Bachelard, Enriques and Weyl: comparing some of their ideas

Giuseppe Iurato
Department of Physics, University of Palermo, IT

E-mail: giuseppe.iurato@unipa.it

Abstract. Some aspects of Federigo Enriques mathematical philosophy thought are taken as central reference points for a critical historic-epistemological comparison between it and some of the main aspects of the thought of other his contemporary thinkers like Gaston Bachelard and Hermann Weyl. From what will be exposed, it will be also possible descry eventual educational implications of the historic-epistemological approach.

1. Introduction

Even in the modern textbooks and treatises on History of Philosophy and Philosophy of Science, both Italian[1] and foreign, there exist neither a whole chapter nor few sections, devoted to the fundamental epistemological work of Federigo Enriques, whose philosophical thought is dismissed in few lines among the subjects related to the modern Italian Philosophy between the end of the 19*th*-Century and the beginning of the 20*th*- one. Make an exception both some prefaces to the various anastatic reprints of the Federigo Enriques works and some remarkable collective and proceeding works mainly edited by the *Centro Studi Federigo Enriques* in Livorno (IT). All that is quite unfair respect to the wide cleverness and acuteness of the forerunner Enriques' thought: he has been remembered only for his high and celebrated contributions to Algebraic Geometry, and only recently a certain further attention has appeared towards this author[2].

An almost same fate has been undergone by Giovanni Vailati, almost to witness that absurd but real (and still effective) kind of reciprocal dislike, by both sides, between philosophers and scientists, which embed its historical roots into the secular dispute between *Geisteswissenschaften* on the one hand, and the *Naturwissenschaften* on the other hand[3]. Historically, many renowned scholars have tried to settle such a dispute, but with very poor results, despite of the immemorial historical course of the scientific culture; in Italy then this problematic situation has been (and still is) much more incisive than abroad, above all after the famous strong disagreement which has had as main protagonists the neo-Hegelian idealistic philosophers (amongst to which G. Gentile and B. Croce) against the neo-positivist ones (amongst to which A. Aliotta, R. Ardigò and U. Spirito).

Federigo Enriques (1871-1946) was one the main Italian scientists, near to the exponents of the logic positivism and of the neo-rationalism, who tried (unfortunately, in vain) to overcome this useless gap between the ones and the others. Just (but not exclusively) on particular aspects of this last wholehearted attempt of reconciliation and the underlying philosophical motivations, it is based this brief note.

2. Enriques and the Italian philosophy of the time

In what follows, we mainly consider the few lines written by Ludovico Geymonat (1908-1991) in (Geymonat, 1976, Volume VII, Capitolo XI and Volume VIII, Capitolo III) who was one of the main exponents of the Italian neo-positivistic current of the 20*th*-Century, on the wake left by his predecessors, amongst to which the same Enriques.

[1] For instance, Ludovico Geymonat, in his celebrated treatise on history of philosophic and scientific thought, devotes a whole chapter to the life and work of Gaston Bachelard but not to those of Federigo Enriques, this resulting to be even more strange because of the fact that the same Geymonat has been a supporter and a follower of the Enriques work. Moreover, this author devotes sections of his treatise to G. Vailati and even to G. Gentile and B. Croce (see (Geymonat, 1976, Volume VII, Capitolo XI), but not to Enriques, who is mentioned here and there in few lines in such a chapter. Likewise for the Nicola Abbagnano treatise.

[2] See, above all, the works of Mario Castellana quoted in References.

[3] See (Dalla Chiara & Toraldo di Francia, 1999, Capitolo 15, § 15.1) and (von Weizsäcker, 1994, Capitolo V, § 6.B).

The main fact that immediately jumps out to the attention is related to the known difficult relationships between the Italian philosophy and the scientific context of the time[4], whose main causes should, yep, be ascribed to the Croce and Gentile[5] neo-idealism but it wouldn't be historically correct neglecting certain other antecedent facts concerning such relationships: indeed, some of the main exponents of the same Italian neo-positivism of the end of 19*th*-Century, amongst to which R. Ardigò, did not give the right relevance that will deserve the fundamental epistemological works made, for instance, by Vailati, Peano and Beltrami as concern the foundations of mathematics. Amongst to them, above Giovanni Vailati (1863-1909) tried to stem this incipient breaking between the Italian philosophers toward the scientific philosophy, having as reference point the recent work and thought manifested by Federigo Enriques since the last years of the 19*th*-Century, who, after the premature died of Vailati, continued himself, in first person, to bring forward this program of reciprocal collaboration. As known, unfortunately both valuable aims, not only intentionally manifested but also put in practice with remarkable works[6], failed with the eclipsing of the anti-idealistic philosophical trends and with all the consequent harmful results still today present into the Italian cultural setting. Almost like a sort of unfair retaliation of the destiny against these benevolent reconciliation and collaborative attempts, to confirmation of the Saint Bernard of Clairvaux maxim according to which *«the good intentions pave the hell's roads»*, Enriques and Vailati were almost neglected by the same Italian culture as regard their philosophical works; only abroad they received major attention, again to confirmation of another Latin maxim according to which *«nemo propheta in patria est*[7]*»*.

Nevertheless, the appreciated collaborative and open perspectives by Enriques, had remarkable parallel attempts in some foreign notable thinkers, among to which Gaston Bachelard (1884-1962) and Hermann Weyl (1855-1955), of which herein we wish to point out certain common aspects of their thought[8].

3. Enriques, Bachelard and Weyl: some comparative attempts

As already said above and as recalls Geymonat in (Geymonat, 1976, Volume VIII, Capitolo III), the causes of the failure of the Enriques philosophical program (with the consequent neglect of the related thought) must not be imputed only to his controversy with Croce and Gentile, albeit it played a pivotal role, but also to the same mathematical community of the time and its own 'internal behavioural deontological codex' according to which was seen with extreme diffidence every attempt turned toward historical, philosophical and foundational (above all logical) questions: it is very strange as such a *mental crease* be still present, as a kind of *internal pure idealism*, into the mathematical sciences, being understood, also nowadays, as totally detached to any type of problematic which did not be of a purely theoretical nature or, at most, technical-applicative. After an initial good consideration, Enriques was yet soon isolated both by the Italian

[4] For a brief outlines concerning the relationships between Mathematics and Philosophy at the beginning of the 20*th*-Century, see also (Berzolari, 1978, Articolo LXI, § 4), where, amongst other, there is a rich related literature.

[5] Just as regards Giovanni Gentile, it is notable to recall as his son Giovannino Gentile Jr. (1906-1942) was a great physicist prematurely died: for some brief biobibliographical notes on him, see (Bernardini & Bonolis, 2002) and (Bernardini, 2007), from which emerges that, after all, the same Giovanni Gentile senior wasn't so adverse to the scientific knowledge as could seem at a first sight, leaving full freedom to the studies of his son, even eulogizing, also publicly (see (Gentile, 1941)), the natural sciences and its Galileian experimental method. Instead, it was above all Croce the main opponent of the scientific knowledge, very likely to counteract a possible advent of the neo-positivistic thought headed mainly by the so-called *Vienna Circle* (but also by the *Berliner Gesellschaft für Wissenschaftliche Philosophie* of H. Reichenbach, near to the former) to whom Enriques was into relation, strong of position conquered by this last within the Italian Philosophical Society. On the other hand, after the died of his son, Gentile senior also published a book entitled *Scritti minori (di scienza, filosofia e letteratura)* which collect all the publications of his son, from which, besides, it clearly emerges an extraordinary eclecticism of Gentile junior quite similar to that of Enriques, even in the undergone fait: indeed, the same Gentile junior was also discriminated, both by scientists and humanists, for his attempts to unify humanistic framework with the scientific one; only Giovanni Polvani and Ettore Majorana were estimators of his singular work, Ettore Majorana having also been a his strict friend (which is quite strange seen his character).

[6] In particular, the important works *I problemi della scienza* (1906) and *Scienza e razionalismo* (1912), despite had been criticized first by Gentile then by Croce, earned to Enriques, for some years, the presidency of the same Italian Philosophical Society. The journal *Scientia, Rivista di sintesi scientifica*, founded in 1907, was the result of the great foresight of Enriques, established just what place of meeting and cultural exchange between philosophers and scientists.

[7] And this maxim reached its highest achievement just relatively to the fate of the journal *Scientia*, whose initial programmatic manifesto was formed by the celebrated book *I problemi della scienza* (1906).

[8] For other aspects, we refer to (Redondi, 1978).

philosophical society and (mainly for these his first interests) by the same mathematical community which was completely unrelated to these type of studies, a human fate this which was also experienced by Bachelard (see (Geymonat, 1976, Capitolo X, § II)).

Enriques was just one of the few ones to try changing this unilateral perspective into the Italian context. In this framework, we want to consider what was the same situation abroad, limiting ourselves to few authors. In France, the general aversion towards the Logic found an influential supporter in Poincaré that opposed the initiative of the logician Louis Couturat to introduce the Peano and Russell ideas in his country, notwithstanding Poincaré himself was one of the greatest French scholar of Epistemology and Philosophy of Science, together to P. Duhem, the latter also a strenuous opponent of the Logicism. Both did themselves paladins of an antidogmatic conception of the science, involving a certain convenctionalism, whose ideas were retaken by their successors as L. Brunscvicg, É. Meyerson, A. Rey and A. Koyré, till to Gaston Bachelard who is considered as the most original thinker in this type of studies. For shortness, we refer also to (Abbagnano, 1995, Capitolo XI, § 799) for exposing the main outlines of Bachelard thought.

With a *license* both in Mathematics and in Philosophy, but first of all historian of science, Bachelard have drawn inspiration sources from his scientific researches for his further epistemological reflection, which yet is different from the neo-positivistic one for a major historicization of the scientific thought, this last being also connected with the various historical, technical, social, cultural and psychological data in which it has evolved. He considers philosophy and science inseparably connected among them, and existing not an only one science but different sciences or an irreducible plurality of knowledge and specific techniques, speaking of an *applied rationalism*, which is very close to the Enriques *experimental* and *critical rationalisms* (see (Redondi, 1978, Capitolo V, Footnote [43]) and (Castellana, 1974)). Both these philosophical trends were substantially motivated by their common interests for physical questions and connected relationships with the mathematics, which, among other things, were also taken as valid and useful educational tool for exact sciences (see (Castelnuovo, 1907)), in particular for mathematics itself; furthermore, both were quite adverse to the idealistic theses despite of they always tried to classify and to compare their studies in the more general framework of the great philosophical systems[9].

Nevertheless, this closeness between their rationalisms does not completely extend to their respective conception and role played by the history of science, that in Enriques coincides with the history of philosophy and goes on from past to future in a continuous manner (see (Enriques, 1938)), whereas in Bacherlard, even if science and philosophy are inseparably connected between them (like in Enriques), nevertheless the history of science is guided only by the current rational values and only minimally is influenced by the past because of discontinuities due to the occurrence of certain breakings[10] (see later). Instead, a common point in their conception of the history of science methodology is findable in certain psychologistic tendencies of both authors: for instance, Enriques, in (Enriques, 1938), states that the study of the historic-phenomenological evolution of scientific ideas may turn out to be useful to understand the *genesis* of the same scientific ideas, from which emerges the necessary inseparable co presence both of rational and empirical factor in the birth and development of it (see (Geymonat, 1976, Volume VIII, Capitolo III, § II)). This last perspective is also considered – hence in agreement too with Enriques – both by Bachelard[11] (see (Geymonat, 1976, Volume VII, Capitolo, X, § III) and by Weyl in (Weyl, 1949, Capitolo 5, § 21) where the latter argues on the formation of scientific theories, reporting the Enriques conception of the continuous epistemic role played by the history of science in the formation of itself; a fundamental tool to pursue this common Bachelard, Enriques and Weyl standpoint being just the history of science intended not as erudite research but as a dynamic and active research of the scientific spirit considered along its diachronic and synchronic development[12].

According to Bachelard, the scientific progress, instead, does not take place through a continuous and unilateral process but by means of *epistemological breakings* respect to the previous theoretical schemes,

[9] For instance, Bachelard, in his conception of philosophy respect to the epistemology and the science, outlines a *philosophical topology* in which to place the various historical philosophical systems and respect to which comparing the same historical evolution of science (see (Geymonta, 1976, Volume VII, Capitolo X, § V)); in this, Bachelard and Enriques are very close.

[10] In this sense, anticipating the T.S. Kuhn thought about scientific revolutions.

[11] Under a certain Husserlian influence.

[12] Even if these authors give a different weight just to these two basic aspects of the historical evolution: for instance, Enriques gives much more importance to the diachronic aspects, whereas Bachelard give more attention to the synchronic ones.

which, in turn, may take place only overcoming the various *epistemological obstacles*[13] that hinder the science path. Nevertheless, Bachelard inherited some of the main themes common to his predecessors, above all Poincaré and Duhem, like the aversion to the logic, the antiempiricism (differently by Enriques), the tendency to link the critics of science with its history, the essential originary creative nature of the theories, and so on. In particular, his contrariness to the logicism and formalism of Peano, Russell and Hilbert, puts him on a same level respect to Enriques who was notoriously into very cold relationships with Peano as noted by Mario Castellana in (Castellana, 1973) – who, inter alia, has also made fundamental epistemological studies just on the authors here considered: see (Castellana, 2004), (Castellana, 2005) and (Castellana, 2010), in which a more in-depth comparative historic-epistemological analysis of these authors is made.

Contrarily to his teacher Brunschvicg who considered the mathematics as a simple language tool, Bachelard instead claims as the mathematics is the *pillar of discovery* which creates the modern physical science, in opposition to the so-called 'doctrinaires of axiomatic' like Hilbert. According to Bachelard (see (Geymonat, 1976, Volume VII, Capitolo X, § III)), every formal thought is an incomplete psychological exemplification since it is a kind of never reached limit-thought, a thought around a some subject, concerning hidden images, which will be auxiliary to build up the related formal framework; the mathematics of the new physics is fed by its own experimental applications, whereas the science, in its educational aspects (to which Bachelard devotes much attention), cannot be exposed in its direct axiomatic form (against the later Bourbakism) but it should first be exposed for being understood, upon which thereafter building up its rigorous theoretical framework. To this purpose, according to Bachelard, the mathematics should be taught with an applicative method oriented toward the sciences, like physics and chemistry, hence together these. From all that, it is evident the common points with the related Enriques thought, which was notoriously opposed to any form of strict and curt formalism as well as favourable to this educational way of teaching.

On the other hand, the Enriques' aversion to logicism is clearly identifiable in some of his fundamental works on Algebraic Geometry: indeed, taking into account the introduction[14] of Guido Castelnuovo – who was one of the closer collaborator of Enriques and, in turn, himself clever mathematician – to the posthumous publication of the first 1942 edition of the basic work (Enriques, 1949), it is possible to glimpse what practical conception of the mathematics had Enriques. From that, Castelnuovo also expresses a his own worry as concerns the new course undertaken by the mathematics of the 20*th*-Century beginning, quite different from that intuitive and imaginative characterizing the very fruitful and advantageous 19*th*-Century mathematical thought. Instead, on the basis of Enriques work, Castelnuovo argues on the new way of doing mathematics in the first half of the 20*th*-Century, more turns toward the technical and logical aspects rather than sight, at first, the general framework of the mathematical question on which only subsequently building up the theory; this is compared too with an analogous situation which was taking place in the artistic context, where even there the imagination and fantasy were dismissed and pejoratively considered as arising from the romantic era, giving instead more consideration to the technical and tool aspects. From all that, it is evident why Enriques did not appreciate the logicistic way of doing mathematics that has gradually taken place ever more, trying to find confirmations to his way of seeing mathematics into the philosophical context, reaching to very original and interesting, innovative explanatory modes concerning a mathematical reasoning.

The intuitive and imagination view in mathematics was that mainly adopted and then carried out by the great mathematicians of the 19*th*-Century, as Gauss, Riemann, Abel, Jacoby, Poincaré and others, and for make the idea of that, we herein report the exact Castelnuovo textual words, which we surely may consider reflecting what Enriques himself believed in this regards

> *«La fantasia, la intuizione che guidavano la ricerca di allora sono oggi guardate con sospetto per il terrore degli errori a cui possono condurre. Le teorie sorgevano per rispondere al bisogno che il matematico provava di delineare e precisare degli oggetti del pensiero che erano già, in forma vaga, presenti alla sua mente. Era l'esplorazione di un ampio territorio intravisto da una cima lontana. Si costruirono così nel secolo scorso quei gioielli che si chiamano teoria delle funzioni analitiche, delle funzioni ellittiche, abeliane, superficie ad area minima, superficie cubiche… Oggi più che il terreno*

[13] Besides, to explain their occurrence, Bachelard appeal, amongst other, to the Freudian and Jungian psychoanalytic theories as well as to the Husserlian phenomenology. The epistemological obstacle theory led thereafter Bachelard to his *Philosophy of No.*

[14] See (Enriques, 1949, pp. V-VIII).

da esplorare interessa la via che vi conduce, e questa via ora vien seminata di ostacoli artificiali, ora si libra tra le nuvole».

As the same Poincaré said in his celebrated work (Poincaré, 1905), a mathematical construction is necessarily composed first by an *intuitive* process, which discovers, then by a *logic* process, which proves, coherently with what Enriques says just above through the Castelnuovo report. Hence, Enriques philosophy of mathematical thought as mainly based first on intuition – and which is common to almost all the celebrated exponents of the Italian algebraic geometry school of the time (among to which E. Beltrami, L. Cremona, F. Severi, E. Castelnuovo, C. Segre, B. Segre, G. Veronese, G. Fano and others) – is in accordance with the Poincaré thought[15] as well as with that of Riemann whose geometric perspective was, amongst other, one of the main common point of the thought of Bachelard, Enriques and Weyl, as witnessed by (Castellana, 2004) and[16] (Redondi, 1978). Furthermore, Enriques, Bachelard and Poincaré were also joined by the common constant doing reference to the psychological sciences, but not in a reductive way; in this regards, the work of Enriques was abundant of suggestions for the subsequent works of Jean Piaget and Pierre Gonseth[17] in epistemology, while the Poincaré philosophical legacy will be, for instance, later retaken by J. Hadamard in his celebrated work (Hadamard, 1945).

From this, it is easy to find interesting historical connections between the Enriques ideas and the work of another as much great mathematician, Hermann Weyl. Both were two among the greatest mathematicians of history whose work on pure and applied mathematics allowed them to be able to understand the various aspects of a mathematical reasoning, so that their thought should be taken into great account. As regards Weyl's thought on the nature of mathematical reasoning, it is enough to recall[18] as, according to him, in the edification of a mathematical theory, the general starting point is represented by what he calls an *operative framework* (*Operationsbereich*), formed by the choice of a number of fundamental categories of entities respect to which are given certain properties and relations, from which, then, to go on for building up the whole theoretical system through the creative iterative application of certain *generative* processes which include two main types, the *logic* process and the *mathematical* process: the former generate new properties and relations (said *derived*) starting from an initial stock of primitive relations and properties related to the entities of certain *initial* categories, applying the common usual elementary logical operations[19]; the latter, instead, allows to constitute new *ideal* entities from a given system of properties and relations related to certain entities already known, identifying a class of entities including only those having such properties. Subsequently, Weyl himself, in his celebrated work (Weyl, 1949, Capitolo 1), represents the same distinction between logic and mathematical process by means of the distinction between *combinatorial* and *creative* definition, the combinatorial one being legitimated by the logic process, whereas the creative one is that legitimated by the mathematical process. The creative iteration of these two inseparable processes lead to the notions of *types* and *orders* through the so-called *expanded* and *limited* processes.

The Weyl philosophical thought, relatively to the properly mathematical context, has evolved in time from the first work *Das Kontinuum* (1918) to the final *Philosophy of Mathematics and Natural Science* (1949) which is a revised and enlarged edition of a first German 1926 paper published in the *Handbuch der Philosophie* and that recently it has been republished in a new 2009 edition with an introduction by the Physics Nobel laureate A.F. Wilczek regarding the parts more properly physical of this Weylian work. We here does not wish to discuss the Weyl philosophical positions and their evolution, but point out only few of their aspects which may be quite close to the Enriques ones. First, both thinkers belonged to the very restrict class of pure scientists which could not do without to consider also the philosophical questions inherent a given mathematical or scientific problem, like Poincaré, Riemann, Einstein, Eddington, Mach, Russell and few other scientists of 19*th*- and 20*th*-Century, cultural tradition, this, that will go ever more to disappear[20].

[15] As known, after Poincaré, in France the logicism and formalism currents attained their highest height with the *Boubakism*, prevailing educational address until few years ago. In this regards, Vladimir I. Arnold, which may be considered as a great intuitive mathematician, was very critical on this, trying to reintroduce many mathematical textbooks oriented towards the intuitive and imaginary way of doing mathematics (from a PhD Seminar lesson held by Prof. Giorgio Bolondi).

[16] To which we refer for a more careful study.

[17] See Sections 1 and 2 of the introductory survey by O. Pompeo Faracovi to the Italian edition of (Enriques, 1938). Moreover, about the relationships among Enriques, Bachelard and Gonseth, see above all (Castellana, 2005).

[18] In what follows, we refer to (Casari, 1972, Capitolo XIII, § 1).

[19] And therefore characterized by a low degree of creativity, differently from the mathematical one.

[20] Today being almost inexistent.

In particular, Weyl himself, in the Preface to (Weyl, 1949), states as it has been no possible to him leave aside from philosophical questions each time that the opportunity will arose, ever trying to put the given mathematical or physical question into comparison with the suitable known philosophical frameworks; on the other hand, it is never enough the importance given to the philosophical thought in motivating and stimulating the same mathematical or scientific production, in this case being sufficient to recall the Riemann[21] and Einstein ideas history. Furthermore, from every part of his book, it gives rise the Weylian idea according to which there is almost always a prevalence of the imaginative components for the occurrence of a mathematical insight (either it concerns a proof or the institution of a new formal object). Moreover, taking also into account the physical science[22], in the work (Weyl, 1932) the author explains the so-called *essence* of the new scientific mind turned towards the contemplation of a pluralistic and dynamic *open world* put into not aggressive but sympathetic relationships with the religious spirit: in this regards, Weyl devotes the first chapter of his work, entitled *God and the Universe*, to discuss just these last aspects, trying to justify the apparent contrasts which can arise if this argument is carried out by a mathematician; and, on the other hand, this type of extreme and romantic philosophical digressions weren't estrange to the same Enriques which, in this regards, so he expresses himself at the end of his celebrated work (Enriques, 1949, Capitolo XI, § 9)

> *«A questo punto ci sia consentito fermarci un istante, come in un'ascensione alpina si ama sostare sul picco conseguito e di là contemplare lo spettacolo della Natura che si offre alla vista.*
>
> *Cinquant'anni or sono s'iniziava in Italia lo studio di queste teorie* [delle superfici algebriche], *appena abbozzate dal genio di un precursore (Max Noether); allora, scherzando sulle difficoltà e le eccezioni che s'incontravano da ogni parte, si soleva dire che, mentre le curve algebriche (già composte in una teoria armonica) sono create da Dio, le superficie invece sono opera del Demonio*[23].
>
> *Ora si palesa invece che piacque a Dio di creare per le superficie un ordine di armonie più riposte ove rifulge una meravigliosa bellezza, e ch'Ei volle in esse – diciamo col Poeta –*
>
> *del creator suo spirito*
> *più vasta orma stampar.*
>
> *La ricchezza delle proprietà e la bellezza, lungamente nascosta, che qui si palesano, non debbono costituire ragione di vano orgoglio per la scuola geometrica italiana o per i geometri stranieri che hanno collaborato a scoprirle, ma piuttosto debbono suscitare un senso di reverenza per quell'ordine meraviglioso degli enti matematici, che il pensiero trova innanzi a sé e quasi raccoglie, al pari delle specie viventi, dalla Natura Madre; e così alimentare la fede dei giovani ricercatori che dietro alle difficoltà, alle eccezioni, alle apparenti incongruenze, c'è realmente in questo mondo di enti, una divina armonia, che gli sforzi concordi degli studiosi riusciranno sempre meglio a mettere in luce».*

The treatise (Enriques, 1949), as recalls the same Castelnuovo, was, amongst others, the most important work of Enriques devoted to Algebraic Geometry, began just after he was graduated from Scuola Normale Superiore of Pisa till to the last years of his life, ever in continuous remaking and revision. However, the main feature of the whole book is just the intuitive and imaginative method in treating geometrical questions therein introduced, but that led himself to undergo various criticisms as regards the proof correctness of certain theorems notwithstanding the importance of the achieved results. This peculiar way of doing mathematics is characteristic of that unique kind of *scientific-humanistic* current which Enriques wanted to pursue and that was partially retaken and kept alive by very few of some of his pupils, amongst to which above all Ludovico Geymonat, Attilio Frajese (1902-1986), Luigi Campedelli (1903-1978) and Lucio Lombardo-Radice (1916-1982): the latter, in his preface to the anastatic reprint of (Enriques, 1938), remembers some of these distinguishing Enriques features first of all his attempt to overcome the reductive

[21] In particular, as regards Riemann, it is enough to recall the basic notable influence exerted on his scientific production by the thought of the anti-idealist German philosopher J.F. Herbart (1776-1841) and of G.Th. Fechner (1801-1887) (see the Introduction by R. Pettoello to (Riemann, 1994)).

[22] And the constant and repeated attention to these is also a further common point of the Weyl and Enriques thought.

[23] In this regards, it circulates too another similar maxim but concerning the integer and complex numbers, in part included in that due to L. Kronecker according to which *«the integer numbers are due to the God action, everything else being due to the human one»*; the addendum, according to which yet *«the complex numbers are due to the Devil action»*, seems instead to be anonymous.

barrier, or *fence*, between the *Geisteswissenschaften* and *Naturwissenschaften*, gap this which was inexistent in him since the beginning of his juvenile studies; the link between philosophical and exact sciences was of an indissoluble and mutual character in Enriques training who was hostile to any form of extreme specialization, nourishing a sense of *circular unity* of the knowledge, mastering a great quantity of cognitions in many fields of knowledge but without never becoming a specialist (with an exception for Algebraic Geometry). Maybe, just for this he undergone the unhappy fate of the beaten and lonely scientist, like Bachelard, even if such a condition did not weight on his spiritual serenity which characterized almost the whole of his life.

In short, there existed notable scientists, like Weyl and Enriques (and, in part, also Bachelard if one takes into account his curriculum vitæ and studiorum), for which their pure scientific work couldn't be disjoined by the philosophical speculation: for them, it is valid what says the same Weyl, namely that there exist men like artists, scientists, technologists or politicians which devote themselves to the construction, whereas instead others devote themselves to the reflection and to the philosophical speculation; the one and the other work should actively integrate among them otherwise the creativity loses itself into the mechanicalness of pure routine, while the reflection becomes abstract and void natter.

4. Conclusions

The intuitive and imaginative manner to approach, in a first phase, an arbitrary mathematical question as understood above all by Enriques and Weyl, but also by Bachelard on the wake left by Poincaré via Brunschvicg (see (Geymonat, 1976, Volume VII, Capitolo X, § III)), might have non-negligible implications from an educational perspectives if one considers the mathematics like an *immanent order of the Nature* or an *intelligible reality* external to our mind (like Plato), revoking the medieval realists against the nominalists; in this regards, according to Enriques and Castelnuovo (see his basic, but little known, paper (Castelnuovo, 1907)), the methodology of Physics might playing a fundamental educational role also from a mathematical viewpoint, above all in Geometry[24].

They surely belong to the so-called *visual* mathematicians rather than the *abstract* ones, above all Enriques that applied this mathematical philosophy to the active geometrical research field of the time, reaching to valuable results: the work (Enriques, 1949) is considered as an unique, valuable source of mathematical ideas as concerns the algebraic geometry of surfaces, although it were found many proofs little correct from a pure formal viewpoint and that the same author tried to remedy with a continuous revision of his work, but substantially without changing its remarkable content of ideas. The same Castelnuovo, in the introduction to (Enriques, 1949), affirms as Enriques was forced to improve his work because of certain criticisms moved by formalists only to the proofs of some of his theorems. On the other hand, it was well-known, and the same Castelnuovo and Enriques were aware of this, in what state was the theory of algebraic surface at that time, hoping in a future improvement of it from a formal viewpoint[25].

In short, from what has been so far, it clearly emerges as almost every creative mathematical process necessarily first should start with an intuitive and imaginative approach which could, in the eventuality, to be corrected or improved by a subsequently formal or abstract revision phase which, in turn, might to provide further results susceptible of possible further physical interpretations (like the discovery of antimatter expected by the physical interpretation of the eigenvalue problem solutions of electron Dirac's relativistic equation deduced from a formal relativistic extension of the Schrödinger equation). In particular, the main theses on the real nature of mathematics by Enriques, come just from the geometric context, that is to say, the Geometry is the main *epistemological paradigm* for a neo-Platonic conception of mathematics: among the contemporary thinkers which agree with such an Enriques view, we recall H. Freudenthal – that, among other things, has tried to apply this program to the educational context[26] – and R. Thom (which acquires the

[24] As regards the experimental character of mathematics, see also the brief but important note of Jean Leray in (Hamburger, 1986).

[25] Which, besides, couldn't take place without these initial results. However, this intuitive way of working was common among the above mentioned exponents of the so-called Italian geometric school (in part, following that of the German tradition dating back to Riemann, Klein and von Helmholtz), which yet attained to original and remarkable results in the geometry field. Only subsequently many other mathematicians, above all not Italians, improved their results from a formal and abstract viewpoint, often arguing against such a School, in particular towards Enriques; among them, it is not possible omit the name of Oscar Zariski, who scientifically growth just within this celebrated school.

[26] Furthermore, the names of H. von Helmholtz and H. Freudenthal are also historically related to some important problems concerning the axiomatic characterization of the so-called Physical Geometry, an important field of studies

related Poincaré ideas legacy). All this is coherent (and prodomal) with the modern cognitive science researches according to which at the basis of the mathematical thought fundamentally there are visual-spatial skills. In any way, nowadays it is almost inexistent this type of reciprocal useful and fruitful relationships between Mathematics, History and Philosophy which might also be useful from an educational standpoint[27].

Hence, in conclusion, by Poincaré, on the French side, and by Riemann, Klein and von Helmholtz, on the German side, Enriques picks up these influences, coeval respectively to the Bachelard and Weyl ideas, to originally develops his thought towards an intuitive view of mathematics – and applying it to his pioneering geometrical researches – which will reach its highest value with the L.E.J. Brouwer *intuitionism*[28], does not never neglecting the related philosophical counterpart: so, this is just the *leitmotiv* that has led the drawing up of this brief note. In our simple opinion, the historical considerations so far done might have some educational implications both for natural sciences and mathematics.

Remark. In this paper, we have limited ourselves to point out those points of Bachelard, Enriques and Weyl thought which overall lead, amongst other, to the revaluation of that line of thought referring to the visual and intuitive conception of mathematics dating back to Plato and Socrates. Between Bachelard and Enriques, via Poincaré, we have tried to identify some common points more oriented toward the philosophy and history of science than toward the relationships between mathematics and physics which yet are also present. Instead, as regard Enriques and Weyl, we have put more attention to these last type of basic relationships, as well as the relationships between philosophy and science, the historical questions being strongly present more into the Enriques thought than the Weyl one.

For other as much interesting common points among the mathematical philosophy thought of these authors, we refer to (Redondi, 1978). In particular, we want to stress out what fundamental educational role may played the philosophical thought in science, the works of Enriques and Weyl being enough to prove this.

References

Abbagnano, N. (1995-1996), *Storia della Filosofia*, 10 Voll., Milano: TEA-Tascabili degli Editori Associati.

Bernardini, C. and Bonolis, L. (2002), Giovannino Gentile, 60 anni dopo, *Il Nuovo Saggiatore*, 18 (1-2), 7-13.

Bernardini, C. (2007), Piccoli attriti tra scuole parallele, *Il Nuovo Saggiatore*, 23 (1-2), 10-12.

Berzolari, L. (Ed) (1972), *Enciclopedia delle Matematiche Elementari e Complementi*, Volume III, Parte 2ª, ristampa anastatica dell'edizione originale del 1949, Milano: Ulrico Hoepli Editore.

Casari, E. (1972), *Questioni di filosofia della matematica*, Milano: Feltrinelli Editore.

Castellana, M. (1973), Enriques e Bachelard: due epistemologie razionalistiche, *Il Protagora*, 85-86, 49-65.

Castellana, M. (1974), *Il surrazionalismo di Gaston Bachelard*, Napoli: Edizioni Glaux.

Castellana, M. (2004), *Razionalismi senza dogmi. Per una epistemologia della fisica matema*tica, Soveria Mannelli (CZ): Rubbettino Editore.

Castellana, M. (2005), F. Enriques, G. Bachelard et F. Gonseth: esquisse d'une tradition épistémologique, *Revue de Synthèse*, 2, 303-316.

linking together basic physical questions (also correlated to General Relativity) and formal geometrical arguments (like the Riemann-Helmholtz-Lie and Yamabe problems), which derive by some of the multiple intersections between Physics and Geometry and whose program is build up along the lines traced by the works made by Riemann, Poincaré, Helmholtz, Klein (in this regards, of this author see above all (Klein, 1926-27)) and Enriques on the same arguments (for more information, see (Schmidt, 1979), (Freudenthal, 1965) and (Moore, 1919)). On the other hand, the same Weyl has had also to do with such questions, as proves his fundamental work (Weyl, 1923), so that, via the Chapter IV of Enriques' *I problemi della Scienza* (1906) related to Geometry, it is possible *en passant* to identify another common point which goes from Helmholtz, Riemann and Lie till to Weyl in considering and treating this *problem of the space*, and that besides will deserve more attention.

[27] Bruno D'Amore, in (D'Amore, 2001, pp.75-76), point out just these possible perspectives arising from a constructive cooperation between historical questions and educational programs, remembering as so far nobody has put into practice this program notwithstanding distinguished historical attempts dating back just to Enriques, Campedelli and Lombardo-Radice. Nevertheless, a modern exception is given by the beautiful work of the theoretical physicists and science philosopher Carl Friedrich von Weizsäcker (1912-2007) among whom related works we mention, for our purposes, only (von Weizsäcker, 1994).

[28] In this regards, see also (Fieschi, 1976, Volume II, Appendice B, III.D).

Castellana, M. (2010), *Federigo Enriques et Hermann Weyl: philosophie et mathématiques*, in: Alunni, C., Castellana, M., Ria, D. and Rossi, A. (Eds), *Albert Einstein et Hermann Weyl 1955-2005, Questions épistémologiques ouvertes*, Manduria: Barbieri-Selvaggi Editore-ENS Rue d'Ulm, 2010, pp. 69-87.
Castelnuovo, G. (1907), Il valore didattico della Matematica e della Fisica, *Rivista di Scienza*, I, 329-337.
Dalla Chiara, M.L. and Toraldo di Francia, G. (1999), *Introduzione alla filosofia della scienza*, Bari: Editori Laterza.
D'Amore, B. (2001), *Scritti di Epistemologia della Matematica*, Bologna: Pitagora Editrice.
Enriques, F. (1934), *La signification de l'histoire de la pensée scientifique*, Paris: Hermann (Italian Edition: (1936), *Il significato della storia del pensiero scientifico*, Bologna: Zanichelli Editore).
Enriques, F. (1938), *La théorie de la connaissance scientifique de Kant à nos jours*, Paris: Hermann (Italian Translation: (1983), *La teoria della conoscenza scientifica da Kant ai nostri giorni*, Bologna: Zanichelli).
Enriques, F. (1982), *La matematica nella storia e nella cultura*, Ristampa anastatica della prima edizione del 1938, Bologna: Nicola Zanichelli Editore.
Enriques, F. (1949), *Le superficie algebriche*, Bologna: Nicola Zanichelli Editore.
Fieschi, R. (Ed) (1976), *Enciclopedia della Fisica*, 2 Voll., Milano: ISEDI.
Freudenthal, H. (1965), Lie Groups in the Foundations of Geometry, *Advances in Mathematics*, 1, 145-190.
Gentile, G. (1941), *Il pensiero di Leonardo*, Firenze: G.C. Sansoni.
Geymonat, L. (1970-1976), *Storia del Pensiero Filosofico e Scientifico*, 9 Voll., Milano: Garzanti Editore.
Geymonat, L. (1985), *Lineamenti di Filosofia della Scienza*, Biblioteca della EST, Edizioni Scientifiche e Tecniche Mondadori, Milano: Arnoldo Mondadori Editore.
Hadamard, J. (1945), *The Psychology of Invention in the Mathematical Field*, Princeton: Princeton University Press (Italian Translation: (1993), *Psicologia dell'invenzione in campo matematico*, Milano: Raffaello Cortina Editore).
Hamburger, J. (Ed) (1987), *La philosophie des sciences aujourd'hui*, Paris: Gauthier-Villars.
Klein, F. (1926-1927), *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Teile I und II, Berlin: Verlag von Julius Springer.
Moore, R.L. (1919), On the Lie-Riemann-Helmholtz-Hilbert Problem of the Foundations of Geometry, *American Journal of Mathematics*, 41 (4), 299-319.
Poincaré, H.J. (1905), *La Valeur de la Science*, Paris: Flammarion (Italian Translation: (1947), *Il valore della scienza*, Firenze: La Nuova Italia Editrice).
Redondi, P. (1978), *Epistemologia e storia della scienza. Le svolte teoriche da Duhem a Bachelard*, Milano: Feltrinelli Editore.
Riemann, B. (1994), *Sulle ipotesi che stanno alla base della geometria, e altri scritti scientifici e filosofici*, Torino: Bollati Boringhieri.
Schmidt, H.J. (1979), *Axiomatic Characterization of Physical Geometry*, Berlin, Heidelberg, New York: Springer-Verlag.
Von Weizsäcker, C.F. (1994), *L'uomo nella sua storia: una sintesi unificatrice del pensiero scientifico, filosofico e politico sullo sfondo della tematica religiosa*, Milano: Edizioni San Paolo.
Weyl, H. (1923), *Mathematische Analyse des Raumproblems*, Berlin: Springer-Verlag (Italian Translation: (1990), *Analisi matematica del problema dello spazio*, Bologna: Zanichelli Editore).
Weyl, H. (1932), *The Open World*, New Haven: Yale University Press (Italian Translation: (1981), *Il mondo aperto*, Torino: Editore Boringhieri).
Weyl, H. (1949), *Philosophy of Mathematics and Natural Science*, Princeton (NJ): Princeton University Press (Italian Translation: (1967), *Filosofia della matematica e delle scienze naturali*, Torino: Paolo Boringhieri Editore). See also the new 2009 edition ever published by Princeton University Press with the contributions of F.A. Wilczek.