Protective Measurement and the Meaning of the Wave Function

Shan Gao[*]

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Abstract

This article analyzes the implications of protective measurement for the meaning of the wave function. According to protective measurement, the mass and charge of a charged quantum system are distributed in space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. It is argued that the mass and charge distributions are not real but effective; they are formed by the ergodic motion of a localized particle with the total mass and charge of the system. Moreover, the ergodic motion is arguably discontinuous and random. Based on this result, we suggest that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and at a deeper level, it represents the property of the particles that determines their random discontinuous motion. In particular, the modulus squared of the wave function (in position space) gives the probability density of the particles *being* in certain positions in space.

1 Introduction

The physical meaning of the wave function is an important interpretative problem of quantum mechanics. Notwithstanding more than eighty years' developments of the theory, however, this is still a debated issue. It has been widely argued that the probability interpretation is not wholly satisfactory because of resorting to the vague concept of measurement - though it is still the standard interpretation in textbooks nowadays (Bell 1990). On the other hand, the meaning of the wave function is also in dispute in the alternatives to quantum mechanics such as the de Broglie-Bohm theory and the many-worlds interpretation (de Broglie 1928; Bohm 1952; Everett 1957; De Witt and Graham 1973). In view of this unsatisfactory situation, it seems that

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[*]Institute for the History of Natural Sciences, Chinese Academy of Sciences, Beijing 100190, P. R. China. E-mail: gaoshan@ihns.ac.cn.

we need a new starting point to solve this fundamental interpretive problem of quantum mechanics.

The meaning of the wave function is often analyzed in the context of conventional impulsive measurements, for which the coupling interaction between the measured system and the measuring device is of short duration and strong. Even though the wave function of a quantum system is in general extended over space, an ideal position measurement will collapse the wave function and can only detect the system in a random position in space. Then it is unsurprising that the wave function is assumed to be related to the probabilities of these random measurement results by the standard probability interpretation. However, it has been known that there exists another kind of measurement that is less directly related to the collapse of the wave function, namely the protective measurement (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993; Aharonov, Anandan and Vaidman 1996). Protective measurement also uses a standard measuring procedure, but with a weak and long duration coupling interaction and an appropriate procedure to protect the measured wave function from collapsing. These differences permit protective measurement to be able to gain more information about the measured quantum system and its wave function, and thus it may help to unveil more physical content of the wave function. In this paper, we will analyze the possible implications of protective measurement for the meaning of the wave function.

The plan of this paper is as follows. In Section 2, we first introduce the principle of protective measurement. It is stressed that protective measurement can measure the expectation values of observables for a single quantum system, and these expectation values are physical properties of the system, not properties of an ensemble of identical systems. Section 3 gives a typical example of such properties, the mass and charge density. According to protective measurement, the mass and charge of a charged quantum system are distributed throughout space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. In Section 4, the physical origin of the mass and charge density is then investigated. It is argued that the mass and charge density of a quantum system is not real but effective; it is formed by the ergodic motion of a localized particle with the total mass and charge of the system. Moreover, the ergodic motion is discontinuous and random. Based on this result, we suggest in Section 5 that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and at a deeper level, it represents the property of the particles that determines their random discontinuous motion. According to this interpretation, the modulus squared of the wave function (in position space) not only gives the probability density of the particles *being found* in certain locations as the probability interpretation holds, but also gives the objective probability density of the particles *being* there. Conclusions are given in the last section.

2 Protective measurements

Protective measurement is a method to measure the expectation values of observables on a single quantum system. A general scheme is to let the measured system be in a nondegenerate eigenstate of the whole Hamiltonian using a suitable protective interaction, and then make the measurement adiabatically so that the state of the system neither changes nor becomes entangled with the measuring device appreciably. In this way, protective measurement can measure the expectation value of an observable on a single quantum system. In the following, we will introduce the principle of protective measurement in more detail (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993)[1].

2.1 Measurements with natural protection

As a typical example, we consider a quantum system in a discrete nondegenerate energy eigenstate $|E_n\rangle$. In this case, the system itself supplies the protection of the state due to energy conservation and no artificial protection is needed.

The interaction Hamiltonian for a protective measurement of an observable A in this state involves the same interaction Hamiltonian as the standard measuring procedure:

$$H_I = g(t)PA, \tag{1}$$

where P is the momentum conjugate to the pointer variable X of an appropriate measuring device. Let the initial state of the pointer at $t = 0$ be $|\phi(x_0)\rangle$, which is a Gaussian wave packet of eigenstates of X with width w_0, centered around the eigenvalue x_0. The time-dependent coupling strength $g(t)$ is also a smooth function normalized to $\int dt g(t) = 1$. But different from conventional impulsive measurements, for which the interaction is very strong and almost instantaneous, protective measurements make use of the opposite limit where the interaction of the measuring device with the system is weak and adiabatic, and thus the free Hamiltonians cannot be neglected. Let the Hamiltonian of the combined system be

$$H(t) = H_S + H_D + g(t)PA, \tag{2}$$

where H_S and H_D are the Hamiltonians of the measured system and the measuring device, respectively. The interaction lasts for a long time T, and $g(t)$ is very small and constant for the most part, and it goes to zero gradually before and after the interaction.

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[1]Although there appeared numerous objections to the validity of protective measurements (see, e.g. Unruh 1994; Rovelli 1994; Ghose and Home 1995; Uffink 1999), these objections have been answered (Aharonov, Anandan and Vaidman 1996; Dass and Qureshi 1999; Vaidman 2009; Gao 2012).

The state of the combined system after T is given by

$$|t = T\rangle = e^{-\frac{i}{\hbar} \int_0^T H(t) dt} |E_n\rangle |\phi(x_0)\rangle. \tag{3}$$

By ignoring the switching on and switching off processes[2], the full Hamiltonian (with $g(t) = 1/T$) is time-independent and no time-ordering is needed. Then we obtain

$$|t = T\rangle = e^{-\frac{i}{\hbar} HT} |E_n\rangle |\phi(x_0)\rangle, \tag{4}$$

where $H = H_S + H_D + \frac{PA}{T}$. We further expand $|\phi(x_0)\rangle$ in the eigenstate of H_D, $\left|E_j^d\right\rangle$, and write

$$|t = T\rangle = e^{-\frac{i}{\hbar} HT} \sum_j c_j |E_n\rangle \left|E_j^d\right\rangle, \tag{5}$$

Let the exact eigenstates of H be $|\Psi_{k,m}\rangle$ and the corresponding eigenvalues be $E(k, m)$, we have

$$|t = T\rangle = \sum_j c_j \sum_{k,m} e^{-\frac{i}{\hbar} E(k,m)T} \langle \Psi_{k,m}|E_n, E_j^d\rangle |\Psi_{k,m}\rangle. \tag{6}$$

Since the interaction is very weak, the Hamiltonian H of Eq.(2) can be thought of as $H_0 = H_S + H_D$ perturbed by $\frac{PA}{T}$. Using the fact that $\frac{PA}{T}$ is a small perturbation and that the eigenstates of H_0 are of the form $|E_k\rangle \left|E_m^d\right\rangle$, the perturbation theory gives

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\begin{aligned}
|\Psi_{k,m}\rangle &= |E_k\rangle \left|E_m^d\right\rangle + O(1/T), \\
E(k, m) &= E_k + E_m^d + \frac{1}{T}\langle A\rangle_k \langle P\rangle_m + O(1/T^2).
\end{aligned} \tag{7}
$$

Note that it is a necessary condition for Eq.(7) to hold that $|E_k\rangle$ is a non-degenerate eigenstate of H_S. Substituting Eq.(7) in Eq.(6) and taking the large T limit yields

$$|t = T\rangle \approx \sum_j e^{-\frac{i}{\hbar}(E_n T + E_j^d T + \langle A\rangle_n \langle P\rangle_j)} c_j |E_n\rangle \left|E_j^d\right\rangle. \tag{8}$$

For the special case when P commutes with the free Hamiltonian of the device, i.e., $[P, H_D] = 0$, the eigenstates $\left|E_j^d\right\rangle$ of H_D are also the eigenstates of P, and thus the above equation can be rewritten as

$$|t = T\rangle \approx e^{-\frac{i}{\hbar} E_n T - \frac{i}{\hbar} H_D T - \frac{i}{\hbar}\langle A\rangle_n P} |E_n\rangle |\phi(x_0)\rangle. \tag{9}$$

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[2]The change in the total Hamiltonian during these processes is smaller than PA/T, and thus the adiabaticity of the interaction will not be violated and the approximate treatment given below is valid. For a more strict analysis see Dass and Qureshi (1999).

It can be seen that the third term in the exponent will shift the center of the pointer $|\phi(x_0)\rangle$ by an amount $\langle A \rangle_n$:

$$|t = T\rangle \approx e^{-\frac{i}{\hbar}E_n T - \frac{i}{\hbar}H_D T} |E_n\rangle |\phi(x_0 + \langle A \rangle_n)\rangle. \tag{10}$$

This shows that at the end of the interaction, the center of the pointer has shifted by the expectation value of the measured observable in the measured state.

For the general case when $[P, H_D] \neq 0$, we can introduce an operator $Y = \sum_j \langle P \rangle_j \left| E_j^d \right\rangle \langle E_j^d |$ and rewrite Eq.(8) as

$$|t = T\rangle \approx e^{-\frac{i}{\hbar}E_n T - \frac{i}{\hbar}H_D T - \frac{i}{\hbar}\langle A \rangle_n Y} |E_n\rangle |\phi(x_0)\rangle. \tag{11}$$

Then by rechoosing the state of the device so that it is peaked around a value x_0' of the pointer variable X' conjugate to Y, i.e., $[X', Y] = i\hbar,$[3] we can obtain

$$|t = T\rangle \approx e^{-\frac{i}{\hbar}E_n T - \frac{i}{\hbar}H_D T - \frac{i}{\hbar}\langle A \rangle_n Y} |E_n\rangle |\phi(x_0')\rangle = e^{-\frac{i}{\hbar}E_n T - \frac{i}{\hbar}H_D T} |E_n\rangle |\phi(x_0' + \langle A \rangle_n)\rangle. \tag{12}$$

Thus the center of the pointer also shifts by $\langle A \rangle_n$ at the end of the interaction. This demonstrates the generic possibility of the protective measurement of $\langle A \rangle_n$.

It is worth noting that since the position variable of the pointer does not commute with its free Hamiltonian, the pointer wave packet will spread during the long measuring time. For example, the kinematic energy term $P^2/2M$ in the free Hamiltonian of the pointer will spread the wave packet without shifting the center, and the width of the wave packet at the end of interaction will be $w(T) = [\frac{1}{2}(w_0^2 + \frac{T^2}{M^2 w_0^2})]^{\frac{1}{2}}$ (Dass and Qureshi 1999). However, the spreading of the pointer wave packet can be made as small as possible by increasing the mass M of the pointer, and thus it will not interfere with resolving the shift of the center of the pointer in principle.

As in conventional impulsive measurements, there is also an issue of retrieving the information about the center of the wave packet of the pointer (Dass and Qureshi 1999). One strategy is to consider adiabatic coupling of a single quantum system to an ensemble of measuring devices and make impulsive position measurements on the ensemble of devices to determine the pointer position. For example, the ensemble of devices could be a beam of atoms interacting adiabatically with the spin of the system. Although such an ensemble approach inevitably carries with it uncertainty in the knowledge of the position of the device, the pointer position, which is the average

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[3]Note that it may not always be possible to physically realize the operator Y, and an operator canonically conjugate to Y need not always exist either. For further discussions see Dass and Qureshi (1999).

of the result of these position measurements, can be determined with arbitrary accuracy. Another approach is to make repeated measurements (e.g. weak quantum nondemolition measurements) on the single measuring device (Dass and Qureshi 1999). This issue does not affect the principle of protective measurements. In particular, retrieving the information about the position of the pointer only depends on the Born rule and is independent of whether the wave function collapses or not during a conventional impulsive measurement.

2.2   Measurements with artificial protection

Protective measurements can not only measure the discrete nondegenerate energy eigenstates of a single quantum system, which are naturally protected by energy conservation, but also measure the general quantum states by adding an artificial protection procedure in principle (Aharonov and Vaidman 1993). For this case, the measured state needs to be known beforehand in order to arrange a proper protection.

For degenerate energy eigenstates, the simplest way is to add a potential (as part of the measuring procedure) to change the energies of the other states and lift the degeneracy. Then the measured state remains unchanged, but is now protected by energy conservation like nondegenerate energy eigenstates. Although this protection does not change the state, it does change the physical situation. This change can be brought to a minimum by adding strong protection potential for a dense set of very short time intervals. Then most of the time the system has not only the same state, but also the original potential.

The superposition of energy eigenstates can be measured by a similar procedure. One can add a dense set of time-dependent potentials acting for very short periods of time such that the state at all these times is the nondegenerate eigenstate of the Hamiltonian together with the additional potential. Then most of the time the system also evolves under the original Hamiltonian. A stronger protection is needed in order to measure all details of the time-dependent state. One way is via the quantum Zeno effect. The frequent impulsive measurements can test and protect the time evolution of the quantum state. For measurement of any desired accuracy of the state, there is a density of the impulsive measurements which can protect the state from being changed due to the measuring interaction. When the time scale of intervals between consecutive protections is much smaller than the time scale of the original state evolution, the system will evolve according to its original Hamiltonian most of the time, and thus what's measured is still the property of the system and not of the protection procedure (Aharonov and Vaidman 1993).

Lastly, we note that the scheme of protective measurement can also be extended to a many-particle system (Anandan 1993). If the system is in a

product state, then this is easily done by protectively measuring each state of the individual systems. But this is impossible when the system is in an entangled state because neither particle is then in a unique state that can be protected. If a protective measurement is made only on one of the particles, then this would also collapse the entangled state into one of the eigenstates of the protecting Hamiltonian. The right method is by adding appropriate protection procedure to the whole system so that the entangled state is a nondegenerate eigenstate of the total Hamiltonian of the system together with the added potential. Then the entangled state can be protectively measured. Note that the additional protection usually contains a nonlocal interaction for separated particles. However, this measurement may be performed without violating causality by having the entangled particles sufficiently close to each other so that they have this protective interaction. Then when the particles are separated they would still be in the same entangled state which has been protectively measured.

2.3 Further discussions

According to the standard view, the expectation values of observables are not the physical properties of a single system, but the statistical properties of an ensemble of identical systems. This seems reasonable if there exist only conventional impulsive measurements. An impulsive measurement can only obtain one of the eigenvalues of the measured observable, and thus the expectation value can only be defined as a statistical average of the eigenvalues for an ensemble of identical systems. However, as we have seen above, there exist other kinds of quantum measurements, and in particular, protective measurements can measure the expectation values of observables for a single system, using an adiabatic measuring procedure. Therefore, the expectation values of observables should be considered as the physical properties of a single quantum system, not those of an ensemble (Aharonov, Anandan and Vaidman 1996)[4].

It is worth pointing out that a realistic protective measurement (where the measuring time T is finite) can never be performed on a single quantum system with absolute certainty because of the tiny unavoidable entanglement in the final state[5]. For example, we can only obtain the exact expectation value $\langle A \rangle$ with a probability very close to one, and the measurement may also result in collapse and its result be the expectation value $\langle A \rangle_\perp$ with a

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[4]Anandan (1993) and Dickson (1995) gave some primary analyses of the implications of this result for quantum realism. According to Anandan (1993), protective measurement refutes an argument of Einstein in favor of the ensemble interpretation of quantum mechanics. Dickson's (1995) analysis was more philosophical. He argued that protective measurement provides a reply to scientific empiricism about quantum mechanics, but it can neither refute that position nor confirm scientific realism, and the aim of his argument is to place realism and empiricism on an even score in regards to quantum mechanics.

[5]This point was discussed and stressed by Dass and Qureshi (1999).

probability proportional to $\sim 1/T^2$, where $\perp$ refers to a normalized state in the subspace normal to the initial state as picked out by the first-order perturbation theory(Dass and Qureshi 1999). Therefore, a small ensemble is still needed for a realistic protective measurement, and the size of the ensemble is in inverse proportion to the duration of measurement. However, the limitation of a realistic protective measurement does not influence the above conclusion. The key point is that the effects of entanglement and collapse can be made arbitrarily small, and a protective measurement can measure the expectation values of observables on a single quantum system with certainty *in principle* (when the measuring time T approaches infinite). Thus the expectation values of observables should be regarded as the physical properties of a quantum system.

In addition, we can also provide an argument against the standard view, independently of the above analysis of protective measurement. First of all, although the expectation values of observables can only be obtained by measuring an ensemble of identical systems in the context of conventional impulsive measurements, this fact does not necessarily entail that they can only be the statistical properties of the ensemble. Next, if each system in the ensemble is indeed identical as the standard view holds (this means that the quantum state is a complete description of a single system), then obviously the expectation values of observables will be also the properties of each individual system in the ensemble. Thirdly, even if the quantum state is not a complete description of a single system and additional variables are needed as in the de Broglie-Bohm theory (de Broglie 1928; Bohm 1952), the quantum state of each system in an ensemble of identical systems is still the same, and thus the expectation values of observables, which are calculated in terms of the quantum state, are also the same for every system in the ensemble. As a result, the expectation values of observables can still be regarded as the properties of individual systems.

Lastly, we stress that the expectation values of observables are instantaneous properties of a quantum system (Aharonov, Anandan and Vaidman 1996). Although the measured state may be unchanged during a protective measurement and the duration of measurement may be very long, for an arbitrarily short period of time the measuring device always shifts by an amount proportional to the expectation value of the measured observable in the state according to quantum mechanics (see Eq. (9)). Therefore, the expectation values of observables are not time-averaged properties of a quantum system defined during a finite period of time, but instantaneous properties of the system defined during an infinitesimal period of time or at a precise instant.

3 On the mass and charge distributions of a quantum system

According to protective measurement, the expectation values of observables are properties of a single quantum system. Two examples of such properties are the mass and charge distributions of a quantum system. In this section, we will present a detailed analysis of these properties.

3.1 A general argument

Consider a quantum system in a discrete nondegenerate energy eigenstate $\psi(x)$. We take the measured observable A_n to be (normalized) projection operators on small spatial regions V_n having volume v_n:

$$A_n = \begin{cases} \frac{1}{v_n}, & \text{if } x \in V_n, \\ 0, & \text{if } x \notin V_n. \end{cases} \tag{13}$$

The protective measurement of A_n then yields

$$\langle A_n \rangle = \frac{1}{v_n} \int_{V_n} |\psi(x)|^2 dv = |\psi_n|^2, \tag{14}$$

where $|\psi_n|^2$ is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region V_n. Then when $v_n \to 0$ and after performing measurements in sufficiently many regions V_n we can measure $\rho(x)$ everywhere in space.

Since the measured state $\psi(x)$ is not changed during the above protective measurement (in the limit $T \to \infty$), the measurement result, namely the density $\rho(x)$, reflects (one part of) the actual physical state of the measured system. What density, then, is $\rho(x)$? If the observable A_n and the corresponding interaction Hamiltonian are physically realized by the electromagnetic or gravitational interaction between the measured system and the measuring device, then the measured density $\rho(x)$ (multiplied by the total charge or mass of the measured system) will be the charge density or mass density of the measured system[6]. In other words, the measurement result will show that the mass and charge of a quantum system such as an electron is distributed throughout space, and the mass and charge density of the system in each position x is proportional to the modulus squared of its wave function there, namely the density $\rho(x)$. In the following, we will give a more specific example to illustrate this important result.

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[6]Strictly speaking, the mass density is $m|\psi(x)|^2 + \psi^* H \psi / c^2$ in the non-relativistic domain, but the second term is very small compared with the first term and can be omitted.

3.2 A specific example

Consider the spatial wave function of a single quantum system with negative charge Q (e.g. $Q = -e$)

$$\psi(x,t) = a\psi_1(x,t) + b\psi_2(x,t), \tag{15}$$

where $\psi_1(x,t)$ and $\psi_2(x,t)$ are two normalized wave functions respectively localized in their ground states in two small identical boxes 1 and 2, and $|a|^2 + |b|^2 = 1$. An electron, which initial state is a Gaussian wave packet narrow in both position and momentum, is shot along a straight line near box 1 and perpendicular to the line of separation between the boxes. The electron is detected on a screen after passing by box 1. Suppose the separation between the boxes is large enough so that a charge Q in box 2 has no observable influence on the electron. Then if the system were in box 2, namely $|a|^2 = 0$, the trajectory of the electron wave packet would be a straight line as indicated by position "0" in Fig.1. By contrast, if the system were in box 1, namely $|a|^2 = 1$, the trajectory of the electron wave packet would be deviated by the electric field of the system by a maximum amount as indicated by position "1" in Fig.1.

We first suppose that $\psi(x,t)$ is unprotected, then the wave function of the combined system after interaction will be

$$\psi(x,x',t) = a\varphi_1(x',t)\psi_1(x,t) + b\varphi_2(x',t)\psi_2(x,t), \tag{16}$$

where $\varphi_1(x',t)$ and $\varphi_2(x',t)$ are the wave functions of the electron influenced by the electric fields of the system in box 1 and box 2, respectively, the trajectory of $\varphi_1(x',t)$ is deviated by a maximum amount, and the trajectory of $\varphi_2(x',t)$ is not deviated and still a straight line. When the electron is detected on the screen, the above wave function will collapse to $\varphi_1(x',t)\psi_1(x,t)$ or $\varphi_2(x',t)\psi_2(x,t)$. As a result, the detected position of the electron will be either "1" or "0" in Fig.1, indicating that the system is in box 1 or 2 *after* the detection. This is a conventional impulsive measurement of the projection operator on the spatial region of box 1, denoted by A_1. A_1 has two eigenstates corresponding to the system being in box 1 and 2, respectively, and the corresponding eigenvalues are 1 and 0, respectively. Since the measurement is accomplished through the electrostatic interaction between two charges, the measured observable A_1, when multiplied by the charge Q, is actually the observable for the charge of the system in box 1, and its eigenvalues are Q and 0, corresponding to the charge Q being in boxes 1 and 2, respectively. Such a measurement cannot tell us the charge distribution of the system in each box *before* the measurement.
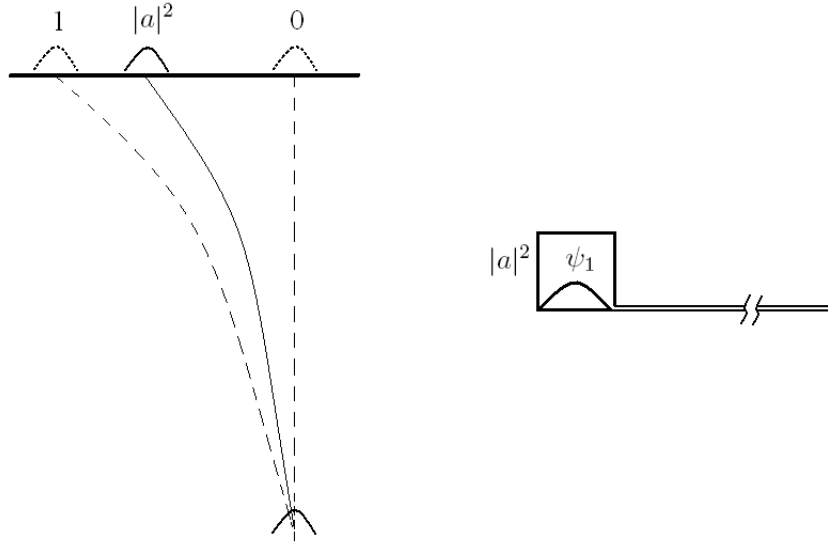
Fig.1 Scheme of a protective measurement of the charge density of a quantum system

Now let's make a protective measurement. Since $\psi(x,t)$ is degenerate with its orthogonal state $\psi'(x,t) = b^*\psi_1(x,t) - a^*\psi_2(x,t)$, we need an artificial protection procedure to remove the degeneracy, e.g. joining the two boxes with a long tube whose diameter is small compared to the size of the box[7]. By this protection $\psi(x,t)$ will be a nondegenerate energy eigenstate. The adiabaticity condition and the weakly interacting condition, which are required for a protective measurement, can be further satisfied when assuming that (1) the measuring time of the electron is long compared to $\hbar/\Delta E$, where ΔE is the smallest of the energy differences between $\psi(x,t)$ and the other energy eigenstates, and (2) at all times the potential energy of interaction between the electron and the system is small compared to ΔE. Then the measurement by means of the trajectory of the electron wave packet is a protective measurement of the charge of the system in box 1. The trajectory of the electron wave packet is only influenced by the expectation value of the charge of the system in box 1, and the wave packet will reach the position "$|a|^2$" between "0" and "1" on the screen as denoted in Fig.1. Since the measurement does not disturb the measured system, its result reflects the actual charge distribution of the system in box 1. Concretely speaking, the result of this protective measurement, namely the expectation value of the charge Q in the state $\psi_1(x,t)$, $|a|^2 Q$, indicates that there exists a charge $|a|^2 Q$ in box 1.

Here it may be necessary to further clarify the meaning of charge distribution as a property of a quantum system. Any physical measurement

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[7]It is worth stressing that the added protection procedure depends on the measured state, and different states need different protection procedures in general.

is necessarily based on some interaction between the measured system and the measuring system. One basic form of interaction is the electrostatic interaction between two electric charges as in the above example, and the existence of this interaction during a measurement, which is indicated by the deviation of the trajectory of the charged measuring system such as an electron, means that the measured system also has the charge responsible for the interaction[8]. Then at least in the sense that any part of a physical entity has electrostatic interaction with another charged system such as another electron, we can say that the physical entity has charge distribution in space[9]. In the above example, the definite deviation of the trajectory of the electron will reflect that there exists a definite amount of charge in box 1, and the extent of the deviation will further indicate how much charge there is there.

It should be noted that the existence of such a charge distribution does not imply that two quantum systems interact directly by way of their charge distributions as in classical mechanics. In other words, the existence of the charge distribution can be consistent with quantum mechanics, in which the interaction between two quantum systems is always described by the interaction potentials in the Schrödinger equation. As we will see in the next section, however, the consistency will restrict and even determine the existing form of the charge distribution of a quantum system.

4   The physical origin of the mass and charge distributions

We have argued that the mass and charge of a quantum system are distributed throughout space, and the mass and charge density in each position

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[8]If one denies this point, then it seems that one cannot obtain any information about the measured system by the measurement. Note that the arguments against the naive realism about operators and the eigenvalue realism in the quantum context are irrelevant here (Daumer et al 1997; Valentini 2010).

[9]This is consistent with the anti-Humean position about laws of nature in contemporary philosophy. According to this view, laws are grounded in the ontology, and the theoretical terms (expressed in the language of mathematics) connect to the entities existing in the physical world. It is essential for a property to induce a certain behaviour of the objects that instantiate the property in question, while the law expresses that behaviour. For example, the parameter we call "charge" in the Schrödinger equation refers to a property of quantum systems. This property is not a pure quality, but a disposition whose manifestation is the electromagnetic interaction between the systems as expressed qualitatively and quantitatively by the Schrödinger equation. In this way, laws are suitable to figure in explanations answering why-questions, and they reveal the real connections that there are in nature. By contrast, according to Humeanism, the laws are mere means of economical description, and they do not have any explanatory function. They sum up what has happened in the world; but they do not answer the question why what has happened did in fact happen, given certain initial conditions. Note that there are a number of substantial philosophical objections against Humeanism (see e.g. Mumford 2004).

is proportional to the modulus squared of the wave function of the system there. In this section, we will further investigate the physical origin of the mass and charge distributions. As we will see, the answer may provide an important clue to the meaning of the wave function.

Historically, the charge density interpretation for electrons was originally suggested by Schrödinger when he introduced the wave function and founded wave mechanics (Schrödinger 1926). Schrödinger clearly realized that the charge distribution cannot be of classical nature because his equation does not include the usual classical interaction between the distributions. Presumably since people thought that the charge distribution could not be measured and also lacked a consistent physical picture, this initial interpretation of the wave function was soon rejected and replaced by Born's probability interpretation (Born 1926). Now protective measurement reendows the charge distribution of an electron with reality. The question is then how to find a consistent physical explanation for it[10]. Our following analysis can be regarded as a further development of Schrödinger's idea to some extent. The twist is that the charge distribution is not classical does not imply its non-existence; rather, its existence points to a non-classical picture of quantum reality hiding behind the wave function.

4.1 The mass and charge distributions are effective

As noted earlier, the expectation values of observables are the properties of a quantum system defined either at a precise instant or during an infinitesimal time interval. Correspondingly, the mass and charge distributions of a quantum system, which can be protectively measured as the expectation values of certain observables, have two possible existent forms: it is either real or effective. The distribution is real means that it exists throughout space at the same time. The distribution is effective means that at every instant there is only a localized, point-like particle with the total mass and charge of the system, and its motion during an infinitesimal time interval forms the effective distribution. Concretely speaking, at a particular instant the mass and charge density of the particle in each position is either zero (if the particle is not there) or singular (if the particle is there), while the time average of the density during an infinitesimal time interval gives the effective mass and charge density. Moreover, the motion of the particle is ergodic in the sense that the integral of the formed mass and charge density in any region is required to be equal to the expectation value of the total mass and charge in the region. In the following, we will determine the existent form of the mass and charge distributions of a quantum system.

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[10]Note that the proponents of protective measurement did not give an analysis of the origin of the charge distribution. According to them, this type of measurement implies that the wave function of a single quantum system is ontological, i.e., that it is a real physical wave (Aharonov, Anandan and Vaidman 1993).

If the mass and charge distributions are real, then any two parts of the distributions, e.g. the two wavepackets in box 1 and box 2 in the example given in the last section, will have gravitational and electrostatic interactions described by the interaction potential terms in the Schrödinger equation[11]. The existence of such gravitational and electrostatic self-interactions for individual quantum systems is inconsistent with the superposition principle of quantum mechanics (at least for microscopic systems such as electrons). Moreover, the existence of the electrostatic self-interaction for the charge distribution of an electron also contradicts experimental observations. For example, for the electron in the hydrogen atom, since the potential of the electrostatic self-interaction is of the same order as the Coulomb potential produced by the nucleus, the energy levels of hydrogen atoms will be remarkably different from those predicted by quantum mechanics and confirmed by experiments if there exists such electrostatic self-interaction. By contrast, if the mass and charge distributions are effective, then there will be only a localized particle at every instant, and thus there will exist no gravitational and electrostatic self-interactions of the effective distributions. This is consistent with the superposition principle of quantum mechanics and the Schrödinger equation.

Since this argument is pivotal for our later discussions, we will give a more detailed analysis here. It can be seen that the existence of the mass and charge distributions poses a puzzle. According to quantum mechanics, two charge distributions such as two electrons, which exist in space at the same time, have electrostatic interaction described by the interaction potential term in the Schrödinger equation, but in the example given in the last section, the two charges in box 1 and box 2 have no such electrostatic interaction. This puzzle is not so much dependent on the existence of mass and charge distributions as properties of a quantum system. It is essentially that according to quantum mechanics, the wavepacket ψ_1 in box 1 has interaction with any test electron (e.g. deviating the trajectory of the electron wavepacket), so does the wavepacket ψ_2 in box 2, but these two wavepackets, unlike two electrons, have no interaction.

Facing this puzzle one may have two choices. The first one is simply admitting that this is a distinct feature of the laws of quantum mechanics, but insisting that the laws are what they are and no further explanation is needed. In our opinion, this choice seems to beg the question and is unsatisfactory in the final analysis. A more reasonable choice is to try to explain this puzzling feature of the evolution of the wave function, which is governed by the Schrödinger equation[12]. After all, there is only one actual

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[11]According to quantum mechanics, two real mass and charge distributions such as two electrons have gravitational and electrostatic interactions described by the interaction potential terms in the Schrödinger equation. Moreover, these two distributions will be entangled and their wave function will be defined in a six-dimensional configuration space.

[12]An immediate explanation may be that why the two wavepackets with charges have no

form of the mass and charge distributions, while there are two possible forms as given above, and we need to determine which possible form is the actual one.

The above argument provides an answer to this question[13]. The reason why two wavepackets of an electron, each of which has part of the electron's charge, have no electrostatic interaction is that these two wavepackets do not exist at the same time, and their charges are not real but effective, formed by the motion of a localized particle with the total charge of the electron. If the two wavepackets with charges, like two electrons, existed at the same time, then they would also have the same form of electrostatic interaction as that between two electrons. The lack of such interaction then indicates that the two wavepackets of an electron exist in a way of time division, and their charges are effectively formed by the motion of a localized particle with the total charge of the electron. Since in this case there is only a localized particle at every instant, there exist no electrostatic self-interactions of the effective charge distribution formed by the motion of the particle. Note that this argument does not assume that real charges that exist at the same time are classical charges and they have classical interaction[14].

To sum up, we have argued that the superposition principle of quantum mechanics requires that the mass and charge distributions of a quantum system such as an electron are not real but effective; at every instant there is only a localized particle with the total mass and charge of the system, while during an infinitesimal time interval the ergodic motion of the particle forms the effective mass and charge distributions, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there.

4.2 The ergodic motion of a particle is discontinuous

Which sort of ergodic motion? This is a further question. If the ergodic motion of a particle is continuous, then it can only form the effective mass and charge density during a finite time interval. But according to quantum mechanics, the effective mass and charge density is required to be formed by the ergodic motion of the particle during an infinitesimal time interval (*not* during a finite time interval) near a given instant. Thus it seems that the ergodic motion of the particle cannot be continuous. This is at least

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electrostatic interaction is because they belong to one quantum system such as an electron, and if they belong to two charged quantum systems such as two electrons, then they will have electrostatic interaction. However, this explanation seems still unsatisfactory, and one may further ask why two wavepackets of a charged quantum system such as an electron, each of which has charge, have no electrostatic interaction.

[13]In some sense, this argument provides an explanation of why there is no gravitational and electrostatic self-interaction terms in the Schrödinger equation.

[14]By contrast, the Schrödinger-Newton equation, which was proposed by Diosi (1984) and Penrose (1998), describes the gravitational self-interaction of classical mass density.

what the existing theory says. However, there may exist a possible loophole here. Although the classical ergodic models that assume continuous motion are inconsistent with quantum mechanics due to the existence of a finite ergodic time, they may be not completely precluded by experiments if only the ergodic time is extremely short. After all quantum mechanics is only an approximation of a more fundamental theory of quantum gravity, in which there may exist a minimum time scale such as the Planck time. Therefore, we need to investigate the classical ergodic models more thoroughly.

Consider an electron in a one-dimensional box in the first excited state $\psi(x)$ (Aharonov and Vaidman 1993). Its wave function has a node at the center of the box, where its charge density is zero. Assume the electron performs a very fast continuous motion in the box, and during a very short time interval its motion generates an effective charge distribution. Let's see whether this distribution can assume the same form as $e|\psi(x)|^2$, which is required by protective measurement[15]. Since the effective charge density is proportional to the amount of time the electron spends in a given position, the electron must be in the left half of the box half of the time and in the right half of the box half of the time. But it can spend no time at the center of the box where the effective charge density is zero; in other words, it must move at infinite velocity at the center. Certainly, the appearance of velocities faster than light or even infinite velocities may be not a fatal problem, as our discussion is entirely in the context of non-relativistic quantum mechanics, and especially the infinite potential in the example is also an ideal situation. However, it seems difficult to explain why the electron speeds up at the node and where the infinite energy required for the acceleration comes from.

Let's further consider an electron in a superposition of two energy eigen-states in two boxes $\psi_1(x) + \psi_2(x)$. In this example, even if one assumes that the electron can move with infinite velocity (e.g. at the nodes), it cannot *continuously* move from one box to another due to the restriction of box walls. Therefore, any sort of continuous motion cannot generate the effective charge distribution $e|\psi_1(x) + \psi_2(x)|^2$. One may still object that this is merely an artifact of the idealization of infinite potential. However, even in this ideal situation, the model should also be able to generate the effective charge distribution by means of some sort of ergodic motion of the electron; otherwise it will be inconsistent with quantum mechanics. On the other hand, it is very common in quantum optics experiments that a single-photon wave packet is split into two branches moving along two well

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[15]Note that in Nelson's stochastic mechanics, the electron, which is assumed to undergo a Brownian motion, moves only within a region bounded by the nodes (Nelson 1966). This ensures that the theory can be equivalent to quantum mechanics in a limited sense. Obviously this sort of motion is not ergodic and cannot generate the required charge distribution. This conclusion also holds true for the motion of particles in some variants of stochastic mechanics (Bell 1986; Vink 1993), as well as in the de Broglie-Bohm theory (de Broglie 1928; Bohm 1952).

separated paths in space. The wave function of the photon disappears outside the two paths for all practical purposes. Moreover, the experimental results are not influenced by the environment and experimental setup between the two paths of the photon. Thus it seems impossible that the photon performs a continuous ergodic motion back and forth in the space between its two paths.

In view of these drawbacks of the classical ergodic models and their inconsistency with quantum mechanics, we conclude that the ergodic motion of particles cannot be continuous. If the motion of a particle is essentially discontinuous, then the particle can readily appear throughout all regions where the wave function is nonzero during an arbitrarily short time interval near a given instant. Furthermore, if the probability density of the particle appearing in each position is proportional to the modulus squared of its wave function there at every instant, the discontinuous motion can also generate the right mass and charge distributions. This will solve the above problems plagued by the classical ergodic models. The discontinuous ergodic motion requires no existence of a finite ergodic time. Moreover, a particle undergoing discontinuous motion can also "jump" from one region to another spatially separated region, no matter whether there is an infinite potential wall between them, and such discontinuous motion is not influenced by the environment and experimental setup between these regions either.

4.3   An argument for random discontinuous motion

We have argued that the ergodic motion of a particle is discontinuous. However, the argument doesn't require that the discontinuous motion must be random. It is possible that the randomness of the result of a quantum measurement is only apparent. In order to know whether the motion of particles is random or not, we need to analyze the cause of motion. For example, if motion has no *deterministic* cause, then it will be random, only determined by a probabilistic cause. This may also be the right way to find how particles move. Since motion involves change in position, if we can find the cause or instantaneous condition determining the change[16], we will be able to find how particles move.

Let's consider the simplest states of motion of a free particle, for which the instantaneous condition determining the change of its position is a constant during the motion. The instantaneous condition can be deterministic or indeterministic. That the instantaneous condition is deterministic means that it leads to a deterministic change of the position of the particle at a given instant. That the instantaneous condition is indeterministic means that it only determines the probability of the particle appearing in each

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[16]The word "cause" used here only denotes a certain instantaneous condition determining the change of position, which may appear in the laws of motion. Our analysis is independent of whether the condition has causal power or not.

position in space at a given instant. If the instantaneous condition is deterministic, then the simplest states of motion of the free particle will have two possible forms. The first one is continuous motion with constant velocity, and the equation of motion of the particle is $x(t + dt) = x(t) + vdt$, where the deterministic instantaneous condition v is a constant[17]. The second one is discontinuous motion with infinite average velocity; the particle performs a finite jump along a fixed direction at every instant, where the jump distance is a constant, determined by the constant instantaneous condition[18]. On the other hand, if the instantaneous condition is indeterministic, then the simplest states of motion of the free particle will be random discontinuous motion with even position probability distribution. At each instant the probability density of the particle appearing in every position is the same.

In order to know whether the instantaneous condition is deterministic or not, we need to determine which sort of simplest states of motion are the solutions of the equation of free motion in quantum mechanics (i.e. the free Schrödinger equation). According to the analysis in the last subsection, the momentum eigenstates of a free particle, which are the solutions of the free Schrödinger equation, describe the ergodic motion of the particle with even position probability distribution in space. Therefore, the simplest states of motion with a constant probabilistic instantaneous condition are the solutions of the equation of free motion, while the simplest states of motion with a constant deterministic instantaneous condition are not.

When assuming that (1) the simplest states of motion of a free particle are the solutions of the equation of free motion; and (2) the instantaneous condition determining the position change of a particle is always deterministic or indeterministic for any state of motion, the above result then implies that motion, no matter whether it is free or forced, has no deterministic cause, and thus it is random and discontinuous, only determined by a probabilistic cause. The argument may be improved by further analyzing these two seemingly reasonable assumptions, but we will leave this for future work.

5   The wave function as a description of random discontinuous motion of particles

In classical mechanics, we have a clear physical picture of motion. It is well understood that the trajectory function $x(t)$ in classical mechanics describes the continuous motion of a particle. In quantum mechanics, the trajectory function $x(t)$ is replaced by a wave function $\psi(x, t)$. If the particle ontology

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[17]This deterministic instantaneous condition is often called intrinsic velocity (Tooley 1988).

[18]In discrete space and time, the motion will be a discrete jump across space along a fixed direction at each time unit, and thus it will become continuous motion with constant velocity in the continuous limit.

is still viable in the quantum domain, then it seems natural that the wave function should describe some sort of more fundamental motion of particles, of which continuous motion is an approximation in the classical domain, as quantum mechanics is a more fundamental theory of the physical world, of which classical mechanics is an approximation. The analysis in the last section provides a strong support for this conjecture. It suggests that a quantum system such as an electron is a localized particle that undergoes random discontinuous motion, and the probability density of the particle appearing in each position is proportional to the modulus squared of its wave function there. As a result, the wave function in quantum mechanics can be regarded as a description of the more fundamental motion of particles, which is arguably discontinuous and random. In this section, we will give a more detailed analysis of random discontinuous motion and the meaning of the wave function (Gao 2011).

5.1   An analysis of random discontinuous motion of particles

Let's first make clearer what we mean when we say a quantum system such as an electron is a particle. The picture of particles appears from our analysis of the mass and charge density of a quantum system. As we have argued in the last section, the mass and charge density of an electron, which is measurable by protective measurement and proportional to the modulus squared of its wave function, is not real but effective; it is formed by the ergodic motion of a localized particle with the total mass and charge of the electron. If the mass and charge density is real, i.e., if the mass and charge distributions at different locations exist at the same time, then there will exist gravitational and electrostatic interactions between the distributions, the existence of which not only contradicts experiments but also violates the superposition principle of quantum mechanics. It is this analysis that leads us to the basic existent form of a quantum system such as an electron in space and time; an electron is a particle[19]. Here the concept of particle is used in its usual sense. A particle is a small localized object with mass and charge, and it is only in one position in space at an instant. However, as we have argued above, the motion of an electron described by its wave function is not continuous but discontinuous and random in nature. We may say that an electron is a quantum particle in the sense that its motion is not continuous motion described by classical mechanics, but random discontinuous motion described by quantum mechanics.

Next, let's analyze the random discontinuous motion of particles. From a logical point of view, for the random discontinuous motion of a particle, the particle must have an instantaneous property (as a probabilistic instantaneous condition) that determines the probability density to appear in every

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[19]However, the analysis cannot tell us the precise size and possible structure of electron.

position in space; otherwise the particle would not "know" how frequently it should appear in each position in space. This property is usually called indeterministic disposition or propensity in the literature[20], and it can be represented by $\varrho(x,t)$, which satisfies the nonnegative condition $\varrho(x,t) \geqslant 0$ and the normalization relation $\int_{-\infty}^{+\infty} \varrho(x,t)dx = 1$. As a result, the position of the particle at every instant is random, and its trajectory formed by the random position series is also discontinuous at every instant[21].

Unlike the deterministic continuous motion, the trajectory function $x(t)$ no longer provides a useful description for random discontinuous motion. In the following, we will give a strict description of random discontinuous motion of particles based on measure theory. For simplicity but without losing generality, we will mainly analyze the one-dimensional motion that corresponds to the point set in two-dimensional space and time. The results can be readily extended to the three-dimensional situation.
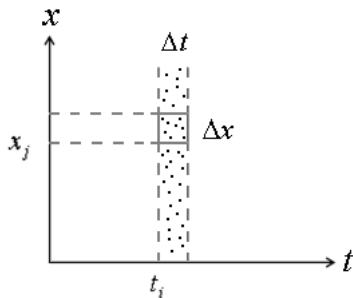

Fig.2 The description of random discontinuous motion of a single particle

We first analyze the random discontinuous motion of a single particle. Suppose the probability density of the particle appearing in position x at instant t is determined by a disposition function $\varrho(x,t)$, which is differen-

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[20]Note that the propensity here denotes single case propensity. For long run propensity theories fail to explain objective single-case probabilities. According to these theories, it makes no sense to speak of the propensity of a single isolated event in the absence of a sequence that contains it. For a helpful analysis of the single-case propensity interpretation of probability in GRW theory see Frigg and Hoefer (2007). In addition, it is worth stressing that the propensities possessed by particles relate to their objective motion, not to the measurements on them. By contrast, according to the existing propensity interpretations of quantum mechanics, the propensities a quantum system has relate only to measurements; a quantum system possesses the propensity to exhibit a particular value of an observable if the observable is measured on the system (see Suárez 2004 for a comprehensive analysis). Like the Copenhagen interpretation of quantum mechanics, it seems that these interpretations cannot be wholly satisfactory because of resorting to the vague concept of measurement.

[21]However, there is an exception. When the probability density function is a special δ-function such as $\delta(x - x(t))$, where $x(t)$ is a continuous function of t, the motion of the particle is deterministic and continuous. In addition, even for a general probability density function it is still possible that the random position series forms a continuous trajectory, though the happening probability is zero.

tiable with respect to both x and t. Consider the state of motion of the particle in finite intervals Δt and Δx near a space-time point (t_i, x_j) as shown in Fig. 2. The positions of the particle form a random, discontinuous trajectory in this square region[22]. We study the projection of this trajectory in the t-axis, which is a dense instant set in the time interval Δt. Let W be the discontinuous trajectory of the particle and Q be the square region $[x_j, x_j + \Delta x] \times [t_i, t_i + \Delta t]$. The dense instant set can be denoted by $\pi_t(W \cap Q) \in \Re$, where π_t is the projection on the t-axis. According to the measure theory, we can define the Lebesgue measure:

$$M_{\Delta x, \Delta t}(x_j, t_i) = \int_{\pi_t(W \cap Q) \in \Re} dt. \tag{17}$$

Since the sum of the measures of all such dense instant sets in the time interval Δt is equal to the length of the continuous time interval Δt, we have:

$$\sum_j M_{\Delta x, \Delta t}(x_j, t_i) = \Delta t. \tag{18}$$

Then we can define the measure density as follows:

$$\rho(x, t) = \lim_{\Delta x, \Delta t \to 0} M_{\Delta x, \Delta t}(x, t)/(\Delta x \cdot \Delta t). \tag{19}$$

This quantity provides a strict description of the position distribution of the particle or the relative frequency of the particle appearing in an infinitesimal space interval dx near position x during an infinitesimal interval dt near instant t, and it satisfies the normalization relation $\int_{-\infty}^{+\infty} \rho(x, t) dx = 1$ by Eq. (18). Note that the existence of the limit relies on the continuity of the evolution of $\varrho(x, t)$, the property of the particle that determines the probability density for it to appear in every position in space. In fact, $\rho(x, t)$ is determined by $\varrho(x, t)$, and there exists the relation $\rho(x, t) = \varrho(x, t)$. We call $\rho(x, t)$ position measure density or position density in brief.

Since the position density $\rho(x, t)$ changes with time in general, we may further define the position flux density $j(x, t)$ through the relation $j(x, t) = \rho(x, t)v(x, t)$, where $v(x, t)$ is the velocity of the local position density. It describes the change rate of the position density. Due to the conservation of probability, $\rho(x, t)$ and $j(x, t)$ satisfy the continuity equation:

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial j(x, t)}{\partial x} = 0. \tag{20}$$

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[22]Recall that a trajectory function $x(t)$ is essentially discontinuous if it is not continuous at every instant t. A trajectory function $x(t)$ is continuous if and only if for every t and every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that whenever a point t_0 has distance less than δ to t, the point $x(t_0)$ has distance less than ε to $x(t)$.

The position density $\rho(x,t)$ and position flux density $j(x,t)$ provide a complete description of the state of random discontinuous motion of a single particle[23].

The description of the motion of a single particle can be extended to the motion of many particles. At each instant a quantum system of N particles can be represented by a point in an $3N$-dimensional configuration space. Then, similar to the single particle case, the state of the system can be represented by the joint position density $\rho(x_1, x_2, ...x_N, t)$ and joint position flux density $j(x_1, x_2, ...x_N, t)$ defined in the configuration space. They also satisfy the continuity equation:

$$\frac{\partial \rho(x_1, x_2, ...x_N, t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial j(x_1, x_2, ...x_N, t)}{\partial x_i} = 0. \tag{21}$$

The joint position density $\rho(x_1, x_2, ...x_N, t)$ represents the probability density of particle 1 appearing in position x_1 and particle 2 appearing in position x_2, , and particle N appearing in position x_N. When these N particles are independent, the joint position density can be reduced to the direct product of the position density for each particle, namely $\rho(x_1, x_2, ...x_N, t) = \prod_{i=1}^{N} \rho(x_i, t)$. Note that the joint position density $\rho(x_1, x_2, ...x_N, t)$ and joint position flux density $j(x_1, x_2, ...x_N, t)$ are not defined in the real three-dimensional space, but defined in the 3N-dimensional configuration space.

5.2   Interpreting the wave function

Although the motion of particles is essentially discontinuous and random, the discontinuity and randomness of motion are absorbed into the state of motion, which is defined during an infinitesimal time interval and represented by the position density $\rho(x,t)$ and position flux density $j(x,t)$. Therefore, the evolution of the state of random discontinuous motion of particles may obey a deterministic continuous equation. By assuming that the nonrelativistic equation of random discontinuous motion is the Schrödinger equation in quantum mechanics, both $\rho(x,t)$ and $j(x,t)$ can be expressed by the wave function in a unique way[24]:

$$\rho(x,t) = |\psi(x,t)|^2, \tag{22}$$

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[23]It is also possible that the position density $\rho(x,t)$ alone provides a complete description of the state of random discontinuous motion of a particle. Which one is right depends on the laws of motion. As we will see later, quantum mechanics requires that a complete description of the state of random discontinuous motion of particles includes both the position density and the position flux density.

[24]Note that the relation between $j(x,t)$ and $\psi(x,t)$ depends on the concrete evolution under an external potential such as electromagnetic vector potential. By contrast, the relation $\rho(x,t) = |\psi(x,t)|^2$ holds true universally, independently of the concrete evolution.

$$j(x,t) = \frac{\hbar}{2mi}[\psi^*(x,t)\frac{\partial \psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial \psi^*(x,t)}{\partial x}]. \qquad (23)$$

Correspondingly, the wave function $\psi(x,t)$ can be uniquely expressed by $\rho(x,t)$ and $j(x,t)$ (except for a constant phase factor):

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{im\int_{-\infty}^{x}\frac{j(x',t)}{\rho(x',t)}dx'/\hbar}. \qquad (24)$$

In this way, the wave function $\psi(x,t)$ also provides a complete description of the state of random discontinuous motion of particles. For the motion of many particles, the joint position density and joint position flux density are defined in the 3N-dimensional configuration space, and thus the many-particle wave function, which is composed of these two quantities, is also defined in the 3N-dimensional configuration space.

Interestingly, we can reverse the above logic in some sense, namely by assuming the wave function is a complete objective description for the motion of particles, we can also reach the random discontinuous motion of particles, independently of our previous analysis. If the wave function $\psi(x,t)$ is a complete description of the state of motion for a single particle, then the quantity $|\psi(x,t)|^2dx$ will not only give the probability of the particle being found in an infinitesimal space interval dx near position x at instant t (as required by quantum mechanics), but also give the objective probability of the particle being there at the instant. This accords with the common-sense assumption that the probability distribution of the measurement results of a property is the same as the objective distribution of the values of the property in the measured state. Then at instant t the particle will be in a random position where the probability density $|\psi(x,t)|^2$ is nonzero, and during an infinitesimal time interval near instant t it will move throughout the whole region where the wave function $\psi(x,t)$ spreads. Moreover, its position density in each position is equal to the probability density there. Obviously this kind of motion is random and discontinuous.

One important point needs to be pointed out here. Since the wave function in quantum mechanics is defined at instants, not during an infinitesimal time interval, it should be regarded not simply as a description of the state of random discontinuous motion of particles, but more suitably as a description of the property of the particles that determines their random discontinuous motion at a deeper level[25]. In particular, the modulus squared of the wave function represents the property that determines the probability density of the particles appearing in certain positions in space at a given instant (this means $\varrho(x,t) \equiv |\psi(x,t)|^2$). By contrast, the position density and position flux density, which are defined during an infinitesimal time interval near a given instant, are only a description of the state of the resulting random

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[25]For a many-particle system in an entangled state, this property is possessed by the whole system.

discontinuous motion of particles, and they are determined by the wave function. In this sense, we may say that the motion of particles is "guided" by their wave function in a probabilistic way.

5.3　On momentum, energy and spin

We have been discussing random discontinuous motion of particles in real space. Does the picture of random discontinuous motion exist for other dynamical variables such as momentum and energy? Since there are also wave functions of these variables in quantum mechanics, it seems tempting to assume that the above interpretation of the wave function in position space also applies to the wave functions in momentum space etc[26]. This means that when a particle is in a superposition of the eigenstates of a variable, it also undergoes random discontinuous motion among the corresponding eigenvalues of this variable. For example, a particle in a superposition of energy eigenstates also undergoes random discontinuous motion among all energy eigenvalues. At each instant, the energy of the particle is definite, randomly assuming one of the energy eigenvalues with probability given by the modulus squared of the wave function at this energy eigenvalue, and during an infinitesimal time interval, the energy of the particle spreads throughout all energy eigenvalues. Since the values of two noncommutative variables (e.g. position and momentum) at every instant may be mutually independent, the objective value distribution of every variable can be equal to the modulus squared of its wave function and consistent with quantum mechanics[27].

However, there is also another possibility, namely that the picture of random discontinuous motion exists only for position, while momentum, energy etc do not undergo random discontinuous change among their eigenvalues. This is a minimum formulation in the sense that the ontology of the theory only includes the wave function and the particle position. A heuristic argument for this possibility is as follows. In quantum mechanics, the definitions of momentum and energy relate to spacetime translation. The momentum operator and energy operator are defined as the generators of space translation and time translation, respectively. By these definitions momentum and energy seem distinct from position. For random discontinuous motion of particles, the position of a particle is its primary property defined at instants, while momentum and energy are secondary properties relating only to its state of motion (e.g. momentum and energy eigenstates), which is

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[26]Under this assumption, the ontology of the theory will not only include the wave function and the particle position, but also include momentum and energy.

[27]Note that for random discontinuous motion a property (e.g. position) of a quantum system in a superposed state of the property is indeterminate in the sense of usual hidden variables, though it does have a definite value at each instant. This makes the theorems that restrict hidden variables such as the Kochen-Specker theorem (Kochen and Specker 1967) irrelevant.

formed by the motion of the particle. In other words, position is an instantaneous property of a particle, while momentum and energy are only manifestations of its state of motion during an infinitesimal time interval. Note that the particle position here is different from the position property described by the position operator in quantum mechanics, and the latter is also a secondary property relating only to the state of motion of the particle such as position eigenstates. Certainly, we can still talk about momentum and energy on this view. For example, when a particle is in an eigenstate of the momentum or energy operator, we can say that the particle has definite momentum or energy, whose value is the corresponding eigenvalue. Moreover, when the eigenstates of the momentum or energy operator are well separated in space, we can still say that the particle has definite momentum or energy in certain local regions.

Lastly, we note that spin is a more distinct property. Since the spin of a free particle is always definite along one direction, the spin of the particle does not undergo random discontinuous motion, though a spin eigenstate along one direction can always be decomposed into two different spin eigenstates along another direction. But if the spin state of a particle is entangled with its spatial state due to interaction and the branches of the entangled state are well separated in space, the particle in different branches will have different spin, and it will also undergo random discontinuous motion between these different spin states. This is the situation that usually happens during a spin measurement.

6 Conclusions

In this paper, we have argued that protective measurement may have important implications for the physical meaning of the wave function. There are three key steps in the argument. First of all, the results of protective measurements as predicted by quantum mechanics show that the mass and charge of a charged quantum system are distributed throughout space, and the mass and charge density in each position is proportional to the modulus squared of the wave function of the system there. Next, the superposition principle of quantum mechanics requires that the mass and charge distributions are effective, that is, they are formed by the ergodic motion of a localized particle with the total mass and charge of the system. Lastly, the consistency of the formed distribution with that predicted by quantum mechanics requires that the ergodic motion of the particle is discontinuous, and the probability density of the particle appearing in every position is equal to the modulus squared of its wave function there. Based on this analysis, we suggest that the wave function in quantum mechanics describes the state of random discontinuous motion of particles, and at a deeper level, it represents the property of the particles that determines their random dis-

continuous motion. In particular, the modulus squared of the wave function (in position space) gives the probability density of the particles being in certain positions in space.

Acknowledgments

References

[1] Aharonov, Y., Anandan, J. and Vaidman, L. (1993). Meaning of the wave function, Phys. Rev. A 47, 4616.

[2] Aharonov, Y., Anandan, J. and Vaidman, L. (1996). The meaning of protective measurements, Found. Phys. 26, 117.

[3] Aharonov, Y. and Vaidman, L. (1993). Measurement of the Schrödinger wave of a single particle, Phys. Lett. A 178, 38.

[4] Anandan, J. (1993). Protective Measurement and Quantum Reality. Found. Phys. Lett., 6, 503-532.

[5] Bell, J. S. (1986). Beables for quantum field theory. Phys. Rep. 137, 49-54.

[6] Bell, J. S. (1990). Against measurement, in A. I. Miller (ed.), Sixty-Two Years of Uncertainty: Historical Philosophical and Physics Enquiries into the Foundations of Quantum Mechanics. Berlin: Springer, 17-33.

[7] Bohm, D. (1952). A suggested interpretation of quantum theory in terms of "hidden" variables, I and II. Phys. Rev. 85, 166-193.

[8] Born, M. (1926). Quantenmechanik der Stoβvorgänge. Z. Phys. 38, 803; English translation in Ludwig, G., eds., 1968, Wave Mechanics, Oxford: Pergamon Press: 206.

[9] Dass, N. D. H. and Qureshi, T. (1999). Critique of protective measurements. Phys. Rev. A 59, 2590.

[10] Daumer, M., Dürr, D., Goldstein, S., and Zangh, N. (1997). Naive Realism About Operators, Erkenntnis 45, 379-397.

[11] de Broglie, L. (1928). in: Electrons et photons: Rapports et discussions du cinquime Conseil de Physique Solvay, eds. J. Bordet. Paris: Gauthier-Villars. pp.105. English translation: G. Bacciagaluppi and A. Valentini (2009), Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference. Cambridge: Cambridge University Press.

[12] Dickson, M. (1995). An empirical reply to empiricism: protective measurement opens the door for quantum realism. Philosophy of Science 62, 122.

[13] Diósi, L. (1984). Gravitation and the quantum-mechanical localization of macro-objects. Phys. Lett. A 105, 199-202.

[14] Everett, H. (1957). Relative state formulation of quantum mechanics. Rev. Mod. Phys. 29, 454-462.

[15] Frigg, R and Hoefer, C. (2007). Probability in GRW Theory. Studies in the History and Philosophy of Modern Physics 38, 371-389.

[16] Gao, S. (2011). Interpreting Quantum Mechanics in Terms of Random Discontinuous Motion of Particles. http://philsci-archive.pitt.edu/9057/.

[17] Gao, S. (2012). On Uffink's alternative interpretation of protective measurements. http://philsci-archive.pitt.edu/9447/.

[18] Ghose, P. and Home, D. (1995). An analysisi of Aharonov-Anandan-Vaidman model, Found. Phys. 25, 1105.

[19] Kochen, S. and Specker, E. (1967). The Problem of Hidden Variables in Quantum Mechanics, J. Math. Mech. 17, 59-87.

[20] Mumford, S. (2004). Laws in nature. London: Routledge.

[21] Nelson, E. (1966). Derivation of the Schrödinger equation from Newtonian mechanics. Phys. Rev. 150, 10791085.

[22] Penrose, R. (1998). Quantum computation, entanglement and state reduction. Phil. Trans. R. Soc. Lond. A 356, 1927.

[23] Rovelli, C. (1994). Meaning of the wave function - Comment, Phys. Rev. A 50, 2788.

[24] Schrödinger, E. (1926). Quantizierung als Eigenwertproblem (Vierte Mitteilung). Ann. d. Phys. (4) 81, 109-139. English translation: Quantisation as a Problem of Proper Values. Part IV, Reprint in

Schrödinger, E. (1982). Collected Papers on Wave Mechanics. New York: Chelsea Publishing Company, pp. 102-123.

[25] Suárez, M. (2004), Quantum Selections, Propensities and the Problem of Measurement, British Journal for the Philosophy of Science, 55(2), 219-55.

[26] Tooley, M. (1988). In defence of the existence of states of motion. Philosophical Topics 16, 225-254.

[27] Uffink, J. (1999). How to protect the interpretation of the wave function against protective measurements. Phys. Rev. A 60, 3474-3481.

[28] Unruh, W. G. (1994). Reality and measurement of the wave function Phys. Rev. A 50, 882.

[29] Vaidman, L. (2009) Protective measurements, in Greenberger, D., Hentschel, K., and Weinert, F. (eds.), Compendium of Quantum Physics: Concepts, Experiments, History and Philosophy. Springer-Verlag, Berlin. pp.505-507.

[30] Valentini, A. (2010). De Broglie-Bohm Pilot-Wave Theory: Many Worlds in Denial? in Saunders, S., Barrett, J., Kent, A. and Wallace, D. (eds.). Many Worlds? Everett, Quantum Theory, and Reality. Oxford: Oxford University Press.

[31] Vink, J. C. (1993). Quantum mechanics in terms of discrete beables. Phys. Rev. A 48, 1808.