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Hidden Determinism, Probability, and Time's Arrow

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Abstract

In present-day physics the fundamental dynamical laws are taken as a time-translation-invariant and time-reversal-invariant one-parameter groups of automorphisms of the underlying mathematical structure. In this context-independent and empirically inaccessible description there is no past, present or future, hence no distinction between cause and effect.

To get the familiar description in terms of causes and effects, the time-reversal symmetry of the fundamental dynamics has to be broken. Thereby one gets *two* representations, one satisfying the generally accepted rules of retarded causality ("no effect can precede its cause"). The other one describes the strange rules of advanced causality. For *entangled* (but not necessarily interacting) quantum systems the arrow of time must have the same direction for all subsystems. But for classical systems, or for classical subsystems of quantum systems, this argument does not hold. As a cosequence, classical systems allow the conceptual possibility of advanced causality in addition to retarded causality.

Every mathematically formulated dynamics of statistically reproducible events can be extended to a description in terms of a one-parameter group of automorphisms of an enlarged mathematical structure which describes a *fictitious hidden determinism.* Consequently, randomness in the sense of mathematical probability theory is only a weak generalization of determinism. The popular ideas that in quantum theory there are gaps in the causal chain which allow the accommodation of the freedom of human action are fantasies which have no basis in present-day quantum mechanics. Quantum events are governed by *strict* statistical laws.

Freedom of action is a constitutive necessity of all experimental science which requires a violation of the statistical predictions of physics. We conclude that the presently adopted first principles of theoretical physics can neither explain the autonomy of the psyche nor account for the freedom of action necessary for experimental science.

1 Determinism Does not Deal with Predictions

A large part of the difficulty with the "determinism versus indeterminism" debate lies in the failure to define the terms of discussion clearly. Many physicists and philosophers do not make the important distinction between determinism and the concept of predictability, and thereby commit a category mistake by claiming that determinism implies the possibility of prediction the future course of the universe.

For example, Max Born considers the distinction between determinism and predictability as hairsplitting and completely superfluous.[1] He maintains that classical point mechanics is not deterministic since there are unstable mechanical systems which are epistemically not predictable.

I think the following remarks by John Earman are appropriate:[2]

> "The history of philosophy is littered with examples where ontology and epistemology have been stirred together into a confused and confusing brew. ... Producing an 'epistemological sense' of determinism is an abuse of language since we already have a perfectly adequate and more accurate term – prediction – and it also invites potentially misleading argumentation – e.g., in such-an-such a case prediction is not possible and, therefore, determinism fails."

That is, *determinism does not deal with predictions.* Nevertheless, ontic descriptions are often confused with epistemic ones.[3] In the philosophical literature there are many examples for such category mistakes. For example Rudolf Carnap says: "Causal relation means predictability."[4] Likewise, Karl Popper maintains: "Scientific determinism is the doctrine that the state of any closed physical system at any future instant can be predicted." [5] Physicists drop similar careless assertions. For example Leon Brillouin: "The Poincaré discontinuities correspond to conditions where prediction is actually impossible and determinism cannot exist."[6]

2 Terminology and Basic Concepts

In order to set the stage for my discussion, I introduce some definitions and notions I will use in the following.

Determinism: First of all, determinism will be taken to refer exclusively to *ontic descriptions*, and should not be confused with statements concerning our knowledge or beliefs. In particular, according to the definition adopted here,

[1]Compare for example Max Born (Born (1955a); Born (1955b), Born (1958)). For a critique of Born's view compare von Laue (1955).

[2]Earman (1986). pp.7–8.

[3]For more details on the ontic/epeistemic distinction compare the contribution by Harald Atmanspacher in this volume.

[4]Carnap (1966), p.192

[5]Popper (1982), p.36.

[6]Brillouin (1964), p.135.

determinism does not to imply predictability. Both, predictability and retrodictability have their proper place only in the framework of *epistemic descriptions.*

Causal relations: A process within which one event is a necessary condition for another event is described by a *causal relation.* The producing event is known as the *cause* and the event produced as its *effect. A causal relationship is an irreflexive, antisymmetric and transitive binary relation between two events.* That is:

- no event can be the cause of itself;
- if a is the cause of b, then b cannot be the cause of a;
- if a is the cause of b, and b the cause of c, then a is the cause of c.

Causal ordering: A causal nexus requires some universal order. A fundamental issue is the relation of causality to time. According to DAVID HUME causal relations have three components: contiguity of time and place, temporal priority of the cause, and constant conjunction.[7] For Hume "all inferences from experience ... are effects of custom, not of reasoning",[8] so that according to Hume's view the idea of cause and effect is not a matter of fact but a mental habit of association, that is, essentially subjectively fabricated. Hume's characterization implies that causal and temporal arrows are related *by definition.* Yet, this merging of the two very different ideas of causal order and temporal order is conceptually not sound. Moreover, it precludes many *logical* possibilities, like a backward causation, or a time-independent ordering of the causal nexus.

Arrowless time in physics: If one wants to characterize causal ordering by temporal ordering, then one has first to introduce *temporal direction.* Yet, the generally adopted first principles of physics do not distinguish the future from the past. First principles are characterized by high symmetries. A corresponding physical law is said to be fundamental if it is as independent as possible of any particular context. For example, we assume that the laws of nature are the same all the time and everywhere.

The assumption that there is neither a favored point of the origin nor a preferred direction in time and space is a basic symmetry postulate required in in all fundamental physical theories. Since a fundamental theorem by EMMY NOETHER theorem implies a deep connection between symmetries and conservation laws, the idea that fundamental laws should be characterized by high symmetries is not just an aesthetic concept.[9] For example, Noether's theorem requires that the time-translation symmetry implies and is implied by the

[7] HUME (1793), book 1, part 3, p.466.
[8] HUME (1748), section V, part 1.
[9] NOETHER (1918). Noether's theorem says that if the action integral of a dynamical system is invariant with respect to a n-parameter continuous group of symmetry transformation, then the equations of motion have n linearly independent conservation laws.

conservation of energy. In fundamental physical theories the basic dynamical laws are not only taken as a time-translation-invariant but also as time-reversal-invariant. In the mathematical jargon we say that a fundamental dynamics is given by a *time-translation-invariant* and *time-reversal-invariant* one-parameter group of automorphisms of the underlying mathematical structure.

If we consider the time-reversal symmetry as primary, then there is no ordering so that we cannot use the concepts of cause and effect. In such a formulation of physics all reality is already pre-existent, and nothing new can come into existence. In order for time and causality to be genuinely active, some degree of freedom is necessary to provide a mechanism by which the events "come into being". Without breaking the time-reversal symmetry nothing new can ever arise. Within special contexts a spontaneous breaking of this symmetry is possible, so that the direction of time has to be considered as *contextual.*

3 Breaking the Time-Reversal Symmetry

Every experiment requires nonanticipative measuring instruments, hence a distinction between past and future. In engineering physics the direction of causation is always assumed to go from past to future. That is, to derive experimental physics from first principles, the time-reversal symmetry of the fundamental laws has to be broken. The anisotropy of time is a *precondition* for any theory of irreversible processes.

The phenomenon of symmetry breaking is well-understood in modern physical theories. It is not an ad-hoc postulate, but follows from the first principles of theoretical physics. Nevertheless, there remains an important problem. The time-reversal symmetry is represented by a group of order two. If the time-reversal symmetry is broken one gets *two* representations, one satisfying the generally accepted rules of retarded causality (forward causation) and the other one the strange rules of advanced causality (backward causation).

That is, if it is possible at all to derive the principle of retarded causality ("no effect can precede its cause"), then the very same methods allow the derivation of processes goverened by advanced causality. The decision which of the two possibilities is appropriate can therefore not be derived from the first principles of physics. *So the conceptual problem is not the breaking of the time-reversal symmetry, but the proper selection of one or the other one-sided realization.*

4 Arrow of Time

Mechanical and electrical input–output systems are non-anticipative. In other words, they are characterized by the fact that any present output values do not depend on future input values. The success of such traditional phenomenological physical descriptions suggests that in epistemic descriptions the time-reversal symmetry is always broken. Also, conscious perception and cognition seems to presuppose the usual forward direction of time, implying a memory of the past, but no anticipation of the future. The one-way property of time which we experience in our everyday life, and which we find in phenomenological physical laws has been called *"time's arrow"* by ARTHUR STANLEY EDDINGTON.[10]

In spite of these empirical facts, advanced causality is a conceptual possibility which is not ruled out by any fundamental physical law. So even on an ontic level we have to distinguish between the following *two* possibilities:

- *Forward determinism*, that is the thesis that, given the past of a physical system, there is a *unique future* (retarded realization).
- *Backward determinism*, that is the thesis that, given the future of a physical system, there is a *unique past* (advanced realization).

The nature and origin of a temporal asymmetry in the physical world is a perplexing problem for a theoretician. What is the *origin* of the arrow of time? Why do most (perhaps all) observable processes show the *same* arrow of time? What is puzzling about this temporal asymmetry is the *universality* of the direction of the arrow of time. The usual choice of retarded causality cannot be explained by a statistical mechanical formulation of the "second law" without an a priori postulate imposing an asymmetric evolution toward increasing time.

Backward causation is usually disregarded in science, but – as indicated abov – should not be excluded rashly. From the viewpoint of physical first principles there are no reasons to select one of the two temporal directions. The question of whether epistemic descriptions in terms of backward causality are rewarding or not, should not be decided by some a-priori argument. It seems to be more reasonable not to hold any prejudices, but first to work out the theory in full mathematical detail and then discuss the consequences.[11]

5 Indeterminism

The phrase "indeterminism" hides a number of different concepts and implications. It can refer to limits of knowledge, inherent or practical unpredictability, unpredictable mechanistic or non-mechanistic causation, uncaused action, or even lawlessness.

First we have to emphasize that *noncausation has to be distinguished from indeterminism.* On the other hand, there are ontically deterministic systems which produce epistemically irreducible random outcomes. In addition, *the existence of strict statistical laws for certain random events suggests that there must be some causation.* If a system produces epistemically irreducible random outcomes which fulfill the statistical laws of Kolmogorov's probability theory,[12] then we speak of *statistical causation.* It may be tempting to exclude indeterministic events which are not ruled by the laws of mathematical probability or any kind of statistical laws. But this would be premature without further arguments. The concept of *noncausation* will be used to refer to such events.

[10] EDDINGTON (1928), p.68.
[11] Compare also the contribution by PHIL DOWE in this volume.
[12] KOLMOGOROFF (1933).

6 Statistical Causality

A broken time-reversal symmetry gives rise to two *epistemic* causal relations: *prediction* and *retrodiction*. Forward causality is logically independent from backward causality. Ontic determinism implies neither epistemic predictability nor epistemic retrodictability.

The usual causal explanations have predictive power and refer to a world to be acted on. Predictions require a *memory of the past*, they refer to probabilistic inferences of the future behavior of a system from empirically estimated past states. In a statistical description, predictive processes are entropy-increasing. Retrodictive explanations require a Carrollian *"memory of the future"*, they refer to probabilistic inferences of the past behavior of a system. In a statistical description retrodictive processes are entropy-decreasing.[13]

Forward causal and backward causal descriptions have the same logical status a priori. Progress in science depends on the discovery of causal connections between events, regardless of whether this nexus is backward or forward. In spite of the fact that memories usually store past events only, *we have to acknowledge that advanced causations are not forbidden by the first principles of physics.*

A theme recurring again and again is the obsession that the second law of thermodynamics is related with the arrow of time, and that it precludes the existence of entropy-decreasing systems. However, a system showing a globally entropy-increasing behavior can nevertheless possess open subsystems with locally retrodictive and entropy-decreasing behavior. There are many everyday physical examples for entropy-decreasing systems. A successful retrodictive teleological description of entropy-decreasing *living systems* is just as legitimate as the predictive description of an entropy-increasing system.

7 Why Can There Be Laws of Chance?

Mathematical probability theory refers to events or processes whose outcomes are *individually random*, but are governed by *strict statistical laws.* This theory has a rich mathematical structure so we have to ask *under which conditions* the usual "laws of chance" are valid. Long ago, Marian von Smoluchowsky pointed out that the concept of probability can be defined, and the laws of probability can be derived from the theory of strictly deterministic but non-robust classical dynamical systems.[14] So it may be tempting to presume that chance events which satisfy the axioms of classical mathematical probability theory always result from the deterministic behavior of an underlying physical system. Such a claim cannot be demonstrated though. What can be proven is the weaker statement that every probabilistic system which fulfills the axioms

[13]For details compare Watanabe (1968), Watanabe (1969a), Watanabe (1970), Watanabe (1975), Watanabe (1986).

[14]Von Smoluchowsky (1918).

of classical mathematical probability theory can be embedded into a fictitious larger deterministic description.

To understand this claim we recall that the logic of classical probability theory is given by a (usually nonatomic) Boolean σ-algebra. Mathematical probability is defined as a σ-additive norm on a Boolean σ-algebra. A theorem by Lynn H. Loomis and Roman Sikorski implies that every probabilistic system has a (nonunique) extension to an *atomic* Boolean algebra, such that every probability can be represented as a mean over two-valued states. *It follows that random variables fulfill the laws of chance if and only if it is possible to find a larger Boolean system such that every expectation value is the mean of a (possibly hidden) variable with respect to a two-valued state of the enlarged system.* Such deterministic embeddings are usually not constructive and nothing substantial can be said about a possible ontic interpretation of hidden variables of the enlarged deterministic system. The important point is that *the strict statistical regularities are due to the mere possibility of a deterministic mathematical description.*

Mathematical details on probability and hidden determinism

Mathematical probabilistic theory[15] can be described in terms of a probability algebra $(\mathcal{B}, p)$ where the probability p is a strictly positive σ-additive normed measure on the elements of a Boolean σ-algebra $\mathcal{B}$ of events. According to the fundamental Loomis–Sikorski representation theorem every Boolean σ-algebra $\mathcal{B}$ is isomorphic to a σ-algebra $\boldsymbol{\Omega}/\boldsymbol{\Delta}$, where $\boldsymbol{\Omega}$ is some point set and $\boldsymbol{\Delta}$ is a σ-ideal $\boldsymbol{\Sigma}$, $\mathcal{B} \sim \boldsymbol{\Omega}/\boldsymbol{\Delta}$.[16] Every statistical state can be represented by a probability p on $\mathcal{B}$, or equivalently, by the restriction of a probability measure $\boldsymbol{\mu}$ on the measurable space $(\boldsymbol{\Omega}, \boldsymbol{\Sigma})$ to the Boolean algebra $\boldsymbol{\Omega}/\boldsymbol{\Delta}$. Equivalently, a statistical state can be represented by the restriction of a probability measure $\boldsymbol{\mu}$ on the measurable space $(\boldsymbol{\Omega}, \boldsymbol{\Sigma})$ to the Boolean algebra $\boldsymbol{\Omega}/\boldsymbol{\Delta}$. In this representation the nonnegative number $\boldsymbol{\mu}(\mathcal{B})$ is the probability of the event $\mathcal{B} \in \boldsymbol{\Sigma}$ in this statistical state.

The power set $\boldsymbol{\Pi}(\boldsymbol{\Omega})$ of all subsets of the set $\boldsymbol{\Omega}$ is a complete and atomic Boolean algebra which can be considered as an extension of the Boolean σ-algebra $\mathcal{B}$ of events. Every element $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ defines a dispersion-free individual state $\chi_{\boldsymbol{\omega}}$ by

$$\chi_{\boldsymbol{\omega}}(\mathcal{B}) \;=\; \begin{cases} 1 & \boldsymbol{\omega} \in \mathcal{B} \\ 0 & \boldsymbol{\omega} \notin \mathcal{B} \end{cases} \quad , \quad \mathcal{B} \in \boldsymbol{\Pi}(\boldsymbol{\Omega}) \quad .$$

The classical system characterized by the Boolean algebra $\boldsymbol{\Pi}(\boldsymbol{\Omega})$ is one of the many possible hidden-variable extension of the probability algebra $(\mathcal{B}, p)$, or equivalently, of the Kolmogorov probability space $(\boldsymbol{\Omega}, \boldsymbol{\Sigma}, \boldsymbol{\mu})$. Every probability measure $\boldsymbol{\nu}$ on $(\boldsymbol{\Omega}, \boldsymbol{\Sigma})$ is a mean over a family $\{\chi_\omega \,|\, \boldsymbol{\omega} \in \boldsymbol{\Omega}\}$ of two-valued states of the extended system with the Boolean algebra $\boldsymbol{\Pi}(\boldsymbol{\Omega})$,[17]

$$\boldsymbol{\nu}(\mathcal{B}) \;=\; \int_{\mathcal{B}} \chi_{\boldsymbol{\omega}}\, \boldsymbol{\nu}(d\boldsymbol{\omega}) \quad , \quad \mathcal{B} \in \boldsymbol{\Sigma} \quad .$$

Usually the sample space $\boldsymbol{\Omega}$ is uncountable so that an atom $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ cannot represent an experimental proposition. In spite of the fact that the points of an uncountable phase space have no epistemic relevance whatsoever, the introduction of such "hidden variables" is a convenient mathematical tool. *This result*

[15] For an outline of the "point-free" approach to probability, compare Halmos (1944); Kolmogoroff (1948); Loś (1955), Kappos (1969).

[16] Loomis (1947). Compare also Sikorski (1969), p.117.

[17] Compare Kamber (1964), §7; Kamber (1965), §14.

implies that random variables fulfill the laws of chance if and only if they can formally be reduced to hidden two-valued variables.

Since the requirement $\mathbf{\Omega}/\mathbf{\Delta} \sim \mathcal{B}$ does not determine the set $\mathbf{\Omega}$ uniquely, any interpretation of individual points of the Kolmogorov probability space $(\mathbf{\Omega}, \mathbf{\Sigma}, \boldsymbol{\mu})$ can be misleading and should be avoided.

The possibility of a fictitious deterministic embedding of the statistical laws of mathematical probability theory explains the possibility of a *frequency interpretation* of probabilistic physics. Similarly, the modern consistent formulations of *subjective* probabilities postulate hat a rational man acts as if he had a deterministic model compatible with his preknowledge.[18] Accordingly, these theories also presuppose a hidden Boolean determinism as an underpinning for a coherent rational behavior.

Conclusion: *A probabilistic system fulfills the axioms of classical mathematical probability theory if and only if it can be embedded into a larger deterministic system. The question whether physical theories are deterministic or merely probabilistic in the sense of mathematical probability theory is empirically undecidable.*

8 Are There Statistically Irreproducible Events?

From a logical point of view one can neither exclude the occurrence of unique irreproducible events nor the existence of chance events for which the traditional "laws of chance" do not apply. Empirically this question is not decidable. Within the framework of the usual frequency interpretation of probability one encounters the well-known difficulty of "small probabilities". Already in 1866 John Venn tried to define a probability explicitly in terms of relative frequencies of occurrence of events "in the long run". He added that "the run must be supposed to be very long, in fact never to stop." Yet, without additional assumptions nothing can be inferred about the value of the limiting frequency of a finite segment, no matter how long it may be. Supplementary decision rules that allow to decide which probability statements we should accept are notoriously difficult to formulate. Even Kolmogorov[19] had to adopt the working rule by Antoine Augustine Cournot:[20]

If the probability of an event is sufficiently small, one should act in a way as if this event will not occur at a solitary realization.

Yet, the theory gives no criterion for deciding what is "sufficiently small". As emphasized by Wolfgang Pauli, no frequency interpretation can avoid a subjective factor:[21]

[18] Compare for example Savage (1954), Savage (1962), Good (1965), Jeffrey (1965), De Finetti (1972), De Finetti (1974), De Finetti (1975). For a convenient collection of the most important papers on the modern subjective interpretation, compare Kyburg & Smokler (1964).

[19] Kolmogoroff (1933), p.4, postulate B.

[20] Cournot (1843).

[21] Pauli (1954), p.114. English translation quoted according to Pauli (1994), p.45.

> In this purely mathematical form, BERNOULLI's theorem is thus not as yet susceptible to empirical test. For this purpose it is necessary somewhere or other to include a rule for the attitude in practice of the human observer, or in particular the scientist, which takes account of the subjective factor as well, namely that the realisation, even on a single occasion, of a very unlikely event is regarded from a certain point on as impossible in practice. Theoretically it must be conceded that there is still a chance, different from zero, of error; but in practice actual decisions are arrived at in this way, in particular also decisions about the empirical correctness of the statistical assertions of the theories of physics or natural science. At this point one finally reaches the limits which are set in principle to the possibility of carrying out the original programme of the rational objectivation of the unique subjective expectation.

WOLFGANG PAULI made the inspiring proposal to characterize unique events by the absence of any type of statistical regularity:[22]

> "The synchronicity phenomena considered by [Jung] elude being captured by *"laws"* of nature, since they are not reproducible, that is to say, they are unique and *blurred* by the statistics of large numbers. In physics, on the other hand, 'acausalities' are *captured* just by statistical laws of large numbers."

9 Hadamard's Principle of Scientific Determinism

For a mathematical formulation the traditional philosophical doctrine that each event is the necessary and unique consequence of prior events has to be sharpened by introducing the concept of the *state* of a physical system. What changes in a *dynamical system* is called the *state* of the system.

In modern system theory a state of a nonanticipative input-output system is represented by a kind of memory. It represents "the minimal amount of information about the past history of the system which suffices to predict the effect of the past upon the future".[23] Intuitively a system-theoretical state can be considered as a kind of memory, representing the relevant history of the system. Under very general conditions there exists a state-space realization for every nonanticipative input-output system. The most compact description is given by the so-called *Nerode state* at time t, defined as the equivalence class of all histories for $t < 0$ of the system which give rise to the same output for all conceivable future.[24] No particular ontic or epistemic interpretation is implied by this definition.

A well-posed nonanticipative dynamical system requires a rule for determining the Nerode state at a given future time from a given present state.

[22]Letter of June 3, 1952, by Wolfgang Pauli to Markus Fierz, quoted from VON MEYENN (1996), p.634. German original: "Die von [Jung] betrachteten Synchronizitätsphänomene ... entziehen sich der Einfangung in Natur-*'Gesetze'* , da sie nicht-reproduzierbar, d.h. einmalig sind und durch die Statistik grosser Zahlen *verwischt* werden. In der Physik dagegen sind die 'Akausalitäten' gerade durch statistische Gesetze (grosse Zahlen) *erfassbar*."

[23]KALMAN (1963), p.154. Compare also KALMAN, FALB & ARBIB (1969), section 1.1.

[24]For the first time this idea was clearly expressed by NERODE (1958) in the context of automata theory. Compare also KALMAN, FALB & ARBIB (1969), chapters 1.3 , 6.3, and 7.2.

Hadamard's principle of scientific determinism requests that in a well-posed forward-deterministic dynamical system every initial state determines all future states uniquely.[25] Hadamard's principle of scientific determinism is not a natural law but a regulative principle which leads to an appropriate choice of the state space. If Hadamard's principle is not fulfilled then it can often be enforced by choosing a larger state space.

Example: Hadamard's principle in classical point mechanics

Newton's second law $m\ddot{\boldsymbol{q}}_t = \boldsymbol{F}(\boldsymbol{q}_t, t)$, $\boldsymbol{q}_t \in \mathbb{R}^3$, for a point particle of mass m under the influence of a force $\boldsymbol{F}$ does not fulfill Hadamard's principle for the configuration space $\mathbb{R}^3$. All the information about the past is given by the specification of the position and the velocity of every point particle at a particular time. The Lagrangian formulation considers the positions $\boldsymbol{q}$ and the velocity $\boldsymbol{v} := \dot{\boldsymbol{q}}$ as *independent* variables. This move allows us to rewrite Newton's second law in a system-theoretic form as a system of two *first order* differential equations, $\dot{\boldsymbol{q}}_t = \boldsymbol{v}_t$, $m\dot{\boldsymbol{v}}_t = \boldsymbol{F}(\boldsymbol{q}_t, t)$, $\boldsymbol{q}_t \in \mathbb{R}^3$, $\boldsymbol{v}_t \in \mathbb{R}^3$. This Lagrangian formulation fulfills Hadamard's principle in the state space $\mathbb{R}^6$. The pair $\{\boldsymbol{q}_0, \boldsymbol{v}_0\} \in \mathbb{R}^6$ represents the Nerode state specifying the initial conditions necessary for the unique determination of the solution of the equations of motion for $t > 0$.

Since a stochastic process is nothing else but a family of random variables,[26] every stochastic process can be dilated to a family of two-valued variables. Moreover, under appropriate continuity conditions, *every stochastic process can be dilated to a Hadamard-deterministic process.* Nevertheless, without further assumptions no ontological implication can be attributed to such a dilation.

Mathematical models for deterministic motions

The mathematical description of any kind of motion uses an uninterpreted concept of time. In fundamental descriptions time is represented by the additive *group* $\{t \,|\, t \in \mathbb{R}\}$ of real numbers. Our ability to distinguish between *before* and *after* requires that time intervals are *oriented*. If the time interval $t_2 - t_1$ between two instance is positive, the time t_2 is said to be *later* than t_1. The physically fundamental equations of motion are not only invariant under translations $t \to t' = t + \tau$, $\tau \in \mathbb{R}$, but also under the time-reversal transformation, an involution which exchanges the time parameter t by $-t$.[27] If the time-reversal symmetry is broken, time is represented either by the additive semigroup $\{t \,|\, t \geq 0\}$, or by the additive semigroup $\{t \,|\, t \leq 0\}$.

In a mathematical description of a dynamical system a *Nerode state* at time t is represented by an element $\boldsymbol{x}_t$ of some topological space, called the *state space* $\mathfrak{X}$. A *motion* of the system is represented by a function $t \mapsto \boldsymbol{x}_t \in \mathfrak{X}$ which maps each instant t on exactly one element $\boldsymbol{x}_t$ of the state space.

For a Hadamard-deterministic system the motion $t \mapsto \boldsymbol{x}_t$ of a system-theoretical state element $\boldsymbol{x}_t$ is ruled by a family $\{\varphi_{r,s} \,|\, 0 \leq s \leq r\}$ of of continuous *state transition maps* $\varphi_{s,t}$, fulfilling

$$\varphi_{r,s} \circ \varphi_{s,t} = \varphi_{r,t} \quad , \quad \varphi_{t,t} = \text{identity} \quad , \quad 0 \leq t \leq s \leq r \quad .$$

[25] Hadamard (1923). Compare also Hille & Phillips (1957), sect. 23.6.

[26] Doob (1953), p.46.

[27] An involution is an operation whose square is the identity. The involution associated with time-reversal does not only change the direction of time but also associated quantities like the velocity, the momentum, the angular momentum, the electrical current and the magnetic field. In elementary particle physics, the invariant involution PCT is associated with time-reversal T ($T^2 = 1$) also involves the space reflection P ($P^2 = 1$) and the charge conjugation C ($C^2 = 1$) (PCT-theorem).

The dynamics of a Hadamard-deterministic system is then given by the functional relation $\boldsymbol{x}_s = \varphi_{s,t}\{\boldsymbol{x}_t\}$ with $s \geq t$.

A dynamical system is called *autonomous* if it is not subject to time-varying external influences. In this case the state transition map $\varphi_{s,t}$ depends only on the difference $s-t$ so that the time evolution is *time-translation-invariant* and given by the *one-parameter semigroup* $\{\varphi_t \,|\, t \geq 0\}$ with $\varphi_t = \varphi_{0,t}$ and $\varphi_s \circ \varphi_t = \varphi_{s+t}$. The semigroup of time-translation invariant state transition maps φ_t is called a *semiflow*. Under very general conditions, a non-autonomous dynamical system can be embedded into a larger autonomous dynamical system.[28]

An autononous dynamical system is said to be *reversible* if for each $t \geq 0$ the inverse state transition map φ_t^{-1} exists. If the family of state transition maps φ_t form a one-parameter group, $\varphi_s \circ \varphi_t = \varphi_{s+t}$, $s, t \in \mathbb{R}$, then $\{\varphi_t \,|\, t \in \mathbb{R}\}$ is called a *flow*. For every initial state element $\boldsymbol{x}_0$ the state element $\boldsymbol{x}_t$ at a later instant t is given by $\boldsymbol{x}_t = \varphi_t\{\boldsymbol{x}_0\}$. For a reversible time-translation-invariant dynamical system the inverse φ_t^{-1} is given by φ_{-t}, so that the trajectory $s \mapsto \varphi_{-s}\{\boldsymbol{x}_s\}$, $0 \leq s \leq t$, arrives back at $\boldsymbol{x}_0$ after the time t, $\boldsymbol{x}_0 = \varphi_{-t}\{\boldsymbol{x}_t\}$. Since the dynamics can be uniquely reversed, we speak of a *time-reversal-invariant dynamics*. Note that a time-reversal-invariant dynamics entails the possibility of a *process reversal*, not the reversal of the direction of time.

10 Experimental Science Requires Freedom of Action

The premise that we often have a choice about what we are going to do is called the *free-will thesis*.[29] Some philosopher claim the laws of physics and the freedom of will are of a different type and operates at a different level so that one can without contradiction subscribe to both determinism and free will. An example for such a "compatibilist viewpoint" is the suggestion that we should abandon the view that the laws of nature act like inviolable prescriptions, and that we should adopt a descriptive view of natural laws so that the problem of free will does not even arise. This descriptive view claims that whatever happens in the world, there are true descriptions of those events, and that whatever you do, there is a true description of what you have done. Other philosophers and scientists opt for another variant, claiming that all behavior is determined, and that "free will" is a "meaningless concept", or an "illusion" .

At present the problem of how free will relates to physics seems to be intractable since no known physical theory deals with consciousness or free will. Fortunately, the topic at issue here is a much simpler one. It is neither our experience of personal freedom, nor the question whether the idea of freedom could be an illusion, nor whether we are responsible for our actions. The topic here is *that the framework of experimental science requires a freedom of action in the material world as a constitutive presupposition.* In this way "freedom" refers to actions in a material domain which are not governed by deterministic first principles of physics.

[28] Compare Howland (1974); Reed & Simon (1975), section X.12; Nickel (1996); Engel & Nagel (2000); section VI.9. Compare also the contribution by Gregor Nickel in this volume.
[29] Van Inwangen (1983), p.222.

To get a clearer idea of what is essential in this argument we recall that the most consequential accomplishment by ISAAC NEWTON was his insight that the *laws of nature* have to be separated from *initial conditions*. The initial conditions are not accounted for by first principles of physics, they are assumed to be "given". In experimental physics it is always taken for granted that the experimenter has the freedom to choose these initial condition, and to repeat his experiment at any particular instant. To deny this freedom of action is to deny the possibility of experimental science.

In other words, we assume that the physical system under investigation is governed by strictly deterministic or probabilistic laws. On the other hand, we also have to assume that the experimentalist stands out of these natural laws. The traditional assumption of theoretical physics that the basic deterministic laws are universally and globally valid for all matter thus entails a pragmatic contradiction between theory and practice. *A globally deterministic physics is impossible.*

11 Quantum Randomness

Often it is claimed that according to quantum mechanics "the basic constituents of matter behave in a fundamentally random way,"[30] or that quantum mechanics allows "uncaused" events.[31] Such statements are misleading since all quantum events are governed by *strict statistical laws*. Probability is an essential element in every complete epistemic description of quantum events. From an epistemic viewpoint, individual quantum events are in general irreducibly random. But this epistemic quantum randomness does neither imply ontic randomness nor that determinism has been refuted by quantum theory. As the following example illustrates, *a link between randomness and determinism, analogous to that discussed in Section 7, holds also for quantum physics.*

Example of a deterministic embedding for quantum measurements

In spite of the fact that no experiment can ever realize a measurement of the first kind, it is worthwhile to discuss this customary idealization in the framework of quantum theory. We consider the simple example of a first-kind measurement of the observable σ_3,

$$\sigma_3 = |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta| \ , \ \alpha,\beta \in \mathbb{C}^2 \ , \ \langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1 \ , \ \langle\alpha|\beta\rangle = 0 \ ,$$

of a two-level quantum object system. We assume that the initial state is pure. It can either be described by a normalized state vector $\Psi_{\text{initial}} = c_1\,\alpha + c_2\,\beta \in \mathbb{C}^2$, or by the density operator $D_{\text{initial}} = |\Psi_{\text{initial}}\rangle\langle\Psi_{\text{initial}}|$.

The result of any measurement is irreversibly recorded by a *Boolean* device. Therefore a full statistical description of a first-kind measurement process has to include the object system, the measuring apparatus, and the classical recording device. In the framework of algebraic quantum mechanics such a full description is possible. Here we discuss only the resulting statistical state transition map $D_{\text{initial}} \to D_{\text{final}}$ connecting the initial with the final state of the object system.

[30] Compare for example BARRETT (1999), p.1
[31] MARGENAU (1957), p.724.

For a first-kind measurement of σ_3 the final density operator of the object system is given by

$$D_{\text{final}} = |c_1|^2\,|\alpha\rangle\langle\alpha| + |c_2|^2\,|\beta\rangle\langle\beta| = \tfrac{1}{2}\sigma_3 \;+\; \tfrac{1}{2}\mathrm{tr}\{D_{\text{initial}}\,\sigma_3\}\;\sigma_3 \quad .$$

Since in quantum mechanics the set of statistical state functionals is not a simplex, the final density operator D_{final} allows infinitely many distinct convex decompositions into pure states. Therefore the linear map $D_{\text{initial}} \to D_{\text{final}}$ *does not imply* that the final state is a classical mixture of pure states described by the density operators $|\alpha\rangle\langle\alpha|$ and $|\beta\rangle\langle\beta|$. Nevertheless, the probabilities $|c_1|^2$ and $|c_2|^2$ are the correct *conditional* probabilities for a measurement of the first kind. The condition is that the classical measuring instrument has factually and irreversibly registered one of the eigenvalues of the observable σ_3. In other words, the density operator D_{final} is the *conditional expectation* of D_{initial}, conditioned by the commutative W*-algebra generated by the observable σ_3.

The simplest dynamical description of the state transition map $D_{\text{initial}} \to D_{\text{final}}$ is given by the linear dynamical semigroup

$$\dot{D}(t) \;=\; \frac{\omega}{2i}\,[\sigma_3, D(t)] \;-\; \frac{\kappa}{4}\,[\sigma_3,[\sigma_3,D(t)]] \quad ,$$

with $D_{\text{initial}} = D(0)$, $D_{\text{final}} = D(\infty)$. The constant $\omega \in \mathbb{R}$ is an angular frequency describing the Hamiltonian dynamics, and $\kappa > 0$ is a relaxation frequency describing the dissipative interaction with the environment. For an arbitrary initial density matrix $D(0)$ we get the asymptotic density matrix $D(\infty)$,

$$D(\infty) \;:=\; \lim_{t\to\infty} D(t) \;=\; \tfrac{1}{2}\sigma_3 \;+\; \tfrac{1}{2}\mathrm{tr}\{D(0)\,\sigma_3\}\;\sigma_3 \quad .$$

The statistical map $D_{\text{initial}} \to D_{\text{final}}$ can be generated in many different ways by an individual Hadamard deterministic motion. Consider for example a solution $t \mapsto \Phi(t)$ of the following nonlinear stochastic Schrödinger equation in the sense of Stratonovich

$$\begin{aligned} i\,d\Phi(t) \;&=\; \tfrac{1}{2}\,\omega\,\sigma_3\,\Phi(t)\,dt \;+\; i\,\kappa\,\langle\Phi(t)\,|\,\sigma_3\,\Phi(t)\rangle\Big\{\sigma_3 - \langle\Phi(t)\,|\,\sigma_3\,\Phi(t)\rangle\Big\}\,\Phi(t) \\ &+\; i\,\sqrt{\kappa/2}\,\Big\{\sigma_3 - \langle\Phi(t)\,|\,\sigma_3\,\Phi(t)\rangle\Big\}\,\Phi(t)\circ dw(t) \;\;,\;\; \Phi(0) = \Psi_{\text{initial}} \;\;. \end{aligned}$$

Then the mean density operator $D_\Phi(t) \;=\; \mathcal{E}\big\{|\Phi(t)\rangle\langle\Phi(t)|\big\}$ satisfies the same dynamical semigroup as the density operator $D(t)$. Moreover, an individual trajectory $t \mapsto \Phi(t)$ describes the individual behavior of the object system within the hypothetical extended description. For an arbitrary initial state vector Ψ_{initial} one gets asymptotically[32]

$$\Psi_{\text{initial}} \;\xrightarrow{t\to\infty}\; \begin{cases} \alpha & \text{with probability } \;|\langle\alpha|\Psi_{\text{initial}}\rangle|^2 \\ \beta & \text{with probability } \;|\langle\beta|\Psi_{\text{initial}}\rangle|^2 \end{cases} \quad ,$$

where α and β are the normalized eigenvectors of σ_3.

The white noise process $t \mapsto dw(t)/dt$ is a generalized stationary Gaussian process. It is well known that the trajectories of every continuous Gaussian processes can be generated by a linear Hamiltonian system.[33] A physically realistic stochastic differential equation is never driven by white noise processes but by some variant of band-limited smooth noise processes . A theorem by Wong and Zakai implies that the white-noise limit of a stochastic differential equation with an essentially band-limited noise process is the solution of the corresponding stochastic differential equations in the sense of Stratonovich.[34] Therefore the stochastic Stratonovich equation mentioned above has a Hadamard deterministic

[32] Gisin (1984).
[33] Compare for example Picci (1988).
[34] Wong & Zakai (1965a), Wong & Zakai (1965b).

dilation to a Hamiltonian equation of motion for the deterministic trajectories of the the stochastic process $t \mapsto \Phi(t)$.

Of course, the sketched construction of a deterministic representation of a quantum measurement process is quite ad hoc and implies nothing about an ontic interpretation. The salient point is to show that the strict statistical regularities of a quantum measuring process can be understood in terms of a hidden determinism. Note that the various theorems which show that in quantum theory it is impossible to introduce hidden variables only say that it is impossible to embed quantum theory into a deterministic *Boolean* theory.

The context-independent laws of an ontic description of quantum mechanics are strictly deterministic but refer to a *non-Boolean* logical structure. The epistemically irreducible probabilistic structure of quantum theory is due to the fact that every communication in terms of an unequivocal language requires a Boolean domain of discourse. The nonpredictable outcome of a quantum experiment is related to the projection of the non-Boolean lattice of the deterministic ontic description to the Boolean algebra of the epistemic description of a particular experiment. The associated epistemic quantum-theoretical probabilities cannot be attributed to the object system; they are *conditional* probabilities for state transitions induced by the interaction of the object system with a classical measuring apparatus. The epistemic probabilities depend on the experimental arrangement, but for a fixed context they are objective since the underlying ontic structure is deterministic.

12 Quantum Mechanics Cannot Explain Free Will

Since the Hadamard determinism of classical mechanics and the freedom of actions collide, some physicists and many philosophers find delight in the belief that the so-called "uncertainty principle" of quantum mechanics can solve the problem of freedom.[35] For instance, the neurophysiologist John Eccles argued that in addition to the material world there is a nonmaterial and nonsubstantial mental world. He speculated that our mind acts on the brain at the quantum level by momentarily increasing the probability of exocytosis in selected cortical areas and thereby controlling "quantum jumps", turning them into voluntary excitations of the neurons that account for body motion.[36]

The main idea of all such fantasies seems to be that in quantum theory there are gaps in the causal chain which allow the accommodation of free will and corresponding action. Yet, quantum events are governed by the strict mathematical laws of Kolmogorov's probability theory which lead to the empirically reproducible statistical rules of statistical physics. Consequently, *quantum randomness is just a weak generalization of determinism.*[37] Human actions are, however, in general not reproducible, hence not subject to the mathematical

[35] Compare for example Jordan (1932); Jordan (1934); Jordan (1956), pp.114ff.; Margenau (1957); Rohs (1996), pp.232–239.

[36] Beck & Eccles (1992); Eccles (1994).

[37] Similarly, Wolfgang Pauli has stressed that "quantum mechanics is a very weak general-

laws of probability theory.[38] *Freedom of action requires a violation of the statistical predictions of quantum theory. Hence present-day quantum theory is neither in a position to explain the autonomy of the psyche nor the existence of free will.*

13 Why Does Time's Arrow Always Point in the Same Direction?

The presupposed possibility of breaking the time-reversal symmetry and defining the arrow of time locally is not yet sufficient to understand the seemingly global character of the direction of time. Why should the arrow of time in two noninteracting physical systems point to the same direction? Classical causality is local and cannot explain the *universality* of the direction of the arrow of time. In quantum physics the omnipresence of Einstein–Podolsky–Rosen correlations even between noninteracting subsystems implies a global character of the time-reversal operation in quantum systems. In an entangled system it is impossible to define a *local* time-reversal operation for one subsystem only. Consequently the arrow of time of *entangled* (but not necessarily interacting) quantum systems must have the same direction. *Quantum causality is holistic and requires the same direction of the arrow of time even for non-interacting entangled subsystems.*

Representing time-reversal symmetry in quantum mechanics

In the C*-algebraic description of arbitrary physical systems a symmetry is represented by a *Jordan automorphism* of the underlying C*-algebra $\mathfrak{A}$. A Jordan-automorphism is a linear *-preserving bijection which respects the symmetrized product,[39]

$$\alpha(A^*) = \alpha(A)^* \ , \ \alpha(AB + BA) = \alpha(A)\alpha(B) + \alpha(B)\alpha(A) \ , \ A, B \in \mathfrak{A} \ .$$

A C*-automorphism α is a Jordan-automorphism which preserves the order of the ordinary product, $\alpha(AB) = \alpha(A)\alpha(B)$. An anti-automorphism α, which reverses the order of the terms in the ordinary product, $\alpha(AB) = \alpha(B)\alpha(A)$, is also a Jordan-*-isomorphism. Every Jordan-automorphism of a C*-algebra is the sum of a C*-automorphism and a C*-anti-automorphism.[40]

ization of the old causality". In a letter to Markus Fierz of November 26, 1949 Pauli writes: "Ich habe keinen Zweifel, daß die quantenmechanische 'statistische Korrespondenz', viel näher auf der Seite des alten Determinismus liegt als auf der Seite des Synchronizitätsphänomens. Vom letzteren aus betrachtet muß die Quantenmechanik als eine *sehr schwache* Verallgemeinerung der alten Kausalität gelten. Und doch scheint mir die Quantenmechanik auch jenen Wegweiser nach der anderen Richtung zu haben, wo von willkürlicher Reproduzierbarkeit keine Rede mehr sein kann. Die Quantenmechanik scheint mir eine Art *Mittelstellung* einzunehmen." Quoted from von Meyenn (1993), p.710.

[38] Curiously, Warren Weaver (1948) (p.33) claims "that individual human decisions, like the individual events of physics, are not ruled by causality; while the statistical behavior of a man, like the statistics of a physical ensemble, is ruled by causality." It may be that *some* human actions are ruled by statistical laws, but certainly there are also singular non-recurring human actions.

[39] Compare for example Emch (1972), p.152.

[40] Kadison (1951), theorem 10.

In the irreducible representation of quantum mechanics on a Hilbert space $\mathcal{H}$, the time reversal is represented by an anti-unitary operator on the algebra $\mathfrak{B}(\mathcal{H})$,[41] so that it implements a C*-anti-automophism on the algebra $\mathfrak{B}(\mathcal{H})$. More generally, for an arbitrary physical system, the time reversal is represented by an involutary anti-automorphism τ of the underlying C*-algebra $\mathfrak{A}$,

$$\tau(A^*) = \tau(A)^* \ , \ \tau(AB) = \tau(B)\tau(A) \ , \ \tau\{\tau(A)\} = A \ , \ A, B \in \mathfrak{A} \ .$$

The time-reversal map τ is positive, but for quantum systems (where $\mathfrak{A}$ is noncommutative) the map τ is *not completely positive.*[42] That is, if $(\mathfrak{A}_1, \tau_1)$ and $(\mathfrak{A}_2, \tau_2)$ describe two noninteracting quantum systems, then the *local* map $\tau_1 \otimes \mathbf{1}_2$ and the *local* map $\mathbf{1}_1 \otimes \tau_2$ on the minimal C*-tensor product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ are not positive, hence they are not time-reversal maps. The time-reversal map for the composite quantum system is given by the *global* positive map

$$(\tau_1 \otimes \mathbf{1}_2)(\mathbf{1}_1 \otimes \tau_2) \ = \ \tau_1 \otimes \tau_2 \quad .$$

This fact implies that even when the two quantum systems do *not interact* in any way, the time-reversal for the first quantum system is *not* given by $\tau_1 \otimes \mathbf{1}_2$. *The map $\tau_1 \otimes \mathbf{1}_2$ represents the time-reversal operator for the first subsystem if and only if the two systems are not entangled with Einstein–Podolsky–Rosen correlations.* This is the case for every state of the combined system if and only if at least one of the subsystems is classical (in the sense that either $\mathfrak{A}_1$ or $\mathfrak{A}_2$ is commutative).[43] In an entangled system with broken time-reversal symmetry the direction of the arrow of time has to point to the same direction for all (even noninteracting) subsystems. This requirement of *global consistency* distinguishes quantum causality from classical causality.

14 Hadamard Determinism Cannot Be Globally Valid

Complex systems can sometimes establish truly novel emergent properties, new properties that the component parts do not have. Often the doctrine of determinism is opposed by the principle of emergence. Yet, if the underlying physical system is deterministic in the sense of Hadamard, then all consistent higher-level theories can be extended to deterministic systems in the sense of Hadamard.

From the point of view adopted here freedom refers to actions in the material world which are not determined by the presently adopted first principles of physics. Such a view violates the usually assumed universal validity of the first principles of physics. Yet, such a violation is not surprising. The postulate that the underlying fundamental time evolution should be an *automorphism* is a queer assumption indeed. An automorphism is by definition *a map which does not change any physically relevant feature.* From a physical point of view the dynamics of interacting systems cannot simply be postulated to be automorphic, but has to be *derived* from first principles. Yet, a mathematically consistent theory of interactions does not yet exist.

[41] Wigner (1932). Compare also Wigner (1959), chapter 26.)

[42] A linear map $\tau : \mathfrak{A} \to \mathfrak{A}$ is said to be *completely positive* if the linear map $\tau \otimes \mathbf{1}_n : \mathfrak{A} \otimes \mathfrak{B}(\mathbb{C}^n) \to \mathfrak{A} \otimes \mathfrak{B}(\mathbb{C}^n)$ is positive for all $n \geq 1$, where $B(\mathbb{C}^n)$ is the C*-algebra of all complex $n \times n$-matrices and $\mathbf{1}_n$ is the identity transformation of $B(\mathbb{C}^n)$ onto itself.

[43] This is a consequence of theorem 4.14 in Takesaki (1979), p.211.

Remarks on the derivation of the dynamics from first principles

Nowadays it is generally accepted that *all known interactions of matter are due to gauge fields.* The dynamics of so-called *bare elementary systems* is known to be automorphic. Here, a bare elementary system is defined as an indecomposable representation of a kinematical group (like the Galilei or Lorentz group), disregarding interactions intermediated by gauge fields. Such elementary systems are are called *bare* since they only characterize the transformation properties under the actions of the kinematical group. Yet, elementary systems with non-zero mass or non-zero electrical charge are inevitably coupled to the gravitational or electromagnetic gauge fields. Due to this interaction, a bare elementary system acquires a complicated structure. This process is referred to as *dressing.* The derivation of the dynamics of a dressed system from the group-theoretically known automorphic dynamics of the bare elementary systems and the bare gauge fields in a mathematically rigorous way involves many difficulties. In particular our understanding of radiation effects and the infinities of self-interactions is far from satisfactory. It is an open question whether the resulting dynamic is automorphic. In spite of the fact that present-day quantum field theory is quite successful, it has only the status of a phenomenological theory.

Notwithstanding the problematic character of automorphic time evolutions, mounting evidence shows that for practically all concrete applications the time evolution of any open subsystem of the material world can be described by the restriction of an automorphic dynamics of a larger system consisting of the object system and its environment. So we have to ask why a hypothetical universal automorphic dynamics leads to contradictions. A possible answer derives from the remark that not everything that is locally true is necessarily true globally. So it may be worthwhile to contemplate about theories which *only locally, but not globally* correspond to the contemporary theories. At present we have no idea how to achieve such a goal. Besides, it is questionable whether a solution of this problem could contribute to an understanding of the problem of freedom of action in the material world.

It is disquieting that *we have no idea where precisely in the material domain the presently adopted first principles of physics do not apply.* Presumably new physical principles will need to be discovered. Since in physics time is used in a rather ad-hoc manner, one could think about a more physical concept of time, say as *time operator* which represents time as a dynamical observable.[44] But it is very much an open question what form a physical theory including the freedom of action will take.

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[44] Compare for example the ideas discussed by Misra (1995).

References

Barrett, J.A. (1999): *The Quantum Mechanics of Minds and Worlds.* New York: Oxford University Press.

Beck, F. & Eccles, J.C. (1992): *Quantum aspects of brain activity and the role of consciousness.* Proceedings of the National Academy of Sciences of the United States of America **89**, 11357–11361.

Born, M. (1955a): *Ist die klassische Mechanik wirklich deterministisch?.* Physikalische Blätter **11**, 49–54.

Born, M. (1955b): *Continuity, determinism and reality.* Danske Videnskabernes Selskab Mathematisk Fysiske Meddelelser **30**, No. 2, pp. 1–26.

Born, M. (1958): *Vorausagbarkeit in der klassischen Physik.* Zeitschrift für Physik **153**, 372–388.

Brillouin, L. (1964): *Scientific Uncertainty, and Information.* New York: Academic Press.

Carnap, R. (1966): *Philosophical Foundations of Physics: An Introduction to the Philosophy of Science. Edited by Martin Gardner.* New York: Basic Books.

Cournot, A.A. (1843): *Exposition de la théorie des chances et des probabilitiés.* Paris.

Doob, J.L. (1953): *Stochastic Processes.* New York: Wiley.

Earman, J. (1986): *A Primer on Determinism.* Dordrecht: Reidel.

Eccles, J.C. (1994): *How the Self Controls Its Brain.* Berlin: Springer.

Eddington, A.S. (1928): *The Nature of the Physical World.* Cambridge: Cambridge University Press.

Emch, G.G. (1972): *Algebraic Methods in Statistical Mechanics and Quantum Field Theory.* New York: Wiley.

Engel, K.-J. & Nagel, R. (2000): *One-Parameter Semigroups for Linear Evolution Equations.* New York: Springer.

Finetti, B.de (1972): *Probability, Induction and Statistics. The Art of Guessing.* London: Wiley.

Finetti, B.de (1974): *Theory of Probability. A Critical Introductory Treatment.* Volume 1. London: Wiley.

Finetti, B.de (1975): *Theory of Probability. A Critical Introductory Treatment.* Volume 2. London: Wiley.

Gisin, N. (1984): *Quantum measurements and stochastic processes.* Physical Review Letters **52**, 1657–1660.

Good, I.J. (1965): *The Estimation of Probabilities. An Essay on Modern Bayesian Methods.* Cambridge, Massachusetts: MIT Press.

Hadamard, J. (1923): *Lectures on Cauchy's Problem in Linear Partial Differential Equations.* New Haven: Yale University Press.

Halmos, P.R. (1944): *The foundations of probability.* American Mathematical Monthly **51**, 493–510.

Hille, E. & Phillips, R.S. (1957): *Functional Analysis and Semi-groups.* Providence, Rhode Island: American Mathematical Society.

Howland, J.S. (1974): *Stationary scattering theory for time-dependent Hamiltonians.* Mathematische Annalen **207**, 315–335.

Hume, D. (1748): *An Enquiry Concerning Human Understanding.* London.

Hume, D. (1793): *A Treatise of Human Nature. Book 1.* London.

Inwangen, P.van (1983): *An Essay on Free Will.* Oxford: Clarendon Press.

JEFFREY, R.C. (1965): *The Logic of Decision.* New York: McGraw-Hill.

JORDAN, P. (1932): *Die Quantenphysik und die Grundprobleme der Biologie und Psychologie.* Naturwissenschaften **20**, 815–821.

JORDAN, P. (1934): *Quantenphysikalische Bemerkungen zur Biologie und Psychologie.* Erkenntnis **4**, 215–252.

JORDAN, P. (1956): *Der gescheiterte Aufstand. Betrachtungen zur Gegenwart.* Frankfurt am Main: Klostermann.

KADISON, R.V. (1951): *Isometries of operator algebras.* Annals of Mathematics **54**, 325–338.

KALMAN, R.E. (1963): *Mathematical description of linear dynamical systems.* SIAM Journal on Control **1**, 152–192.

KALMAN, R.E., FALB, P.L. & ARBIB, M.A. (1969): *Topics in Mathematical System Theory.* New York: McGraw-Hill.

KAMBER, F. (1964): *Die Struktur des Aussagenkalküls in einer physikalischen Theorie.* Nachrichten der Akademie der Wissenschaften, Göttingen. Mathematisch Physikalische Klasse **10**, 103–124.

KAMBER, F. (1965): *Zweiwertige Wahrscheinlichkeitsfunktionen auf orthokomplementären Verbänden.* Mathematische Annalen **158**, 158–196.

KAPPOS, D.A. (1969): *Probability Algebras and Stochastic Spaces.* New York: Academic Press.

KOLMOGOROFF, A. (1933): *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Berlin: Springer.

KOLMOGOROFF, A. (1948): *Algèbres de Boole métriques complètes.* VI. Zjazd Matematyków Polskich. Annales de la Societe Polonaise de Mathematique **20**, 21–30.

KYBURG, H.E. & SMOKLER, H.E. (1964): *Studies in Subjective Probability.* New York: Wiley.

LAUE, M.VON (1955): *Ist die klassische Physik wirklich deterministisch?* Physikalische Blätter **11**, 269–270.

LOOMIS, L.H. (1947): *On the representation of σ-complete Boolean algebras.* Bulletin of the American Mathematical Society **53**, 757–760.

ŁOŚ, J. (1955): *On the axiomatic treatment of probability.* Colloquium Mathematicum (Wroclaw) **3**, 125–137.

MARGENAU, H. (1957): *Quantum mechanics, free will, and determinism.* Journal of Philosophy **54**, 714–725.

MEYENN, K. VON (1993): *Wolfgang Pauli. Wissenschaftlicher Briefwechsel, Band III: 1940–1949.* Berlin: Springer.

MEYENN, K. VON (1996): *Wolfgang Pauli. Wissenschaftlicher Briefwechsel, Band IV, Teil I: 1950–1952.* Berlin: Springer.

MISRA, B. (1995): *From time operator to chronons.* Foundations of Physics **25**, 1087–1104.

NERODE, A. (1958): *Linear automaton transformations.* Proceedings of the American Mathematical Society **9**, 541–544.

NICKEL, G. (1996): *On evolution semigroups and nonautonomous Cauchy problems.* Dissertation Summaries in Mathematics **1**, 195–202.

NOETHER, E. (1918): *Invariante Variationsprobleme.* Nachrichten von der Gesellschaft der Wissenschaften, Göttingen. Mathematisch Physikalische Klasse **1918**, 235–257.

PAULI, W. (1954): *Wahrscheinlichkeit und Physik.* Dialectica **8**, 112–124.

PAULI, W. (1994): *Writings on Physics and Philosophy.* Edited by C. P. Enz and K. von Meyenn. Berlin: Springer.

PICCI, G. (1988): *Hamiltonian representation of stationary processes.* In: GOHBERG, I., HELTON, J. W. & RODMAN, L. (ed.): *Operator Theory: Advances and Applications.* Basel: Birkhäuser. Pp.193–215.

Popper, K. (1982): *The Open Universe: An Argument for Indeterminism.* London: Hutchinson.

Reed, M. & Simon, B. (1975): *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness.* New York: Academic Press.

Rohs, P. (1996): *Feld, Zeit, Ich. Entwurf einer feldtheoretischen Transzendentalphilosophie.* Frankfurt am Main: Klostermann.

Savage, L.J. (1954): *The Foundations of Statistics.* New York: Wiley.

Savage, L.J. (1962): *The Foundations of Statistical Infereence. A Discussion.* London: Methuen.

Sikorski, R. (1969): *Boolean Algebras.* Third edition. Berlin: Springer.

Smoluchowsky, M. von (1918): *Über den Begriff des Zufalls und den Ursprung der Wahrscheinlichkeitsgesetze in der Physik.* Naturwissenschaften **6**, 253–263.

Takesaki, M. (1979): *Theory of Operator Algebras I.* New York: Springer.

Watanabe, S. (1968): *"Teleological" explanations in biophysics.* Supplement of the Progresss of Theoretical Physics **Extra Number**, 495–497.

Watanabe, S. (1969a): *Knowing and Guessing. A Quantitative Study of Inference and Information.* New York: Wiley.

Watanabe, S. (1970): *Creative time.* Studium Generale **23**, 1057–1087.

Watanabe, S. (1975): *Norbert Wiener and cybernetical concept of time.* IEEE Transactions on Systems, Man, and Cybernetics **SMC-5**, 372–375.

Watanabe, S. (1986): *Separation of an acausal subsystem from a causal total system.* In: Namiki, M. (ed.): *Proceedings of the 2nd International Symposium. Foundations of Quantum Mechanics in the Light of New Technology.* Physical Society of Japan. Pp.276–281.

Weaver, W. (1948): *Statistical freedom of the will.* Reviews of Modern Physics **20**, 31–34.

Wigner, E.P. (*1932):* Über die Operation der Zeitumkehr in der Quantenmechanik. *Nachrichten von der Gesellschaft der Wissenschaften, Göttingen. Mathematisch Physikalische Klasse* **31***, 546–559.*

Wigner, E.P. (1959): *Group Theory.* New York: Academic Press.

Wong, E. & Zakai, M. (1965a): *On the convergence of ordinary integrals to stochastic integrals.* Annals of Mathematical Statististics **36**, 1560–1564.

Wong, E. & Zakai, M. (1965b): *On the relation between ordinary and stochastic differential equations.* International Journal of Engineering Science **3**, 213-229.