

# Three Merry Roads to T-violation

Bryan W. Roberts  
University of Southern California  
[www.usc.edu/bryanroberts](http://www.usc.edu/bryanroberts)

March 17, 2013

ABSTRACT. This paper is a tour of how the laws of nature can distinguish between the past and the future, or be *T-violating*. I argue that, in terms of the basic argumentative structure, there are basically just three approaches currently being explored. The first is an application of Curie’s Principle, together with the *CPT* theorem. The second route makes use of a principle due to Pasha Kabir which allows for a direct detection. The third route makes use of a Non-degeneracy Principle, and is related to the energy spectrum of elementary particles. I show how each provides a general template for detecting *T*-violation, illustrate each with an example, and discuss their prospects in extensions of particle physics beyond the standard model.

## 1. INTRODUCTION

Unlike thermal physics, the physics of fundamental particles does not normally distinguish between the past and the future. For example, a typical classical mechanical system never makes such distinction, although one can imagine strange systems that do<sup>1</sup>. And there was a time in the mid-20th century when this “invariance under time reversal” or *T*-invariance was *demand*ed, for example, even by Weinberg (1958), because of the great simplification it provided in the description of weakly interacting particles.

Well, a lot has changed since then, and a great deal of evidence has been accumulated which shows that, contrary to the early views of particle physicists, fundamental physics can be *T-violating* — it *does* distinguish between the past and the future! I do not wish to retell that story here. There are many sources<sup>2</sup>, which are really much better than me, that will explain to you all about the gritty but ingenious detections

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<sup>1</sup>For an overview of the classical case, see (Roberts 2013).

<sup>2</sup>For just one of my favorite recent book-length overviews, try (Bigi and Sanda 2009).

of  $T$ -violating interactions, the deep and beautiful theory underlying them, and how we can expect that theory to develop from here.

At this conference, I would like to attempt a different project, which is to draw out the basic analytic arguments underlying the various approaches to  $T$ -violation. I would like to cast these arguments into their bare skeletal form; to think about what makes them alike and distinct; and to ask how they may fare as particle physics changes is extended beyond what we know today. In sum, what I would like is to take a lighthearted tour – from a birds eye view, if you like – of the existing roads to  $T$ -violation.

What’s helpful about this perspective, I think, is that it makes clear that there are really only three distinct roads to  $T$ -violation where we stand today. They can be summarized as follows.

- (1) *T-Violation by Curie’s Principle.* Pierre Curie declared that there is never an asymmetric effect without an asymmetric cause. This idea, together with the so-called  $CPT$  theorem, was the road to the very first detection of  $T$ -violation in the 20th century.
- (2) *T-Violation by Kabir’s Principle.* Pasha Kabir pointed that, whenever the probability of an ordinary particle decay  $A \rightarrow B$  differs from that of the time-reversed decay  $B' \rightarrow A'$ , then we have  $T$ -violation. This provides a second road.
- (3) *T-Violation by a Non-degeneracy Principle.* Certain kinds of matter, such as an elementary electric dipole, turn out to be  $T$ -violating whenever their energy spectrum is non-degenerate<sup>3</sup>. This provides the final road, although it has not yet led to a successful detection of  $T$ -violation.

In the next three sections, I will explain each of these three roads to  $T$ -violation. Some of these roads are very exciting and surprising, especially if you have not travelled down them before, and I will try to keep things light-hearted. My explanations will begin with a somewhat abstract formulation of an analytic principle, and then an illustration how it provides a way to test for  $T$ -violation. I’ll end each section with a little discussion about the prospects for extensions of particle physics beyond the standard model, and in particular extensions in which the dynamical laws are not unitary.

Let’s start at the beginning.

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<sup>3</sup>A self-adjoint operator  $A$  in finite dimensions is *degenerate* if it has two orthogonal eigenvectors with the same eigenvalue. I will discuss this property in more detail below.

## 2. $T$ -VIOLATION BY CURIE'S PRINCIPLE

The first evidence that the laws of a certain kind of “weakly interacting” system are  $T$ -violating, rather incredibly, was produced in 1964, *with little knowledge of the laws themselves*. How was this possible? It was through a clever mode of reasoning first pointed out by the great French physicists Pierre Curie, and adopted by James Cronin and Val Fitch in a very surprising discovery. Here is that story.

**2.1. Curie's principle.** In 1894, Pierre Curie argued that physicists really ought to be more like crystallographers, in treating certain symmetry principles like explicit laws of nature. He emphasized one symmetry principle in particular, which has come to be known as *Curie's principle*:

When certain effects show a certain asymmetry, this asymmetry must be found in the causes which gave rise to them. (Curie 1894)

To begin, we'll need to sharpen the statement of Curie's Principle, by replacing the language of “cause” and “effect” with something more precise. An obvious choice in particle physics is to take an “effect” to be a quantum state. What then is a cause? A natural answer is: the *other* states in the trajectory (e.g. the states that came before), together with the law describing how those states dynamically evolve. So, Curie's principle can be more clearly formulated:

If a quantum state fails to have a linear symmetry, then that asymmetry must also be found in either the initial state, or else in the dynamical laws.

This is a common interpretation of Curie's principle<sup>4</sup>. In fact it can be sharpened even more, and we will do so shortly. But first let's now see how it applies to the history of symmetry violation.

**2.2. Application to  $CP$ -violation.** In the study of symmetry violation, Curie's Principle appears to have first been used by Gell-Mann and Pais (1955). They did not refer to it in this way, but I think it will be clear that this is what they were using. Let's start with the example of *charge conjugation* (CC) symmetry, which has the effect of transforming particles into their antiparticles and vice versa. Suppose we have two particle states  $\theta_1$  and  $\theta_2$ ; their interpretation is not important for this point<sup>5</sup>. And suppose the state  $\theta_1$  is “even” under

<sup>4</sup>C.f. (Earman 2004), (Mittelstaedt and Weingartner 2005, §9.2.4).

<sup>5</sup>Gell-Mann and Pais used  $\theta_1^0$  and  $\theta_2^0$  refer to the neutral kaon states  $K_1$  and  $K_2$  discussed in Footnote 6 below.

charge conjugation, in that  $C\theta_1 = \theta_1$ , while the state  $\theta_2$  is “odd,” in that  $C\theta_2 = -\theta_2$ . Then, Gell-Mann and Pais observed,

according to the postulate of rigorous CC invariance, the quantum number  $C$  is conserved in the decay; the  $\theta_1^0$  must go into a state that is even under charge conjugation, while the  $\theta_2^0$  must go into one that is odd. (Gell-Mann and Pais 1955, p.1389).

Given  $C$ -symmetric laws, a  $C$ -symmetric state must evolve to another  $C$ -symmetric state. Or, reformulating this claim in another equivalent form: if a  $C$ -symmetric state evolves to a  $C$ -asymmetric state, *then the laws themselves must be  $C$ -violating*. That’s a neat way to test for symmetry violation. And it’s a simple application of Curie’s Principle.

Although Gell-Mann and Pais were discussing  $C$ -symmetry, the same reasoning applies to any linear symmetry whatsoever. In particular, it applies to  $CP$ -symmetry, which is the combined application of charge conjugation with the parity ( $P$ ) or “mirror flip” transformation. James Cronin and Val Fitch exploited Curie’s Principle when they made the shocking discovery of  $CP$ -violation in 1964, for which they won the 1980 Nobel Prize in physics. In fact, Cronin later wrote that the Gell-Mann and Pais article “sends shivers up and down your spine, especially when you find you understand it,” pointing out that it suggests a statement that is an unmistakable application of Curie’s Principle (although Cronin does not call it that way):

You can push this a little bit further and see how CP symmetry comes in. The fact that CP is odd for a long-lived  $K$  meson means that  $K_L$  could not decay into a  $\pi^+$  and a  $\pi^-$ . If it does — and that was our observation — then there is something wrong with the assumption that the CP quantum number is conserved in the decay. (Cronin and Greenwood 1982, p.41)

When you create a beam of neutral  $K$  mesons or “kaons,” the long-lived state  $K_L$  is all that’s left after the beam has traveled a few meters<sup>6</sup>. It had been discovered eight years earlier in the same Brookhaven laboratory by Lande et al. (1956). And it was known that  $K_L$  is *not* invariant

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<sup>6</sup> The study of strong interactions had led to the identification of kaon particle and antiparticle states  $K_0$  and  $\bar{K}_0$  that are eigenstates of a degree of freedom called *strangeness*. In the study of  $CP$ -violation, it is easier to study the superpositions  $K_1 = (K^0 + \bar{K}^0)/\sqrt{2}$  and  $K_2 = (K^0 - \bar{K}^0)/\sqrt{2}$ , since the lifetime of the latter is orders of magnitude longer. At the time,  $K_2$  was identified as the “long-life kaon state  $K_L$ .”

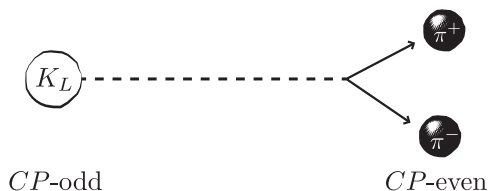


FIGURE 1. The  $K_L \rightarrow \pi^+\pi^-$  decay. By Curie’s Principle, this asymmetry between an initial state and a final state implies an asymmetry in the laws.

under the  $CP$  transformation, whereas a two pion state  $\pi^+\pi^-$  is invariant under  $CP$ . The observation of such an asymmetric decay, Cronin suggests, could only be the result of a  $CP$ -violating law.

Amazingly, when Cronin and Fitch analyzed the photographs of a  $K_L$  beam in a spark chamber at Brookhaven National Laboratory, they found clear evidence that some of the long-lived kaons were decaying into a pair of pions,  $K_L \rightarrow \pi^+\pi^-$ . Their conclusion, by a simple application of Curie’s Principle, was that the laws must be  $CP$ -violating. They told Pais about their discovery at Brookhaven over coffee. Pais later wrote, “After they left I had another coffee. I was shaken by the news.” (Pais 1990)

Of course, there were *many* deep insights that led to the discovery of  $CP$ -violation. They included the discovery of the strangeness degree of freedom, the prediction of kaon-antikaon oscillations, the discovery of the long-lived  $K_L$  state, the understanding of kaon regeneration, and many other things. But I hope to have shown here that, in skeletal form, the first argument for  $CP$ -violation is really a simple application of Curie’s Principle.

**2.3. The conclusion of  $T$ -violation.** The final step to the conclusion of  $T$ -violation now follows from the so-called  $CPT$ -theorem. Virtually all known laws of physics are invariant under the combined transformation of charge-conjugation ( $C$ ), parity ( $P$ ), and time reversal ( $T$ ). Of course, the precise law of unitary evolution governing the decay of the neutral kaon was not known in 1964. But there was a theorem to assure us that, at least for quantum theory as we know it — describable in terms of local (Wightman) fields, and a unitary representation of the Poincaré group — the laws must be invariant under  $CPT$ . This result, called the *CPT theorem*, was first proved in this form by Jost

(1957). And it straightforwardly implies that if  $CP$  is violated,  $T$  must be violated as well<sup>7</sup>.

Thus, insofar as the  $CPT$  theorem applies to our world, the Cronin and Fitch application of Curie’s principle provides immediate evidence for  $T$ -violation as well.

**2.4. Mathematical underpinning.** The statement of Curie’s principle described above is not just a helpful folk-theorem. It can be given precise mathematical expression. Let me now try to make the mathematics more clear. I’ll begin with a very simple mathematical statement of Curie’s Principle in terms of unitary evolution, and then show how it can be carried over to scattering theory.

To begin, recall what it means for a law to be invariant under a linear symmetry transformation  $R$ .

**Definition** (invariance of a law). A law of physics is *invariant* under a linear transformation  $R$  if whenever  $\psi(t)$  is an allowed trajectory according to the law, then so is  $R\psi(t)$ .

In the standard model of particle physics, interactions are assumed to evolve unitarily over time, by way of a continuous unitary group  $\mathcal{U}_t = e^{-itH}$ , where  $H$  is the Hamiltonian generator of  $\mathcal{U}_t$ . In this context, the above definition of invariance is equivalent to

$$[H, R] = 0$$

where  $H$  again is the Hamiltonian and  $R$  is linear (Earman 2002). In these terms, we can give a first formulation of Curie’s Principle as follows<sup>8</sup>.

**Fact 1** (Unitary Curie Principle). *Let  $\mathcal{U}_t = e^{-itH}$  be a continuous unitary group on a Hilbert space  $\mathcal{H}$ , and  $R : \mathcal{H} \rightarrow \mathcal{H}$  be a linear bijection. Let  $\psi_i \in \mathcal{H}$  (an “initial state”) and  $\psi_f = e^{-itH}\psi_i$  (a “final state”) for some  $t \in \mathbb{R}$ . If either*

- (1) *(initial but not final)  $R\psi_i = \psi_i$  but  $R\psi_f \neq \psi_f$ , or*
- (2) *(final but not initial)  $R\psi_f = \psi_f$  but  $R\psi_i \neq \psi_i$*

*then,*

- (3) *(R-violation)  $[R, H] \neq 0$ .*

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<sup>7</sup> $CPT$ -invariance says that  $(CPT)H = H(CPT)$ , and thus that  $(CP)THT^{-1} = H(CP)$ . So, if we have time reversal invariance  $THT^{-1} = H$ , then we must also have  $CP$ -invariance  $CP(H) = H(CP)$ . Equivalently, if  $CP$  invariance fails, then so does time reversal invariance.

<sup>8</sup>A version of this fact was pointed out by Earman (2004, Prop. 2).

*Proof.* Suppose that  $[R, H] = 0$ , and hence (since  $R$  is linear) that  $[R, e^{-itH}] = 0$ . Then  $R\psi_i = \psi_i$  if and only if  $R\psi_f = Re^{-itH}\psi_i = e^{-itH}R\psi_i = e^{-itH}\psi_i = \psi_f$ .  $\square$

This, again, is just a helpful first formulation. We have not yet arrived at a principle that is appropriate for the description of  $CP$ -violation. The claim of Cronin and Fitch was that in a neutral kaon scattering event, *there is a particular decay mode*  $K_L \rightarrow \pi^+\pi^-$  that occurs only if the laws are  $CP$ -violating  $[CP, H] \neq 0$ . We have not yet given a rigorous formulation of *that* application of Curie's Principle.

To get there, we first observe that it is enough for  $CP$  to fail to commute with the  $S$ -matrix,  $[CP, S] \neq 0$ . For, if a symmetry  $R$  commutes with the "free" part of the Hamiltonian  $[R, H_0] = 0$  (which is true of most familiar symmetries, including  $CP$ ), then by the definition of the  $S$ -matrix<sup>9</sup>,

$$[R, S] \neq 0 \text{ only if } [R, H] \neq 0.$$

Thus, by showing that the scattering matrix is  $CP$ -violating, one equally shows that the unitary dynamics are  $CP$ -violating as well. With this in mind, we can now state Curie's Principle in a form that is more appropriate for scattering theory.

**Fact 2** (Scattering Curie Principle). *Let  $S$  be a scattering matrix, and  $R : \mathcal{H} \rightarrow \mathcal{H}$  be a unitary bijection. If there exists any decay channel  $\psi^{in} \rightarrow \psi^{out}$  such that either,*

- (1) *(in but not out)  $R\psi^{in} = \psi^{in}$  but  $R\psi^{out} = -\psi^{out}$ , or*
- (2) *(out but not in)  $R\psi^{out} = \psi^{out}$  but  $R\psi^{in} = -\psi^{in}$ ,*

*then,*

- (3)  $[R, S] \neq 0$ .

*Moreover, if  $\mathcal{U}_t = e^{-it(H_0+V)}$  is the associated unitary group, and if  $R$  commutes with the free component  $H_0$  of the Hamiltonian  $H = H_0 + V$ , then ( $R$ -violation)  $[R, H] \neq 0$ .*

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<sup>9</sup> One easy way to prove this is to just look at the explicit Dyson expression of the  $S$ -matrix,

$$(1) \quad S = \mathcal{T} \exp \left( -i \int_{-\infty}^{\infty} dt V_I(t) \right),$$

where  $V_I$  is the interacting part of the Hamiltonian  $H = H_0 + V_I$ , and  $\mathcal{T}$  is the time-ordered multiplication operator (Sakurai 1994, p.73). If  $H = H_0 + V_I$ , then  $[R, H_0] = 0$  and  $[R, H] = 0$  implies that  $[R, V_I] = [R, H - H_0] = [R, H] - [R, H_0] = 0$ . Thus, since  $R$  is linear, we can pass it through the integral above to get that  $RSR^{-1} = S$ .

*Proof.* We prove the contrapositive; suppose that  $[R, S] = 0$ . Since  $R$  is unitary,  $\langle \psi^{out}, S\psi^{in} \rangle = \langle R\psi^{out}, RS\psi^{in} \rangle = \langle R\psi^{out}, SR\psi^{in} \rangle$ . So, if either the “in but not out” or the “out but not in” conditions hold, then,

$$\langle \psi^{out}, S\psi^{in} \rangle = \langle R\psi^{out}, SR\psi^{in} \rangle = -\langle \psi^{out}, S\psi^{in} \rangle.$$

Hence,  $\langle \psi^{out}, S\psi^{in} \rangle = 0$ , which means that there can be no decay channel  $\psi^{in} \rightarrow \psi^{out}$ . Finally, we note that if  $[R, H_0] = 0$ , then and  $[R, S] \neq 0$  implies that  $[R, H] \neq 0$  by the definition of the  $S$ -matrix.  $\square$

This, finally, is the precise mathematical statement of Curie’s Principle that was applied by Cronin and Fitch. Taking  $\psi^{in} = K_L$  and  $\psi^{out} = \pi^+\pi^-$ , they discovered a scattering event  $\psi^{in} \rightarrow \psi^{out}$  that satisfies “out but not in” for the transformation  $R = CP$ . It follows that the laws are  $CP$ -violating. And given  $CPT$  invariance, it follows that they are  $T$ -violating as well.

**2.5. Advantages and limitations.** An obvious advantage of this approach to  $T$ -violation is that you don’t have to know the laws to know that they are  $T$ -violating. At the time of its discovery in 1964, many of the structures appearing in the modern laws of neutral kaon decay were absent: there were no  $W$  or  $Z$  bosons, no Kobayashi-Maskawa matrix, and certainly no standard model of particle physics. All that came later. Nevertheless, Curie’s Principle provided a surprisingly simple test that the laws are  $T$ -violating, even without knowing the laws themselves.

A more subtle advantage is that, as a test for  $CP$  violation, Curie’s Principle will likely continue hold water in non-unitary extensions of quantum theory<sup>10</sup> Although unitary evolution is assumed in some of the background definitions, nothing about the argument from Curie’s Principle requires the evolution be unitary. For example, the “scattering version” of Curie’s principle in no way depends on the unitarity of the  $S$ -matrix; indeed, the conclusion that  $[R, S] \neq 0$  holds when  $S$  is any Hilbert space operator whatsoever that connects  $\psi^{in}$  and  $\psi^{out}$  states. In this sense, the argument from Curie’s principle is very general indeed.

The limitation is that it is an indirect test for  $T$ -violation, and one that we might not trust as we attempt to extend particle physics beyond the standard model. In particular, the reliance on the  $CPT$  theorem is troubling. It is not implausible that  $CPT$  invariance could fail as particle physics is extended beyond the standard model. For example, we might wish to consider a representation of the Poincaré

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<sup>10</sup>C.f. Weinberg (1989).



group that is not completely unitary. In such cases, the *CPT* theorem can fail, and thus so would the link between *CP*-violation and *T*-violation. It would be preferable to have a direct test of *T*-violation instead.

One might respond to this concern by trying to apply Curie's Principle directly to the case of *T*-violation. Unfortunately, that doesn't work. Recall in the statement of Curie's Principle above assumed the symmetry transformation was linear. This turns out to be a crucial assumption; Curie's Principle fails badly for antilinear symmetries like time reversal<sup>11</sup>. So, this road to *T*-violation is *essentially* indirect. One can check directly for *CP* violation, but only recover *T*-violation by the *CPT* theorem. A direct test of *T*-violation will have to follow a completely different argument. That is the topic of the next section.

### 3. *T*-VIOLATION BY KABIR'S PRINCIPLE

New progress has recently been made in the understanding of *T*-violation. We now have evidence for the phenomenon that appears to be much more direct. The first such evidence began with an experiment by Angelopoulos et al. (1998), performed at the CPLEAR particle detector at CERN. Like the original *T*-violation experiment, this discovery involved the decay of neutral kaons. But unlike previous tests of *T*-violation, this experiment did not make use of Curie's Principle, and in this way managed a *direct* detection of *T*-violation. Things got even better when, just a few months ago now, yet another direct detection of *T*-violation was announced by the BaBar collaboration at Stanford (Lees et al. 2012). This experiment involved the decay of a different particle, the neutral *B* meson. It's an exciting time for the study of *T*-violation!

What I would like to point out is that both recent detections of *T*-violation hinge on a common principle. It is not Curie's Principle, for we have seen that this does not allow for the direct detection of *T*-violation. Let me call it *Kabir's Principle*, since it was pointed out in an influential pair of papers by Kabir (1968, 1970). Unlike the Curie Principle approach to symmetry violation, this one is really built to handle antilinear transformations like time reversal. Here is how it works.

**3.1. Kabir's Principle.** To begin, let me summarize the simple idea behind Kabir's Principle somewhat roughly.

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<sup>11</sup>See Roberts, "The simple failure of Curie's Principle," manuscript of July 21, 2012, <http://philsci-archive.pitt.edu/9249/>

If a transition  $\psi^{in} \rightarrow \psi^{out}$  occurs with different probability than the time-reversed transition  $T\psi^{out} \rightarrow T\psi^{in}$ , then the laws describing those transitions must be  $T$ -violating.

This suggests a straightforward technique for checking whether or not an interaction is governed by  $T$ -violating laws. We set up a detector to check how often a particle decay  $\psi_i \rightarrow \psi_f$  occurs (called its *branching ratio*), and compare it to how often a the decay  $T\psi_f \rightarrow T\psi_i$  occurs. Easier said than done, naturally. But if one occurs more often than the other, then we have direct evidence of  $T$ -violation.

In the next subsection, I will sketch briefly how such a procedure was first carried out at CERN. I'll then discuss the precise mathematical formulation of Kabir's Principle.

**3.2. Application to  $T$ -violation.** The first direct detection of  $T$ -violation involved the decay of our friend the neutral kaon. So, let's return to the strangeness eigenstates  $K^0$  and  $\bar{K}^0$ , which have strangeness eigenvalues  $\pm 1$ , respectively. It is generally thought that, if strong interactions were all that governs the behavior of these states, then strangeness would be conserved. So, by the kind of arguments discussed above, you could never have a particle decay like  $K^0 \rightarrow \bar{K}^0$ , because these states have different values of strangeness. However – and this is another thing pointed out in the remarkable article by Gell-Mann and Pais (1955) – when weak interactions are in play, there is no reason not to entertain decay channels that fail to conserve strangeness.

In fact, in the presence weak interactions, it makes sense to consider both  $K^0 \rightarrow \bar{K}^0$  and  $\bar{K}^0 \rightarrow K^0$  as possible decay modes. These particles could in principle bounce back and forth between each other,  $K^0 \rightleftharpoons \bar{K}^0$ , by a phenomenon called *kaon oscillation*. This is a very exotic property, which applies to only a few known particles (one of them being the  $B$  meson), and it is part of what makes neutral kaons so wonderfully weird.

Now, the convenient thing about oscillations between  $K^0$  and  $\bar{K}^0$  is that they are very easy to time reverse. In particular,

$$TK^0 = K^0, \quad T\bar{K}^0 = \bar{K}^0.$$

This allows us to apply Kabir's Principle in a particularly simple form: if we observe  $K^0 \rightarrow \bar{K}^0$  to occur with a different probability than  $\bar{K}^0 \rightarrow K^0$ , then we have direct evidence for  $T$ -violation! This is precisely what was found at the CPLEAR detector, in showing that there is “time-reversal symmetry violation through a comparison of the probabilities



FIGURE 2. Application of Kabir's Principle. If the decay  $K^0 \rightarrow \bar{K}^0$  happens more often than the time-reversed decay  $\bar{K}^0 \rightarrow K^0$ , then the interaction is  $T$ -violating.

of  $\bar{K}^0$  transforming into  $K^0$  and  $K^0$  into  $\bar{K}^0$ " (Angelopoulos et al. 1998).

At this level of abstraction, it was the very same strategy that was used in the Stanford  $T$ -violation experiment with  $B$  mesons. It turns out that neutral  $B$  mesons can also oscillate between two states,  $B^0 \rightleftharpoons B_-$ . Bernabéu et al. (2012) pointed out that if these transitions were to occur with different probabilities, then we would have  $T$ -violation. And this is just what was recently detected by Lees et al. (2012) at Stanford. Thus, both the Stanford detection and the original CPLEAR detection  $T$ -violation were made possible by the abandonment of Curie's Principle, in favor of the more direct principle of Kabir.

**3.3. Mathematical Underpinning.** As with Curie's Principle, Kabir's Principle has a rigorous mathematical underpinning. But before getting to that, it's important to note the special way that unitary operators like the  $\mathcal{U}_t = e^{-itH}$  and the  $S$ -matrix transform under time reversal. The point where many get stuck is on the fact that  $T$  is *antiunitary*. This means that it conjugates the amplitudes,  $\langle T\psi, T\phi \rangle = \langle \psi, \phi \rangle^*$ . It also means that it is *antilinear*, in that it conjugates any complex number that we pass it over:

$$T(a\psi + b\phi) = a^*T\psi + b^*T\phi.$$

As a consequence, the condition of time reversal invariance that  $[T, H] = 0$  does *not* imply that the unitary operator  $\mathcal{U}_t = e^{-itH}$  commutes with  $T$ . Instead, the complex constant picks up a negative sign. That is, for time reversal invariant systems,  $T\mathcal{U}_tT^{-1} = e^{-(-itTHT^{-1})} = e^{itH} = \mathcal{U}_{-t} = \mathcal{U}_t^*$ . Similarly, a unitary  $S$ -matrix describes a time-reversal invariant system if and only if  $TST^{-1} = S^*$ .

We can formulate a mathematical statement of Kabir's Principle. Note that, as discussed in Section 2.4, the failure of the  $S$ -matrix to be time reversal invariant ( $TST^{-1} \neq S^*$ ) implies  $T$ -violation in the ordinary sense ( $T\mathcal{U}_tT^{-1} \neq \mathcal{U}_t^*$ ).

**Fact 3** (Kabir's Principle). *Let  $S$  be a unitary operator (the  $S$ -matrix) on a Hilbert space  $\mathcal{H}$ , and let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an antiunitary bijection. If,*

$$(1) \text{ (unequal amplitudes) } \langle \psi^{in}, S\psi^{out} \rangle \neq \langle T\psi^{out}, ST\psi^{in} \rangle,$$

*then,*

$$(2) \text{ (} T\text{-violation) } TST^{-1} \neq S^*.$$

*Proof.* We argue the contrapositive. Suppose  $TST^{-1} = S^*$ . Since  $T$  is antiunitary,  $\langle \psi^{out}, S\psi^{in} \rangle = \langle T\psi^{out}, TS\psi^{in} \rangle^*$ . But  $TS = S^*T$  by time reversal invariance, so,

$$\langle \psi^{out}, S\psi^{in} \rangle = \langle T\psi^{out}, S^*T\psi^{in} \rangle^* = \langle T\psi^{in}, ST\psi^{out} \rangle,$$

where the last equality just applies properties of the inner product.  $\square$

**3.4. Advantages and limitations.** Like Curie's Principle, Kabir's Principle provides a way to establish  $T$ -violation of the laws without assuming very much at all about those laws. But even better, it does so without recourse to the  $CPT$  theorem. In this sense, Kabir's Principle stands a better chance of remaining valid in  $CPT$ -violating extensions of the standard model.

A limitation is that, unlike the Curie's Principle approach, Kabir's Principle only seems to work when the dynamics is unitary. As in Section 2.5, suppose we consider some non-unitary extension of the standard model. Unfortunately, an essential part of the Kabir Principle argument involves the assumption that time reversal invariance has the effect,

$$T\mathcal{U}_t T^{-1} = \mathcal{U}_{-t} = \mathcal{U}_t^*.$$

When  $\mathcal{U}_t$  is a *unitary* group, this is a simple mathematical fact. However, if we wish to consider a one-parameter group  $\mathcal{U}_t$  that is *not* unitary, then the concept of time reversal invariance  $T\mathcal{U}_t T^{-1} = \mathcal{U}_{-t}$  does not necessarily imply that  $T\mathcal{U}_t T^{-1} = \mathcal{U}_t^*$ . But this latter fact is (crucially) applied in the proof of Kabir's Principle.

Thus, although the Kabir Principle applied by Angelopoulos et al. (1998) and Lees et al. (2012) has the advantage of providing a direct test, they are not general enough to apply without modification to the context of a non-unitary dynamics.

#### 4. $T$ -VIOLATION BY A NON-DEGENERACY PRINCIPLE

I'd like to finish with one final road to  $T$ -violation. It is perhaps the most direct and yet the least well-known of all the approaches. In simplest terms, this route involves the search for exotic new kinds of matter. Let me begin with a toy model of how this can happen.

I'll then turn to the general reasoning underpinning this approach to  $T$ -violation, and finally show how this reasoning has been applied (unsuccessfully so far) in empirical tests.

**4.1. A toy example.** An electric dipole moment typically describes the displacement between two opposite charges, or within a distribution of charges. But suppose that, instead of describing a distribution of charges, we use an electric dipole moment to characterize a property of just one elementary particle. This particle might be referred to as an “elementary” electric dipole moment.

The existence of such particles has been entertained, though none have yet been detected. Let  $H_0$  be the Hamiltonian describing the particle in the absence of interactions; let  $J$  represent its angular momentum; and let  $E$  represent an electromagnetic field. Then these “elementary” electric dipoles are sometimes<sup>12</sup> characterized by the Hamiltonian,

$$H = H_0 + J \cdot E.$$

Since time reversal preserves the free Hamiltonian  $H_0$  and the electric field  $E$ , but reverses angular momentum  $J$ , this Hamiltonian is manifestly  $T$ -violating:  $[T, H] \neq 0$ . Therefore, an elementary electric dipole of this kind would constitute a direct detection of  $T$ -violation. No need for Kabir’s Principle. No need for Curie’s Principle. No need for the  $CPT$  theorem.

Like the  $T$ -violating  $K_L \rightarrow \pi^+\pi^-$  and  $K^0 \rightleftharpoons \bar{K}^0$  decays, there are general principles underpinning this example of  $T$ -violation, too. In this case, they stem from the relationship between  $T$ -invariance and the degeneracy of the energy spectrum. The relevant relationship can be summarized as follows.

**4.2. A Non-degeneracy Principle.** A system is called *degenerate* if its Hamiltonian has distinct energy states with the same energy eigenvalue. An intuitive example is the free particle on a string, which is degenerate: the particle can either move to the left, or to the right, and have the same kinetic energy either way. Kramers (1930) showed that an odd number of electrons can be expected to have a degenerate energy spectrum, and for this his name remains attached to that effect: *Kramers Degeneracy*<sup>13</sup>. But it was Wigner (1932) showed the much deeper relationship between degeneracy and time reversal invariance.

<sup>12</sup>(See Khriplovich and Lamoreaux 1997)

<sup>13</sup>The reason people were interested in the first place, it seems, is that degeneracy was a key part of knowing how to studying very low temperature phenomena using paramagnetic salts (Klein 1952).

For the purposes of understanding  $T$ -violation, the relevant relationship can be summarized as follows.

**Fact 4** (Non-degeneracy Principle). *If (1) time reversal acts non-trivially on states, in that  $T\psi \neq e^{i\theta}\psi$  for some eigenvector  $\psi$  of  $H$ ; and (2) the Hamiltonian  $H$  is non-degenerate; then we have  $T$ -violation, in that  $[T, H] \neq 0$ .*

We will see shortly how this fact has a simple prove deriving from the work of Wigner. But first, let me point out how it can be used to provide evidence of  $T$ -violation, if we were to detect a particular kind of electric monopole.

**4.3. Application to  $T$ -violation.** We observed above that an appropriate system can provide an explicit and direct example of  $T$ -violation. The properties that these systems tend to share, it turns out, are just the properties of the Non-degeneracy Principle above. There are various examples that one could study here to illustrate. But let me spare the reader and give just one that is rather important, the elementary electric dipole moment.

The thing that is not obvious is that the elementary electric dipole moment is that it always satisfies part (1) of the Non-degeneracy Principle. That is, time reversal always acts non-trivially on such systems, in that there is some eigenvector  $\psi$  of  $H$  that is transformed non-trivially,  $T\psi \neq e^{i\theta}\psi$ . We'll show why that is in the following. But to get from there to  $T$ -violation, notice that we need only make the plausible assumption that an elementary particle in a stable ground-state is non-degenerate. It then follows by the Non-degeneracy Principle that the system is  $T$ -violating.

To begin, let's introduce the elementary electric dipole moment<sup>14</sup>. It is normally taken to be a system characterized the following three properties.

- (1) (*Permanence*) There is an observable  $D$  representing the dipole moment is "permanent", in that  $\langle \psi, D\psi \rangle = a > 0$  for some eigenvector  $\psi$  of the Hamiltonian  $H$ . That is, the dipole is a permanent feature of the particle, like its charge or spin-type.
- (2) (*Isotropic Dynamics*) Since it is an elementary particle, its simplest interactions are assumed to be isotropic, in that time evolution commutes with all rotations,  $[e^{-itH}, R_\theta] = 0$ . Note that if  $J$  is the "angular momentum" observable that generates the rotation  $R_\theta = e^{i\theta J}$ , then this is equivalent to the statement that  $[H, J] = 0$ .

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<sup>14</sup>C.f. (Ballentine 1998, §13.3), (Messiah 1999, §XXI.31), or (Sachs 1987, §4.2).

- (3) (*Time Reversal Properties*) Time reversal, as always, is an antiunitary operator. It has no effect on the electric dipole observable ( $TDT^{-1} = D$ ), which is basically a function of position. But it does reverse the sign of angular momentum ( $TJT^{-1} = -J$ ), since spinning things spin in the opposite orientation when their motion is reversed.

A system with these three properties turns out to satisfy condition (1) of the Non-degeneracy principle, that  $T\psi \neq e^{i\theta}\psi$  for some eigenvector  $\psi$  of  $H$ . To see why, assume (for reductio) that it does not, and thus that for each eigenvector  $\psi$  of the Hamiltonian, there is a unit  $e^{i\theta}$  such that  $T\psi = e^{i\theta}\psi$ . We will show that the assumption that the dipole moment is “permanent” then fails, contradicting our hypothesis.

Since  $[H, J] = 0$ , there is a common eigenvector for  $H$  and  $J$ , which we will denote  $\psi$ . By the Wigner-Eckart Theorem<sup>15</sup>, each eigenvector of  $H$  and  $J$  will satisfy,

$$(2) \quad \langle \psi, D\psi \rangle = c\langle \psi, J\psi \rangle$$

for some  $c \in \mathbb{R}$ . Now, an antiunitary operator  $T$  satisfies  $\langle T\psi, T\phi \rangle = \langle \psi, \phi \rangle^*$  for any  $\psi, \phi$ . And a self-adjoint operator satisfies  $\langle \psi, A\psi \rangle^* = \langle \psi, A\psi \rangle$  for any  $\psi$ . Applying these two facts to Equation (2), we get that  $\langle T\psi, TD\psi \rangle = c\langle T\psi, TJ\psi \rangle$ . But  $T$  commutes with  $D$  and anti-commutes with  $J$ , so this equation may be written,

$$(3) \quad \langle T\psi, D(T\psi) \rangle = -c\langle T\psi, J(T\psi) \rangle$$

Finally, we assume the distinct ray condition fails, so  $T\psi = e^{i\theta}\psi$  for some  $e^{i\theta}$ . Applying this to Equation (3), we get

$$\begin{aligned} (e^{-i\theta}e^{i\theta})\langle \psi, D\psi \rangle &= -(e^{-i\theta}e^{i\theta})c\langle \psi, J\psi \rangle \\ \Rightarrow \langle \psi, D\psi \rangle &= -c\langle \psi, J\psi \rangle. \end{aligned}$$

Combined with Equation (2), this implies that  $\langle \psi, D\psi \rangle = 0$ , contradicting our hypothesis that  $D$  is permanent.

So, the elementary electric dipole has at least one energy eigenvector  $\psi$  such that  $T\psi \neq e^{i\theta}\psi$ . That’s premise (1) of the non-degeneracy argument. To get to  $T$ -violation, we need only convince ourselves of premise (2), that such a system is described by a non-degenerate Hamiltonian. Constructing such a system is part of an active search for  $T$ -violation.

There are many interesting things to say about this research; for a book-length treatment, see Khriplovich and Lamoreaux (1997). All

<sup>15</sup>A special case of this theorem states that the components of any vector observable are proportional to the components of angular momentum. (See Ballentine 1998, §7.2, esp. page 195).

I would like to point out for now is that this approach to  $T$ -violation hinges on a simple Non-degeneracy Principle, which is distinct from all the other approaches to  $T$ -violation discussed so far.

**4.4. Mathematical Underpinning.** As suggested above, Fact 4 basically arises out of Wigner’s discovery of a connection between time reversal and degeneracy for systems with an odd number of fermions. Here is how that connection leads to a principle for understanding  $T$ -violation.

Wigner began by noticing a strange fact that two successive applications of the time reversal operator  $T$ . When applied to a system consisting of an odd number of electrons, it does not exactly bring an electron back to where we started. Instead, it adds a phase factor of  $-1$ . Only by applying time reversal twice more can we return an electron to its original vector state. This is a curious property indeed! But there is no getting around it. It is effectively forced on us by the definition of time reversal and of a spin-1/2 system (Roberts 2012).

This led Wigner to the following argument that the electron always has a degenerate Hamiltonian<sup>16</sup>.

**Proposition 1** (Wigner). *Let  $H$  be a self-adjoint operator on a finite-dimensional Hilbert space, which is not the zero operator. Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an antiunitary bijection. If*

- (1) (electron condition)  $T^2 = -I$ , and
- (2) ( $T$ -invariance)  $[T, H] = 0$

then,

- (3) (degeneracy)  $H$  has two orthogonal eigenvectors with the same eigenvalue.

That’s a fine argument for degeneracy, when we are confident about time reversal invariance. But what if we are interested in systems that are  $T$ -violating? No problem. We can just interpret Wigner’s result in the following equivalent form.

**Corollary.** *Let  $H$  be a self-adjoint operator on a finite-dimensional Hilbert space, which is not the zero operator. Let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be an antiunitary bijection. If*

- (1) (electron condition)  $T^2 = -I$ , and
- (2) (non-degeneracy)  $H$  has no two orthogonal eigenvectors with the same eigenvalue

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<sup>16</sup>Wigner’s assumption of a finite-dimensional Hilbert space can be relaxed, as generalizations exist for Hamiltonians with a continuous energy spectrum as well (Roberts 2012).



then,

$$(3) \text{ (} T\text{-violation) } [T, H] \neq 0.$$

This means that Wigner’s result is actually a toy strategy for testing  $T$ -violation in disguise! Suppose we discover an electron described by a non-degenerate Hamiltonian. Then we will have achieved a direct detection of  $T$ -violation.

There is a more general sort of reasoning at work here. It turns out that the  $T^2 = -I$  condition is stronger than is really needed to prove the result. The following generalization, which otherwise follows Wigner’s basic argument, is available.

**Proposition 2.** *Let  $H$  be a self-adjoint operator on a finite-dimensional Hilbert space, which is not the zero operator. Let  $T$  be an antiunitary bijection. If*

- (1) *(distinct ray condition)  $T\psi \neq e^{i\theta}\psi$  for some eigenvector  $\psi$  of  $H$ , and*
- (2) *(non-degeneracy)  $H$  has no two orthogonal eigenvectors with the same eigenvalue*

then,

$$(3) \text{ (} T\text{-violation) } [T, H] \neq 0$$

*Proof.* We prove the contrapositive, by assuming (3) fails, and proving that either (1) or (2) fails as well. Let  $H\psi = h\psi$  for some  $h \neq 0$  and some eigenvector  $\psi$  of unit norm. Since  $T$  is antiunitary,  $T\psi$  will also have unit norm.

Suppose (3) fails, and hence that  $[T, H] = 0$ . As we saw in the proof of Proposition 1, this implies that if  $\psi$  is an eigenvector of  $H$  with eigenvalue  $h$ , then so is  $T\psi$ . By the spectral theorem, the eigenvectors of  $H$  form an orthonormal basis set. So, since  $\psi$  and  $T\psi$  are both unit eigenvectors, either  $T\psi = e^{i\theta}\psi$  or  $\langle T\psi, \psi \rangle = 0$ . The latter violates non-degeneracy (2). And, since  $\psi$  was arbitrary, the former violates the distinct ray condition (1). Therefore, either (1) or (2) must fail.  $\square$

This simple generalization is now more than a “toy” experimental test. It is the mathematical grounds for the Non-degeneracy Principle stated in Section 4.2, and part of an active search for  $T$ -violation.

## 5. CONCLUSION

We have seen three routes to  $T$ -violation, of distinctly different forms. The first route, which employs Curie’s Principle and the  $CPT$  theorem, is by necessity indirect. The reason is the curious result

that Curie’s Principle fails for time reversal in quantum mechanics. As a consequence, one can only use this principle to test for linear symmetries like  $CP$ -violation. Insofar as the premises of the  $CPT$  theorem are correct,  $T$ -violation can then be derived as a consequence of  $CP$ -violation. But for a more direct test, one can take the second route and apply “Kabir’s Principle,” which restores the possibility of a direct detection of  $T$ -violation. For another direct test, one can take a third route and apply the Non-degeneracy Principle. This allows for a direct test of  $T$ -violation, which is not contingent on the premises of the  $CPT$  theorem, although it requires knowing more about the form of the Hamiltonian.

Curiously, the former two approaches (the only successful approaches) both ultimately rely, in their own different ways, on the assumption of unitary time evolution. The first approach does so not with Curie’s Principle – it doesn’t require unitarity – but in the application of the  $CPT$  theorem. The second approach does so in the application of Kabir’s Principle. This suggests that, in extensions of the standard model that relax the assumption of unitarity, we may lose our best existing evidence for  $T$ -violation. Of course, there will always be a limiting case in which unitary evolution is justified, and so there will be a limiting case where we have  $T$ -violation. But moving forward, the question of whether  $T$ -violation will remain an explicit feature of the *fundamental* laws is, for the moment, an open one.

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