very case.[36] Other properties are non-extensional: Whether or not they apply to α in case γ makes reference to other cases. For example, when cases are times, simple tensed statements such as *Jack was at home* refer to other cases (past time periods), as do simple modals such as *Jack might be at Caroline's* (alternative possibilities). One-word examples of non-extensional English predicates would be *soluble* and *aggressive*, which are naturally taken to involve reference to possible cases. Among non-extensional predicates, of paramount importance are "sortals" such as *natural number* and *horse*. We will characterize CIFOL sortals via a pair of intensional properties, meaning thereby to contribute to the clarification of the interface between logic and metaphysics without ourselves waxing metaphysical. A critical feature of sortals is that they naturally enable tracing across cases as discussed in §4.3 below. (You will do no wrong if you read "sortal property" as "tracing property.")

English tends to carry qualities with adjectives or verbs, while generally marking sortals with common nouns. Gupta 1980 is a careful and illuminating formal working-out of this policy. Montague 1973 also treats qualities

[36]We are thinking of a rough-and-ready notion of "is lean." If one prefers to analyze leanness against a background comparative, "is leaner than," or "is leaner for a boy than," CIFOL can handle an analysis that would render "is lean" non-extensional.

and sortals as different categories of basic expressions. CIFOL, however, unlike English and Montague and Gupta, does not distinguish qualities and sortals syntactically. Instead, following Bressan, the distinction is marked by the properties with which each primitive predicate is endowed by suitable axioms.[37] Whether or not a predicate constant has one of these properties naturally depends on the interpretation, $\mathcal{I}$. We confine our discussion to one-place properties, leaving to the reader the easy but tedious generalization to n-place properties, including the possibility that a relational property might be of a special type with respect to one argument but not another. It will be clear that throughout we are ignoring vagueness, just as one does in applying standard extensional logic.

4.1 Existence and non-existence

We shall treat qualities and sorts separately. There is, however, one property of properties that might be thought common to every property. Although case-intensional logic needs to allow for individual intensions representing that α does not exist in a certain case (by setting $\mathcal{M}, \gamma \models \alpha = *$; see Def. 1), no matter the application, it would be coherent to insist that every interesting candidate for a primitive predicate, P, be "existence-implying" in each case:

Definition 12 (Existence-implying) *A property* Θ *is existence-implying* $\leftrightarrow_{df}$ $\mathcal{M} \models \Box\forall x(\Theta x \to Ex)$*. For this, we write* $\textit{Exist-imp}(\Theta)$*.*

The restriction to "primitive" would be essential. Being existence-implying is, for example, not closed under complementation: If Q is introduced by the definition $\forall x(Qx \leftrightarrow_{df} \neg Px)$ (with P existence-implying), Q is not existence-implying. Even so, the policy of requiring every primitive predicate to be existence-implying, although coherent, would create more problems than it solves. Is it natural to suppose that since Socrates is necessarily a *Man*, he must exist in every case? Certainly not if the application is temporal, with the cases being time intervals. Therefore, it is not a good idea to force *Man* to be existence-implying: We want to treat cases or times in which Socrates doesn't exist in a way that allows that he is a *Man* in those cases. We want to say that the *Man*, Socrates, didn't exist before he was born nor after his death.

[37] Such axioms are in spirit second order, but technically each is first order.

This seems more puzzling than it has to be. Keep firmly in mind that (1) Socrates is represented in CIFOL by an individual intension living from birth to death if the cases are time intervals, (2) that *Man*, being a sortal, is non-extensional, applying to the whole intension, whereas (3) existence is extensional and case-bound. The times when Socrates doesn't exist are marked by setting Socrates $= *$ at those times. Only in this way can we easily interpret the present truth of "Socrates is a *Man* who doesn't (now) exist, but formerly existed." In contrast, it may appear suitable to suppose that a natural quality such as "is six feet tall" be existence-implying: It seems unnecessary to countenance "Socrates is six feet tall (now) but doesn't exist (now)."[38]

The upshot is that we want the CIFOL representation of Socrates to have an extension at every case, that extension being $*$ when he doesn't exist. Let us emphasize and re-emphasize that the CIFOL representation of Socrates is a matter of choice, not of metaphysics.[39] For instance, science and metaphysics are hardly likely to totter when faced with the view that at the present moment, Socrates = Plato, meaning that they have the same extension, namely, $*$. As logicians, however, we do have to take special account of the fact that our representations of Socrates and Plato at the present moment are identical. See our treatment of "modal separation" in §4.3. We also need to be thoughtful about sentences or terms containing the symbol, $*$. Doubtless if f represents "the average distance of $(\ldots)$ from the Sun," then we should have $\mathcal{M}, \gamma \models f(*) = *$. If, however, f represents "the mother of," then supposing γ is a case or time in which JFK doesn't exist but his mother does, even though $\mathcal{M}, \gamma \models \text{JFK} = *$, one would want $\mathcal{M}, \gamma \models f(\text{JFK}) = \text{Rose Kennedy} \neq *$ ("the-mother-of" is non-extensional).

[38] Still, in our general formal definition of a CIFOL quality, Def. 14 below, we do not include the requirement that such a quality be existence-implying. We can easily enter such a requirement explicitly if that is called for in a specific case.

[39] Bressan 1972 makes a different choice. We won't go into the matter except to register our belief that every way of representing the workings of case-dependently empty names requires some unexpected complication. Note that on our choice, but not on many others, we can have names *for* things of a certain kind in a uniform way: If α is a name for a P, then $P\alpha$ is true in any case. (We don't have to check whether the bearer of the name still exists in order to use it properly.)

4.2 CIFOL qualities are extensional

In a typical application of CIFOL, the vast majority of primitive predicates will represent "qualities" carried in English by descriptive constructions such as "smells sweet" or "is six feet to the north of Mary." The formal mark of a quality predicate in CIFOL is extensionality ("a rose by any other definite description would smell as sweet"). Run-of-the-mill extensionality is a local property; that is, extensionality is case-dependent. Properties that are extensional in some but not all cases can be useful, but they are comparatively rare. For this reason, we reserve plain "extensional" for what might be called "everywhere extensional."[40]

Definition 13 (Extensionality, extensional, $(extnl\ x)$) *A property Θ is* extensional *iff it is extensional at every $\gamma \in \Gamma$, i.e., iff the following holds:*

$$\mathcal{M} \models \Box \forall x \forall y\, (x = y \rightarrow (\Theta(x) \leftrightarrow \Theta(y))).$$

A context Φ, presumably with free x, is extensional with respect to x *iff the following holds:*

$$\mathcal{M} \models \Box \forall y\, (x = y \rightarrow (\Phi \rightarrow [y/x]\Phi)).$$

$(extnl\ x)\Phi$ is to be read as "Φ is extensional with respect to x" and is defined as follows:

$$(extnl\ x)\Phi \leftrightarrow_{df} \Box \forall y\, (x = y \rightarrow (\Phi \rightarrow [y/x]\Phi)).$$

For conceptual bookkeeping, it is convenient to introduce the following bit of redundancy.

Definition 14 (CIFOL quality) *A property, Θ, is a* CIFOL quality $\leftrightarrow_{df}$ *Θ is extensional.*

We are now prepared to state a thesis connecting natural qualities with CIFOL qualities.

[40] One use of the case-dependent notion that comes to mind is in the characterization of a property as a "set" when some case, call it γ_R, is intended as "the real case." Then it is good to say that a property is a *set* only if it is both modally constant and also extensional in the real case. See Bressan 1973. Another use crops up in connection with the At_n: construction, as discussed in §5.2.

Thesis 1 (Qualities) *Up to an idealization, natural qualities are CIFOL qualities; that is, they are extensional.*

We said that almost all primitive predicates are extensional. The prime examples of non-extensional predicates are the absolute predicates, to which idea we soon turn. First, however, we define the *extensionalization* of Θ as the weakest extensional "super-predicate" of Θ, and then we define the result of forcing an existence-implying version of Θ by deleting from this all the individual intensions that do not exist (Def. 1).

Definition 15 (Extensionalization and existence: $\Theta^{(e)}$ and $\Theta^{(e!)}$)

$$\Box\forall x[\Theta^{(e)}x \leftrightarrow_{df} \exists y[y = x \ \wedge \ \Theta y]],$$

$$\Box\forall x[\Theta^{(e!)}x \leftrightarrow_{df} [\Theta^{(e)}x \ \wedge \ Ex]].$$

When Θ represents a sort, it seems that we often want both Θ and either $\Theta^{(e)}$ or $\Theta^{(e!)}$, each to do its own job. We postpone illustration until we have introduced absolute properties, a task to which we now turn.

4.3 CIFOL sortals: modally constant and modally separated

We will define absoluteness as a conjunction of two other non-extensional properties of properties, modal constancy and modal separation. This unique development is an entirely original contribution of Bressan.[41] The first of the two ideas is modal constancy, defined as follows.

Definition 16 (Modal constancy) *A property* Θ *is* modally constant *iff*

$$\textit{(MConst)} \qquad \mathcal{M} \models \forall\, x\, (\Diamond\Theta x \rightarrow \Box\Theta x).$$

Given non-extensional predication, this is "rigidity" of predicates or properties. (The displayed definition doesn't work properly if Θ is extensional. In fact in CIFOL there are only two properties that are both extensional and modally constant: the modally empty property and the modally universal property.) Back in the day of Kripke and Putnam, rigidity of singular

[41] Those familiar with Bressan 1972 will observe that in reworking, adjusting, and simplifying Bressan's account, we omitted his idea of a "quasi-absolute" property, which is not useful in CIFOL.

terms and predicates was often treated as essentially the same property, distinguished only by grammatical category. Recall, however, that modal constancy or "rigidity" of singular terms or individual intensions has no use in CIFOL, and indeed is not expressible (§2.3). In contrast, modal constancy of properties is exactly the idea of "rigidity" applied to properties: If Θ is modally constant, then $ext_{\mathcal{M},\gamma}(\Theta) = ext_{\mathcal{M},\gamma'}(\Theta)$ for all cases $\gamma, \gamma' \in \Gamma$.[42]

The second leg on which absoluteness stands is modal separation:

Definition 17 (Modal separation) *A property* Θ *is* modally separated *iff*

(MSep) $$\mathcal{M} \models \Box \forall x \forall y \, ((\Theta x \wedge \Theta y) \rightarrow (\Diamond(Ex \wedge Ey \wedge x = y) \rightarrow \Box x = y)).$$

A property is modally separated iff, roughly, individual intensions falling under it at any case are either everywhere identical or never. The logical point is that if α falls under a modally separated property at a case, then, in this special situation, as long as α exists in the case, the extension of α determines its intension, so that except for cases of non-existence, $\alpha_1 = \alpha_2$ being true in some case suffices for replacement of α_1 by α_2 in any and every CIFOL context.[43] Putting modal constancy together with modal separation yields absoluteness:

Definition 18 (Absoluteness) *A property* Θ *is* absolute *iff it is modally constant and modally separated. Where* Θ *is a predicate, possibly a* λ*-predicate, we let* $Abs(\Theta)$ *abbreviate "*Θ *is absolute."*

To serve as a convenient companion to "CIFOL quality" (Def. 14), we introduce "CIFOL sortal" as definitionally equivalent to "absolute property," in order to emphasize the difference between the formal notion of a CIFOL sortal and the informal notion of a natural sortal.

[42] Oddly enough, in CIFOL we use the *universal* identity of extensions of a property, Θ (that is, modal constancy), without having any use for the *retail* identity of individual extensions of Θ at a pair of distinct cases γ, γ'.

[43] The concept of modal separation is, perhaps, the single most original idea of Bressan's contribution to quantified modal logic. Parks 1972 invoked Bressan's concept to some purpose, and Gupta 1980 not only used the concept but significantly elaborated on it. Aside from these two publications, even as deep and important as is modal separation, we have been unable to find anything remotely like it in the literature. (Montague's **IL** is powerful enough to define modal separation, but Montague didn't exercise this power.) *Exercise*: Explain why Bressan's profound ideas have been ignored for four decades.

Definition 19 (CIFOL sortal) *A property, Θ, is a* CIFOL sortal $\leftrightarrow_{df}$ Θ *is an absolute property.*

Modal absoluteness provides an analysis of the case-intensional aspects of natural-language sortal predicates in CIFOL, spelling out formally specifiable necessary conditions for a predicate to be a natural sortal. There is, however, no suggestion that absoluteness—or any other formalizable property—fully captures those properties that a property must have to be a "substance concept" useful in either scientific or metaphysical applications; we do not aim at giving a sufficient condition. Just for starters, there are just far too many trivial absolute properties, and absoluteness taken alone omits the pragmatics of scientific properties, along with much else. Bressan speaks of *natural* absolute properties, and so shall we.

A sortal property is meant to answer the Aristotelian question of "what a thing is." As Aristotle explains in his *Categories* by the example of Socrates, to say that he is white (or pale), is not yet to say what he is: That is a merely contingent ("accidental") property. In fact, being white is quite plausibly taken to be an extensional property in the sense of Def. 13: It applies or doesn't apply, given an individual and a case, and that is all there is to that.[44] The property of being white is ascribed to Socrates purely by considering the property's *principle of application* at a case. To say that Socrates is a man, however, *is* to answer the "what is it?" question, according to Aristotle: It is to say what sort the thing belongs to. A sortal property, such as being a horse, or a man, supplies not just a principle of application, but also a *principle of identity*, or, as we like to say, a *tracing principle*.[45] If we have said *what* Socrates is (namely, a man), we have thereby specified something that he could not *not* be: This is precisely modal constancy. Being white, on the other hand, is obviously not modally constant: If Socrates becomes

[44] *Running* is another favorite illustration of an accidental property of Socrates. There is a debate in philosophy of physics whether one can ascribe instantaneous velocities to things in an intrinsic manner. For an argument to the effect that one cannot, as well as for some background references to that debate, see Butterfield 2006. We will not need to enter that debate here, being assured that on the one hand there are certainly at least *some* extensional properties, and on the other hand, case-intensional semantics welcomes non-extensional properties.

[45] The now common distinction between a principle of application and a principle (or criterion) of identity (see, e.g., Dummett 1973, p. 546 and Gupta 1980, p. 2) comes from Geach 1962, §31, who in turn follows Aquinas in "distinguishing general terms as *substantival* and *adjectival*."

sunburned and turns red, he will cease to be white, but he will not thereby cease to be Socrates.[46] There is also a clear motivation for requiring the property of being a man to be modally separated: Because Socrates is a man, he can be traced under that concept from time to time or case to case, and there cannot exist a man who coincides with him in one case but not in another.[47] Sortal concepts are absolute. Thus, we are prepared to defend the following thesis:

Thesis 2 (Sortals) *Up to an approximation, every natural sortal is a CIFOL sortal; that is, every natural sortal is absolute, which is to say, is modally constant and modally separated.*

It is instructive to prove that if Θ is a CIFOL sortal, then necessarily, if $\alpha \in \Theta^{(e!)}$, there is an unique member of Θ that is extensionally identical to α; that is, to prove

Fact 1 (From absoluteness to necessary identity)

$$Abs(\Theta) \to \Box[\Theta^{(e!)}\alpha \to \exists x[x = \alpha \land \Theta x \land \forall y[(y = \alpha \land \Theta y) \to \Box x = y]]].$$

Note that Fact 1 contains three occurrences of the absolute predicate, Θ, in predicative position: Once Θ is used in the context of an extensionalization, $\Theta^{(e!)}$, that is used for reporting the local fact that in the case at hand, α is (extensionally) identical to an intension falling under Θ—recall that Def. 15 defines $\Theta^{(e!)}\alpha$ as $\exists z(z = \alpha \land \Theta z) \land E\alpha$. And twice Θ is used as a tracing principle that, in effect, traces the individual falling under Θ in the case at hand to "the same individual" in each case invoked for the necessity statement.

[46]It is obvious that the compound, "white horse," definable as the intersection of a quality with a sortal, is neither extensional nor absolute. Along the same lines, we analyze phased sortals such as "boy", which do arguably answer the Aristotelian question of what a thing is but aren't modally constant, into a true sortal ("man") and a quality ("under-age"); see Wiggins 2001, p. 30ff. In the other direction, in §5.4 we definitely classify "natural number" as a sortal because it is important that numbers can be traced from case to case, whereas it would be absurd to think of natural numbers as metaphysically akin to Aristotelian substances such as horses.

[47]This is our brief comment on the question of relative identity as raised by Geach 1962: We hold that Geachean examples necessarily mix true sortal, absolute properties (such as being a man) with non-sortal properties. For a diagnosis similar in spirit, see Lowe 2009.

Proof.

1	$Abs(\Theta)$	hyp
2	$\Theta^{(e!)}\alpha$	hyp
3	$x_0 = \alpha \wedge \Theta x_0 \wedge E\alpha$	2, def, choose x_0
4	$y_0 = \alpha \wedge \Theta y_0$	hyp, choose y_0
5	$x_0 = y_0 \wedge Ex_0$	3, 4
6	$\Theta x_0 \wedge \Theta y_0$	3, 4
7	$\Diamond(Ex_0 \wedge Ey_0 \wedge x_0 = y_0)$	5, **S5**
8	$\Box(x_0 = y_0)$	6, 7, 1 ($MSep(\Theta)$)
9	$\forall y[(y = \alpha \wedge \Theta y) \rightarrow \Box(x_0 = y)]$	4–8, (y_0/y)
10	$\exists x[x = \alpha \wedge \Theta x \wedge \forall y[(y = \alpha \wedge \Theta y) \rightarrow \Box x = y]]$	3, 9, (x_0/x)
11	$\Box[\Theta^{(e!)}\alpha \rightarrow \exists x[x = \alpha \wedge \Theta x \wedge \forall y[(y = \alpha \wedge \Theta y) \rightarrow \Box x = y]]]$	2–10, 1, **S5**
12	$Abs(\Theta) \rightarrow \Box[\Theta^{(e!)}\alpha \rightarrow \exists x[x = \alpha \wedge \Theta x \wedge \forall y[(y = \alpha \wedge \Theta y) \rightarrow \Box x = y]]]$.	1–11 ■

Put semantically, the relevant logical fact is this: If $Abs(\Theta)$,

$$\text{then if } \mathcal{M}, \gamma \models \Theta\alpha_1 \wedge \Theta\alpha_2 \wedge E\alpha_1 \wedge E\alpha_2, \text{ then}$$
$$ext_{\mathcal{M},\gamma}(\alpha_1) = ext_{\mathcal{M},\gamma}(\alpha_2) \rightarrow int(\alpha_1) = int(\alpha_2).$$

So the following definition is acceptable whenever Θ is absolute and $\Theta^{(e!)}\alpha$: $\overline{\alpha}_\Theta$ is the intensionally unique individual intension, x, such that $x = \alpha \wedge \Theta x$.

Suppose Θ is carried in English by the common noun, *horse*, and suppose α is taking the place of *Carlotta's favorite Christmas present*. Then one might read Fact 1 as follows: If *horse* is an absolute property [which, of course, it is], then if Carlotta's favorite Christmas present is a horse in a certain case, then Carlotta's favorite Christmas present is identical (in that case) to the necessarily unique traceable individual falling under the property of being a horse. That is, however, such a mouthful that we introduce an informal device. In English, we permit ourselves to say that α is an *absolute horse*, which makes no literal sense, meaning thereby that α falls under the absolute property of being a horse. In the same spirit, we occasionally say that some α is an *extensional horse*, meaning thereby that α is identical, in the case at hand, to an existing absolute horse.

5 Illustrations

5.1 The horses

As evidence of its usefulness, we illustrate applications of some central ideas of CIFOL using an imaginary finite version of "is a horse," represented by the predicate, H, which should certainly be an absolute property. We want to work through the example in detail and with diagrammatic pictures. For this reason, we imagine that there are five horses, Andy, Doris, Gale, Hal, and Jack, and four cases, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. The four cases may be thought of as one-hour time periods, with γ_i starting at, say, i o'clock p.m. on a specific day.[48] Let us suppose that the extensional domain, D, contains at least $\{\mathrm{a, b, c, d, e, f, g, h, i, j, k, l, n}, *\}$. It will help the imagination to think of the lettered extensions as being distinct "horse stages" of one or another of the five horses; and of course all distinct from $*$.[49] It is convenient to represent an individual intension over Γ by a sequence of length four, one entry for each case; for example, ab$*$d, for the individual intension, $\overline{z}$ (i.e., function in $\Gamma \mapsto D$), such that $\overline{z}(\gamma_1) = \mathrm{a}$, $\overline{z}(\gamma_2) = \mathrm{b}$, $\overline{z}(\gamma_3) = *$, and $\overline{z}(\gamma_4) = \mathrm{d}$. With that convention in mind, let us suppose that the individual intensions of the five horses are as follows: int(Andy) = ab$**$, int(Doris) = $*$def, int(Gale) = g$***$, int(Hal) = $*$hi$*$, and int(Jack) = jkln. Since we have no information concerning the nature of the "horse stages," this display of intensions only tells us about Andy that he exists at times γ_1 and γ_2, and then no longer exists.[50] In the same way, the intension of Doris tells us that she

[48]Choosing a set of time periods for the set of cases, Γ, illustrates the flexibility of CIFOL, and reminds us that CIFOL is logic rather than metaphysics. (We agree that calling time periods "cases" sounds awkward; see note 14.)

[49]"Stages," which Quine 1960 glosses as "brief temporal segments" of, e.g., rabbits (§12), play a prominent role as objects in metaphysical literature such as Sider 2000. CIFOL, however, provides only a logic, sans metaphysics. At the risk of sounding parochial, we wonder whether the metaphysical discussions are too loose to be of help.

[50]Nothing *logically* useful comes of comparing extensions across cases. In particular, there is nothing logical to be gained by using an intension aaaa, nor by insisting that a and b in the intension ab$**$ be distinct: Identity/distinctness of extensions across cases is information that CIFOL cannot use. That should count as a *discovery* by Bressan. It follows, as Bressan noted, and as we reported in note 15, that a semantic system in which each case, γ, is given a distinct domain, D_γ (with or without overlaps, but all equinumerous), would yield the same logic. (Perhaps it is worth saying explicitly that comparing extensions at a certain case between individual intensions is quite essential; that is the work of the sign, $=$, of case-dependent identity.)

begins to exist at time γ_2, and then persists through times γ_3 and γ_4. Gale has a short but happy life, as indicated by her intension; Hal commences existing at time γ_2, continuing to exist at time γ_3, and then no longer exists. And Jack lives through all four time periods. What, now, about the representation of the absolute property, H = is a horse? Although the official type of a one-place predicate such as H has the form $\Gamma \mapsto ((\Gamma \mapsto D) \mapsto \mathbf{2})$, intensions of this type are easier to visualize if put in a conceptually equivalent form by construing that type as $\Gamma \mapsto \wp(\Gamma \mapsto D)$, which enables listing, for each case, the individual intensions (horses) falling under that case. Table 1 therefore faithfully represents H as modally constant by listing each of the five horse intensions under every case: If an individual is a horse in any case, then it

$H\backslash$Case	γ_1	γ_2	γ_3	γ_4
Andy	ab**	ab**	ab**	ab**
Doris	*def	*def	*def	*def
Gale	g***	g***	g***	g***
Hal	*hi*	*hi*	*hi*	*hi*
Jack	jkln	jkln	jkln	jkln

Table 1: Horse intension

is a horse in every case, including cases (as marked by $*$) in which it fails to exist. This representation allows that horses come to be and pass away, but rules out transubstantiation for horses. Horses cannot turn into other things: Once a horse, always a horse.

Being a horse is also modally separated: If $H(\alpha)$ and $H(\beta)$, then if α and β share some non-$*$ horse stage in any one case, then they are identical individual intensions; which is to say, if horses α and β are distinct horses, then there is no single case in which they share an extension, other than possibly $*$ (the sign of non-existence). That is to say, there *can* be a case in which two horses both fail to exist. The convenient visual sign of modal separation in any one case γ_i is this: If you look at the five intensions listed in Table 1 under γ_i, there is no column containing a repetition of anything other than possibly $*$. Of course because of modal constancy, the same intensions fall under each γ_i, so that it suffices to check only a single case.

Let us dwell a bit on the issue of a principle of application vs. a principle of identity or tracing principle, introduced in §4.3. Any property has to separate what falls under it from what doesn't. (Reminder: We make no attempt to illuminate vagueness.) Confined to a single case, the best we can do to check whether an absolute property Θ holds or doesn't hold of a thing, α, is to check extensionally (see Def. 13), with due attention to existence: If α falls under $\Theta^{(e!)}$ in case γ, then in that case there is an existing thing $\overline{\alpha}_{\Theta}$ that is a true Θ and that is identical (extensionally) with α in case γ. (Recall the remark following the proof in §4.3.) Thus, if α is "Mary's favorite thing," α should not be represented by an individual intension *for* a horse (because, for example, there is a case in which α is a mobile phone—or, which is just as bad, she has different horses as her favorite in different time periods), but it could be Jack in case γ, and then, $H^{(e!)}\alpha$ would be true, provided he exists in that case.[51]

$H^{(e!)}\backslash$Case	γ_1	γ_2	γ_3	γ_4
Andy	a---	-b--	$\varnothing$	$\varnothing$
Doris	$\varnothing$	-d--	--e-	---f
Gale	g---	$\varnothing$	$\varnothing$	$\varnothing$
Hal	$\varnothing$	-h--	--i-	$\varnothing$
Jack	j---	-k--	--l-	---n

Table 2: Existence-implying extensionalization of H

Table 2 describes $H^{(e!)}$, relying on the convention that a hyphen in a certain position indicates, in a manageably brief way, that we are picturing a (very large) set of individual intensions constructible by putting each member of D in place of the hyphen. Thus, opposite Andy and under γ_i, we list every intension that is identical to Andy in γ_i, omitting those that have a $*$ in the γ_i position.[52]

Now the work of absoluteness can be stated pictorially. Using Table 2, pick any intension falling under (for example) γ_1 in the extensional Table 2, perhaps a$*$c$*$ or gdea. Observe that we have *extensional-in-*γ_1 informa-

[51] We should not think of H vs. $H^{(e!)}$ as exposing an ambiguity in the English use of "horse", but rather as a convenient way of handling certain quantificational phenomena involving "horse." The lexicon of English has just one entry under "horse", viz., "a solid-hoofed plant-eating domesticated mammal with a flowing mane and tail, ...," but on our analysis, a sentence containing "... is a horse" can exhibit quantifier scope ambiguity reminiscent of the *de re*/*de dicto* distinction. See also §5.3 and §5.4 below.

[52] The a--- entry under γ_1 represents $14^3 = 2744$ entries. By our count, under γ_1 there should therefore be 8232 entries.

tion only; that is, information that is closed under identity in γ_1. Now for Bressan's profound observation: This extensional information suffices for determining a *unique intension* falling under H, either ab** or g***. Given any intension falling under $H^{(e!)}$, and given the case,[53] exactly one horse (one intension falling under H) is determined. Since H is absolute, then given a case, extension determines a unique intension. This illustrates the force of the proof we gave in §4.3.

5.2 The paddocks

To help illustrate how absolute and extensional properties interact, we temporarily enrich CIFOL by adding one-place connectives At_n: to be read "at γ_n," for $n = 1, 2, 3, 4$.[54] The semantics of At_n: is given as follows:

$$\mathcal{M}, \delta, \gamma \models At_n{:}\Phi \leftrightarrow \mathcal{M}, \delta, \gamma_n \models \Phi.$$

Evidently, if $At_n{:}\Phi$ is true at a case γ, it is true at every case (since the At_n: connective always switches evaluation to the specific case γ_n). At the level of predicates, this means that a lambda predicate with a leading At_n: will be modally constant, as we will illustrate.

Imagine now that there are two paddocks, a north and a south, into which horses are put at various times. Suppose that Andy and Jack are in the north paddock at γ_1, but are moved to the south paddock at γ_2, and are then put back in the barn, while Doris is put in north paddock (only) at γ_3 and Jack is put in the north paddock (again) at γ_4. Table 3 represents this supposal, with N and S representing, extensionally, the occupancy of the north and south paddocks at each time interval.

Table 4 contains the representations of $\lambda x At_1{:}Nx$ and $\lambda x At_2{:}Sx$, each of which is modally constant.

We've come this far in order to conjoin the two properties represented in Table 4, yielding Table 5; it's easy.

Given that in our story, it is only Andy and Jack who are in the north paddock at γ_1 and in the south paddock at γ_2, this result is an unexpectedly weird collection of intensions: Most of them are not among the proper horse-intensions of Table 1. What has gone wrong? Mere extensional properties

[53] You have to be given the case because modal separation is determined case by case.

[54] Bressan shows that the effect of quantifying over cases is *already* available in ML^{ν} at a higher order; see note 33 above.

	γ_1	γ_2	γ_3	γ_4
N	a- - - j- - -	$\varnothing$	- - e-	- - - n
S	$\varnothing$	- b- - - k- -	$\varnothing$	$\varnothing$

Table 3: Respective occupants of north and south paddocks

	γ_1	γ_2	γ_3	γ_4
$\lambda x At_1{:}Nx$	a- - - j- - -	a- - - j- - -	a- - - j- - -	a- - - j- - -
$\lambda x At_2{:}Sx$	- b- - - k- -	- b- - - k- -	- b- - - k- -	- b- - - k- -

Table 4: Applying $At_{\gamma_1}{:}$ and $At_{\gamma_2}{:}$

simply do not give us anything we can use if based on more than one case. They are fine one-case-at-a-time, but the whole thing goes to pieces when you try to mix multiple cases with extensional properties. What is missing is the ability to trace individuals between cases, leading to (at best) the gerrymandered horses of Table 5. What is needed is the absolute property of being a horse. Then it all comes right: If you meet what we have so far with H, you arrive at Table 6. The tracing power of the absolute property of being a horse returns us to sanity.

5.3 Essential properties: sex

Let's look at how a definite description works, say, $\imath x Nx$ (the occupant of the north paddock), in connection with "essential" properties. Keep in mind that according to Table 3, N is extensional. First we identify the description's intension according to Def. 10 by consulting that table, recalling that we get a $*$ in any case without exactly one entry (because we are counting distinct extensions):

$$int(\imath x Nx) = {*}{*}\text{en}.$$

That is, in some cases $\imath x Nx$ is a female, Doris in γ_3, and sometimes a male, Jack in γ_4, and sometimes "the occupant of the north paddock" doesn't exist

	γ_1	γ_2	γ_3	γ_4
$\lambda x(At_1{:}Nx \wedge At_2{:}Sx)$	ab-- ak-- jb-- jk--	ab-- ak-- jb-- jk--	ab-- ak-- jb-- jk--	ab-- ak-- jb-- jk--

Table 5: Gerrymandered horses found in north paddock at γ_1 and in south paddock at γ_2

	γ_1	γ_2	γ_3	γ_4
$\lambda x(At_1{:}Nx \wedge At_2{:}Sx \wedge Hx)$	ab$**$ jkln	ab$**$ jkln	ab$**$ jkln	ab$**$ jkln

Table 6: Absolute horses found in north paddock at γ_1 and in south paddock at γ_2

(either because there is no occupant of the north paddock, in γ_2, or more than one, in γ_1). This seems outrageous until one recalls that the range of x includes all individual intensions, including gerrymandered horses.

There are male horses and there are female horses, properties that may be represented with the extensional accounts pictured in Table 7.

	γ_1	γ_2	γ_3	γ_4
M = Male	a--- $\varnothing$ j---	-b-- -h-- -k--	$\varnothing$ --i- --l-	$\varnothing$ $\varnothing$ ---n
F = Female	$\varnothing$ g---	-d-- $\varnothing$	--e- $\varnothing$	---f $\varnothing$

Table 7: The extensional, existence-implying properties Male and Female.

It is common coin that the sex of each horse is a property that is "essential" to it, qua horse.[55] "Horse" in the preceding sentence is used once in the context, $H^{(e!)}$, which produces an extensional construct, and once in

[55]We are using "essential" in a minimal sense here, meaning just that the sex of a horse is a property such that in all cases in which the horse exists, it has that property.

the absolute sense, H. The first occurrence is extensionalized; a horse, for instance, has a sex no matter how you describe it. It is equally true that each particular horse (at the case in question) has its sex as an essential property, *qua horse*, but certainly not qua occupant of the north paddock, which is male in some cases and female in others. The second occurrence is not extensional: You need to use "horse" (H) as a tracing principle as you move from case to case. Following that trace, you find out that either necessarily the horse is male in every case in which it exists, or necessarily the horse is female in every case in which it exists. Even though the sex of the occupant of the north paddock, i.e., the sex of $\imath x N x$, varies case by case (because of gerrymandering), its sex qua horse does not vary as you trace it from case to case.

As many philosophers have noted, English is not altogether convenient for sorting out the matter. CIFOL notation helps.

$$\mathcal{M} \models \forall x[H^{(e!)}x \to \forall y[(x = y \wedge Hy) \to (\Box(Ey \to My) \vee \Box(Ey \to Fy))]].$$

The use of $H^{(e!)}$ in this "constant sex" principle tells us that it applies to each existing horse, "under every description," as the saying goes, which is precisely what is wanted. The use of the absolute notion of "horse" represented by Hy is required because the extensional characterization of x by $H^{(e!)}x$ is not enough to support the use of x inside a modal context. If all you know of x is that $H^{(e!)}$ applies to it, you do not have enough information to allow you to trace a value of x from case to case. That is the point of the "qua horse" clause. When y is characterized by an absolute property, you know that you can safely trace the value of y between cases.

What about the sex of the horse in the north paddock? Essential or not? The syntactic ambiguity of the question is well known. Here is how it goes in CIFOL, where, as in many other treatments, the ambiguity is one of scope illustrating *de re* vs. *de dicto*.

***De re*, true.** Necessarily, the occupant of the north paddock is, qua horse, either essentially[56] male or essentially female:

$$\Box\forall x[(Hx \wedge x = \imath y N y \wedge Ex) \to (\Box(Ex \to Mx) \vee \Box(Ex \to Fx))].$$

You might think that the antecedent forces us to take the intension of x as $**$en—which is indeed the intension of $\imath y N y$; but that's not an intension falling under the absolute concept "horse" (as opposed to falling under

[56]I.e., necessarily-if-existent.

$H^{(e)}$). Instead, the antecedent asks us to consider—case by case—any horse-intension existing in that case and (extensionally) identical to $**$en, which comes down to $*$def (Doris) in case γ_3 and jkln (Jack) in case γ_4. And each of these intensions is (extensionally) either male in all cases in which it exists or female in all cases in which it exists. It figures: CIFOL's treatment is exactly right.

***De dicto*, false.** Necessarily, either the occupant of the north paddock is essentially male or the occupant of the north paddock is essentially female:

$$\Box(\Box(E(\imath yNy) \to M(\imath yNy)) \vee \Box(E(\imath yNy) \to F(\imath yNy))).$$

Here we are looking at only the one intension, $**$en. But now the disjunction comes out false, since that intension is neither male in every case in which it exists, nor female in every such case. Again, the CIFOL treatment is entirely accurate.

Although it would be possible to present the *de re*/*de dicto* contrast in CIFOL using the Russellian account of definite descriptions, it seems to us that the present formulation, taking definite descriptions as categorematic in the Frege-Bressan way, is cleaner and more intuitive.

5.4 Natural numbers

Bressan 1972 shows how one obtains a natural absolute concept, $\mathbb{N}$, of "natural number" (= non-negative integer) in case-intensional higher-order type theory (ML^{ν}), by verbatim adoption of the usual extensional theory, but using $\Box(x = y)$ in place of $x = y$ everywhere except in the Peano axiom $0 \neq n + 1$ (p. 100), which should retain its extensional meaning. In this development, each natural number is identified with an extensional second-order property. For example, the number 2 is identified with the second-order property of first-order properties of applying to exactly two things (counting extensionally):

$$2 =_{df} \lambda P(\exists x \exists y(x \neq y \ \wedge \ Px \ \wedge \ Py \ \wedge \ \forall z(Pz \to (z = x \ \vee \ z = y)))).$$

In this context, it is more idiomatic to say "class" instead of "property" (observing that CIFOL, being innocent of metaphysics, does not distinguish the two): The number 2 is identified with the second-order class of first-order classes that contain exactly two things; the class 2 will be extensional. The absolute third-order class, $\mathbb{N}$, of natural numbers is identified with the class containing 0 and all its successors.[57] By this construction, it is only right and proper that the extension of the term (numeral) "2" in a case, γ, can differ from the extension of "2" in another case, γ', depending on which first-order properties happen to apply to exactly two things in these cases. *So even numerals shouldn't be "rigid designators" that have the same extension in every case!* What is logically important, is that the third-order property of being a natural number, $\mathbb{N}$, is absolute, so that we can trace numbers across cases. Let's see by example how $MSep$ works for $\mathbb{N}$. For instance, since 1 and 2 each fall under $\mathbb{N}$, $MSep$ requires, in effect, that there is no possible case in which they agree: using $\mathcal{M}^\nu$ for a model appropriate for ML^ν, $\mathcal{M}^\nu \models \neg\Diamond(1 = 2)$. But fortunately it cannot be that $\mathcal{M}^\nu, \gamma \models 1 = 2$ for some γ, for that would require the impossibility that the class of classes having exactly one member in γ be the same as the class of classes having exactly two members in γ (counting extensionally).

It seems a shame to apply Bressan's sophisticated theory of numbers to Quine's over-familiar example of the number of planets, but in truth it can't hurt. Now, in order to stay within the confines of CIFOL, we suppose that $\mathbb{N}$ is brought down to the first order as an absolute property, and that each numeral is brought down to the first order as a term, n, with the standard Peano properties. Everybody agrees that the sentence,

The number of planets is necessarily greater than seven,

has two readings, and is true under one and false under the other. The CIFOL analysis is as follows. We use a term, α, for the definite description, "the number of planets" (remember from §3.1.7 that definite descriptions are just terms, with an intension and with an extension at each case, so abbreviating the description as a general term α does not beg any questions). We use a one-place predicate, G, for "is greater than seven." In the case at hand, γ, the number of planets is eight (recently downgraded from nine); in another possible case, γ', it is four.

***De dicto*, false.** The first reading ("*de dicto*"), on which the sentence is false, is:

$$\Box\forall x((\mathbb{N}x \ \wedge \ x = \alpha) \to Gx),$$

[57] We omit details.

"in every case, if there is a natural number that is the number of planets in that case, then that that number is greater than seven." This is false because in case γ', the natural number four is equal to the number of planets, and four is not greater than seven.

***De re*, true.** The second reading ("*de re*"), on which the sentence comes out true in case γ (even though false in case γ'), is:

$$\forall x((\mathbb{N}x \ \wedge \ x = \alpha) \rightarrow \Box Gx),$$

"for any absolute natural number that is equal to the number of planets in the case at hand, in every case, that number is greater than seven." This is true because eight is greater than seven in any case.

On the *de re* reading, the absolute concept, $\mathbb{N}$, plays an important role: Given the case at hand, γ, the extension of α ("the number of planets") at γ determines a unique absolute natural number, eight, due to modal separation. There are many individual intensions that have the same extension-at-γ as α ("Mary's favorite number" may be among them), but there is only one absolute natural number-intension among them. If we change the reading to demand not true natural numbers but only intensions that are extensionally equal to a natural number in the case at hand, we obtain a reading on which our sentence is false even in case γ:

$$\forall x((\mathbb{N}^{(e!)}x \ \wedge \ x = \alpha) \rightarrow \Box Gx),$$

"for any intension that is equal to an absolute natural number and that is equal to the number of planets in the case at hand, in every case, that intension is greater than seven." This is false because, for example, the intension of α itself fulfills the antecedent, but not the consequent. CIFOL delivers the verdict that even the sentence,

The number of planets is a natural number,

has two readings, under one of which it is false: It is true that $\mathbb{N}^{(e!)}\alpha$, but false that $\mathbb{N}\alpha$.[58]

[58]To repeat, it is the sentence that has two readings, not $\mathbb{N}$. See note 51.

5.5 Testing horses for the X aberration

Next we go through an important kind of application of CIFOL, namely, to an ineliminable use of possibility in scientific discourse. (Bressan 1972 offers a detailed argument for the necessity of using possibility and absolute properties in hard science.)

Horses sometimes have chromosomal aberrations (Bugno et al. 2009). For some of these there is an accurate test. We assume that one of these is "the X aberration," which can always be revealed by "the X test," the result of which is stated in terms of an appropriate natural number, "the X number," which we may think of as 1 when the horse has the X aberration and 0 when not. Let us assume that the X test is seldom administered, but it always *can* be administered (it's possible that it is administered) for any extensionally characterized horse, and it is always accurate. In other words, it is not only possible to administer the X test to any horse, securing an X number, but in addition any further possible X test on that horse will give the same X number.

We must use absolute properties for tracing individuals between diverse cases, such as those encountered in speaking of possible tests. In the example at hand, it is horses and natural numbers that must be traced. In terms of notation, let $X(\alpha, y)$ be a predicate meaning that the X test is carried out on α with resulting X number y. So our existence-and-uniqueness assumption is this, assuming that variables are chosen so as to avoid conflict:

$$(\%) \quad H^{(e!)}(\alpha) \to \exists x[x = \alpha \wedge Ex \wedge Hx \wedge \\ \exists y[\Diamond(\mathbb{N}y \wedge Ey \wedge X(x,y)) \wedge \\ \forall z[\Diamond(\mathbb{N}z \wedge Ez \wedge X(x,z)) \to y = z]]].$$

"Given any extensionally identified existing horse, α, there is an absolute horse, x, identical to α, such that it is possible to carry out the X-test on x yielding absolute numerical result y, and such that every possible X-testing of x yields a numerical result extensionally identical to y."[59] It is critical that values of each variable be characterized as falling under some absolute

[59]It is obvious that the limitation Ex is needed, since it makes no sense to test a horse, α, at a case in which α doesn't exist. On the other hand, Ey and Ez are redundant, assuming the theory of the natural numbers, $\mathbb{N}$, adumbrated in §5.4 according to which they all exist in every case; that is, *Exist-imp*($\mathbb{N}$) (see Def. 12).

property. In the present example, (%), we can verify that the values of each of x, y, and z are so characterized.

In order to strengthen our claim to the usefulness of CIFOL, we wish to prove that, given the absoluteness of H and $\mathbb{N}$, the final case-dependent identity, $y = z$, can be raised to a strict identity, $\Box(y = z)$; that is, letting $X'(v, w)$ abbreviate $\Diamond(\mathbb{N}w \wedge Ew \wedge X(v, w))$ and let $Y(w)$ abbreviate $w = \alpha \wedge Ew \wedge Hw$:

Fact 2 (Raising to a strict identity)

$$(Abs(\mathbb{N}) \wedge Abs(H)) \rightarrow$$
$$[H^{(e!)}(\alpha) \rightarrow [\exists x[Yx \wedge \exists y[X'(x, y) \wedge \forall z[X'(x, z) \rightarrow y = z]]]] \rightarrow$$
$$[H^{(e!)}(\alpha) \rightarrow [\exists x[Yx \wedge \exists y[X'(x, y) \wedge \forall z[X'(x, z) \rightarrow \Box y = z]]]]]$$

PROOF.

1	$Abs(H) \wedge Abs(\mathbb{N})$	hyp
2	$(\%)$	hyp
3	$H^{(e!)}\alpha$	hyp
4	$x_0 = \alpha \wedge Ex_0 \wedge Hx_0 \wedge \exists y[X'(x_0, y) \wedge$ $\forall z[X'(x_0, z) \rightarrow y = z]]$	2,3, choose x_0
5	$X'(x_0, y_0) \wedge \forall z[X'(x_0, z) \rightarrow y_0 = z]$	4, choose y_0
6	$X'(x_0, z_0)$	hyp, choose z_0
7	$\Diamond(\mathbb{N}y_0 \wedge Ey_0 \wedge X(x_0, y_0))$	5, def
8	$\Diamond(\mathbb{N}z_0 \wedge Ez_0 \wedge X(x_0, z_0))$	6, def
9	$\Diamond\mathbb{N}y_0 \wedge \Diamond\mathbb{N}z_0$	7, 8, **S5**
10	$\Box\mathbb{N}y_0 \wedge \Box\mathbb{N}z_0$	9, 1 ($MConst(\mathbb{N})$)
11	$\mathbb{N}y_0 \wedge \mathbb{N}z_0$	10, **S5**
12	$Ey_0 \wedge Ez_0$	11, $Exist\text{-}imp(\mathbb{N})$
13	$y_0 = z_0$	5 (z_0/z), 6
14	$\Diamond(Ey_0 \wedge Ez_0 \wedge y_0 = z_0)$	12, 13, **S5**
15	$\Box(y_0 = z_0)$	1, 11, 14 ($MSep(\mathbb{N})$)
16	$\forall z[X'(x_0, z) \rightarrow \Box(y_0 = z)]$	6–15 (z_0/z)
17	$\exists y[(X'(x_0, y) \wedge \forall z[X'(x_0, z) \rightarrow \Box(y = z)]]$	5, 16 (y_0/y)
18	$\exists x[x = \alpha \wedge Ex \wedge Hx \wedge \exists y[X'(x, y) \wedge$ $\forall z[X'(x, z) \rightarrow \Box(y = z)]]]$	4, 17 (x_0/x)
19	$H^{(e!)}(\alpha) \rightarrow \exists x[x = \alpha \wedge Ex \wedge Hx \wedge$ $\exists y[X'(x, y) \wedge \forall z[X'(x, z) \rightarrow \Box(y = z)]]]$	3–18
20	$1 \rightarrow (2 \rightarrow 19)$	1, 2–19 ■

As promised, line 19 is just like (%), except that $y = z$ is promoted to a strict identity. It is worth noting how the various simple features of CIFOL work together to produce this result.

Fact 2 amounts to showing that the result of the testing is invariably intensionally unique (same individual intension). This in turn would justify a Mach-style definition of "the X number" of each horse, parallel to the Mach definition of "the mass of body b" as detailed in Bressan 1972, §§N18–N23. As far as we can see, none of the extensionalist first order quantified modal logics with only extensional predication possesses sufficient sophistication to handle the argument that is the backbone of the example.[60]

6 Summary

Let us look back at what has been achieved, and forward to what remains to be done. In this essay, we argued for ideals of ease of use, uniformity, expressive power, and usefulness at which a successful combination of modal logic with first order quantification theory should aim (§1). We reviewed several systems on the market in §1.3 and complained that they fail to live up to our ideals largely because (a) they treat the issue of tracing individuals across cases as a matter of logic, (b) are thereby forced to rely on choices with detrimental effects on ease of use and uniformity, and (c) disallow non-extensional predication. We introduced our system CIFOL, a legitimate descendant of Bressan 1972, with the express aim and promise that these problems in combining modality and quantification can be overcome. In our introduction of the basics of the system in §2, we stressed that a key conceptual ingredient in our approach is to depart from focusing on possible worlds, and to welcome the broader notion of possible cases. This notion may be given both a purely modal, a temporal, or a temporal-modal interpretation, or indeed any other useful interpretation. We stressed in §2.5 that this change of perspective on the underlying possibilities brings with it a liberating change of perspective on the values of variables. Yes, values of variables can be individuals of the concrete world, but no, these values are not extensions, but intensions.

In §3 we introduced the formal semantic machinery of CIFOL. The main points are (a) a fully general application of Carnap's method of extension and intension, whereby each expression has an extension in each case, and

[60] Gupta 1980 escapes this verdict by building Bressanian tracing via absolute concepts directly into his quantifier semantics. See §1.3 above.

an intension that is the pattern of extensions across cases, (b) as a result, the uniform semantical treatment of individual expressions, including definite descriptions, (c) a single, uniform semantical rule for non-extensional predication which adds expressive power (§3.1.3), and (d) the specific requirement that identity be extensional (§3.1.5). On that basis, a simple treatment of modal operators as quantifiers over possible cases and of first-order quantification was possible.

§3, which constitutes the full definition of the "creative" part of CIFOL as a logical system, contains no discussion of individuals, substances, "rigid designators," "trans-world identity," sortal terms, or any other issues of modal metaphysics. While this may on a first view look like a restriction of the usefulness of CIFOL as a quantified modal logic, in reality it only shows that CIFOL is properly a logic and not a logic-cum-metaphysics. As far as logic is concerned, it is simply not settled whether there are any individuals, substances, or kinds. The usefulness of CIFOL is shown, in our view, by the fact that it includes a simple and general definitional interface that allows for the *extra-logical* discussion of issues of individuals and the like. We described this interface in §4, in which we introduce the CIFOL-definable notions of extensional CIFOL quality vs. absolute CIFOL sortal (Def. 14 and Def. 19, respectively). We illustrated the use of absolute properties for tracing individuals across cases in §5, commenting on the issues of essential properties, modality *de dicto* vs. *de re*, and the testing of individuals for traits. To repeat, these are extra-logical matters; the fact that they can be successfully and perspicuously discussed in CIFOL, however, is a strong point in favor of the system as an easy-to-use, uniform, expressively powerful, and useful intensional logic.

In the present Part I of this essay, we have had recourse to a temporal, or temporal-modal, interpretation of the set of cases where that seemed appropriate for purposes of illustration. We have not given a formal theory for the temporal application—the length of the present essay forces us to defer this discussion to Part II. Briefly, the issue is as follows. A proper treatment of temporal-modal cases in the formal framework of branching histories (Thomason 1970) brings with it new challenges both of a formal and of a metaphysical nature. Formally, we will need to generalize the notion of an absolute property in order to deal with possible variations of extensions across different cases belonging to the same moment in a branching structure. Metaphysically, we will need to extend the notion of an individual to a branching framework. Many arguments against branching as a representation of real future possibility rely on intuitive, sketchy complaints about the notion of a "branching individual" and its possible futures. We believe that the formal framework of CIFOL provides the necessary stable logical background and formal interface to address these difficult challenges in Part II. It is certainly worth the effort: A successful formal model of individuals in a branching framework will make possible a perspicuous formal perspective on what it is to be like one of us: a proper individual facing a future of real possibilities.

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