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Circular Discernment in Completely Extensive Structures and How to Avoid such Circles Generally

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Abstract. In this journal, D. Rizza [2010: 176] expounded a solution of what he called "the indiscernibility problem for *ante rem* structuralism", which is the problem to make sense of the presence, in structures, of objects that are indiscernible yet distinct, by only appealing to what that structure provides. We argue that Rizza's solution is circular and expound a different solution that not only solves the problem for completely extensive structures, treated by Rizza, but for nearly (but not) all mathematical structures.

Keywords: structuralism, structure, identity-criterion, discernibility.

According to the view of mathematics called *ante rem structuralism*, (i) the subject-matter of mathematics consists of abstract structures, as Bourbaki famously promulgated; and (ii) the objects of these abstract structures that are related to each other in various ways (by the relations, mappings and operations that define the structure) have no properties by and from themselves; their nature, in so far as they have one, is wholly determined by the structure they inhabit and by nothing else besides (see, for example, Benacerraf [1965], Shapiro [1997], Resnik [1997], Hellman [2001], Dummett [1991: 295], who calls this *mystical structuralism*). These 'objects' are sometimes called 'positions in the structure' and sometimes 'pure relata'.

Keränen [2001] argued that if relata cannot be individuated while respecting the symmetries of the structure, they are not distinct individuals and therefore have to be identified, which is absurd. If a line, or a plane, is a set of relata characterised by axioms in terms of the ternary in-betweenness-relation, then no point has a property that any other point lacks, and therefore they have to be identified, leading to the absurd conclusion that a line, and a plane, consist of a single point. D. Rizza [2010: 175] adds that if the distinctness (non-identity) of the points is taken as a primitive mathematical fact, their distinctness is not accounted for by the structure, which conclusion is an unwelcome guest in the house of *ante rem* structuralism.

"The issue then is to make sense of the simultaneous and seemingly contradictory presence of distinctness and indiscernibility: I will call this the indiscernibility problem of *ante rem* structuralism", Rizza [2010: 176] writes. He expounds a solution at the hand of a *complete extensive structure*:

$$\mathfrak{S} \;=\; \langle \mathcal{X},\, <,\, + \rangle\,, \tag{1}$$

where $\mathcal{X}$ is a non-empty set with a linear ordering ($<$) and an associative, commutative operation ($+$) that is strictly monotonic wrt $<$, and further $\mathcal{X}$ is Dedekind-complete wrt $<$.

We shall be a little more explicit in discerning syntactics from semantics than Rizza. Let $\mathcal{L}(\mathfrak{S})$ be the language of $\mathfrak{S}$, $\boldsymbol{\varphi}(\cdot)$ an open sentence of $\mathcal{L}(\mathfrak{S})$ with one free variable, and let $\mathbf{T}(\mathfrak{S})$ be the theory of complete extensive structures; the background logic is classical elementary predicate logic with identity. Then $\mathfrak{S}$ (1) constitutes the standard model of $\mathbf{T}(\mathfrak{S})$, with assignment: $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$, $v \mapsto V$, and the satisfaction-relation. One proves that set $\mathcal{X}$ is *homogeneous*, which is to say that the following schema holds:

$$\mathfrak{S} \;\models\; \forall\, x, y :\; \boldsymbol{\varphi}(x) \;\longleftrightarrow\; \boldsymbol{\varphi}(y)\,, \tag{2}$$

which is to say that we can prove in the meta-theory (some set-theory):

$$\forall\, X, Y \in \mathcal{X} :\; \boldsymbol{\varphi}^{\mathfrak{S}}(X) \;\longleftrightarrow\; \boldsymbol{\varphi}^{\mathfrak{S}}(Y)\,. \tag{3}$$

The members of a homogeneous set are defined to be *indiscernible*, for what one can say about any given member, one can say about every other member too. Let $\mathrm{Aut}(\mathfrak{S})$ be the set of automorphisms of $\mathfrak{S}$ (1). One proves that for every automorphism $f : \mathcal{X} \to \mathcal{X}$:

$$\forall\, X \in \mathcal{X} :\; \boldsymbol{\varphi}^{\mathfrak{S}}(X) \;\longleftrightarrow\; \boldsymbol{\varphi}^{\mathfrak{S}}\big(f(X)\big)\,. \tag{4}$$

One also proves that every two members of $\mathcal{X}$ can be connected by some automorphism:

$$\forall\, X, Y \in \mathcal{X},\; \exists\, f \in \mathrm{Aut}(\mathfrak{S}) :\; f(X) = Y\,. \tag{5}$$

Rizza [2010: 179–180] explains that homogeneous structures such as $\mathfrak{S}$ (1) pose an indiscernibility problem for *ante rem* structuralism. The key to solving it, Rizza continues, is to let the automorphisms of $\mathfrak{S}$ determine a co-ordinate chart over $\mathfrak{S}$ with unit X. This comes about as follows.

A *co-ordinate chart with unit* X, denoted by Φ_X, is a mapping that sends every $Y \in \mathcal{X}$ (1) to an automorphism $f_Y \in \mathrm{Aut}(\mathfrak{S})$ that maps unit X to Y:

$$\Phi_X :\; \mathcal{X} \to\; \mathrm{Aut}(\mathfrak{S}),\;\; Y \mapsto \Phi_X[Y] = f_Y, \quad \text{such that:}\;\; X \mapsto f_Y(X) = Y\,. \tag{6}$$

Consider next $Z \in \mathcal{X}$, different from both X and Y, and let $\Phi_Y[Z] = g_Z$, where $g_Z(Y) = Z$. Since $\mathrm{Aut}(\mathfrak{S})$ is an Abelian composition-group, we obtain:

$$Z \;=\; g_Z(Y) \;=\; g_Z\big(f_Y(X)\big) \;=\; f_Y\big(g_Z(X)\big)\,, \tag{7}$$

whence:

$$\Phi_X[Z] \;=\; \Phi_X[g_Z(Y)] \;=\; \Phi_X[Y] \circ g_Z \;=\; f_Y \circ \Phi_Y[Z] \, . \tag{8}$$

Rizza [2010: 186] then draws the moral from chain (8): if $X \neq Y$, then automorphism f_Y marks the transition from the co-ordinate chart with unit X to the chart with unit Y. "This happens for any two distinct elements of $\mathcal{X}$, so these elements are discriminated by the different co-ordinate systems [charts] they give rise to." And, trivially, if $X = Y$, then f_Y in (8) becomes the identity-mapping on $\mathcal{X}$, $I_{\mathcal{X}} : X \mapsto X$, and the co-ordinate charts coincide:

$$\Phi_X[Z] \;=\; I_{\mathcal{X}} \circ \Phi_Y[Z] \;=\; \Phi_Y[Z] \, . \tag{9}$$

Thus in standard logical terminology, Rizza's has proposed the following *identity-criterion* for the members of $\mathcal{X}$ is:

$$X = Y \quad \text{iff} \quad \Phi_X = \Phi_Y \, . \tag{10}$$

But when are two of these co-ordinate charts different? What is the identity-criterion for these co-ordinate charts? Well, in full generality, mappings are identical iff they send the same thing to the same other thing; this becomse for the special case of the co-ordinate charts:

$$\Phi_X = \Phi_Y \quad \text{iff} \quad \forall\, Z \in \mathcal{X} :\; \Phi_X[Z] = \Phi_Y[Z] \quad \text{iff} \quad \forall\, Z \in \mathcal{X} :\; f_Z = g_Z \, . \tag{11}$$

The identity statement at the far right side of (11) is one between automorphisms on $\mathfrak{S}$. Since they too are mappings, their identity-criterion is similar:

$$f_Z = g_Z \quad \text{iff} \quad \forall\, V \in \mathcal{X} :\; f_Z(V) = g_Z(V) \, . \tag{12}$$

Since $f(V) \in \mathcal{X}$ and $g(V) \in \mathcal{X}$, the proposed identity-criterion for members of $\mathcal{X}$ (10) relies on the identity between the objects in $\mathfrak{S}$ (12), and hence (10) is circular. The co-ordinate charts Φ_X and Φ_Y account for the numerical diversity of $\mathcal{X}$, that is, for $X \neq Y$, iff $X \neq Y$, which account thus is circular.

The very idea of having an identity-criterion is to have an open sentence in $\mathcal{L}(\mathfrak{S})$ with two free variables *in which '=' does not occur*, say $\boldsymbol{\psi}(\cdot,\cdot)$, such that it is demonstrably logically equivalent to the identity-relation:

$$\mathbf{T}(\mathfrak{S}) \;\vdash\; \forall\, x, y :\; \boldsymbol{\psi}(x,y) \;\longleftrightarrow\; x = y \, , \tag{13}$$

and therefore:

$$\forall\, X, Y \in \mathcal{X} :\; \boldsymbol{\psi}^{\mathfrak{S}}(X,Y) \;\longleftrightarrow\; X = Y \, . \tag{14}$$

We must conclude that Rizza's proposal (10) is *not* an identity-criterion.

The key to a solution to Rizza's (and Keränen's) problem is to distinguish between *absolute discernibility*, which is discerning by means of properties, expressed by sentences of one free variable, as in schema (2), and *relational discernibility*, which is discerning by means of relations, expressed by sentences of two free variables. As soon as we have an identity-criterion $\boldsymbol{\psi}(\cdot,\cdot)$ (13), we have criterion for discernment, namely its negation, $\neg\boldsymbol{\psi}(\cdot,\cdot)$, wherein the identity-relation does not occur either. Can we find such a sentence $\boldsymbol{\psi}(\cdot,\cdot)$ in $\mathcal{L}(\mathfrak{S})$?

Yes we can. Here comes such a relation. Definition:

$$D(x,y) \quad \text{iff} \quad x < y \ \vee \ y < x \ , \tag{15}$$

where we have, for the sake of clarity, confused '$<$' in $\mathcal{L}(\mathfrak{S})$ with its standard interpretation in $\mathfrak{S}$, which we have *ab initio* also called '$<$' (1), rather than '$<^{\mathfrak{S}}$'. If $D(x,y)$, then $x \neq y$, because relation $<$ is irreflexive and anti-symmetric, and conversely. So we have a distinctness-criterion:

$$\mathbf{T}(\mathfrak{S}) \ \vdash \ \forall\, x,y: \ D(x,y) \ \longleftrightarrow \ x \neq y \ , \tag{16}$$

and therefore:

$$\forall\, X,Y \in \mathfrak{X}: \ D^{\mathfrak{S}}(X,Y) \ \longleftrightarrow \ X \neq Y \ . \tag{17}$$

Hence the distinctness of X from Y is grounded in the relation $<$, which is part and parcel of the structure $\mathfrak{S}$ (1). So, after all, the members of $\mathfrak{X}$ are distinguished by the structure, moreover in an automorphic manner, due to: $X < Y$ iff $f(X) < f(Y)$, for every $f \in \mathrm{Aut}(\mathfrak{S})$. An identity-criterion for the objects in $\mathfrak{X}$ then is $\neg D$ (15), which is of course also automorphic and does not rely on '$=$'.

Everything is fine for structuralism. So what went wrong with Rizza? His definition of indiscernibility below (2) is unacceptably restrictive: it is a definition of *absolute* indiscernibility at best. True and utter indiscernibility encompasses both absolute and relational discernibility, not only one of them, as Quine [1976] knew, but Rizza ignored, and Keränen [2001] as well. To spell it out, in case of a complete extensive structure, $X, Y \in \mathfrak{X}$ are *relationally indiscernible* iff the following schema holds for all sentences $\boldsymbol{\rho}(\cdot,\cdot)$ in $\mathcal{L}(\mathfrak{S})$ with two free variables:

$$\forall\, Z \in \mathfrak{X}: \ \boldsymbol{\rho}^{\mathfrak{S}}(Z,X) \ \longleftrightarrow \ \boldsymbol{\rho}^{\mathfrak{S}}(Z,Y) \ \ \wedge \ \ \forall\, V \in \mathfrak{X}: \ \boldsymbol{\rho}^{\mathfrak{S}}(X,V) \ \longleftrightarrow \ \boldsymbol{\rho}^{\mathfrak{S}}(Y,V) \ . \tag{18}$$

Restricting schema (18) to autmorphic sentences is not needed, because the automorphic character of $\boldsymbol{\rho}(\cdot,\cdot)$ is automatically guaranteed due to (4). Extensions beyond binary relations are possible (for futher kinds of discernibility, their logical interrelations, and the role of symmetries, see Caulton and Butterfield [2011], and Ladyman, Linnebo and Pettigrew

[2012]). In nearly all abstract structures, one can find a relation that grounds the numerical diversity of their objects. For example, in the second case that Rizza [2010: 187–189] treats, which is affine plane geometry, every pair of points in the plane are discerned by the following relation: there is a point in between the points on the one line connecting them. The two points then are identical iff there is no such point.

The rare cases where even relational discernibility is not possible are certain unlabelled graphs, as pointed out by Ladyman & Leitgeb [2008], and certain categories, such as a pre-order category with two objects and four arrows, two of which identity-arrows: the nodes and objects in these respective structures are utterly indiscernible. Not only the *ante rem* structuralist is unable to ground the *quantitative* numerosity of objects in these rare structures *qualitatively* by the means provided by these structures *and* respecting their symmetries, but no one is able to do this, because in *these* structures it is impossible to do so. Therefore an *ante rem* structuralist view of these rare structures seems very difficult, notwithstanding the fact that for nearly all other structures, the view seems impeccable — which is not to deny that certain features of this stand in need of clarification, as pointed out by e.g. Linnebo (2008).

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