On the role played by the work of Ulisse Dini on implicit function theory in the modern differential geometry foundations: the case of the structure of a differentiable manifold, 1

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1. Introduction

The structure of a differentiable manifold defines one of the most important mathematical object or entity both in pure and applied mathematics[1].

From a traditional historiographical viewpoint, it is well-known[2] as a possible source of the modern concept of an affine differentiable manifold should be searched in the Weyl's work[3] on Riemannian surfaces, where he gave a new axiomatic description, in terms of neighborhoods[4], of a Riemann surface, that is to say, in a modern terminology, of a real two-dimensional analytic differentiable manifold. Moreover, the well-known geometrical works of Gauss and Riemann[5] are considered as prolegomena, respectively, to the topological and metric aspects of the structure of a differentiable manifold.

All of these common claims are well-established in the History of Mathematics, as witnessed by the work of Erhard Scholz[6]. As it has been pointed out by the Author in [Sc, Section 2.1], there is an initial historical-epistemological problem of how to characterize a manifold, talking about a ‘’dissemination of manifold idea’’, and starting, amongst other, from the consideration of the most meaningful examples that could be taken as models of a manifold, precisely as submanifolds of a some environment space $\mathbb{R}^n$, like some projective spaces $\mathbb{P}(\mathbb{R}^m)$ or the zero sets of equations or inequalities under suitable non-singularity conditions, in this last case mentioning above all of the work of Enrico Betti on Combinatorial Topology [Be] (see also Section 5) but also that of R. Lipschitz on Analytical Mechanics. Scholz states that this last originary conception of a *number manifold* as zero sets of equations or inequalities under suitable non-singularity conditions, was the most general one reached at the time, although the Betti work was limited to the global case and not to the local one. It was only thanks to the Dini's work on implicit function theory that such a primary character was explicitly and rigorously stated, becoming generally known[7], putting in evidence what role has played this Dini's work, but nevermore quoting it below, making seem it little relevant for the following developments.

Instead, in this first paper, we want to point out other, besides little investigated, viewpoints on the same historical question, as regards the fundamental role played just by the works of Ulisse Dini on implicit function theory[8] and its geometrical applications, in the (‘implicit’) institution of the modern structure of a differentiable manifold. Exactly, we want first to show, without entering into a detailed formal discussion, as

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[1] See [Mu, Chapter 5, Section 23].
[2] See, for instance, [Sc] and [Ma].
[3] See [We1].
[4] Mainly following Klein's work on the same subject.
[5] Nevertheless, taking into account what said in the *Introduction* of [MR], we may state: «[...] *that, for a modern reader, it is very tempting to regard his* [that is, by Riemann] *efforts as an endeavor to define a ‘’manifold’’, and it is precisely the clarification of Riemann's ideas, as understood by his successors, which led gradually to the notions of manifold and Riemannian space as we know them today*». In this regards, see also what will be said in next Section 9 of this paper.
[6] See [Sc], where, however, the Whitney's work is not cited.
[7] Scholz thereupon mention [GP] and [Jo] as main dissemination sources of the Dini's work on implicit function theory: as we will see later, he has made right considerations as regard the Italian edition of [GP], but not for [Jo] which does not in any way quote the Dini name in the related Numéros 90-95 devoted to the implicit function subject.
[8] And, in general, by the role played by the whole doctrine field of implicit function theory along his historical development. Moreover, as concerns the case properly herein considered, a very fundamental role is played by the Dini's work on systems of implicit functions in the real field.

the Dini's works on implicit functions provide pivotal and crucial syntactic tools which were at the foundations of the modern theory of differentiable manifolds (and of other mathematical aspects of differential and algebraic geometry), in the sense that will be briefly outlined in Section 9. Indeed, according to [KP, Chapter 1, Section 1.3], the whole doctrine field of implicit function theory has constituted an essential and powerful *formal paradigm* in Mathematics, of which here we want to give only some of its possible applicative implications concerning certain aspects of the Foundations of Modern Differential Geometry.

Following this last viewpoint, in this paper we want only to start with putting historiographical[9] in evidence the emergence of the Dini's work on implicit function theory[10] in the birth of the concept of a differentiable manifold. Without this fundamental analytic work by Dini, it is very likely that such an important mathematical structure wouldn't have seen light in the realm of the mathematical objects, that it to say, such a structure wouldn't have been explicitly stated in the form which we know nowadays[11]: to see this, it is enough what follows. According to what said in [So, Chapter 3, Section 1.1], we can indeed state that

Although a differentiable manifold is described as an abstract geometrical object, the [explicit] *idea of a manifold derives from rather concrete examples*[12]*, namely nonsingular varieties of* $\mathbb{R}^n$*, defined as follows*

A non-empty set $V \subseteq \mathbb{R}^n$ *is called a p-dimensional differentiable variety of* $\mathbb{R}^n$ *if, to each point* $x \in \mathbb{R}^n$*, there exist a neighborhood U of x in* $\mathbb{R}^n$ *and* $m = n - p$ *differentiable functions* $\Phi_1,\dots,\Phi_m$ *on U, such that*

i) *the matrix* $\left\|\frac{\partial \Phi_i}{\partial x_j}\right\|$, $i=1,\dots,m$, $j=1,\dots,n$, *has rank m at every point* $x \in U$,

ii) $V \cap U = \{x;\ x \in U,\ \Phi_1(x) = \cdots = \Phi_m(x) = 0\}$, *if non-empty.*

Let Ω *be a non-empty open subset of* $\mathbb{R}^n$ *and let* $f_1,\dots,f_m$ *be m functions which are differentiable on* Ω*. Then, the set*

$$S = \{x;\ x \in \Omega,\ f_1(x) = \cdots = f_m(x) = 0,\ \mathrm{rank}\left\|\frac{\partial f_i(x)}{\partial x_j}\right\| = m\},$$

if not empty, is a p-dimensional differentiable variety of $\mathbb{R}^n$ *as just defined above: in fact, we can choose the same functions* $f_1,\dots,f_m$ *at every point* $x \in S$ *and observe that the set*

$$U = \{x;\ x \in \Omega,\ \mathrm{rank}\left\|\frac{\partial f_i(x)}{\partial x_j}\right\| = m\}$$

is open. Then, every $x \in S$ *has a neighborhood U such that* rank $\left\|\frac{\partial f_i(x)}{\partial x_j}\right\| = m$ *for all* $x \in U$ *and*

$$S \cap U = \{x;\ x \in U,\ f_1(x) = \cdots = f_m(x) = 0\}.$$

It now follows, by [Dini's] *implicit function theorem, that, if* Ω *is an open subset of* $\mathbb{R}^n$ *and* $f_1,\dots,f_r$ *are r differentiable functions on* Ω*, then the set*

$$V = \{x;\ x \in \Omega,\ f_1(x) = \cdots = f_r(x) = 0\}$$

is a p-dimensional differentiable variety of $\mathbb{R}^n$ *if every point* $x \in V$ *has a neighborhood in* $\mathbb{R}^n$ *where the matrix* $\left\|\frac{\partial f_i(x)}{\partial x_j}\right\|$ $i = 1,\dots,r$, $j = 1,\dots,n$ *has constant rank*[13] $m < n$ *and m is the same at every* $x \in V$.

[9] In the sense that here shall limit us to mention some works and papers related to this subject, without any other formal in-depth study (that, on the other hand, should also be necessary, even from an historical viewpoint).

[10] Above all, but not only, as concern his contributions to the theory of the systems of implicit functions.

[11] In the sense of what we will be stated in the final Section 9.

[12] Build up, as we will see, just by means of the Dini's implicit function theorem.

[13] In general, *rank* $\left\|\frac{\partial f_i(x)}{\partial x_j}\right\| = m \leq \min\{r, n\}$ for all $x \in V$.

By the local parametrization theorem, which is a consequence of the implicit function theorem, a p-dimensional differentiable variety V of $\mathbb{R}^n$ *has the following two properties*

1. *to every point* $x \in V$*, there exist an open subset* $\Omega \subseteq \mathbb{R}^n$ *and an injective map* $\Phi: \Omega \to \mathbb{R}^n$ *such that* $\Phi(\Omega)$ *is an open neighborhood of x in V,*
2. *there exist an open subset U of* $\mathbb{R}^n$*, with* $\Phi(\Omega) \subseteq U$*, and a differentiable map* $\psi: U \to \mathbb{R}^n$ *such that the map* $\psi \circ \Phi$ *is the identity on* Ω.

Obviously, there are several choices for the set Ω *and the map* Φ*. Every pair* $(\Phi(\Omega), \Phi)$ *is called a coordinate neighborhood of V. The map* Φ *and its inverse, the restriction of* ψ *to* $\Phi(\Omega)$*, determine a homeomorphism of* Ω *onto* $\Phi(\Omega)$*. Finally, the generalization of the above properties 1. and 2. to topological spaces, not necessarily subsets of some Euclidean space, led to the general structure of a differentiable manifold.*

On the other hand, in [Wl, Chapter I, p. 10], after having expounded a modern version of Whitney's theorem, (see [Wl, Chapter I, Section 1, Theorem 1.10], the Author states that

> «*This theorem* [of Whitney] *tells us that all differentiable manifolds (compact and non-compact) can be considered as submanifolds of Euclidean space, such submanifolds having been the motivation for the definition and concept of manifold in general*».

Taking into account possible hypotheses on the origins of mathematical entities briefly recalled in the final Section 9 of this paper, it is clear as all of these considerations about the origins of the concept of the strutture of a differentiable manifold, may be seen as the result of that *objectivation process* there mentioned, applied to the proof procedures of Dini's implicit function theorems. On the other hand, and this is an uncommon case in mathematics, the degree of generality and abstractness of the given mathematical object so obtained is not higher than those of the $\mathbb{R}^n$ as proves the fundamental Whitney's work (see next Section 2), so that we may think the Dini's theory on implicit functions as a theory, in a certain sense, deductively equivalent to the modern abstract theory of differentiable manifold, via the basic works of Hassler Whitney[14] which will be briefly sketched in the next section.

2. The papers of Hassler Whitney

Hassler Whitney[15] (1907-1989) was graduated in Mathematics from Yale University in 1928 and received his doctorate degree from Harvard University in 1932. He gave, amongst other, fundamental contributions to the topology of manifolds with many related papers of which here we recall only those pertinent to our purposes.

In [Wh1], the Author prove some important theorems on the differentiable (and, when possible, even analytic) extension, to $\mathbb{R}^n$, of a given continuous function *f* defined on a bounded or unbounded nonempty closed set A of $\mathbb{R}^n$, starting from the notion of derivative which arise naturally from the consideration of Taylor's formula and reaching to considerable results also through the use of some approximation methods of Real Analysis, amongst to which the Weierstrass' polynomial approximation theorems, mainly to obtain possible analytic extensions[16].

The results obtained in the above mentioned papers have been extensively used in proving the fundamental theorems stated in the main Whitney's paper [Wh5]. In *Introduction* to [Wh2], the Author states as follows

> «*A differentiable manifold is generally defined in one of two ways;* [according to O. Veblen and J.H.C. Whitehead[17]] *as a point set with neighbourhoods homeomorphic with Euclidean space* $\mathbb{R}^n$,

[14] For a modern treatment of the theory of differentiable manifolds strictly related to Dini's and Whitney's theorems (and for other interesting imbedding results), see, for example, [Na].

[15] See, for example, [Ar] for the brief biobibliographic notices herein remembered.

[16] The arguments treated in [Wh1] have been further investigated by the Author in subsequent papers, amongst to which [Wh2], [Wh3] and [Wh4].

[17] These Authors give an abstract definition of a manifold that seems disregards the previous work made by H. Weyl on Riemann surfaces and mentioned in the above Section 1 (see also next Sections 6 and 7 for more information).

coordinates in overlapping neighbourhoods being related by a differentiable transformation, or [according to Dini's implicit function theory] *as a subset of* $\mathbb{R}^n$*, defined near each point by expressing some of the coordinates in terms of the others by differentiable functions*.

The first fundamental theorem is that the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space. In fact, it may be made into an analytic manifold in some $\mathbb{R}^n$*»*.

Subsequently, he improved and extended part of these results, mainly as regard the imbedding results: for instance, his celebrated imbedding theorem was first stated in [Wh5] for compact manifold, and extended to every manifold in [Wh7, II.8], together to further improvements carried out on the dimension of the real imbedding space. As already said, in [Wh5] Whitney use many results of [Wh1] and, especially, of[18] [Wh6], thanks to which, in [Wh5, II.8, Theorem 1], he proves (a first version of) the following, celebrated *(Whitney's) imbedding theorem*

«Any C^r*-manifold of dimension m (with* $r \geq 1$ *finite or infinite) is* C^r*-homeomorphic with an analytic manifold in Euclidean space* $\mathbb{R}^{m+1}$*»*.

It has been possible to achieve many results of [Wh5] on the basis of the fundamental paper [Wh6], and vice versa. In particular, and this is a crucial point for our purpose, in [Wh6, II.6] the Author recall some fundamental lemmas related to the implicit function theory, which are indeed forms of implicit function theorem with uniformity and continuity conditions, as the same Author says at the end of Section II.6. These last results, amongst other things, will be used in proving the fundamental Theorem I of [Wh6, I.3].

Finally, we recall what he says in [Wh6, I.1]

«Let $f_1, ..., f_{n-m}$ *be differentiable functions defined in an open subset of* $\mathbb{R}^n$*. At each point p at which all* f_i *vanish, let the gradient* $\nabla f_1, ..., \nabla f_{n-m}$ *be independent. Then the vanishing of the* f_i *determines a differentiable manifold M of dimension m. Any such a manifold we shall say is in ''regular position'' in* $\mathbb{R}^n$*. Only certain manifolds are in regular position* [...]*. The purpose of the paper is to show that any m-manifold M in regular position*[19] *in* $\mathbb{R}^n$ *may be imbedded into a* $(n-m)$*-parameter family of homeomorphic analytic manifold; these fill out a neighborhood of M in* $\mathbb{R}^n$*. We may extend the above definition as follows: M is in regular position if, roughly, there exist* $n-m$ *continuous vector functions in M which, at each point p of M, are independent and independent of vectors determined by pairs of points of M near p. If M is differentiable, the two definitions agree; the* ∇f_i *are the required vectors. The theorem holds also for this more general class of manifolds»*.

Clearly, again the recalls to the Dini's work on implicit functions are evident. Moreover, the Lemmas 4 and 5 of [Wh6, II] (which, as already said, are equivalent forms of Dini's implicit function theorems) are largely used in proving the fundamental theorem I of [Wh6, I.1], given in [Wh6, II.7] for differentiable manifolds, on the basis of the previous Lemmas 2 and 3 which give sufficient conditions for the various related Jacobians be nonzero with suitable coordinate changes. In [Wh6, II.7], Whitney redefines a differentiable manifold just through these Lemmas 3 and 4, hence according to the Dini's work on implicit functions.

However, the book [VW] is quoted, for instance, in [Wh5, p. 645, Footnote [2)]]. Further, since [VW] is intended to be a companion book of [Ve1], it might be of a some interest, from the historical viewpoint, to remark as the latter, in Section 3 of Chapter V, treat with questions inherent to spaces immersed into an Euclidean environment, mentioning the related previous partial works of H. Maschke, L. Ingold and others; in [Ve1], on the other hand, Weyl is, for instance, quoted in Section 16 of Chapter II and in Section 21 of Chapter III, for which it is very unlikely that, at least Veblen, didn't know the celebrated work [We1].

[18] Although this paper immediately follows [Wh5] in the publication on *The Annals of Mathematics*, nevertheless it is very likely that it was already known, at least in the main contents, to the author before preparing [Wh5], because its results are used and quoted in the latter. Vice versa, results of [Wh5] are also used and cited in [Wh6], so that it is presumable that both papers have been composed almost together, even if a certain precedence should be given to [Wh6] than to [Wh5], because the first was received, by the redaction of *The Annals of Mathematics*, on February 26, 1935 (and in revised form on February 10, 1936), while the second was received on February 10, 1936 (the same submission date of the revised form of [Wh6]) and immediately accepted.

[19] The Author then specifies that any differentiable manifold of $\mathbb{R}^n$ necessarily is in regular position therein.

Furthermore, as it has subsequently been done after the Whitney's work[20], the Theorem 2 of [Wh5, II.8], nowadays called *regular value theorem*, may be re-expressed and simplified through the implicit function theorem, starting from the original Whitney's proof, with few modifications. Further, the implicit function theorems, via the Whitney's work, are at the basis of the important notion of *transversality*, a modern differential topology tool[21] that specifies the already mentioned intuitive concept of ''generic position'' (drawn from Algebraic Geometry) of a manifold[22]. However, on the basis of what established at the end of Section 2, we are mainly interested to the above fundamental Theorem 1 of [Wh5, II.8], because this imbedding theorem, amongst other things, provides a certain logical-syntactic equivalence between the theory of differentiable manifolds according to the explicit definition given by Weyl (that is to say, the modern one) and that implicitly deducible by the above mentioned Dini's work.

It will be necessary to return later on the aspects of Hassler Whitney work related to the use of approximation methods.

3. The Implicit Function Theorem: a brief historical sketch

The most complete work on the history of implicit function theory, which we mainly follow, is[23] [KP]; furthermore, a very interesting short history of implicit function theory, in particular that that goes from Cauchy to Dini, is findable in [HR, Section IV], which we also follow for some respects, together to [KP].

The prolegomena to the idea for the implicit function theorem can be traced both in the works of I. Newton, G.W. Leibniz, J. Bernoulli and L. Euler on Infinitesimal Analysis, and in the works of R. Descartes on algebraic geometry. Later on, in the context of analytic functions, J.L. Lagrange found a theorem that may be seen as a first version of the present-day inverse function theorem[24]. Subsequently, A.L. Cauchy gave, in 1831 (when he was in Turin – see [KP, Chapter 2, Section 2.1] and next Footnote [61]) and in 1852, the first rigorous formulations of the previous results on implicit functions, providing existence and uniqueness results in the case of regularity C^1 and only for one variable (real or complex), assuming that such functions were expressible as power series, a restriction first removed later by Dini[25]. It follow some works of A.A. Briot and C. Bouquet[26] but only for one variable (see [BB]), until the work of Ulisse Dini, in 1876, with his lecture notes [Di1], where he states the Implicit Function Theorem in the rigorous form found in most textbooks today, with basic results of existence, regularity and uniqueness for functions of n (≥ 1) variables and of regularity class C^r ($r \geq 1$), but only in the real scalar field[27]; further, he also gave, for the first time, formulations of this theorem for systems of two or more functions of this type, which constitutes just the main mathematical tool thereupon to the *leitmotiv* of this paper[28].

After the work of Dini, it follow some partial improvements of his results by other authors. We here *in primis* recall the little known work [Sd] of Elcìa Sadun[29], who first remembers the importance of the Dini's work on implicit functions made in [Di1], hence he exposes some further improvements and generalizations to it, together with interesting geometrical applications also relatively to the complex scalar field. Then, we also recall those achieved, as regards the scalar field (from ℝ to ℂ), by C. Jordan in the 1893 second revised

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[20] See, for instance, [Hi, Section 1.1], [Hi, Theorem 3.2] and [Na].

[21] See [Hi].

[22] This last remark about the origins of the notion of *transversality* (from implicit function theorems) is a further example confirming what will be said in Section 9 about the origins of a mathematical entity.

[23] For some aspects of this history, see also [MR]. Moreover, as already said, very interesting historical notes on implicit function theorems are given in Section IV of [HR] where, amongst other things, it is also affirmed which important role have played such a theory in Mathematics: the Authors, indeed, say that the «[…] *classical Implicit Function Theorem is imbedded in many parts of mathematics, including differentiable manifolds and optimization theory* […]» (see [HR, Section III, Remark 7, e)]). On the basic role played by implicit function theory in mathematics, see [KP] and also [HG] and references therein.

[24] See also [Kr, Section 2] for certain limitations to this theorem.

[25] See [Ca, p. 431] and [Hb, Volume I, Chapter V, Section 316].

[26] See [HR, Section IV], and references therein, for a more complete discussion of this, till to the modern treatments.

[27] Nevertheless, in [Di1, Chapter XIII, Section 166], Dini himself states that the complex case is easily deducible from the real one considering the usual decomposition of a complex function in the form $f(x,y)=g(x,y)+ih(x,y)$, with g and h real functions.

[28] Indeed, it is just the work of Dini on systems of implicit functions to constitute one of the main formal tool used by Whitney in his works and, in general, in the theory of differentiable manifold.

[29] This work is almost never cited about history of implicit function theory and hence it would deserve further attention.

edition of [Jo], but these does not seem to go beyond those of Dini. It follow, then, the contributions by E. Lindelöf in 1899, who also considered analytic functions of many variables in the complex scalar field and proved the implicit function theorem by means of power series expansion which, as said above, was a restriction first removed by Dini. Finally, other contributions were given by W.F. Osgood in 1901, É. Goursat in 1903 and E.H. Young in 1909; in particular, Young's Theorem has a weaker assumption than Dini's one, since it requires a lower order differentiability[30].

In conclusion, we may state that the classical form of implicit function theorem is that given by Dini, with results of existence, uniqueness and regularity C^r ($r \geq 1$), in the real scalar field[31], for functions and above all for systems of functions of one or more variables, having, at least, regularity C^1; similar smoothness assumptions have then formed the backbone of most proofs since then of the various forms of implicit function theorems[32]. All of the subsequent works on implicit functions start from this fundamental Dini's work which, however, remains the central pillar of all implicit function theory as concerns the real scalar field and functional systems.

On the other hand, following [KP, Chapter 2, Section 2.1] and as already said at the beginning of this section, the main historical contributions to the history of implicit function theory, on which Krantz and Parks focusing their attention, are those due to Newton, Cauchy and Lagrange, the main mathematicians upon whose works is centered the whole Chapter 2 of their book, affirming then that «[…] *the real variable form of the implicit function theorem was not enunciated and proved until the work of Ulisse Dini (1845-1918) that was first presented at the University of Pisa in the Academic Year 1876-1877. In the remainder of this Chapter, we will describe the contributions of Newton, Lagrange and Cauchy mentioned above. The real-variable approach, going back to Dini, is pervasive throughout the rest of this book*».

From the whole book of these Author, it clearly emerges the fundamental *paradigmatic* role played by implicit function theorem in Mathematics.

3.1 An intersection with the history of the Calculus of Variations

To our historical ends, it is necessary, here, to recall some historical aspects related to the connection between the history of calculus of variations and the Dini's work on implicit function theory, mainly following [Go]. In fact, among other things, the implicit function theorems have played a fundamental role in the calculus of variations, so that many interesting historical notices on these may be traced in the history of the calculus of variations.

At the turn of the 19*th*-Century, a number of mathematicians (among to which O. Veblen, W.F. Osgood and G.A. Bliss), who later became leaders in American mathematical community, went to Germany to study. Among these were W.F. Osgood and E.R. Hedrick, who made available, in the United States of America, the ideas of Weierstrass and Hilbert in the calculus of variations. William Fogg Osgood[33] (1864-1943) was graduated from University of Harvard, going subsequently to study in Göttingen and Erlangen with Felix Klein, and having taught at Harvard from 1890 to 1933, whereas Oswald Veblen[34] (1880-1960) received the major mathematical training from University of Chicago by O. Bolza, H. Maschke and E.H. Moore, and taught at Princeton from 1905 to 1932. Oskar Bolza (1857-1942), amongst other, taught at the University of Chicago from 1892 to 1910 and had Gilbert Ames Bliss (1876-1951) as doctoral student at Chicago. The latter studied in Göttingen during the year 1902-1903 and, subsequently was also a strict collaborator of Veblen at Princeton from 1905 to 1908; in various summer or autumn terms from 1906 to 1911, he gave courses at Wisconsin, Chicago, Princeton and Harvard (see [Gr2]). Next to George D. Birkhoff[35], Veblen was the most influential force in American mathematics and, although not of Birkhoff's status, he had a personality which was much more appealing to Europeans, perhaps due to his past training in Germany.

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[30] See [HR] and references therein.

[31] This might seem one of the main limitations to the Dini's work on implicit functions if one neglected what said in the previous Footnote [27], whereas as concerns the hypothesis of C^1 regularity, see also what will be said in the next Section 4.2.

[32] However, for further remarks on the regularity assumptions made by Dini thereupon his theorems on implicit functions, see also what will be said in the next Section 4.2.

[33] See [Ar] and [Re].

[34] See [Ar] and [Re].

[35] See [Re].

Following what says H.H. Goldstine in [Go, Sections 5.13, 7.2 and 7.3], in the basic work[36] [Os1] the Author, besides to give a clear and succinct survey of Weierstrass' ideas, for proving some field local existence results in the calculus of variations uses an implicit function theorem due to Dini which was given in 1877-1878 Pisa lectures and appeared in the first of Dini's two known volumes *Analisi Infinitesimale* (that is to say, [Di1]). Nevertheless, by the references given in [Os1] and [Os2], Osgood surely known the Dini's work on implicit function theory, besides the original Dini's work [Di1], also from the related chapters devoted to this subject by the well-known treatise on Infinitesimal Calculus, and monographs on its applications, of that time, amongst to which [GP] and [Bl2, Chapter I, Section 1, p. 7, Footnote *] which[37], in turn, refer to the original Dini's treatise [Di1]. Furthermore, among the references quoted at the beginning of [BFHW, Band II, Zweiter Teil, Artikel B.1] and [BFHW, Band II, Erster Teil, Erste Hälfte, Artikel A.1 und A.2], it is mentioned many Italian treatises on Infinitesimal Calculus, amongst which above all[38] [GP] and [Pa], both with German translation: in particular, in [Pa] are quoted the Dini's lessons [Di1] and [Di3], the former as regards his work on implicit functions[39], whereas the first printed version of the Dini's work on implicit function theory is in [GP] (see [HR, Section IV, p. 25]).

Moreover, an important historical survey of implicit function theorem can be found in [BFHW, Band II, Zweiter Teil, B.1.a), Section I.5, p. 19, Footnote [30)] and B.1.b), Section IV.44, pp. 103-105, Footnotes [247), 248), 249)]] and [BFHW, Band II, Erster Teil, Erste Hälfte, A.2.c), Section II.9, pp. 71-73, Footnotes [69), 69a)]], in which[40] it is clear as well-known were these Dini's works. In particular, amongst others, Osgood, Bolza and Bliss (see also next section for this last author) were mathematicians whose works have made[41] large use of the Dini's implicit function theorems. For instance, in[42] [Bo2, German Translation, Vierte Kapitel, Abschnitt 22, a), p. 159, Footnote [1)]], Bolza explicitly mention, for first, the lessons [Di1] in a chapter[34] devoted to recall the main principles on real function theory in view of their applications to the calculus of variations.

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[36] See also the treatise [Os2] which, as an enlarged and revised edition of the first Osgood's Article published in [BWFH, Band II, Zweiter Teil, Artikel B.1], was one of the main work on function theory of the time (see [Ko]).

[37] See also [Jo, Tome I, Chapitre VII, Numéros 90-95], where, yet, there is no any explicit reference to the Dini's work.

[38] As regards [GP], see also what said in the immediately following footnote.

[39] See [Pa, Parte I, Prefazione, p. IX] in which the Author mention all of the references used in drawing this his first volume, among to which the Dini's one [Di1] (there called *Lezioni di Calcolo – litografate*) and the same [GP]. Instead, the Italian edition of [GP] largely mention both [Di1] and [Di3]: precisely, in the introductory Annotations No. 1 to No. 200 (see [GP, pp. VII-XXXII], in which each number refers to the corresponding section of the main text, albeit those total, there present, be 216), where, amongst other things, it is also contained some interesting historical-bibliographical notes, the lessons [Di1] are cited in the Annotations No. 103 and in No. 110 to No. 120, these last being just those including the corresponding sections (from No. 110 to No. 117, included among the *Abschnitten* 5-8 of the *Viertes Kapitel* of the German edition) treating the implicit function theory. Nevertheless, and this is quite strange, the German translation of [GP], contain neither the above mentioned Annotations of the Italian edition nor any other reference, referring to the Italian edition for any other thing; moreover, some other Italian treatise on Infinitesimal Calculus, like, for example, those of E. Cesàro ([Ce]) and G. Vivanti ([Vi]), widely quote [Di3] but not [Di1]. All this, together to the lack of an official in print publication of the lessons [Di1], maybe contributed to the misunderstanding of the Dini's work here discussed.

[40] The first reference is related to the Chapter, of the Volume II.2, entitled *B.1 Allgemeine Theorie der analytischen Funktionen a) einer und b) mehrere komplexen Größes*, which belongs to the Article written by F.W. Osgood with heading title *B. Analysis der Komplexen Größen*, whereas the second reference is relative to the Chapter, of the Volume II.3.1, entitled *9. Funktionen von mehreren Variabein*, which belongs to the Section *C. Differentialrechnung* of the Article written by Aurel E. Voss (1845-1931) with heading title *A.2 Differential- und Integralrechnung*.

[41] In a certain sense, almost paradoxically, it might be possible to say that they have done ''implicitly'' use of the Dini's work on implicit functions. It is as if, almost for a historical accident, this Dini's work had undergone that sneering fate inflicted by its name itself: *rerum sunt consequentia nominum* rather than the Justinian I *nomina sunt consequentia rerum* (*Corpus Iuris Civilis*, *Institutiones Iustiniani*, Libro II: *Res*, VII-3), if one wishes to use, in this case, an adapted inverse form of a well-known Latin rhetorical phrase. But, yet, one of main aim of the *historical-scientific eponymy* (in this case, related to the History of Mathematics), is just that trying to remedy to this sort of unpleasant historical drawbacks (see also *Remark 2* of Section 9).

[42] The German translation of the initial American version, is an enlarged and revised edition (of thirteen chapters and about 700 pages) of this last (of seven chapters and about 270 pages): for instance, the Dini's lessons [Di1], in correspondence to his work on implicit function theory, is quoted (together with others, among to which that of Genocchi and Peano [GP]) in the *Viertes Kapitel*, *Hilfssätze über reelle Funktionen reeller Variabeln*, at the *Abschnitt 22*, *Ein Satz über eindeutige Abbildung, und seine Anwendungen*, which is not included in the American edition where, in the Preface, the Author quotes only the Dini's work [Di3] in addition to [BFHW], [GP], [Jo] and others.

On the other hand, seen that Whitney was student at Harvard in the1930s, it is very likely that he however had attended at the various lectures (mainly, but not only, on the calculus of variations) of Veblen as professor in that University or in the various summer or autumn terms, hence known his works, whence, in particular, also the above mentioned Dini's ones[43] since, without doubts, his lecture notes [Bl2] were largely known among the American mathematical community, as results to be from what stated in [Gr2].

Furthermore, the treatise of E.W. Hobson[44] (which was one of the main English language treatises since its first 1907 edition) deals with the Dini's work on implicit function theorems, with possible their extensions[45], after the Cauchy's work. The Dini's work on implicit functions is also cited in the well-known paper [Yo1] and in the textbook [Yo2, Chapter XI, Section 38, p. 48]. In any case, therefore, the whole work of Ulisse Dini was surely known and widely appreciated, at least in the American mathematical community, as witnessed by the obituary of W.B. Ford[46], read before the American Mathematical Society in September 3, 1919, in which it is clearly recalled the main mathematical contributions of Dini, among to which those just related to implicit functions.

In this regards, at first, Ford states that

> *«If we inquire in a more detailed sense what his relations to mathematics actually were and what his special achievements, it may be noted in the first place that his earliest researches lay in the field of infinitesimal geometry; more specifically, in the determination of the form and properties of certain partial differential equations which arise in the theory of applicable surfaces. His work in this connection, though extending over no more than six years (1864-1870), gave rise to some eighteen memoirs dealing chiefly with general problems in the theory of curvature and geodesies, some of which had been proposed earlier by Beltrami. At this early period of his life, however, Dini had not yet begun the researches for which he is to be regarded as famous nor had he in fact even entered seriously into that broad field, namely pure analysis, in whose development he was destined soon to play an active part. The transition of his interest and labours to this latter field took place about 1870. At this comparatively early date it will be recalled that the newer and more rigorous analysis (the so-called ''modern analysis'' of today) was but little known to the world at large, the spirit of its methods being virtually confined to the limited school of pupils immediately surrounding Weierstrass in Germany. Nevertheless, once Dini had turned his efforts in this direction, he appears to have reached within a remarkably short time a full appreciation of these newer ideas and methods. In fact, he straightway acquired such a critical insight into their significance and developed such ability and confidence in their use that he was soon independently at work carrying out for himself their manifold consequences, especially their bearing upon those concepts which lie at the foundation of analysis. Thus, by the year 1877, or seven years from the time he began he published the treatise, since famous, entitled ''Foundations for the Theory of Functions of Real Variables'' (Fondamenti per la teorica delle funzioni di variabili reali –* [Di3]*). Much of what Dini here sets forth concerning such topics as continuous and discontinuous functions, the derivative and the conditions for its existence, series, definite integrals, the properties of the incremental ratio, etc., was entirely original with himself and has since come to be regarded everywhere as basal in the real variable theory. The book has, in fact, served as a model the world over and even at this date, which is more than forty years since its publication, it still affords one of the best available expositions of the basal concepts of analysis as regarded from the standpoint of modern rigor, evidence of which may be found, for example, in the fact that as late as 1902 an authorized translation of the work was published in German».*

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[43] On the other hand, from an online examination of the related databases of the various American university libraries here mentioned (amongst which Princeton, Harvard, Yale and Chicago), it turns out to be that all of the Dini's works are present therein, just including [Di1], [Di2] (above all, these last) and [Di3], but it has not been possible go back to their effective acquisition date; nevertheless, since the autographed lessons [Di1] went ran out in 1905-1906 (according to what Dini himself says in [Di2, Volume I, Prefazione, p. V]), it is very likely that such works were acquired in a period ranging about their publication date. In any case, the lessons [Di2] were present almost everywhere.

[44] See [Hb, Vol. I, Chapter V, Section No. 316 to No. 319], where the work [Di1] is explicitly mentioned.

[45] Such extensions were made possible thanks to some preliminary lemmas due to O. Bolza (see [Bo2]) and Hobson himself (see [Hb, Vol. I, Chapter V, Section 317] and references therein).

[46] See [Fo].

Then, Ford proceeds saying as follows

«With the assurance once gained that he was working upon well-grounded principles and definitions [of Infinitesimal Calculus[47]], *Dini next proceeded to apply them in an extended and detailed sense. Thus, during 1877-78 he reworked and treated in his lectures a wide variety of topics taken from the usual course in higher analysis. In particular, he here gave for the first time a rigorous treatment of the general theory of implicit functions. His numerous researches at this period were left unpublished, however, being preserved only in lithograph form* (*Analisi Infinitesimale* – [Di1]). *Not until the later years of his life did he undertake the considerable task of arranging the whole for publication, but it may now be found, together with much supplementary material, in his four large volumes published as late as 1915 entitled Lessons in Infinitesimal Analysis (Lezioni di Analisi Infinitesimale* – [Di2]*)».*

However, Dini's theorems on implicit functions have also played a fundamental role in proving many important results in the calculus of variations through suitable reformulations and/or extensions of them as, for instance, those given by W.F. Osgood (see [Os1]) and O. Bolza (see [Bo1], [Bo2] and [Bl2, Chapter I, Section 4]), some of which are briefly recalled in [Go], so that it has been almost forced to recall, although sketchily, the main points of the history of calculus of variations properly related to this subject. From this, it followed as these Dini's works on implicit function theorems surely were already known to the German mathematical community between the 19*th*- and 20*th*-Century as witnessed, for instance, by the well-known fundamental *Encyklopädie der Mathematischen Wissenschaften* (see [BFHW] as regard the volumes devoted to Analysis) and reference quoted therein[48].

Moreover, as we will see in the next Section 3.2, the implicit function theorems have been one of the main research argument of the first decades of the 20*th*-Century, among many mathematicians of the American mathematical community of the time, as witnessed by the numerous publications made on this, so that these last discussions about the knowledge of implicit function theory must hold also as concerns the general framework of the American mathematical research of that time.

3.2 On a work of Gilbert A. Bliss

Although it might seem little known from an historiographical viewpoint, the 1909 Gilbert Ames Bliss paper[49] [Bl2] – which plays, as we shall see, a not negligible role for our historical purposes – was instead well-known among the mathematical community of the time[50]. It contain the lectures delivered by the Author before members of the American Mathematical Society, in connection with the summer meeting held at the Princeton University in September 15 to 17, 1909, and published in revised and enlarged form in 1913, within the series of *Colloquia* inaugurated by the American Mathematical Society since 1896.

In the *Introduction*, the Author states as follows

«The existence theorems to which these lectures are devoted have been the subject of a long sequence of investigations extending from the time of Cauchy to the present day, and have found application at the basis of a variety of mathematical theories including, as perhaps of especial importance, the theory of algebraic functions and the calculus of variations. If a single solution $(a;b) = (a_1,\ldots, a_m; b_1,\ldots, b_n)$ *of a set of equations*

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[47] In drawing up the *Fondamenti* [Di3], which were written before [Di1], though in a short time one from the other.

[48] As regard the volumes of this monumental and basic work, relatively to the Mathematical Analysis (*Zweiter Band in Drei Teilen*, precisely II.1.1, II.1.2, II.2, II.3.1 and II.3.2), in the bibliographies related to each Article of it, the Italian works and papers on this subject (as, for instance, those of F. Casorati, G. Vivanti, G. Peano, U. Dini, V. Volterra, L. Tonelli, E. Pascal, S. Pincherle, G. Morera, E. Cesàro and others) are widely quoted, together their works. On the other hand, Dini himself, in [Di2, Volume I, Parte I[a], Prefazione, p. 5], remembers as his autographed lessons [Di1] have been more and more times recalled by the various Authors of this *Encyklopädie* ([BFHW]), in spite of such lessons were not in print published edition (which would have facilitated their circulation in literature).

[49] Evidently, the drawing up of this paper has been surely made before September 15-17, 1909 (that are the dates in which was held this Princeton Colloquium), hence, very likely, when Bliss was research joined with Veblen and others, at Princeton (for what said in Section 3.1).

[50] For instance, it was cited by many remarkable and well-known works and papers of that time (as, for instance, by [Cm], [Et], [La], [Ln], etc, many of which read before the meetings of the American Mathematical Society), so that the contents of [Bl2], together to the references therein quoted, were well-known.

(1) $f_\alpha(x_1, x_2, \ldots, x_m; y_1, \ldots, y_n) = 0 \qquad (\alpha = 1, \ldots, n)$

is known, then in a neighborhood of (a;b) there is one and only one other solution corresponding to each set of values z in a properly chosen neighborhood of the values a, and in the totality of solutions (x;y) so defined the variables y are single-valued and continuous functions of the x's. If a set of initial constants $(\xi;\ \eta_1, \eta_2, \ldots, \eta_n)$ *is given, then in a neighborhood of these values there is one and but one continuous arc*

$$y = y_\alpha(x) \qquad (\alpha = 1, \ldots, n)$$

satisfying the differential equations

$$\frac{dy_\alpha}{dx} = g_\alpha\ (x;\ y_1, \ldots, y_n) \qquad (\alpha = 1, \ldots, n)$$

and passing through the initial values η *when* $x = \xi$*. The formulation and first satisfactory proofs of these theorems, at least for the case where only two variables x, y are involved, seem to be ascribed with unanimity to Cauchy. For the implicit functions his proof rested upon the assumption that the function should be expressible by means of a power series, and the solution he sought was also so expressible, a restriction which was later removed with remarkable insight by Dini. For a differential equation, on the other hand, Cauchy assumed only the continuity of the function g and its first derivative for y, and his method of proof, with the well-known alteration due to Lipschitz, retains today recognized advantages over those of later writers Lipschitz, retains to-day recognized advantages over those of later writers.*

In the following pages (§§ 1-16) the two theorems stated above are proved with such alterations in the usual methods as seemed desirable or advantageous in the present connection. The proof given for the fundamental theorem of implicit functions is applicable when the independent variables x are replaced by a variable p which has a range of much more general type than a set of points in an m-dimensional z-space[51]*. It is not necessary always to know an initial solution in order that others may be found. In the treatment of Kepler's equation, for example, which defines the eccentric anomaly of a planet moving in an elliptical orbit in terms of the observed mean anomaly, one starts with an approximate solution only and determines an exact solution by means of a convergent succession of approximations. This procedure is closely allied to a method of approximation due to Goursat (§ 3), suggested apparently by Picard's treatment of the existence theorem for differential equations. One of the principal purposes of the paragraphs which follow, however, is to free the existence theorems as far as possible from the often inconvenient restriction which is implied by the words ''in a neighborhood of'', or which is so aptly expressed in German by the phrase ''im Kleinen''. It is evident from very simple examples that the totality of solutions (x;y) associated continuously with a given initial solution of a system of equations f = 0 of the form described above, cannot in general have the property that the variables y are everywhere single-valued functions of the variables x, and the result of attempting, perhaps unconsciously, to preserve the single-valued character of the solutions has been the restriction of the region to which the existence theorems apply. In order to avoid this difficulty and to characterize to some extent the totality of solutions associated continuously with a given initial one in a region specified in advance, the writer has introduced (§ 5) the notion of a particular kind of point set called a* sheet *of points. In a suitably chosen neighborhood of a point (a;b) of the sheet there corresponds to every set of values x sufficiently near to the values a, exactly one point (x;y) of the sheet, and the single-valued functions y so determined are continuous and have continuous first derivatives. This condition does not at all imply that there are no other points of the sheet outside the specified neighborhood of the point (a;b) and having a projection x near to a. With the help of the notion of a sheet of points, it can be concluded that with any initial solution (a;b) of the*

[51] The notion of a 'general range' has been elucidated by E.H. Moore in *The New Haven Mathematical Colloquium*, page 4, the special cases which he particularly considers being enumerated on page 13. An application of the method of § 1 of these lectures when the range of *p* is a set of continuous curves, has been made by C.A. Fischer, *A generalization of Volterra's derivative of a function of a line,* PhD Dissertation, Chicago (1912).

equations $f = 0$ there is associated a unique sheet S of solutions whose only boundary points are so-called exceptional points where the functions f either actually fail, or else are not assumed, to have the continuity and other properties which are demanded in the proof of the well-known theorem for the existence of solutions in a neighborhood of an initial one. It is important oftentimes to know whether or not a sheet of solutions is actually single-valued throughout its entire extent, and a criterion sufficient to ensure this property has also been derived (§ 7).

On the basis of these results some important theorems concerning the transformation of plane regions into regions of another plane by means of equations of the form

$$(2) \qquad x_1 = \psi_1 (y_1,y_2), \quad x_2 = \psi_2 (y_1,y_2)$$

as in the theory of conformal transformation, have been deduced (§ 8). If the functions ψ have suitable continuity properties and a non-vanishing functional determinant in the interior of a simply closed regular curve B in the y-plane, and if B is transformed into a simply closed regular curve A of the x-plane, then the equations define a one-to-one correspondence between the interiors of A and B, and the inverse functions so defined have continuity properties similar to those of ψ_1 and ψ_2. This is but a sample of the theorems which may be stated. Others are also given (§ 8) which apply to the transformation of regions not necessarily finite, and to systems containing more than two equations.

The theory of the singularities of implicit functions is of considerable difficulty and has been but incompletely developed. For a transformation of the form above in which the functions ψ_1, ψ_2 are analytic, the singular point to be studied, at which the functional determinant $D = \partial(\psi_1,\psi_2)/\partial(y_1,y_2)$ vanishes, as well as its image in the x-plane, may both without loss of generality be supposed at the origin. The most general case under these circumstances is that for which the determinant D does not vanish identically and the equations $\psi_1 = 0$, $\psi_2 = 0$ have no real solutions in common near the origin except the values $y_1 = y_2 = 0$ themselves. It is found that the branches of the curve $D = 0$ bound off with a suitably chosen circle about the origin a number of triangular regions. Each of these regions is transformed in a one-to-one way into a sort of Riemann surface on the z-plane which winds about the origin and is bounded by the image of the boundary of the triangular region (see § 11, Fig. 6). If the signs of D in two adjacent triangular regions are opposite, then their images overlap along the common boundary; otherwise they adjoin without overlapping. At any point of one of the Riemann surfaces the inverse functions defined by the transformation are continuous and in the interior of the surface they have everywhere continuous derivatives. These results are obtained by means of applications of the theorem described above for the transformation of the interior of a simply closed curve B; and the same method of procedure would undoubtedly be of service when the curves $\psi_1 = 0$, $\psi_2 = 0$ have real branches through the origin in common, which must occur whenever they have common points in every neighborhood of the values $y_1 = y_2 = 0$. The case where the determinant D vanishes identically is also considered (§ 12).

For the singularities of implicit functions defined by a system of equations $f = 0$ there is a generalization of the preparation theorem of Weierstrass (§ 9) suggested to the writer by some remarks in the introduction of Poincare's Thesis, and by a study of the elimination theory of Kronecker for algebraic equations. The theorem is presented here (§ 13) for two equations and two variables y_1, y_2 in the form originally given at the time of the Princeton Colloquium, but the method of proof is similar to that of a later paper and applies with suitable modifications to a system containing more equations and independent variables. These results can not by any means be said to afford a complete characterization of the singularities of implicit functions, but it is hoped that they may be useful in paving the way for researches of a more comprehensive character.

The writer published some years ago a paper[52] *concerning the extensibility of the solutions of a system of differential equations, of the form specified above, from boundary to boundary of a finite closed region R in which the functions g_α are supposed to have suitable continuity properties. In the last chapter of these lectures the character of the region has been generalized so that no restrictions as to its finiteness or closure are made, and it is shown that the approximations of Cauchy converge to a solution over an interval in the interior of which the limiting curve is continuous and interior to R, while at the ends of the interval the only limit points of the curve are at infinity or else are on the*

[52] G.A. Bliss, The solutions of differential equations of the first order as functions of their initial values, *Annals of Mathematics*, **6** (1904) pp. 49-68.

boundary of the region. The solutions so defined are continuous and differentiable with respect to their initial values, a property which once proved is of great service in many of the applications of the existence theorems. One situation in which these results have an important bearing is related to a partial differential equation of the first order

$$F(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0.$$

When this equation is analytic, any analytic curve C, which is not a so-called integral curve, defines uniquely an analytic surface containing the curve and satisfying the differential equation. The uniqueness in this case is a consequence, in the first place, of the fact that an analytic surface is completely determined when an initial series defining its values in a limited region is given, and, in the second place, of the theorem that at a given point and normal of the initial curve C satisfying the differential equation there is but one series defining an integral surface including the points of C and having the given initial normal. It is not self evident in what sense a solution of a non-analytic equation is uniquely determined by an initial curve, as may be seen by very simple examples. An initial curve which is not an integral curve will in general have associated with it, however, a strip of normals which satisfy the partial differential equation, and whose elements as initial values determine a one-parameter family of characteristic strips simply covering a region R_{xy} of the xy-plane about the projection of the initial curve C. There is one and but one integral surface of the differential equation with a continuously turning tangent plane and continuous curvature, which is defined at every point of the region R_{xy} and contains the initial curve C and its strip of normals (§ 19)».

Subsequently, in [Bl2, Chapter I, Section 1], the Author says as follows

«The fundamental theorem of the implicit function theory states the existence of a set of functions

$$y_\alpha = y_\alpha (x_1, \ldots, x_m) \qquad (\alpha = 1, \ldots, n)$$

which satisfy a system of equations of the form

$$(1) \qquad f_\alpha (x_1, \ldots, x_m; y_1, \ldots, y_n) = 0 \qquad (\alpha = 1, \ldots, n)$$

in a neighborhood of a given initial solution (a;b). Dini's method[53] *for the case in which the functions f are only assumed to be continuous and to have continuous first derivatives, is to show the existence of a solution of a single equation, and then to extend his result by mathematical induction to a system of the form given above, a plan which has been followed, with only slight alterations and improvements in form, by most writers on the theory of functions of a real variable. In a more recent paper*[54] *of Goursat has applied a method of successive approximations which enabled him to do away with the assumption of the existence of the derivatives of the functions f with respect to the independent variables x. One can hardly be dissatisfied with either of these methods of attack. It is true that when the theorem is stated as precisely as in the following paragraphs, the determination of the neighborhoods at the stage when the induction must be made is rather inelegant, but the difficulties encountered are not serious. The introduction of successive approximations is an interesting step, though it does not simplify the situation and indeed does not add generality with regard to the assumptions on the functions f. The method of Dini can in fact, by only a slight modification, be made to apply to cases where the functions do not have derivatives with respect to the variables x. The proof which is given in the following paragraphs seems to have advantages in the matter of simplicity over either of the others. It applies equally well, without induction, to one or a system of equations, and requires only the initial assumptions which Goursat mentions in his paper».*

[53] ''U. Dini, *Lezioni di Analisi infinitesimale*, vol. 1, chap. 13. For historical remarks, see Osgood, Encyclopädie der Mathematischen Wissenschaften, II, B 1, § 44 and footnote 30'': this is the original footnote * of [Bl2, Chapter I, Section 1, p. 7], and we think that Bliss did refer to [Di2] instead of [Di1] when cites Dini (see also [BWFH] and the above Section 3.1, for exact references); in this regards, see also what has been said in the previous footnote [43].

[54] É. Goursat, Sur la théorie des fonctions implicites, *Bulletin de la Société mathématique de France*, **31** (1903) pp. 184-192.

From what said above by Bliss, it is evident the primary role played by the implicit function theorems, from Cauchy to Dini, in the proof of the various existence theorems given by the Author in his Lectures; above all, the Dini's work on implicit function theory, generalizing and extending the previous results achieved by his predecessors (amongst to which, A.L. Cauchy), has played a very crucial role in proving these and, in general, in proving many existence theorem of differential equation theory with related geometrical applications[55]. In particular, for trying to extend the univocity of the solutions of a system of equations of the type *(1)* from a neighborhood to a wider region, the Author introduces, on the one hand, the new notion of a *sheet* of points in relation to ordinary points of implicit functions, studying also the related coordinate changes *(2)*, whereas, on the other hand, he introduces a construction quite similar to that of a Riemann surface but in relation to singular points of implicit functions[56], studying the related geometrical properties under coordinate transformations of the type *(2)*, and also through some local approximation theorems (as the Weierstrass' preparation theorem) as regard some types of algebraic functions[57].

However, this work of Bliss, of which here we have only briefly recalled those points closely related to our purposes, would need for further historiographical recognitions also as regard its possible historical connections with other aspects of the History of Mathematics (as, for example, those concerning the history of Riemann surfaces delineated in [We1]). Nevertheless, as said in the above Section 3.1, Bliss was a strict research collaborator of Veblen at Princeton, just in the period in which he wrote and presented these Princeton Colloquium Lectures, so that it is surely possible to affirm as the arguments and subjects treated by them were well-known within the American mathematical community (and not only this) of that time, if nothing else for having delivered these lecture before members of the American Mathematical Society. These last discussions about Bliss work presentation are also valid as concerns the paper [Bl1] (from which shall derive [Bl2]), in which is quoted too the above Dini's work on implicit functions since 1911, because this communication was read before the American Mathematical Society on October 28, 1911.

In conclusion, from all what has been said so far, it emerges as the Dini's work on implicit function theorem was well-known (albeit not always explicitly mentioned) in the international mathematical community, in particular in the American Mathematical Society as witnessed by the many research papers published[58] just on this argument, in the first decades of the 20*th*-Century, since at least the Dini's printed lessons [Di2] were surely well-known and materially available[59].

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[55] Besides [Bl1] and [Bl2], many other interesting geometrical applications of implicit functions (which might also have a some historical interest as regard the argument of this paper), are considered, for instance, in [Cm], [Dn] and [Lo].

[56] Just these last considerations should be retaken in another place in relation to the work of Hermann Weyl on Riemann surfaces and its possible relationships with implicit function theory.

[57] Besides, the Author give, in [Bl] and [Bl2, Chapter II, Sections 8, 9], a proof of the Weierstrass preparation theorem (published in 1886 but known in lectures since 1860 – see [HR, p. 28]) which is independent by the theory of complex variable just by means of the Dini's theorems. In this regard, see also [Bl1] and [Cl], in which the implicit function theorems are largely used as tools in proving generalized versions of this Weierstrass' result, and vice versa (for some respects), so that there exist not negligible relationships among these theorems (see also [KP, Chapter 5] and Section 8 of this paper). For instance, in the papers [Cm] and [Dn] are showed basic formal liaisons between certain forms of Weiertrassian implicit function theorem (today known as *Weierstrass Preparation Theorem*, and exposed in the Chapter *Einige auf die Theorie der analytischen Funktionen mehrerer Veränderlichen beziehende Sätze* of K. Weierstarss, *Abhandlungen aus der Funktionenlehre*, Verlag Von Julius Springer, Berlin, 1886) and those related to the Dini's theory, obtaining interesting comparison results having also useful geometrical implications, above all taking into account the Bliss work [Bl1]. Above all in [Cm] are mentioned interesting references (also the Cauchy's *Turin Memoir*) which refer both to the implicit function theorems of the Theory of Real Functions and the various forms of Weierstrass preparation theorems, which, besides, would deserve further historical attentions.

[58] See [Bl], [Bl1], [Bl2], [Cm], [Dn], [Et], [Gr1], [Ht], [HG], [Lm], [Ln], [Mm] and [Mr], almost all in the main publications of the American Mathematical Society (that is to say, the *Transactions* of the AMS and the *Bulletin* of the AMS) as communications first read before the various meetings of this celebrated institution (this increasing their knowledge within the mathematical community).

[59] See Footnote [43].

4. On the work of Ulisse Dini[60]

Ulisse Dini was born in Pisa on November 14 (α), 1845 and died in his hometown on October 28, 1918 (Ω). He was a student both of Ottavio Fabrizio Mossotti (1791-1863) and Enrico Betti (1823-1892). The first was a physicist and a mathematician, deeply influenced by the works of J.L. Lagrange[61], that taught Geodesy at the University of Pisa in the days of Dini student. The second was professor of Mathematical Physics at the University of Pisa and supervisor of the Dini's thesis[62]. In 1864, Dini started to publish with a paper on an argument of his graduation thesis suggested by Betti; this first paper was followed by many other works on Differential Geometry and Geodesy. In that period, Dini was into a scientific friendship with E. Beltrami which taken the Geodesy Chair of the late Mossotti; at the same time, Dini was into direct touch with G.F.B. Riemann, at the time visiting professor in Pisa under Betti's interests, in the period 1862-1865.

In 1865, Dini spent one year of postgraduate research under the supervision of J. Bertrand and C. Hermite in Paris[63]: this was a period of high mathematical activity for him and many publications came out of the research he undertook during Parisian period of study, where he further developed his graduated thesis arguments with more researches above all in Differential Geometry and Geodesy, but also in Algebra and Analysis. In 1866, Dini came back to Pisa, where he started his teaching career at the Royal University of Pisa, as professor of Geodesy and Advanced Analysis[64]. Nearly seventy, notwithstanding many practical difficulties and political troubles, also due to his profound nationality sense which seen him actively involved in the historical events of the time, for his teaching assignments, Dini settled important and original works on a rigorous revision of the mathematical foundations of Analysis, with his celebrated *Lezioni di Analisi Infinitesimale* (see [Di1] and [Di2]) and, above all, with the *Fondamenti per la teorica delle funzioni di variabili reali* (see [Di3]): as we shall say, the latter were written before the former, which were drawn up also taking into account them. As recalled in [Bi1] and [DBI], according to Luigi Bianchi (who was one of the best students of Dini, in turn himself world-renowned mathematician), these last Dini's works were amongst the early devoted to a rigorous setting of Infinitesimal Calculus: in particular, the in print published

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[60] For the various biographical notes recalled in this Section and hereafter, see [Bi], [Fo] and [DBI, Vol. XL, pp. 162-165]. See also [Sn] and [Tr]. Furthermore, Dini biobibliographical notes are also given by J.C. Poggendorff in [BLH] and by C.C. Gillespie in [Gl, Vol. IV, p. 102], to further testify the knowledge of Dini's work abroad since the beginnings of the 20*th*-Century.

[61] On the other hand, O.F. Mossotti was a close colleague and collaborator of G.A.A. Plana at Torino, and the latter was a pupil of J.L. Lagrange at the Parisian École Polytechnique. Moreover, seen his fundamental work on Mathematical Analysis (also as concern the implicit functions), it also might be of a certain historical interest, in this particular context, to recall as A.L. Cauchy was also professor of Mathematical Physics at the Royal University of Turin from 1831 to 1833 (see [Te]), in the same period in which taught Plana, where he held the Chair of *Sublime Physics* (this having been the name of the time given to Mathematical Physics) that was of A. Avogadro since 1820. Besides, Cauchy gave the first rigorous formulation of existence results for implicit functions just in the Turin period (see [KP, Chapter 2, Section 2.1] and also what said in Section 3), in the paper *Résumé d'un Mémoire sur la mécanique cèleste et sur un nouveau calcul appelé calcul des limites* (also known as *Turin Memoir*), presented in a meeting of the *Accademia delle Scienze di Torino* on October 11, 1831, starting from the previous work of his former teacher Lagrange on this subject, and whose results will be retaken in the subsequent paper *Mémoir sur l'application du Calcul infinitésimal à la détermination des fonctions implicites* presented at *L'Académie des Sciences de Pàris* on February 23, 1852 (besides, this celebrated Cauchy memory is also mentioned in [Os2, II.1, Zweites Kapitel, Abschnitt 1, Anmerkung 1), p. 83] as *Turiner Abhandlimg vom 11 Oktober 1831*). Hence, without doubt, there was a scientific (even human, amongst some of them, age permitting) reciprocal knowledge, at least as concern the works developed by these Authors, and namely among Dini, Mossotti (via Plana, with), Lagrange and Cauchy.

[62] There are really few doubts on the educational importance played, among others, by the work and teaching of Mossotti and Betti on the formation of Dini's scientific outlook. Further, a considerable educational and intuitive role is also played by the university curricula of that time, especially that of Geodesy teaching for developing geometrical intuition and visual and imaginative skills (which are among the main functions through which the human mind thinks). Nevertheless, these last considerations cannot be correlated, in a rigorous and direct manner, to the subject treated in this paper (at least, from what has been said so far) as regard the origins of the notion of a differentiable manifold.

[63] See [Lo].

[64] It wouldn't be historically irrelevant to know the exact subject program of this particular matter of the time, which, nowadays, it might be considered corresponding to a teaching of name 'Mathematical Analysis and Geometry'. Indeed, following what said by Enea Bortolotti at the point 9. of his *Introduzione ai lavori geometrici di Ulisse Dini* (see [Di4, Volume I, pp. 195-209]), the Geodesy teaching hold by Dini was mainly centred on the first elements of Differential Geometry of Surfaces, hence on the geometrical applications of differential calculus.

edition of the *Lezioni di Analisi Infinitesimale* ([Di2]) were widely known in the whole international mathematical community, constituting one of the main references on this matter[65]. In all of these works, it have been inserted many original unpublished results and contributions by the Author: of fundamental importance were the new results achieved in the differential calculus by Dini and, amongst these, in particular the (so-called Dini's) theory of implicit functions (in [Di1] and [Di2]) as regards his *Lezioni*, whereas another paper would be necessary for a minimal historical recognition of the Dini's work regarding the *Fondamenti*. Following [DBI], the *Lezioni* – where, for the first time, was exposed, inter alia, the rigorous theory of implicit functions – were known in Italy and abroad, and provided to the diffusion of the modern analysis.

For our purposes, in the following subsections we briefly expose some of the main historical accounts on such celebrated Dini's works, which constitute milestones of the international mathematical literature concerning the Infinitesimal Calculus and its Geometrical Applications.

4.1 On the autographed Dini's lessons ([Di1])

We are properly interested both to the first autographed lessons *Analisi Infinitesimale* ([Di1]) and to the in print published version of them [Di2]. As regards the former, some first valuable historical notes are furthermore findable in the *Preface* to [Di2, Volume I] by the Author himself, these last lessons being reviewed in the next section.

Their exact original title is *Analisi Infinitesimale, Lezioni dettate nella Regia Università di Pisa dal Prof. Cav. Ulisse Dini, Anno Accademico 1877-78*. These are the lessons given by the Author in the Academic Year[66] 1876-1877 at the Royal University of Pisa, of which there exist two contemporaneous autographed (or lithographed) editions: the edition published by the printing-works *Bertini*, and that published by the printing-works *Gozani*, both editions being in an unique volume, divided into two parts: the first devoted to the Differential Calculus (with Chapters I to XXXII), the second devoted to the Integral Calculus (with Chapters I to XXIII). The theory of implicit functions according to Dini, is expounded, for the first time, in the Chapter (of [Di1]), of *Parte I^a^*, entitled

XIII. Derivate e differenziali dei vari ordini di funzioni implicite di una o più variabili indipendenti,

which comprise the Sections No. 138 to[67] No. 166, whereas in the Chapter (of [Di1]) of *Parte I^a^*, entitled

XV. Cangiamento delle variabili indipendenti,

Dini deal with certain first forms (even if incomplete) of the so-called *inverse function theorem*[68]. In the Sections No. 138 to No. 166, Dini introduce the notions of an implicit function of two or more variables, discussing their uniqueness, existence and C^1-regularity properties first for a single implicit function with their derivatives of order greater or equal to the first, then extending these results to systems of implicit functions, together to some interesting geometrical applications of these results. Finally, in the following Chapters (of [Di1]) of *Parte I^a^*, precisely the Chapters XVII to XXXII, Dini expose the main geometrical and analytical[69] applications of some theorems of the previous Chapters XIII to XV of *Parte I^a^*: just to give an example, it is noteworthy what Dini states in Sections from No. 375 to No. 377 of Chapter XXX (and also in

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65 In this regards, Giovanni Sansone, in the Preface to [Di4, Vol. I], said that *«the 'Lezioni di Analisi Infinitesimale'* ([Di2]) *were a work universally known which, since their first lithographed editions, had great diffusion both in Italy and abroad»*.

66 As recalled in the subtitle to the main one, of the title page of [Di2, Volume I, Parte I^a^].

67 In particular, in Section No. 140, he says that his treatment is limited to the real case – even if, in Section No. 166, he concludes affirming that the complex case is easily reducible to the real one – whereas in Sections No. 157 to No. 162, he considers the first treatment of systems of implicit functions of two or more real variables.

68 We recall the equivalence between the implicit function theorem and the inverse function one (see, for instance, [Pr] and also [KP]).

69 Where, amongst other things, the Author also uses the famous *Dini's numbers* of Mathematical Analysis, which were introduced, for the first time, in the *Fondamenti per la teorica delle funzioni di variabili reali* ([Di3]) that, as already said, were drawn up before [Di1].

Chapter XXXI) on some problems inherent the osculation[70] sphere of an hump curve, in which he continuously use his results on implicit functions, established in the *Parte I^a^*, in deducing the main properties of this sphere approximating the given hump curve in one of its point. Precisely, in Section No. 375, he uses the implicit function theorem to prove the existence of explicit equations representing a piece of a curve passing through a prefixed point (of this hump curve) and always having a determined tangent in it, resembling the characteristic procedure of *local* linear approximation[71], typical of the manifold structure.

Nevertheless, new and remarkable geometrical applications of the theory developed in *Parte I^a^* (in particular, as concerns the theory of implicit functions developed in Chapter XIII and the theory of variable changes of Chapter XV), will be inserted in the subsequent in print publication of these 1877 lectures, which will take place at the first of 1900s, as we shall see in the next section. Indeed, amongst other, further interesting historical accounts on these first autographed Dini's lessons [Di1] – and on [Di3] – will be given in the discussions of the next section related to [Di2].

4.2 On the Dini's in print published lessons ([Di2])

At the beginning of the 20*th*-Century, Dini published a new revised and enlarged edition of these previous lessons [Di1], into two volumes (and each volume, into two parts), and, in this regards, it is fundamental what he says in the *Preface* to [Di2, Volume I]. What follows is mainly drawn from it[72].

In this Preface, Dini first expose the main motivations which were at the basis of the drafting of the initial lessons[73] [Di1]. He recalls what was the state of the Infinitesimal Calculus in that period which, notwithstanding the first rigorousness attempts already made, among others, by N.H. Abel, A.L. Cauchy and P.G.L. Dirichlet, it still persisted not negligible gaps in this respect, as also pointed out by some remarkable authors like H.A. Schwartz, R. Dedekind, G. Cantor, G.F.B. Riemann, H. Hankel, E. Heine, K. Weierstrass and E.H. Du Bois-Reymond. Taking into account the works of these last mathematicians, Dini started to drawing up his celebrated *Fondamenti per la teorica delle funzioni di variabili reali* ([Di3]) in the 1870s, and, almost at the same time, to drawing up his lessons [Di1]: the former were in print published in 1877, but, for various reasons, the latter were left in the autographed form. In this 1907 Preface, Dini himself regret of this failed publication since, in his opinion, these lessons had many interesting and original new points which would have could draw the attention of other scholars: this is what he says in the Footnote (*) of [Di2, Volume I, pages IV-V], outlining some of these new points. Dini himself, in such a Preface, moreover says that already these his lithographed lessons had a wide diffusion among the Italian mathematicians who always quoted such lessons in their work, but also complaining, yet, of the fact that such references to his work were quite scarce in the foreign treatises on Infinitesimal Calculus[74], except the already mentioned *Encyclopädie der Mathematischen Wissenschaften*. On the other hand, the same Dini affirms that in 1879, the French mathematician M.G. Fauré-Bignet, by means of their common friend Ernesto Padova, asked to him if it was possible to have a French translation of these Pisa lessons[75] that, for Italian Editor's troubles, was delayed until the beginning of the 20*th*-century, with the initial aim to publish the original 1877 lessons without changes or additions but that it will be kept as such, partially[76], for the *Parte I^a^* of *Volume I*, making changes and insertions in *Parte II^a^* of *Volume I,* so obtaining a revised and enlarged edition as regard both the parts of *Volume II*. Finally, Dini drawn up the *Volume I* placing, before the old *Parte I^a^* of [Di1], an

[70] Roughly, the osculation falls into the surface contact class of order major or equal to two, whereas the tangency is a surface contact of order one.

[71] The original Dini's discussions and argumentations developed in Section No. 375 (and, in general, all those related to Chapters XVIII to XXXII) are quite particular and they would require a more in-depth analysis.

[72] Precisely, the whole Dini's treatise in two volumes, include only two Prefaces, one for the first volume (see [Di2, Volume I, Parte I^a^, pp. III-VII]), the other for the second one (see [Di2, Volume II, Parte I^a^, pp. III-IV]).

[73] In this regards, see also [Bt] as concern the Dini's contributions to the Foundations of Analysis in the period from 1860s through 1870s.

[74] Nevertheless, here we have tried to refute, in part, this Dini's bitterness, itemizing instead many foreign treatises which quote the Dini's lessons [Di1] (but with some reserve regarding the French literature).

[75] And this proves as these autographed lessons were also well-known in the French mathematical community, in spite of they had been scarcely cited in the relative French literature of the time.

[76] Adding many footnotes to integration of the text. Dini decided to start the printed version of [Di1], initially without carrying out changes or further additions, because, above all, motivated by the will of giving an historical witness of what was the Pisa teaching of Infinitesimal Calculus in the 1870s, but subsequently making substantial enlargements mainly (but not only) concerning the geometrical applications of differential calculus and the integral calculus.

Introduzione formed by some preliminary chapters of [Di3], numbered as Chapters I to VII. It follow the part entitled *Calcolo Differenziale*, formed by the Chapters I to XVII, just as the *Parte I*[a] of [Di1], and the part entitled *Applicazioni Geometriche del Calcolo Differenziale*, with Chapters XVII to XXXVI, that, as it see, is enlarged respect to the same part of [Di1], which contained the Chapters XVII to XXXII.

The related Dini's editorial plan expected a treatise on Infinitesimal Calculus in two volumes, the first devoted to Differential Calculus, the second to the Integral Calculus. The first volume was already expected into two parts, the first devoted to the exposition of the doctrine field of the Differential Calculus, the second to its geometrical applications. From the properly typographical point of view, on the one hand, materially there exists a unique first volume of a total of 720 pages, in which it is not explicitly neither specified nor mentioned, in nowhere the text, this division into two parts, while, on the other hand, materially there also exist two separated tomes for the *Volume I*, the first containing the *Parte I*[a], recalled in the title page as

Vol. I – Calcolo Differenziale
(1[a] *PARTE)*

and in the external back cover, whereas the second one has no title page, starting with a page having only the central headline

APPLICAZIONI GEOMETRICHE
DEL CALCOLO DIFFERENZIALE

without any reference to *Parte II*[a], which is written only in the external back cover, like in the *Parte I*[a] case.

On the other hand, the Author himself, in the Preface to *Volume I*, often speak of two parts of this volume, while in other place of the same work (as in the second volume, when he refers to the first one), he does not mention any part of this[77]. Thus, a certain ambiguity may arise about the originary typographical setting of this first volume which, however, is little relevant from an historical viewpoint because this *Volume I* was conceived in the same period and published, as a whole, in the 1907, both as a unique volume and as a volume into two parts: the only utility of this distinction may turn out to be useful at level of citations, the *Parte I*[a] quoted as concerns the theory of Differential Calculus, and the *Parte II*[a] quoted as concern its geometrical applications. Instead, it is quite different the situation as regards the *Volume II* because its *Parte I*[a], of pages 1-468, was published in 1909, with Chapters I to XVII, whereas its *Parte II*[a], of pages 471-1056, was published in 1915, with Chapters XVIII to XXXII; the *Parte II*[a] of the old lessons [Di1], devoted to the integral calculus, contained the Chapters I to XXIII.

To our purposes, it is notable to analyze some of the new contents added both to Chapters XIII and XV of [Di2, Volume I] and to the various subsequent chapters related to the geometrical applications, compared to the same chapters of [Di1]; in particular, the applications of many results of Chapters XIII and XV are pervasive throughout the rest of this *Parte II*[a] of *Volume I*. As properly regards the Chapter XIII of [Di2, Volume I], the new additions are mainly inserted as footnotes, and concern the regularity and uniqueness conditions for the existence of the derivate of first order of an implicit function, as well as local approximations of an implicit function through Taylor's developments: for instance, it is fundamental to point out as Dini himself, contrarily to what commonly affirmed[78], had already stated both weakest regularity conditions respect to the C^1 one, supposing only continuity and finiteness of the related implicit function without any other conditions to its derivate, and further improvements on uniqueness conditions[79]. Moreover, many interesting geometrical examples are reported as an application of the theoretical concepts and methods there developed, which will be further retaken in the part concerning the geometrical applications (that is to say, in the *Parte II*[a] of *Volume I*).

[77] Moreover, Giovanni Sansone, in the *Prefazione* to the Ulisse Dini's mathematical works (see [Di4, Volume I, Prefazione, p. 2]), speaks of the *Lezioni di Analisi Infinitesimale* ([Di2]) as a work in four volumes, that is to say, the *Volume I* and *Volume II* with their respective *Parte I*[a] and *Parte II*[a]. Furthermore, these printed version of the original Dini's autographed lessons, in the *Elenco delle Pubblicazioni di Ulisse Dini* of [Di4, Volume I, pp. 12-16], are quoted at the publication No. 58) as follows: *Lezioni di Analisi Infinitesimale*, Vol. I, p. I, Pisa, Nistri, 1907, pp. CI+372; Vol. I, p. 2[a], Pisa, Nistri, 1907, pp. 373 a 720 (besides mistaking the number of chapters of p. I). See also [DBI, Vol. XL, pp. 162-165] where are quoted four volumes for the *Lezioni di analisi infinitesimale*.

[78] In [Di2, Volume I, Chapter XIII, Section 149, Footnote (*), p. 204]. See, for instance, [HR] as regard the attempts to weaken the C^1-regularity conditions of the Dini's implicit function theorem.

[79] See [Di2, Volume I, Chapter XIII, Section 150, Footnote (*), p. 205].

Generally, Dini follows his own original ways of treating the applications of differential calculus to geometrical questions, and one of these consists just in considering repeatedly geometrical entities defined implicitly which distinguishes himself by the other well-known treatise on differential geometry (like [Bi3]). In this case, so to adduce some examples appropriate to our purposes, it is remarkable to observe as in the Sections 275 of Chapters XIX and in the Section 366 of Chapter XXVI (of the geometrical applications), the implicit function theorems are applied for the existence and uniqueness of the tangent line and plane at a regular or singular point respectively of a curve or of a surface given in implicit form, from which it follows that, in such cases, only *local* results may be obtained[80], this being just the essence of the *implicit function theorem paradigm* as a main formal tool to study the *local* nature of a geometrical entity, given in implicit form, by means of other geometrical entities (determined by it) given in an explicit form just thanks to this tool[81]. As already said, Dini used it in a pervasive and original manner throughout his *Parte II^a* of *Volume I*.

According to what said in [En, Libro I, Capitolo IV, § 41], among the main international treatises on Infinitesimal Calculus[82], it is just mentioned the two related works of Dini, that is, his *Lezioni di Analisi Infinitesimale* ([Di1]) and the *Fondamenti per la teorica delle funzioni di variabile reale* ([Di3]), recalled as the ones that have provided a first (and, in a certain sense, definitive) systematic and rigorous critical treatment of the Calculus (above all in the printed version [Di2]). Moreover, in [En, Libro III, Capitolo II, § 10], it is said that, in the second half of the 19*th*-Century, the main authors who have set up a new and fully rigorous theoretical rearrangement of the Calculus, were Weierstrass in Germany and Dini in Italy, their works – affirms Enriques – becoming rapidly known in all of the world.

5. The paper of Henry Poincaré

Following E. Scholz[83], in the paper [Po] it may be found a possible source of the concept of a manifold. In fact, Henry J. Poincaré, in [Po, Sections 1-4], also following part of the fundamental work[84] of E. Betti [Be], gave a constructive definition of (unilateral/bilateral[85]) manifold as follows.

If $x_1, ..., x_n$ are generic variables of $\mathbb{R}^n$ ($n \geq 2$), then he considers the following system of p equalities and q inequalities

$$(1) \qquad F_1(x_1, ..., x_n) = 0, ..., F_p(x_1, ..., x_n) = 0, \varphi_1(x_1, ..., x_n) > 0, ..., \varphi_q(x_1, ..., x_n) > 0,$$

with F_i, φ_j continuous and uniform functions, with continuous derivatives in such a way that the Jacobian determinant $J = \det \|\partial F_i/\partial x_k\|$ be non-zero in each point of the common definition domain of the F_i. When rank $\|\partial F_i/\partial x_k\| = p$, then the system (1) defines a *manifold* of dimension $n - p$. If $p = 0$, we have a so-called *domain*.

Subsequently, Poincaré proves[86] as a manifold of dimension $m = n - p$ defined by the system (1), is equivalent to the formal object defined by a system of equations of the following type

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[80] It is above all noteworthy the fact that this *Parte II^a* of [Di2, Volume I] is one of the few work of the time, devoted to the differential geometry, in which are defined and treated the main geometrical elements (as tangent, normal, osculations, etc) for geometrical entities defined implicitly (for instance, differently from the well-known treatise [Bi3], which considers only entities explicitly defined, except the case of enveloping entities – likewise for its next editions).

[81] In this regards, the Hadamard work [Ha] has been written in order to may extend, as far as possible, this point of view, from the local to the global, starting from the known results on implicit functions of the time, and reaching to the celebrated *Hadamard global inverse function theorem* (see [KP, Chapter 4, Section 6.2]). From this last viewpoint, see also the work [Gu1] for interesting geometrical applications of implicit function theory.

[82] In addition to those of Dini, it is recalled other well-known treatises as those of A. Genocchi (as edited by G. Peano), C. Jordan, J. Tannery, O. Stolz, C. de la Vallée Poussin, C. Carathéodory, E. Cesàro, C. Arzelà and E.W. Hobson.

[83] See [Sc] and also [Mc].

[84] Which, besides, it would deserve further attentions for its historical importance relatively to the Combinatorial Topology, together to the related little known work of Alberto Tonelli (1849-1920), entitled ''Osservazioni sulla teoria delle connessioni: nota'', *Atti della Reale Accademia dei Lincei*, Serie 2, Annata 272, Volume 2, 1874-1875, pp. 594-601 (besides cited, amongst others, by Weyl in [We1, Chapter I, Section 6, p. 33, Footnote [14)]] about some interesting geometrical immersion problems).

[85] The distinction between unilateral and bilateral manifolds is given in [Po, Section 8]; we herein refer to the bilateral case. Further, in this context the recalls to the constrained systems of Analytical Mechanics might also be recognized.

[86] See [Po, Section 3].

(2) $$x_1 = \theta_1(y_1, ..., y_m), ..., x_n = \theta_n(y_1, ..., y_m),$$

plus a certain number of inequalities of the form $\psi_i\ (y_1,..., y_m) > 0$, making use of the implicit function theorem (but without explicitly mentioning it). Further, Poincaré himself, at the beginning of [Po, Section 3], says that an *m*-dimensional manifold may be defined only by the system (2), without additional inequalities, when the variables $y_1,...,y_m$ are independent among them.

In any way, the recalls to the implicit function theory (or to its equivalent forms as inverse function theorem) are evident. However, the main historical interest of the paper [Po] is related to the sources of Algebraic Topology, and not, in general, to the birth of a (possible) concept of differentiable manifold[87].

6. On the work of Hermann Weyl

According to the official history of mathematics, the definition of a complex two-dimensional topological manifold, as today known, was exposed, for the first time, in [We1, Section 4], whereas in [We1, Section 6], the Author gives the notion of a differentiable structure on such a manifold type.

The Weyl's analysis starts from a geometrical representation of an analytic form (according to Weierstrass and Riemann), and attaining to a particular structure of (Riemann) surface[88], through the new topological developments achieved by D. Hilbert and others. In particular, the new local Hausdorff's concept of ''neighborhood'' of a point, played a crucial role in the Weyl's construction of a topological manifold[89].

Moreover, some geometric and analytic function theory aspects of Complex Analysis of the time[90], have also played a very important formal role in the Weyl's work, first of all, the works of Weierstrass[91] on analytic function theory, and on the related concept of analytic form, according to the theoretical approach for studying hyperbolic and elliptic functions due to his teacher Christoph Gudermann (1798-1852), which was mainly centered on the use of power series expansion techniques, in turn based on the well-known work of Lagrange on analytic functions that, as briefly mentioned in Section 3, is also closely related to the history of implicit functions[92]. Besides, the notion of analytic form follows from the consideration of those points in which an analytic function is not regular, for example having there a pole or a branch point of finite order, and in this seems evident a methodological analogy with what said in the previous section 3.2, as concerns the work of Bliss on the singular points of implicit functions (see [Bl1] and [Bl2]).

The central Weyl's idea is that of local homeomorphism of a manifold with $\mathbb{R}^n$. Subsequently, Weyl introduces a differentiable structure on a topological manifold by means of such a local homeomorphism of this manifold with $\mathbb{R}^n$, taking into account some fundamental works of Felix Klein, who wrote a fundamental monograph[93] on Riemann surfaces, on the basis of a definition of it more general than the formulation used by Riemann in his studies on the theory of analytic functions.

As said by the same Weyl, these Klein's works seem to assume an important role in the (Weyl's) definition of a differentiable structure on a manifold. Furthermore, the Klein's Erlanger Program viewpoint seems also to be at the basis of the Weyl's definition of ''compatibility relations'' among ''local coordinate systems'' of a generic point of the manifold, since he introduces a group of local coordinate transformations Γ that leave fixed the origin of $\mathbb{R}^2$; such a group characterizes the manifold, and Weyl, in this regards, talks about *surface of type Γ*.

Later on[94], in [We2], the Author shall make use of the main notions and structures introduced in [We1], in the applicative context of General Relativity, reaching to remarkable results in this field.

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[87] See [Sa]. On the other hand, in this paper, none differential structure is introduced, so that, if one want just to put it into historical relationship with the origins of the notion of manifold, then it should refer to the notion of affine topological manifold rather than the differential one.

[88] Which is not a surface in the sense of *Analysis Situs* (see, for instance, [Ve2]).

[89] For an historical account related to the passage from the notion of *local* to the *global* one in Geometry, see [Co] (where, yet, Dini is not quoted).

[90] In this regards, see also what will be said in Section 8.

[91] Indeed, the Section 1 of the first Chapter of [We1] starts just with the Weierstrass' concept of an analytic function.

[92] Hence, we return again to the history of implicit function theory.

[93] See [Kl1], and also [Kl2].

[94] In [We2].

7 On the works of O. Veblen and J.H.C. Whitehead

In the paper[95] [VW1], Oswald Veblen and his former student John Henry Constantine Whitehead introduce the definition of an *n*-dimensional (regular) affine manifold through the basic notion of *coordinate system* and related transformations, through to which it is possible to reach to the fundamental notion of *allowable* coordinate systems on the basis of the analytical properties of the transformations between them, one of the main mathematical tool to establish these being just given by the implicit function theorems, that, in turn, leads to the characterizing notion of *regular* transformation. Precisely, in the *Introduction*, the Authors define

> *«a manifold as a class of elements, called points, having a structure which is characterized by means of coordinate systems»,*

where the notion of (local) coordinate system is the same of the Weyl's one[96]. Then, they introduce the notion of *regular transformation* by means of the Dini's implicit function theorems[97]. This notion is puts at the foundation of a definition of *regular* manifold, through the further notion of *pseudo-group of transformations* [98] via three groups of axioms that, on the whole, characterize the concept of manifold[99]. The subsequent sections of [VW1] are devoted to the consistency and independence of the previous groups of axioms, to some topological considerations and to few analytic applications (some of which are, for instance, related to the differential absolute calculus). Even in this case, the Dini's implicit function theorems play a crucial role in the definition of manifold – as well as that of a fundamental syntactic tool to be largely used – since this last object is characterizable (according to Weyl) as an abstract entity locally diffeomorphics to $\mathbb{R}^n$, via allowable (through regular transformations) local coordinate systems[100], hence indirectly reconnecting, from a formal viewpoint, to these Dini's works.

From these few considerations, it seems therefore reasonable to affirm as one of the main characterizing idea[101], that gradually led to the structure of a differentiable manifold, was that related to the local (and not global, in general) invertibility of a regular coordinate transformation between allowable coordinate systems, whose main formal tool is just provided by the implicit function theory as formulated in the Dini's sense, of which especially importance is played by those chapters (of [Di1] and [Di2]) related to the systems of implicit functions or two or more variables[102].

In [VW2, Chapter III, Section 1], the Authors argue about allowable coordinate systems, asserting as follows

> *«In general, it is desirable to use a much larger class of coordinate systems, so that the transformation of coordinates shall be as general as they can be without destroying the significance of the analytic expressions which are to be used. The theory of transformations which we shall use depends upon the implicit function theorem in much the same way that algebra contained in Chapter I* (of [VW2]) *depends upon Cramer's rule for solving linear equations».*

And the next Section 2 of Chapter III of [VW2], is just devoted to summarize the main facts about the implicit function theorems, considered as a tool analogous to Cramer's rule, following the Chapitre II of the Tome I of Goursat's treatise[103] [Gu]. Nevertheless, Veblen mentions and uses[104] the Dini's works on implicit

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[95] And, more extensively, in [VW2], the latter being considered, by Veblen himself, as a companion to [Ve1].

[96] Even if Weyl is not explicitly quoted, nevertheless it is, for instance, mentioned at page 557 of [VW1] as concerns his book [We1], so that, despite this, it is unthinkable that Veblen and Whitehead did not know the Weyl's work [We1] on these arguments, also in relation to the drawing up of [VW2]. See also Footnote [17].

[97] See [VW1, p. 552].

[98] See [KN] or [Ch].

[99] See the above Section 5.

[100] Besides, the Authors devote the Section 2 of Chapter III of [VW2] to explain the implicit function theorem as a fundamental tool that will be used in the remaining text.

[101] And, in this regards, again we refer to what will be said in Section 9.

[102] And Dini's was really the first (from an historiographical viewpoint) author to give a rigorous treatment of systems of implicit functions in [Di1].

[103] Also this treatise, as that of C. Jordan (see [Jo]), does not explicitly mention the related Dini's work.

functions both in [Ve1, Chapter II, Section 4, p. 15, Footnote †] and in [Ve1, Chapter V, Section 7, pp. 74-75, Footnote *] in which he follows, respectively, the already mentioned Goursat's treatise [Gu] and the Osgood's treatise [Os2] where, as we will see in the next section, the Dini's work [Di1] is clearly mentioned to this purpose.

On the other hand, any systematic work on tensor calculus cannot to leave aside from questions inherent coordinate transformations and related arguments (as functional determinants and implicit function theorems[105]) and, in this regards, Veblen does not make exception (see [Ve1, Chapter I, Section 5]). To this point, therefore, it is necessary to make some useful historical remarks on Osgood's treatise [Os2], which, we repeat, was one of main treatise on function theory of that time.

In conclusion, the work of Veblen and Whitehead [VW2], albeit important per se, does not seems to be so original and fundamental from an historiographical point of view in the search for the real origins of the concept of a differentiable manifold, since, as seen, it is directly and indirectly correlated, with few substantial changes, to the previous Weyl's work on Riemann surface [We1]. These last conclusions have would been possible only by a careful formal analysis directly carried out on their text and not by means of what is said by the secondary historical and historiographical literature on it (see *Remark 2* of Section 9).

7.1 On W.F. Osgood 'Lehrbuch der Funktionentheorie'

In [Os2, Band I, Zweites Kapitel, Abschnitt 4, pp. 47-52], it is recalled the main results established by Cauchy and Dini as concern implicit functions (mentioning[106] the Dini's work [Di1] in Footnote **) of page 47), saying, besides, as Dini, in [Di1], have generalized and extended – and, in some respects, simplified – the previous works on implicit functions of his predecessors, above all Cauchy, and for not having, for first, made use of the power series expansions as done by this last. Moreover, Osgood gave a very clear rigorous reformulation, with the aid of graphical representations, of the original Dini's proof of his theorem on the existence and unicity of implicit functions, with numerous illustrative examples.

On the other hand, reconnecting us to what said in the previous section about the Weyl's works on Riemann surfaces, Osgood himself, in [Os2, Band I, Achtes Kapitel, Abschnitt 11, pp. 345-347], as regards the *Grundlagen der allgemeinen Theorie der Funktionen einer komplexen Größen*, consider just the results achieved on implicit function theory as basic tools for studying Riemannian surfaces deduced from analytic forms: precisely, given a complex algebraic function of the type $G(w,z) = 0$, he considers its decomposition into the sum of two real algebraic functions as follows $G(w,z) = P(u,v; x,y) + iQ(u,v; x,y)$, with P and Q real polynomials, then apply Dini's theorems on systems of implicit functions, just to the systems formed by these two real polynomials to obtain, via Cauchy-Riemann monogenic conditions[107], the following surface equations[108] $u = \varphi\,(x,y)$ and $v = \psi\,(x,y)$.

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[104] Moreover, in [Ve1, Chapter III, Section 22, p. 48, Footnote *], it is mentioned the fundamental work of A. Capelli, *Lezioni sulla teoria delle forme algebriche*, Pellerano, Napoli, 1902. Thereafter, in [Gr1, page 520, Footnote †], it is reported the following footnote *«With the definition of compactness phrased in this way, it is possible to avoid any use of the postulate of Zermelo. Cfr. Cipolla, Atti della Accademia Gioenia in Catania, vol. 6 (1913), Memoir V,* Sul postulalo di Zermelo e la teoría dei limiti delle funzioni», so that even the acts of the *Accademia Gioenia* di Catania were known in the American mathematical community. This, of course, does not have nothing to do with the historical questions here treated, but, from an historiographical viewpoint, it indirectly proves, as in part already said in the above Section 3.1, as the works of Italian mathematicians of the time were well-known in the international mathematical literature, even in the original Italian language. This might already be seen by a simple look of the various references quoted in any related *Litteratur* (or reference list) placed before each *Kapitel* of the well-known *Encyclopädie der Mathematischen Wissenschaften* (see [BFHW]). Hence, the same discourse should also hold for the Dini's works [Di1] and [Di2] which nevertheless didn't have had any foreign translation.

[105] See, for instance, the *Capitolo I* of *Parte Prima* of the celebrated work T. Levi-Civita, *Lezioni di Calcolo Differenziale Assoluto*, Alberto Stock Editore, Roma, 1925, where, amongst other things, the Author largely use the formal tools therein introduced for immersion problems of manifolds (defined according to Riemann) into Euclidean spaces. However, here it is opening another historical question, precisely that relative to the possible historical role played by the implicit function theory in tensor calculus development, which cannot be treated in this place.

[106] Also mentioning the work [GP].

[107] Through which it is possible to get a nonzero value of the relative functional determinant (Wronskian) computed at an arbitrary point of $\mathbb{C} \setminus \{(0,0)\}$.

[108] Also in this case we can identify the action of main *paradigmatic role* played by the implicit function theorem, in finding these *local* explicit surface equations from the *implicit* one.

Furthermore, the implicit function theory in complex scalar field, is also widely used in many parts[109] of [Os2, Band II.1], which concern the *Grundlagen der allgemeinen Theorie der Funktionen mehrerer komplexen Größen*, especially as regards the formal study of algebraic functions from the complex analysis viewpoint.

However, from these last interesting points here mentioned about such fruitful correlations of implicit function theorems with algebraic function theory (and that will be further discussed in the next Section 8), it is possible to observe what crucial role has also played the implicit function theory in developing some basic chapters of the theory of complex algebraic surfaces.

7.2 On O. Veblen and J.H.C. Whitehead 'Foundation of Differential Geometry'

To our purposes, it is notable, above all in view of what will be briefly said in the next Section 9, to state what follows relatively to a formal (above all, syntactic) equivalence between different possible definitions of a constrained system of Analytical Mechanics which, in turn, is closely connected to that of a differentiable manifold.

With extreme concision, but in a rigorous manner (as his style), Vladimir I. Arnold, in his celebrated textbook [An], has proved a kind of rigorous formal equivalence between the usual modern notion of a differentiable manifold according to Weyl, that deducible by the Dini's work on systems of implicit functions (in the sense that we want emphasizes in this place) and the *principle of virtual work* of analytical mechanics in one of its equivalent formulations known as *D'Alembert-Lagrange principle*. All this is simply meaningful of, at least, one secure point: the last two formal tools just mentioned, syntactically provide the central idea of a certain, new *local*[110] mathematical property of an as much new given formal object, in general not globally thinkable: from here, it follows just the notion of a differentiable manifold, according to a general, possible mechanism of mathematical creativity that will be briefly delineated in Section 9.

On the other hand, the notion of *parallel displacement*, which is strictly related to that of the principle of virtual work (after Tullio Levi-Civita original paper) is also a geometrical argument treated in many sections of the Chapter V of [VW2] (as well as by the major part of the treatises on differential geometry and tensor calculus of the time, as well as, of course, by almost all of the textbooks on analytical and rational mechanics) with interesting historical notes, also in connection[111] with the paper [Bl2] as concern some existence theorems for certain differential equations involved in the resolution of the equations related to such a notion of parallel displacement, in the general framework of the geometry of tangent space to a manifold in a given point of it. These last discussions, in short, point out as a complete and organic historical recognition of the origins of the notion of a differentiable structure cannot leave aside from certain basic aspects of the history of Analytical Mechanics (as, for example, those related to the Lagrange and Lipschitz works).

8. The role of Dini's theory on implicit functions in differential geometry: first conclusions

In this paragraph, we want, in conclusion, put in evidence the existence of some possible, relevant logical (and historical – see next *Remark 2*) links between the theory of implicit functions, in particular as settled after the work of Ulisse Dini, and the construction of the abstract theory of a (topological and differentiable) affine manifolds, although they are not immediately evident from the historical viewpoint[112].

As already said in Section 1, it is possible to build up a theory of affine manifolds in $\mathbb{R}^n$, by means of the Dini's implicit function theorems and the related inverse function theorems: see, for instance, the excellent and organic treatment of this given in[113] [Pi2, Parte I^a, Capitolo 2, Paragrafo 2 and Parte II^a, Capitolo 7, Paragrafo 3]) or the as much good exposition given in [De, 2° Volume, Cap. V]; see also [Vl, Volume IV,

[109] As in the *Erstes Kapitel*, *Abschnitten* 6 *und* 7, whereas the whole *Zweites Kapitel* has title *Implizite Funktionen, Teilbarkeit*.

[110] And not global, in general, as can be, for example, that given by a system of linear equations formally regulated by the Cramer's rule, and of which the Dini's work on implicit functions would be a certain generalization and extension of it (to the non-linear case).

[111] See [Bl2, Chapter V, Section 13].

[112] See the discussion of *Remark 2* of Section 9.

[113] By one of the most thorough and complete Italian treatise on Mathematical Analysis, that of Bruno Pini.

Articolo I, § 2]. See also [Sh, Chapter 1, Section 2, pp. 23-28] in which it speaks just of submanifolds described implicitly.

The implicit function theorems and the inverse function theorems, characterize the local structure of any manifold[114]. Moreover, a manifold (in $\mathbb{R}^n$) may be think, in a certain sense, as the zero values of a given system of functions of the type (1) (equivalent to (2)), discussed in the previous Section 5, and here we does not develop the detailed calculations connected with these well-known claims, since we have mainly historical interests[115]. Therefore, from what has been said so far, there are very few doubts on the fundamental formal role played by the Dini's work on implicit functions theory in contributing to establish the structural foundations of the modern concept of an affine differentiable manifold, at least in the mathematical philosophy sense which will be briefly developed in the next Section 9.

Nevertheless, it would be an historical mistake to think that Ulisse Dini had *explicitly* in mind such a manifold theory (although only in $\mathbb{R}^n$): in fact, he set, say *implicitly*, only the fundamental formal tools need for the subsequent modern and *explicit* axiomatic construction of an abstract affine manifold, as, for instance, showed by the formal examples of Section 1, even if[116] it is possible to presume that some eventual, partial hints would have also could to arise by the original geometrical applications of differential calculus made in *Parte II*^{a} of [Di2, Volume I]. In herein current state of the research, we may only to say that between the Dini's work on implicit functions and the theory of manifolds, as showed too by the discussions of Sections 2 and 7.2, there only exist strong links of formal nature[117] through to which it has been possible to identify some typical *local* (and, in general, not global) analytical properties which, in turn, have led – through that general *objectivation process* (see next Section 9) which constitutes a sort of abstract reification (hypostatization[118]) of a mathematical entity providing a new object – to the institution of a new formal object called *manifold*, according to one of the main characteristics of the already mentioned *paradigmatic role* of implicit function theorem.

Further, we have already mentioned the possible role played by Algebraic Geometry[119] and Complex Analysis regards to the formation of the modern concept of a differentiable manifold and we wish to outline some further few words about these last important aspects. As concerns the algebraic geometry context, in Section 3 we have only said as some works of R. Descartes might be considered as a sort of prolegomena to the birth of implicit function theorem, but this historical link seems to be quite little influent from the syntactical viewpoint. Instead, a very fundamental role is played by complex analysis in connection with the geometric theory of complex functions, hence from the complex algebraic geometry viewpoint, since the works of H. Weyl. In fact, the Weyl's work [We1], as said in Section 6, is mainly centered on the rigorous study of the geometrical representation of certain analytic forms, in turn in relationship with the previous studies on Riemann surfaces made (in [Kl1]) by Felix Klein on the same subject. The work of Weyl, as said, is mainly centered on the theory of analytic functions according to Karl Weierstrass, applied to the Riemann's theory of algebraic functions and their Abelian integrals, towards the construction of a Riemann surface, reaching to a particular method of construction of it, by Weyl himself called the *Riemann-Klein approach*.

In the *Preface* to [We1], Weyl outlines a brief historical account of the motivations which were at the source of his work. Stemmed by a winter semester course held at Göttingen in 1911-1912, three events have had a decisive influence on the form of his book: the 1909 F. Brouwer works on topology, the P. Koebe proof of the uniformization theorem and the Hilbert's establishment of the foundation on which Riemann had built his structure and which was now available for uniformization theory, that is to say, the Dirichlet principle. Besides this, as already said, the previous Klein's works on Riemann surfaces have also played a

[114] See, for instance, the *parametrization* technique given by [Se, Capitolo 5], [He, Chapter 5] and [ST, Chapter 5, Sections 3 and 4]). In particular, in [Th, Chapter 15], the Author establish two main theorems which show that, locally, *n*-surface and parametric *n*-surface are the same thing, in order to do this being necessary to use the inverse function theorem; moreover, we recall that the inverse of a parametrization is defined to be a *chart* of this *n*-surface.

[115] Even if such a type of study would result to be necessary if one must to identify historical connections of the *epistemological* type in the sense of *Remark 2* of Section 9. However, to this purpose, see also [Gi, Capitoli VII e VIII] and [Pr, Capitolo 4].

[116] And this presumption would be very hazarded at this stage of the historical research herein undertaken if one, for instance, did not make an historical recognition in the sense delineated in the *Remark 2* of Section 9.

[117] Which, despite all, have also a their own (far from negligible) historical importance, as it will result to be from the discussions of Section 9.

[118] For this, see for instance [Cl, Chapter 1, p. 12].

[119] See, for instance, [KP].

fundamental role in the development of Weyl's work. On the other hand, taking into account what said in Sections 3.2 and 7.1, it has seen as the theory of implicit functions has also played a fundamental role in the theory of algebraic functions and related geometrical applications as witnessed by the fundamental Osgood's articles in [BFHW], which surely couldn't be unknown to Weyl[120].

From all this, it is possible to guess[121] some non-negligible influences also played by the 19*th*-Century Algebraic Geometry, especially as regards the theory of algebraic functions, in the development of some crucial aspects of the theory of differentiable manifolds, above all in $\mathbb{C}$, because many algebraic geometry tools and methods are applied to the study of the so-called *Riemannian surfaces* of an algebraic function. On the other hand, a posteriori, these historical conjectures could find some further partial (syntactical) confirmations by the so-called *Nash-Tognoli imbedding theorems* of Algebraic Geometry[122], a sort of algebraic geometry analogous of the Whitney's imbedding theorems, which prove as any compact smooth manifold is diffeomeorphic to a well-defined nonsingular real algebraic manifold. Moreover, certain extended forms of implicit and inverse function theorems to real algebraic geometry, have played a fundamental role both in developing a so-called ''semi-algebraic differential geometry'' (see [BCR, Chapter 2]) and to prove the above mentioned Nash-Tognoli imbedding theorems (see [BCR, Chapter 14]). Hence, in conclusion, also some fundamental works of 19*th*-Century algebraic geometers, mainly regarding algebraic functions and the theory of algebraic surfaces, might be considered to have a some influence (in the sense specified in Section 9) on the possible sources of the main formal aspects of a differentiable manifold structure, as witnessed by the Weyl's work.

Nevertheless, just the comparison with the Nash-Tognoli theorems mentioned above, does not have any historical importance in the precise question related to the born of modern theory of differentiable manifolds, differently by the case of the Dini's and Whitney's works: this last discussion is, simply, a remark that has a some sense only in regards to the fact related to a some possible historical influence played by ideas, methods and tools developed by the already mentioned wide geometric immersions problematic[123] which. On the other hand, a sort of historical influence, in a certain sense inverse to that just mentioned above, may be identified relatively to one of the many definitions of algebraic variety of Algebraic Geometry, that given by André Weil in[124] [Wi, Chapter VII, Section 3] in which the Author gives a definition of abstract algebraic variety following that of a differentiable manifold.

9. On the nature of mathematical objects: first brief outlines

The book [Gs] has been written by one of the leading mathematician in the field of Calculus of Variation and its Applications, to whom active experience of pure mathematician he has successfully joined an outstanding historical competence: making valuable use of his former research work in pure mathematics, the Author is one of the most suitable scholars to give a direct evidence of how a mathematical thought may be creative and original in thinking and finding new objects, or ideas, which have a some slightest scientific foundation.

According to what says Enrico Giusti in [Gs] after having passed into examination the main historical cases thereupon such a context, one of the main way to reach a new mathematical object or entity, is that consisting in a sort, say, of *objectivation process* which is a type of *abstract reification* (*hypostatization*) of the mathematical proof procedures already established – for instance coming from resolutions of previous

[120] To claim this, it is enough to recall as, for instance, the Osgood's article [BFHW, Band II, Zweiter Teil, Artikel B.1] is, amongst other, quoted by Weyl in [We1, Chapter I, Section 1, p. 1, Footnote [2)]]. On the relationships between implicit function theory and algebraic functions, see above all [Os2, II.1] (in particular, its *Zweites Kapitel*) and also [Bi2, Capitolo VII, §§ 70-80] and [Pi, Capitolo VIII, §§ 154-157 e Capitolo XIII, §§ 243-250]. Moreover, the arguments of Chapter VII of [Bi2] are preliminary to those of Chapter VIII devoted to the introduction of the concept of a Riemann surface, following a line of thought in some respects analogous to the Weyl's one. See also [Ju] for interesting relationships between the theory of implicit functions and the theory of algebraic functions oriented toward Riemann surfaces.

[121] Above all, from the Weyl's work ([We1]); see also [Mc].

[122] See [BCR, Chapter 14]. On the other hand, the same John Nash, together – but independently each from the other – to Jürgen Moser, proved a more general and abstract form of implicit function theorem, known as *Nash-Moser implicit function theorem* (see [KP]), for his researches on the imbedding of Riemannian manifolds (see [HH]).

[123] In this regards, it might be of a some interest to see also what has said Ford (in [Fo]) about the works of Dini on infinitesimal geometry related to problems inherent partial differential equations on given applicable surfaces (see also the above Section 3.1).

[124] See also [Fu, Chapter 6, Section 2].

problems – from which it can arise, at a certain point, a clear and conscious insight institutionalizing this new formal object or entity. Now, the case related to the structure of differentiable manifold, is just one of this, according to what has been said above, confirming, therefore, what hypothesized by Giusti: indeed, the notion of such a geometrical structure can be seen as the result of such an objectivation process related to the proof (and, in lesser part, to the eventual geometrical meaning) of the implicit function theorems, above all according to the Dini's work on systems of implicit functions (albeit in $\mathbb{R}^n$), as well as, for certain aspects, to the mathematical properties concerning the principle of virtual works, as already said in Section 7.2.

Another, fundamental example of the same type[125], also confirming the above Giusti's hypothesis[126], is that related to the institution of the formal topological notion known as *Stone-Čech compactification* of natural numbers $\mathbb{N}$ – namely denoted by $\beta\mathbb{N}$ – because, following what said in [Eg, Chapter 3, p. 179], its, as say, 'implicit' constitution has had place along the proof of the celebrated A. Tychonov theorem on the compactness of the topological product of an arbitrary family of compact spaces, then 'explicited'[127]

[125] Maybe more meaningful of that here considered.

[126] Which, besides, is very close to the Weyl's conception of mathematical construction (see [Cs, Capitolo XII, § 1]) as due to the inter-relationships between *logic* and *mathematical* processes which are the two fundamental *generative processes* of any mathematical construction.

[127] Hence, we might, in a certain sense, to think such an *objectivation process* as due, roughly speaking, to a sort of ''passage from the implicit *towards* the explicit''(with the proper precautions as concern the possible meaning of these last terms). On the other hand, the adjectives 'implicit' and 'explicit' are often used in mathematics but in a roughly sense and in different contexts: for instance, the same Weyl quotes that *«The most general concept* [of surfaces imbedded in spaces] *is probably found explicitly first in Koebe's work* […]» (see [We1, Chapter I, Section 6, p. 33, Footnote 14)]); then, another example of this passage but as concerns the aspect more properly formal, in [Et, page 316, Footnote †], as regards the application of a given theorem, the Author states that *«The theorem* [of K. Carathéodory, *Vorlesungen über réelle Funktionen*, Teubner, Leipzig, 1918, Satz 5, p. 678] *made use of here is not explicitly stated by Carathéodory, but is implicit in the existence theorem for differential equations cited above* [Theorem I of [Et]]», for more details referring to the other his paper H.J. Ettlinger, On Continuity in Several Variables, *Bulletin of the American Mathematical Society*, 33 (1) (1927) pp. 37-38, where, at the very beginning, it is stated that *«The following theorem on the continuity of a function of several variables is contained implicitly in a theorem on the existence of solutions of differential equations by Carathéodory. It is of a general nature and independent of the context in which it is found. It is, therefore, worth while isolating and signalizing it»*. Nevertheless, another, very emblematic example of the opposition «*implicit vs explicit*» in Mathematics from which it may arises a mathematical entity, is given by the definition of probability. Indeed, following [Cs, Capitolo 1, § 3], the birth of the notion of probability has undergone a similar process: from an implicit definition used by B. Pascal in the half of the 17*th*-Century, both G.W. Leibniz on the one hand and P.S. Laplace on the other, have later gave an explicit definition of probability on the basis of the previous implicit definition by Pascal, reaching to the *classical* definition of probability as ratio between favourable cases and possible cases which, in turn, led to the explicit *frequentist* definition of probability according to J. Venn (1866). Thereafter, it have seen the explicit *subjectivist* definition of probability according to B. De Finetti and F.P. Ramsey (1920) and the criticisms moved by A.N. Kolmogorov to them, wishing only an axiomatic definition of probability – hence implicit, according to what states Weyl in [We3, Capitolo 1, § 4, p. 34] – which he shall give in 1933 (see [Cs, Capitolo 2, § 6]); Kolmogorov stated that the explicit definitions of probability aren't of pertinence of mathematical probability theory but of mathematical philosophy; see also [Cs, Capitolo 4, § 1; Capitolo 6, § 1] as regard further discussions upon the explicit definitions. Therefore, the dialectic process *«implicit versus explicit»* has the *status* of a real general *paradigmatic process* of creation of a mathematical entity according to what has been said above. From a properly psychological stance, this process might be put into correspondence with the so-called *Gestaltic switches* inter alia recalled by T.S. Kuhn to explain certain aspects of scientific revolutions. Finally, as regards the historical importance played by this process, A. Weil in [Wi1, Capitolo I, § II], arguing about the history of factorization of prime numbers, recalls some particulars concerning the historical antecedents of the Euclidean algorithm in finding the greatest common divisor of two integers and stating as it was the result of a typical process (of mathematical creativity), that according to which a given mathematical entity has been (implicitly of necessity) discovered in different contexts much time before the substantial and formal identity of these discoveries began to be perceived (explicitly) as an unique mathematical entity or object. Moreover, Weil written a fundamental paper (see [Wi2]) on the relationships between mathematics and its history, covering many points common to the Federigo Enriques thought on a certain identification of these two disciplines; in the same paper, then, he states that *«the ability to recognize the dark and/or incipient form of the mathematical ideas* [that is to say, in implicit manner present] *as well as to follow their traces in the many disguises that them may assume before manifest themselves in the full light of day* [that is to say, their explicitation], *is very likely to be combined with a mathematical talent better than the average one; but, even more than this last, such an ability is an essential component of this talent. That that makes the mathematics very interesting is just the first appearing of concepts and methods devoted to emerge only subsequently into the conscious mind of mathematicians; the main scope of the historian is just that of freeing them as well as retracing their factual or missing influences on the successive*

contemporaneously, but independently one from the other, by Eduard Čech and Marshall H. Stone, who have based their works (above all, the first author) just on this Tychonov's theorem proof.

Remark 1. It could seem incoherent and inhomogeneous the historical recognition here made about some historical aspects related to the history of implicit function theory and its formal methods, in particular oriented towards the Dini's work thereupon made. Indeed, such tools have been identified in the many, quite different mathematical contexts, and here only recalled from the historiographical point of view (implicit function theorem *paradigm*). Nevertheless, if one points out what has been the real intention which has motivated the drawn up of this paper, then it is possible to avoid such an initial criticism.

Indeed, the main aim of this first paper is that of having tried to put the attention on another possible methodological viewpoint in doing History of Mathematics, precisely that consisting in examining, in a detailed manner, the properly mathematical works carried out by the various authors under historical inquiry, above all in consideration of what established in Section 9 (mainly based on the meaningful work [Gs]). Following this principle of the method, it has been possible to identify as the whole theory of implicit functions has played a fundamental role both in differential and algebraic geometry, in this first paper having pointed out only few historiographical facts: precisely, such tools have played a fundamental role in the birth of the notion of affine differentiable manifold both through the Whitney's works (this aspect having been the main subject of this first paper) and through the Weyl's work on Riemann surface (this aspect having been only briefly mentioned here).

On the other hand, it is just this last Weyl's work that led us towards the context of algebraic geometry: in fact, as already said in Section 6, this Author attained to his notion of a (two-dimensional) differentiable manifold by means of geometric and analytic properties of analytic forms which, in turn (as said at the end of Section 8), are also closely related to the implicit function theory in the complex scalar field. Now, only following such a historical research methodology it has been possible to identify such new types of *historical connections* (see also next *Remark 2*) among mathematical contexts quite far among them and not always directly correlated through the usual historical connections.

Remark 2. From a properly historical point of view, perhaps this paper identifies another possible *historical connection*. Precisely, from an historiographical viewpoint, there exist, roughly, various types of historical connections (or correlations), which are the main study objects of the historical sciences, if one does not consider their mere chronicle-historical aspects: a *direct historical connection* is given, for example, by an explicit and direct mention or quotation, by the author under examination, of a given historical source (work, paper, thought, person, etc) belonging, for instance, to another given source, so that we refer to the so-called *primary literature* of the cited source; an *indirect historical connection* is given, instead, by the same type of the previous historical recall but by means of the use of *secondary literature* through which the quoted source is not explicitly and directly mentioned, but we alludes to it, as say, for 'interposed person' or else for *relata refero,* albeit, in any case, it is always possible, with a certain degree of accuracy, date back to the given source.

Finally, there also exists another possible historical connection – which we might call *epistemological* or *theoretical connections* – not included in the previous ones, an example of which is just that given by the case here considered. Indeed, due to the various reasons above exposed[128], the implicit function theorem, in the Dini's form, is, yep, widely used in a given context (in this case, that related to the origins of the structure of a differentiable manifold), but often without put the right emphasis on the relevant syntactic role played by this recurrent formal tool within the various proof processes in which it is involved, sometimes even without in no way mentioning it. Therefore, in this first paper, we have tried to rebuild up the possible historical and historiographical paths of the recurrent *attendance*[129] of this formal tool in proving many pivotal results from which derives the notion of a differentiable manifolds. The evidences thereafter of the

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developments». On the other hand, such a typical formal process «implicit towards explicit» it is also present in the formal development of Theoretical Physics where, for instance, it speaks, following D. Bohm, of an *explicit order* and of an *implicit* one (see [Le, Section 8, pp. 95-96]). Finally, on implicit definitions in Mathematics from an epistemological and education standpoint, see also [Pk].

[128] The main of which being that due either to the lack of an in print published version of the original autographed lessons [Di1] or to the negligence by both of some authors (even Italians) and of some translators/curators in quoting the Dini's work on implicit functions (see, for instance, the case of the German translation of [GP]).

[129] Almost ''ghostlike'', if it is improperly allowed to use a metapsychic term but that 'makes the idea'.

fact that such a formal tool didn't can be unknown at the time[130], as well as its relationships with the Dini's work, have also emerged by particular reconstructions inherent both the *internal* and *external* history, assisted by historiographical basis[131]: in fact, from an internal history viewpoint, we have tried to identify the various teaching liaisons among the authors under examination which were possible, from an external history viewpoint, through the identification of the possible interpersonal relationships which have occurred among them. For instance, as regards this last point, we have identified certain *scientific-human clusters*[132] in which it has been possible to establish direct and indirect relationships (both scientific and/or human, the latter when possible) among their members: two of these[133] are, on the one hand, that involving Lagrange, Cauchy, Plana, Mossotti and Dini as made in Section 4, and, on the other hand, that involving Bolza, Osgood, Bliss, Veblen and Whitney as seen in Section 3.1. Sometimes, this last type of historical/historiographical connections may be reduced to a long series of direct and indirect historical connections, but it may occur the case that such a reduction is not feasible (or it results to be quite unreliable), so that it is possible build up it only at the theoretical level, analyzing the related formal developments inherent the historical subject under consideration[134], reaching exactly to the emergence of an epistemological connections, like that here identified as regards the structure of a differentiable manifold. This last historiographic methodology, from a general standpoint, is also supported by C.N. Yang[135] according to which *«a concept, especially a scientific concept, does not have a full meaning if it is not defined respect to that context of knowledge from which it has derived and developed»*.

It is just this last type of historical connections identified according to this last viewpoint, those that can be put into close correlation with what has been said at the beginning of this section about a possible origin of the mathematical entities.

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[130] As clearly it results from what has been said so far.

[131] As, for instance, those given by the various bibliographical researches.

[132] Which, besides, seem resembles the various graphs of the recent, so-called *mathematical genealogies* that have much in common with the above mentioned historical connections.

[133] Another possible and interesting scientific-human cluster of this type has glimpsed as regards the Weyl's work on Riemann surfaces (and its correlations with algebraic function theory and the theory of algebraic surfaces), involving, amongst others, Riemann, Klein, Weierstrass and others.

[134] Often, this type of historical studies can lead to new theoretical results, as witnessed by what say the Authors of [HR] at the beginning of Section IV of their paper, precisely: *«We originally developed the general theorems* [on implicit functions] *to solve an applied problem* [see the paper [23] of [HR]]. *However, attempts to ascertain their novelty led to an historical study, and we present a few of the highlights that may be of interest to some readers»*. Another, in some respects analogous, example of this type of methodology is given by the celebrated studies conducted by C.L. Siegel – who was also a profound scholarly of History of Mathematics – on the 1850s Riemann unpublished manuscript, the so-called *Riemanns Nachlaß*, from which he deduced, in 1932, a fundamental result of the theory of Riemann zeta-function, namely the well-known *Riemann-Siegel formula*. Furthermore, a quite similar method has been, in a very originally manner, fruitfully adopted by the Authors in drawing up most parts of their book [GPV], directly basing on the original sources; accordingly, it also contain many interesting historical notes.

[135] Precisely, see the *Preface* to [Ya].

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[136] Of this, there exists an Italian translation due to Massimo Galuzzi.