On the mathematical nature of logic

Featuring P. Bernays and K. Gödel

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Abstract

The paper examines the interrelationship between mathematics and logic, arguing that a central characteristic of each has an essential role within the other. The first part is a reconstruction of and elaboration on Paul Bernays' argument, that mathematics and logic are based on different directions of abstraction from content, and that mathematics, at its core it is a study of formal structures. The notion of a study of structure is clarified by the examples of Hilbert's work on the axiomatization of geometry and Hilbert et al.'s formalist proof theory. It is further argued that the structural aspect of logic puts it under the purview of the mathematical, analogously to how the deductive nature of mathematics puts it under the purview of logic. This is then linked, in the second part, to certain aspects of Gödel's critique of Carnap's conventionalism, that 'mere syntax' cannot capture the full content of mathematics, which is revealed to be closely related to the characteristic of mathematics argued for by Bernays. Finally, this is connected with Gödel's latter-day views about two kinds of formality, intensional and extensional (corresponding to logic and mathematics), and the relationship between them.

1 Introduction

It is obvious that logic and mathematics are intimately related; the difficult matter is to say how. Mathematics is a deductive science *par excellence*, and thus surely falls under the jurisdiction of logic, the very science of deduction. However, for the last one hundred years at least, logic has developed as a largely mathematical science, suggesting that logic is essentially mathematical in nature. How exactly, then, are they connected? This is the question examined in this paper. Its starting point is provided by a number of articles written and published in the years 1922–1930 by Paul Bernays[1], who argued that each of logic and mathematics is *essentially* operative in the other. These arguments are considered in §2–§5. Bernays' remarks are brief and not entirely clear; the present paper offers a reconstruction of what I take those remarks to mean, then elaborates on them to put forward a more detailed argument.

Bernays' arguments on the interrelations between logic and mathematics were put forward partly as a critique of logicism[2], arguing that the failure of logicism was not ultimately due to specific problems in how the programme was developed by Frege or by Russell and Whitehead, but is rather conceptually misplaced, since, while there is a sense in which logic is more general than mathematics, there is also a sense in which mathematics is more general than logic. Thus, Bernays argues, as did Hilbert[3] in lectures and papers of the same years, that because of their very nature, logic and arithmetic (or mathematics more generally) cannot be derived one from the other, but must be developed side-by-side. Logicism is one important element of the context of Bernays' arguments, but also important are the development of mathematical logic by Hilbert and Bernays (mainly in lectures), and the development of Hilbert's Programme, inaugurated publicly in Hilbert's paper *Neubegründung der Arithmetik* of 1922.

The various contexts, I think, are important. First, because they can help us understand

[1] Bernays (1888 – 1977) was among the principal contributors to the project in the foundations of mathematics known as "Hilbert's Programme" (see below), and later became a logician and philosopher in his own right.

[2] Logicism is a family of views according to which all of mathematics is ultimately reducible to or expressible in terms of definitions and rules of inference given in purely logical terms, and thus that questions about mathematical truth and knowledge can be answered by an appeal to logical truth and knowledge of logic.

[3] David Hilbert (1862 – 1943) was among the world's leading mathematicians in the late 19th and early 20th centuries, and a pioneer and influential advocate of the axiomatic approach to mathematics which was to shape the way modern mathematics is pursued. In the 1920's he undertook the foundations of mathematics as his main project, resulting in a series of papers and lectures known as "Hilbert's Programme" (see §4.2 for details).

Bernays' remarks. But they are also important for another reason. In the second part of the paper (§6), I consider an argument Gödel puts forward against Carnap. On the face of it, Gödel is discussing something very different from the subject of Bernays' papers. Nevertheless, they are closely connected. The Carnap which Gödel discussed is the Carnap of the period of *Logische Syntax der Sprache* (1934), in which he argued that mathematics is merely 'syntax of language', and that there is therefore nothing that mathematics could be about, no mathematical content. Gödel argues strongly against the Carnapian view, countering that mathematics does have a content beyond its formal presentation and is therefore not just 'mere syntax'. The core of Gödel's argument is that the mathematical 'syntax' *itself* has mathematical content which is outside the capacity of the syntactical structure to capture. This is closely related to the view Bernays puts forward that there is mathematical content to be found even in logical form itself.

Although Gödel is criticizing Carnap, in arguing for the necessary failure of Carnap's reductionist programme, he is also speaking to the failure of Hilbert's Programme and the reasons for it. Indeed, Gödel is explicit about the connection between Carnap's view and the formalist methods of the 'Hilbert school', as is clear in the third draft of his critique of Carnap (Gödel, 1959, version III, §4). Gödel's point is that the finitary syntactic-combinatorial account of logical form *necessarily* fails to capture actual mathematics. Thus, it is analogous to (but in some ways an extension of) Bernays' claim that there is always mathematics present in formal, structural logical presentation. Carnap's position is outlined briefly in §6.1, together with Gödel's critique of it. I will not be concerned here with the question of whether Gödel's presentation of Carnap is a fair treatment of Carnap's views; my interest will be mainly in Gödel's presentation itself. Gödel's critique is linked, in §6.2, with his latter-day views on the relationship between logic and mathematics, as documented by Hao Wang (in his 1987 and 1996). The ideas on the nature of mathematics which are common to Bernays and Gödel are traced back to Hilbert's remarks regarding the 'extra-logical' conditions of thought. Finally, it is considered how Bernays' division of labour between logic and mathematics compares with Gödel's distinction between intensional and extensional formal theories.

As a final prefatory remark: though Bernays' points were made as part of a critique of logicism and (one assumes) with the goal of elaborating Hilbert's formalist programme, they are of interest quite independently of these contexts, and the same can be said for Gödel's argument for mathematical content. Their interest is therefore not just historical, but concerns the general question of the relationship between logic and mathematics and the nature of each. Moreover, the arguments discussed here regarding the relationship between logic and mathematics are indifferent to the question whether a sharp dividing line can be drawn between what one calls 'logic' and 'mathematics', and I take no position on this last matter.

2 Overview of Bernays' argument

The following passage from Bernays (1922b) contains a concise summary of the position he puts forward:

> Mathematical logic does not achieve the goal of a logical grounding of arithmetic. And it is not to be assumed that the reason for this failure lies in the particular form of Frege's approach. It seems rather to be the case that the problem of reducing mathematics to logic is in general wrongly posed, namely, because mathematics and logic do not really stand to each other in the relationship of particular to general.
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> Mathematics and logic are based on two different directions of abstraction. While logic deals with the contentually most general [*das inhaltlich Allgemeinste*], (pure) mathematics is the general theory of the formal relations and properties, and so on the one hand each mathematical reflection is subject to the laws of logic, and on the other hand every logical construct of thought falls into the domain of mathematical reflection on account of the outer structure that is necessarily inherent in it. (Bernays, 1922b, p. 217)[4]

Bernays is making three closely connected points in this passage. First, *logic is not more general than mathematics*. Or, as Bernays puts it, the relation between logic and mathematics is not that of general and particular. Second, *logic is not more abstract than mathematics*; indeed, both are abstract disciplines, but crucially, Bernays maintains that *each of them is an abstraction in a different direction*. Since the directions of abstraction

[4] All page numbers given for quotations, here and below, refer to the respective English translations.

are, at least in part, non-overlapping, there can be no comparison under which either is more abstract than the other. This connects with the third point, perhaps the most substantive: that *just as the logical figures essentially in mathematics, so the mathematical figures essentially in logic*. Just as mathematics, by virtue of being a deductive science, falls under the purview of logic, so logic, by virtue of its instantiating certain kinds of structure, falls under the purview of mathematics. What Bernays seems to be saying is that neither logic nor mathematics can *subsume* the other under itself, since each *essentially* makes use of the other, and thus in a sense *presupposes* some aspects of the other. However, such an essential use does not entail that either logic or mathematics is *reducible* to the other. This will be laid out in more detail in the relevant section below. The following three sections (§3–§5) trace out and elaborate on each of these points in order, although it should be noted that the division between them is rather loose; they are all facets of a single viewpoint.

3 The relative generality of logic and mathematics

Let us start by asking what the claim that Bernays is rejecting might mean. What plausible motivation could we have to say that mathematics and logic *do* stand to one another as particular to general? There are grounds to think that logic is the most general of all possible disciplines. After all, logic is, deliberately and by its very nature, topic-neutral: it applies across all possible subjects of discourse. It is the study of valid deductive reasoning in general, and wherever one finds deductive reasoning, one is answerable to logic. Logic alone, arguably, has this privileged status. Moreover, the relationship between logic and mathematics is *especially* close, closer than between logic and any other discipline, since the very language of logic is arguably designed to capture the conceptual structure of what we express and prove in mathematics. Moreover, mathematics is more than any other science a *deductive* science, a paradigmatic instance of logical and deductive reasoning.[5] In its axiomatic and systematised form, mathematics is governed by logic in each deductive

[5] The claim here is not that mathematics is *entirely* a deductive science; to do so is to ignore the actual process of discovery and development in mathematics which is far more complex than deriving consequences from initial postulates, but that takes us away from the present concerns.

step of every proof.[6] It is a *profoundly* deductive, and therefore logical, pursuit.

Logicism attempted to claim, at least in its Fregean form, that in addition to this, it isn't merely that mathematics is permeated by logic, it just *is* logic, plus appropriate definitions: mathematical content is just logical content. If the logicist project had been successful, we would have a good reason for saying that mathematics is a specialised form of logic, and therefore logic is more general. But logicism should not be thought of as equivalent to the claim that logic is more general than mathematics, with the difficulties faced in carrying logicism through thereby *de facto* refutations of the claim. These two claims, the greater generality of logic and the reducibility of mathematical content to logical content, should be kept distinct: one is about the content of mathematics, the other about its presentation and execution.

In any case, all we have so far is some grasp of what the view is that Bernays is rejecting; we have yet to see why he rejects it. It isn't that Bernays denies the central role that logic plays in mathematics; it is rather that for him, the dependence between logic and mathematics is *bi-directional*. That is, what Bernays identifies as the characteristic of mathematical reasoning is operative *in* logic in a way analogous to how logic is operative in mathematics. This is a substantive claim, and we will come to it in §5. Prior to that, the next section takes up a different aspect of the generality question: is it true that logic is *the most abstract* of all disciplines?

4 Two kinds of abstraction

Let us look again at part of the passage quoted above:

> Mathematics and logic are based on two different directions of abstraction. While logic deals with the *contentually* most general [*das inhaltlich Allgemeinste*], (pure) mathematics is the general theory of the *formal* relations and properties, and so on the one hand each mathematical reflection is subject to the laws of logic, and on the other hand every logical construct of thought falls

[6] This is not meant to limit the scope of mathematics as a whole to *axiomatic* mathematics. The idea, rather, is that while a field of mathematics may be in flux with respect to its foundations early on, in its mature state, a field of mathematics is typically systematised and presented axiomatically.

> into the domain of mathematical reflection on account of the outer structure that is necessarily inherent in it. (Bernays, 1922b, p. 217)

I will explain the notion of 'outer structure' in §5. For now, I begin with the question: what does it mean to talk of 'directions of abstraction'? To understand Bernays' view, we need to consider them in the context of Hilbert's approach to the foundations of mathematics circa 1922 and the search for a formal consistency proof for arithmetic. This, in turn, builds on Hilbert's work on the axiomatisation of geometry and the idea of a relative consistency proof. I start with an overview of this background in §4.1, then return to Bernays' notion of logic and mathematics as different directions of abstraction in §4.2.

4.1 Logical structure and logic as an abstraction

One of the main goals of Hilbert's axiomatisation of geometry (1899) was to explore the logical relations between the axioms and axiom-groups. To show that an axiom is necessary, one needs to prove that it cannot be derived from any combination of the other axioms, i.e., that it is independent from them. Hilbert's method for proving an axiom independent was to show that an axiom system in which all other axioms are assumed, in conjunction with the *negation* of the axiom in question, is consistent. The method for proving consistency is what we now call a *relative consistency proof*: One constructs a procedure for systematically translating the propositions of the axiomatic system whose consistency is in question into propositions of another theory whose consistency is considered beyond reproach; in the case of geometry, Hilbert constructed models in the real number system, translating all geometric terms into real-number coordinates, equations, etc.[7] Thereby, if there were a contradiction in the axiom system under consideration, there must also be a contradiction in the real-number model given for it, i.e., in the real number system itself. By *modus tollens*, this proves the consistency of the theory which we were investigating, but only relative to, or conditional on, the consistency of the theory into which we translated.

[7] For example: a point in the plane is expressible as a pair of (x, y) coordinates; a line is expressible as an equation of the form $y = ax + b$; and so on.

This was only made possible by means of an abstraction away from content, in the sense that the original meaning of the basic terms was allowed to vary and to be reinterpreted freely. To see what a 'change of meaning' consists in, consider for example that a Euclidean point by definition has no parts, but a coordinate in a two- or three-dimensional real number field certainly does. All that matters for the purpose of this abstraction was that the axioms were still recognizably satisfied in whatever model was constructed for them; this mathematical technique made possible to exhibit the logical relations of the axioms, by preserving it across all models of the theory.

Consider the following passages, in which Bernays, in 1922, is reflecting back on Hilbert's work on geometry as a kind of abstraction:

> The task of geometry was understood in broader terms. Geometrical concepts became more general and freed themselves more and more from the subordination to spatial representation. (Bernays, 1922a, p. 189)
>
> Hilbertian axiomatics goes even one step further in the elimination of spatial intuition. Reliance on spatial representation is completely avoided here, not only in the proofs but also in the axioms and concepts. The words "point", "line", "plane" serve only as names for three different sorts of objects, about which nothing else is assumed directly except that the objects of each sort constitute a fixed determinate system. Any further characterization is carried out only through the axioms. In the same way, expressions like "the point A lies on the line a" or "the point A lies between B and C" will not be associated with the usual intuitive meanings; rather these expressions will designate only certain, at first indeterminate, relations which *are implicitly characterized* only through the axioms in which these expressions occur. (Bernays, 1922a, p. 192)

This gives us some idea of the sense in which mathematics is an abstraction: content is abstracted from (in the case of geometry, content as provided by the link to spatial representation), while preserving a certain structure: the structure defined by the axioms. Despite other disagreements between Frege and Hilbert, Hilbert's conception of logic during his work on axiomatising geometry was still quite close to Frege's, and the mathematical abstraction he engaged in was similar to Frege's view of the way in which logic is formal and abstract: " ...as far as logic itself is concerned, each object is as good as any other, and each concept of the first level [i.e., a predicate ranging over objects] as good as any other and can be replaced by it; etc." (Frege, 1906, p. 109). Frege goes on to note, though, that the abstraction which yields logic may only go so far. In opposition to treating logic in the same way that Hilbert (1899) treats geometry, Frege asserts the following:

> But this would be overly hasty, for logic is not as unrestrictedly formal as is here [i.e., in Hilbertian approaches] presupposed. If it were, then it would be without content. Toward what is thus proper to it, its relation is not at all formal. No science is completely formal; but even gravitational mechanics is formal to a certain degree, insofar as optical and chemical properties are all the same to it. To be sure, so far as it is concerned, bodies with different masses are not mutually replaceable; but in gravitational mechanics the difference of bodies with respect to their chemical properties does not constitute a hindrance to their mutual replacement. To logic, for example, there belong the following: negation, identity, subsumption, subordination of concepts. And here logic brooks no replacement. . . . One can express it metaphorically like this: About what is foreign to it, logic knows only what occurs in the premises; about what is proper to it, it knows all. (Frege, 1906, pp. 109–110)

Frege's message is that logic can be treated by the kind of abstraction that Hilbert carries out for geometry, but only up to a point; logic itself (considered here as that which regulates inference, not necessarily as that which has logical axioms giving rise to all of mathematics) has content. No abstraction is made from the logical primitives themselves: negation, implication, and so on. These things have some (presumably intuitively understood) meaning. Let us now consider how logic could be construed in this kind of abstraction, where content is abstracted away from, but logical content is untouched. This would be an abstraction which strips away all detail irrelevant to logic as the study of inference. For example: one needs to ignore the specific content or meaning of any of the names and predicates which appear in a given sentence, but to leave the connectives, quantifiers, etc., unchanged. Logic would be the study of those inferences that remain valid under such an abstraction. For example, consider one of the most elementary (and ubiquitous) forms of valid reasoning, *modus ponens*. Using capital letters to stand for propositions, given that two propositions hold, "If P then Q" as well as "P", we may validly infer that "Q" is the case. There is no need to know what the specific propositions are. This is what

Bernays meant, in the passage quoted earlier, in saying that logic is 'contentually the most abstract'.

Whereas logical abstraction leaves the logical terms constant, mathematical abstraction as Bernays understands it aims to leave *structural* properties constant. Here is how he characterises Hilbert's axiomatic approach:

> According to this conception, the axiomatic treatment of geometry amounts to separating the purely mathematical part of knowledge from geometry, considered as a science of spatial figures, and investigating it on its own. The spatial relationships are, as it were, mapped into the sphere of the abstract mathematical in which the structure of their interconnections appears as an object of pure mathematical thought. This structure is subjected to a mode of investigation that concentrates only on the logical relations and is indifferent to the question of the factual truth, that is, the question whether the geometrical connections determined by the axioms are found in reality (or even in our spatial intuition). (Bernays, 1922a, p. 192)

Despite calling them 'logical relations', what is being considered is not logic in the sense of a science of valid inference, but rather the study of structural relations. In the early axiomatic work to which the passage above refers, the distinction was not made clear, and it will be much easier to discern in the context of the proof theory of Hilbert's Programme. Consider now the following objection: if logic abstracts away from all content, and if further in mathematical abstraction as exemplified in Hilbert's work on geometry logic is left untouched, could it not be argued that logic *is* more abstract than mathematics, even if each of the disciplines abstracts with different invariants in mind? To answer that, we need to consider how logic itself is treated structurally under mathematical abstraction.

4.2 Hilbert's Programme and mathematical abstraction

For the purpose of Hilbert's investigation and axiomatization of geometry, relative consistency proofs were quite sufficient, but the goals of Hilbert's Programme are further-reaching: to provide a formal consistency proof of mathematics as a whole, starting with number theory and analysis. To prove that a certain kind of reasoning cannot lead to a contradiction, it isn't enough to look at typical examples. We require a general proof. The crucial step in the Programme was this: to look not for a *philosophical* argument why such reasoning is safe, but for a *mathematical proof*. And in order to have a mathematical proof about a certain kind of reasoning, this reasoning must be made into the sort of thing which can be treated mathematically: proofs themselves must be transformed into mathematical objects. Reasoning must be transformed into a formal, abstract structure. One needs to map the theory into a formal image of itself, and this in return means a precise characterisation not only of the basic terms of the theory, but also of the language and the forms of inference which are used in these formal images. Hence, the axiomatic approach must now be applied to logic itself. Hilbert takes seriously what he had done with geometry: if one looks at the symbol '$\vee$', say, then there are rules followed in the application of inferences involving it; these are then isolated and taken as axioms governing '$\vee$'. We include additional axioms accounting for how it is related to the other logical constants, and so on. In so doing, Hilbert is repeating what went on with respect to 'point', line', 'plane' etc. The axioms chosen for the logical primitives preserves their behaviour by encoding it in the formal axioms, but the conceptual content is deliberately left behind. The reason this must be done is that part of the critique of 'classical' mathematics by Brouwer and Weyl concerned precisely the applicability of the law of excluded middle to infinite totalities, and the permissibility of quantification over infinite domains.[8] By explicitly incorporating axioms for logic as part of the axioms of the formal theory, those parts of the formal language which are images of the logical primitives are open to reinterpretation. Their only properties are structural: namely, the formal axioms governing the rules for the use of these symbols in the strings of the formal language, operations that can be carried out on such strings, and so on.

To understand Bernays' point, that this is a *structural* direction of abstraction, and the sense in which this is a mathematical treatment of logic, it is useful to compare this to abstract algebra. The algebra familiar to everyone from our school days abstracts away from particular calculations, and discusses the rules that hold generally (the invariants, in mathematical terminology) while the variable letters are allowed to stand for any numbers whatsoever. Abstract algebra goes further, and 'forgets' not just which number the vari-

[8] See, e.g., Brouwer (1921), Brouwer (1923), Weyl (1921). For Hilbert's reply, see the main two papers of his Programme, Hilbert (1922) and Hilbert (1926).

ables stand for, but also what the basic operations standardly mean. The sign '+' need not necessarily stand for addition. Rather, the sign '+' stands for anything which obeys a few rules; for example, the rule that $a + b = b + a$, that $a + 0 = a$, and so on. Remember that the symbol 'a' need not stand for a number, and the numeral '0' need not stand for the number zero, merely for something that plays the same role with respect to the symbol '$+$' that zero plays with respect to addition. By following this sort of reasoning, one arrives at an *abstract algebra*; a mathematical study of what happens when the *formal rules* are held invariant, but the meaning of the signs is deliberately 'forgotten'. This leads to the study of general structures such as groups, rings, and fields, with immensely broad applicability in mathematics, not restricted to operations on numbers.

In a similar way, Hilbert and Bernays sought to develop a general theory of proofs: an algebra, or calculus, of deductive procedures. The word 'calculus' is used here in its literal sense: formal rules govern the composition and manipulation (or, synonymously: formation and transformation) of given objects, i.e., they characterise the operations that can be done with (or on) the objects, and how the objects and operations interact. We care not what these objects are; we are only interested in what we may learn about the system of rules in question. As Bernays explains:

> In adopting the procedure of mathematical logic, Hilbert reinterpreted it as he had done with the axiomatic method. Just as he had formerly stripped the basic relations and axioms of geometry of their intuitive content, he now eliminates the intellectual content of the inference from the proofs of arithmetic and analysis that he makes the object of his investigation. He obtains this by taking the systems of formulas that represent those proofs in the logical calculus, detached from their contentual-logical interpretations, as the immediate object of study, and by replacing the proofs of analysis with a purely formal manipulation that takes place with certain signs according to definite rules.
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> Through this mode of consideration, in which the separation of what is specifically mathematical from everything contentual reaches its high point, Hilbert's view on the nature of mathematics and on the axiomatic method then finds its actual conclusion. For we recognize at this point that the sphere of the mathematical-abstract, into which the methods of thought of mathematics translate all that is theoretically comprehensible, is not that of the contentual-

> logical [*inhaltlich Logisches*] but rather that of the domain of pure formalism. Mathematics then turns out to be the general theory of formalisms, and by understanding it as such, its universal meaning also becomes clear. (Bernays, 1922a, p. 196)

When the notion of 'pure formalism' and of mathematics as the general study of such formalisms is applied to logic and to mathematical proofs, now taken as objects of what Hilbert called 'proof theory', we deliberately forget that, e.g., the sign '$\rightarrow$' stands for implication (i.e., that '$P \rightarrow Q$' stands for 'If P, then Q'). Correspondingly, we assign it purely formal rules, e.g., allowing us to write 'Q' as long as at some previous point we have come across both 'P' and '$P \rightarrow Q$', which are simply strings of symbols in our calculus. Since these are symbols governed by certain rules, we can prove theorems *about* the resulting system. For example, we can prove that strings of symbols such as $0 = 1$ or $1 \neq 1$ are not derivable in the system in question, thereby proving it formally consistent. This will then correspond to informal consistency of the theory we mapped into this formal image. Note, though, that there is no claim that mathematics is equivalent or reducible to its abstract, formal image.

At this point, an objection may arise, along the following line. — It is true that in proofs, considered as the objects of study of proof theory, one can say that logic is not used at all. Rather, a *formal image* of the rules of inference appears as the rules of the calculus of proof. Nevertheless, Hilbert's proof theory *itself* is not a formal theory in the precise sense used above, for if it were formal in that sense, we would be trapped in infinite regress. That is, Hilbertian proof theory it is not a purely formal calculus for the manipulation of strings of symbols; it is a 'contentual' meta-theory with respect to the formal object-theory being investigated. In this meta-theory, deductive reasoning is employed, thus giving logic, construed not as its formal image but rather contentually as dealing with notions of validity, etc., an essential role.

The reply is that there is no need for Bernays to deny this. He is not making the case for a priority of mathematics with respect to logic, nor is he denying that logic is used ubiquitously in mathematics, including the mathematical study of logical proof. Rather, it is his view that logical and mathematical reasoning are thoroughly *interwoven*, and so this is quite in keeping with his view. The logical is not eliminable from the mathematical, nor reducible to it; and, crucially, so is the case vice-versa. This connects the discussion so far, of the distinction between the manner in which logic and mathematics are each *abstract* disciplines, to the next step: the argument why each discipline unavoidably presupposes and employs the reasoning essential to the other.

5 'The formal' and its essential employment in logic

It should at this point be reasonably clear what Bernays considers to be essential to the way in which mathematics is abstract, and how this is different from the way in which logic (as normally practised) is abstract. In order to see why he thinks that something which is fundamentally mathematical enters *essentially* into logic, we need to elaborate on what he takes to be a central part of the fundamental nature of mathematical reasoning: pared down to its very core, what *is* 'the mathematical' for Bernays? We already have the answer from the discussion above: mathematics is, at its heart, a study of *abstract structures* as such. He reinforces this point elsewhere:

> If we pursue what we mean by the mathematical character of a consideration, it becomes apparent that the typical characteristic is located in a certain mode of abstraction that comes into play. This abstraction, which may be called *formal* or *mathematical abstraction*, consists in emphasizing and exclusively taking into account the structural elements of an object — "object" here meant in its widest sense — that is, the manner of its composition from its constituent parts. One may, accordingly, define mathematical knowledge as that which rests on the *structural* consideration of objects. (Bernays, 1930, pp. 238–239)

First, the expression 'abstract structures' used here should be clarified. Abstract structures are given to us by means of the rules, procedures and principles that govern them, and mathematics can be said to be a systematic study of such structures. Indeed, the characterisation of mathematics as a 'science of structures' is not new with Bernays; arguably it is implicit already in Hilbert's work on geometry in the 1890's, among others.[9] The question is, though, how does this relate to the role of mathematics in logic, and Bernays'

[9] For a detailed history of the notion of structure used here and its especially close connection with algebra, see Corry (2004).

insistence that one cannot eliminate ‘the formal’ (by which he is referring to what is essential to mathematics) from logic? We could consider logic ‘contentually’, that is, with respect to its intended meaning and use, i.e., as a study of valid deductive reasoning, or as the study of those logical concepts which Frege identified in the passage quoted above (§4). But we can also see logic as a set of rules and principles. When we are considering logic *formally*, the fact that they are rules of *inference* is beside the point. That is, we could treat logic *algebraically*, that is to say, treat it as we treat abstract algebra, as discussed in previously (§4.2).

This is not a hypothetical suggestion, nor is it something undertaken for the sole purpose of proof-theory. At the same time as Frege was working on his ground-breaking *Begriffsschrift*, others such as Ernst Schröder[10] were treating logic as a kind of algebra, as an abstract rules-based calculus. In a different way, Bernays’ own *Habilitationsschrift* (1918) was based on treating propositional logic as a formal calculus, which can be given various interpretations for various purposes. To prove the independence of the axioms of propositional logic as he formulated them, Bernays interpreted them by constructing algebraic structures, with operations corresponding to the various propositional connectives etc., devised for the sole purpose of proofs of independence, it being quite irrelevant whether or not these structures would ‘naturally’ arise elsewhere in mathematics or how well they accord with the typical meaning of the axioms of propositional logic.[11] The point is that logical inference *can* be treated mathematically, fruitfully so, and in different ways for different purposes. One intellectual contribution of Bernays’ argument is to provide foundations for proof theory, in that it explains how proof theory is at all possible.

A further, more modern example is found in category-theoretical approaches to logic.[12] This is significant, since logic itself can be captured in terms of category theory.[13] While

[10] Schröder (1841 – 1902) was a 19th century mathematician and logician.

[11] This is comparable with the various gerrymandered models of geometrical structures which Hilbert constructed in his *Grundlagen der Geometrie* (1899).

[12] Under the title ‘Category Theory’ we find a number of theories, arising initially in the 1940’s as a branch of abstract algebra, but since then developed, *inter alia*, to serve as a algebraic foundational theory for mathematics. Some of its proponents advocate it as an alternative to the more standard set-theoretic foundations.

[13] There is a considerable body of literature on logic and category theory. See for example Bell (2005) and Goldblatt (1984).

category-theoretic foundations for mathematics and logic are far from being mainstream, the very *possibility* of such a thoroughly algebraic approach is instructive. We know it is possible to start with logic and some extra principles and definitions, and reconstruct arithmetic, analysis and abstract algebra from those building-blocks, as is done in, e.g., set theory or versions of what we now know as 'neo-logicism'. However, category theory shows that one could proceed in the opposite direction: to reconstruct logic (and set theory) in algebraic terms. Bernays' point is not that one of the two directions is superior to the other; quite the contrary, neither is superior. For while the category-theoretic approach avoids any explicit appeal to logical concepts, the very formulation, articulation and explanation of the theory (at a 'meta-' level, so to speak) necessarily involves logical inference, and so does any proving of theorems in the theory. How could it not do so? The same would be true of the formulation and presentation of *any* theory at its 'meta-' level, even if the theory *itself* is purely formal. That one is 'reducible' to the other does not imply that the essential nature of one is rooted in or exhausted by the other. Moreover, in the case of arithmetic and logic, reducibility holds to some extent in *both* directions, reinforcing the point that neither is more fundamental than the other.

Another way to see the role which Bernays sees for the mathematical (formal, structural) in logic is by analogy with the role of mathematics in physics.[14] In physics, mathematics appears in a familiar guise: geometrical structures, systems of differential equations, etc., in terms of which the physical theory is given. The physical structures are instances of mathematical structures. In logic, the mathematical element figures in the very symbolic formalism which is the language and basis of modern logic. To complete the analogy, logical systems are seen as instances of mathematical structures in the same way. It is instructive to consider precisely how Bernays draws the analogy between the role of mathematics in each case:

> [J]ust as the mathematical lawlikeness of theoretical physics is contentually specialized by means of its physical interpretation, so the mathematical relationships of theoretical logic also experience a specialization through their contentual logical interpretation. *The lawlikeness of the logical relationships appears here as a special model for a mathematical formalism.*

[14] Bernays (1922a), p. 196; Bernays (1930), pp. 239-240.

> This peculiar relationship between logic and mathematics, that is, that not only can one subject mathematical judgements and inferences to logical abstraction, but also the logical relationships to a mathematical abstraction, has its reason in the special position the area of 'the formal' [*des Formalen*] occupies vis-à-vis logic. Namely, whereas in logic one can usually abstract from the specific determinations of any domain of logic, this is not possible in the area of the formal, because *formal elements enter essentially into logic.* (Bernays, 1930, p. 239)

To understand Bernays' analogy, it's helpful to reflect on the relationship between mathematics and physics. Some mathematical theories were developed originally in service of physics, others quite independently of it. But the circumstances of ancestry are beside the point: mathematical theories are viable, as objects of study, quite independently of whether or not they find application in some empirical science or practical pursuit. To give a simple example: the concept of the derivative of a function is defined formally in mathematics without reference to anything in the world. If we take as an example distance traveled as a function of time, then the first derivative of the function is the velocity of the object moving according to this relationship, and the second derivative, its acceleration. This is a 'contentual interpretation' of a formal structure. Bernays suggests that the same is the case for logic: it can be thought of as a special case of a certain calculus, as has been discussed above. Physics, from Galileo and Newton to our days, employs mathematics *essentially*; mathematics is ingrained in the very fabric of physics. Bernays argues that mathematics is ingrained in the very fabric of (modern, mathematical) logic in an analogous way, and for an analogous reason: logic is spelled out by means of a formal calculus, and so its very structure is mathematical. Just as physics can be seen as giving an interpretation to certain mathematical structures, so logic can be seen as an interpretation of others. There are different ways to spell out mathematical structures which are operative in logic, depending on the purpose of one's mathematical investigation, but this would not be possible at all (let alone in several ways) if it were not for the mathematical character of logic.

Moreover, Bernays argues in the passage quoted above, the relation is *bi-directional*: the development of mathematics is everywhere guided by logic. The fundamental principle of logic is to abstract away from all specific content in order to distill the most general patterns of reasoning and logical connection, and indeed to isolate in a precise way what is essential in a proof. However, since mathematics, too, is based on an abstraction away from content (though in a different 'direction'), the 'formal' can remain after content is stripped away. The *structural*, rules-based nature of logic is essential to it; and *that* is the point in which characteristically mathematical reasoning operates in logic itself.

6 Mathematics as formal *and* contentual

Gödel's comments on Carnap, considered below, are not, on the face of it, directly concerned with the same matters as Bernays. Nevertheless, I believe there is a deeper connection. It was emphasised above that part of the context for Bernays' argument in 1922 was the failure of logicism, the other major element being philosophical clarification of what would become Hilbert's Programme. Of course, at that time it was not clear that the logicist project could not be carried through; that became clear only after Gödel had shown that no single axiom system of any sort could encompass the whole of mathematics, let alone a system of logical axioms. In §6.1, I consider Gödel's arguments against Carnap's attempt, in *Logical Syntax of Language* (1934), to argue for what can be seen as a variant of logicism, namely the claim that mathematics is analytic and without any specific content, presenting what Carnap thought to be the syntactic framework for the statement of empirical, physical truths about the world. Gödel was at great pains to show that this view of mathematics cannot work. It is the reason *why* Gödel believed it cannot work that is of interest here: namely, Gödel thinks that Carnap's programme cannot work because logic, even when considered 'merely syntactically', has an irreducible mathematical content. The gist of it has to do with the reasons why Gödel thought that Hilbert's Programme could not be carried through: although Hilbert, Bernays, and others active in the programme allowed the use of quite sophisticated and powerful mathematical tools in the investigation of logic (e.g., for the proof of the completeness of first-order logic), Hilbert believed that one could give a consistency proof for the unrestricted use of logic in non-finitary arithmetic based only on a very narrow, finitary syntactic-combinatorial ground. Gödel showed that this is wrong, and employed essentially the same reasoning against Carnap.

In §6.2 I show how this argument by Gödel connects with Bernays' views on what is characteristic of mathematics, through examining Hilbert's argument for an extralogical presupposition necessary for logical reasoning. This 'extralogical' element, I argue, is very similar to Bernays' 'mathematical' and to the ineliminable mathematical content that Gödel asserts against Carnap. Finally, in §6.3, I compare Bernays' characterisation of logic and mathematics with Gödel's division of formal theories into two kinds, intensional and extensional, which approximately correspond to logic and mathematics.

6.1 Is mathematics 'logical syntax of language'?

In 1934, Carnap published *Logische Syntax der Sprache*, with an expanded version in English, *Logical Syntax of Language*, appearing in 1937.[15] The goals of this book, as Carnap describes in his "Intellectual Autobiography" (1963), were twofold. First, to make logic, a paradigmatically *a priori* discipline, unproblematic for an empiricist epistemology. Second, to resolve the foundational disputes in logic, especially the dispute between proponents of classical and intuitionistic logic. Given that Carnap assumed, circa 1934, that the logicist programme is feasible, i.e., that mathematics is reducible to logic, accomplishing the first goal would at the same time make mathematics unproblematic to empiricist accounts of knowledge. Both goals were to be accomplished by adopting a *conventionalist* approach to logic: there is no single Logic, but rather many logics. A logic is the syntactic framework by which a (regimented, scientific) language is given. Different frameworks could be adopted, with the choice between them a matter of pragmatic consideration, based on the goals and *desiderata* of one's present pursuit.

In the terms set out in previous sections, Carnap blends together the logical and the formal; mathematics is taken as reducible to logic, and a logical framework can be thought of as a 'syntax', i.e., a calculus of formation and transformation rules for expressions. Setting aside Carnap's broader epistemological purpose, I would like to consider only his construal of logic as formal, pure syntax. Carnap adapts to his needs the manner in which logic was treated in Hilbert's Programme, but it's worth pointing out that he goes further

[15] It is worth noting that the syntactic view of logic is one that Carnap held in the 1930's. Carnap's later views on logic took a semantic turn, but that period of Carnap's philosophical development is not relevant for the present purpose.

than either Hilbert or Bernays: Carnap seeks to *reduce* logic to pure syntax, whereas Hilbert's Programme did not. In Hilbert's Programme, the mapping of contentual logic to its formal image was a means to a particular end, namely, an eventual consistency proof for (in the first place) arithmetic. Certain *properties* of logic can be captured in its formal image; consistency of a formal system, for example, is the property of the non-derivability of $0 = 1$ in it. Nevertheless, it was nowhere argued that logic is equivalent to its formal image, except for certain purposes. Moreover, as discussed above, the Programme was based on a combination of the formal theory with a contentual meta-theory.[16]

Gödel, in a lecture he gave in 1951 and in his commissioned contribution to Carnap's volume in the Library of Living Philosophers series (ultimately abandoned and unpublished), was at pains to refute conventionalism about logic and mathematics in general, and especially Carnap's views in *Logical Syntax of Language*.[17] A main line of argument for Gödel relies on his incompleteness theorems. To maintain a sharp dividing line between stipulated linguistic conventions and empirical facts, it must be the case that whatever principles are stipulated as part of the linguistic 'framework', their consequences are limited to this framework; the logical syntax of the language must not make a difference to empirical statements expressed in that language. For this to be the case, the logico-mathematical framework must at a minimum be demonstrably consistent; otherwise, being inconsistent, anything at all could be derived from it as a consequence, including empirical statements, which goes against the very idea of conventionalist 'logical syntax'. Therefore there is a need for a *consistency proof* for the logical framework, but by Gödel's second incompleteness theorem, if this framework is sufficient to express basic arithmetic (and it would have to be, to be of any scientific usefulness), its consistency can only be proved in a system which in some sense goes beyond it. The argument that Carnap must require a proof of consistency for the framework has, in recent years, received considerable attention, and a number of commentators have worked out a line of defence, the gist of which is that Gödel's argument underestimates the radically pragmatist nature of Carnap's conventionalist 'Principle of Tolerance'.[18] There is much to be said on this matter, but it falls outside

[16] Bernays' views have been discussed above. For Hilbert's views, see especially Hilbert (1922) and Hilbert (1926).

[17] These texts were published posthumously as Gödel (1951), Gödel (1959).

[18] Ricketts (2008) is a good representative of this line of defence.

the immediate concerns of this paper.

Another line of argument Gödel takes against Carnap parallels that of Quine (1936). Quine argued that to derive logic from syntactic conventions, one must already have in place principles by which to derive consequences, i.e., some logical princples. Similarly, Gödel finds Carnap's conventionalism viciously circular in that it presupposes some of the very mathematical principles it is meant to account for:

> Now it is actually so, that for the symbolisms of mathematical logic, with suitably chosen semantical rules, the truth of the mathematical axioms *is* derivable from these rules; however (and this is the great stumbling block), in this derivation the mathematical and logical concepts and axioms themselves must be used in a special application, namely, as referring to symbols, combinations of symbols, sets of such combinations, etc. (Gödel, 1951, p. 317)

Syntax consists ultimately of rules for the composition and manipulation of strings of symbols, rules of formation and transformation of sets of discrete objects. It is therefore a mathematical theory, in everything but name. Carnap's notion of a syntax is merely a special case of the general notion of a finite manifold (op. cit., p. 320, footnote 29). Gödel also makes the same case another way: for Carnap's conventionalism to succeed, he needs to define mathematical truth in terms of what is derivable from syntactical conventions; that is, to show that once such conventions are stipulated, all mathematical truths are tautologies. This can be done, trivially, for finite numerical systems; one can easily define '$+$' for numbers up to 1,000 in such a way that all equations of the form '$5 + 7 = 12$' come out to be tautologies. The question is, can this be done generally, for the operation '$+$'? To prove that *all* equations of the form $a + b = c$ come out as tautologies under the syntactic stipulations, Carnap would have to invoke the principle of mathematical induction. However, there can be no syntactic justification of this principle itself (Gödel, 1951, pp. 317–318).[19]

In short, in order to argue that Carnap presupposes mathematical principles in his syntactic conventionalism, Gödel identifies the theory of finite manifolds and the principle

[19] This is, as Gödel acknowledges, a variant of Poincaré's argument against logicism (represented by Louis Couturat) and the early version of Hilbert's formalist account of the nature of number; for Poincaré's original argument, see Poincaré (1906). For Gödel's reference to Poincaré, see (Gödel, 1951, p. 319 footnote 28).

of mathematical induction (or something equivalent to it) as fundamental and essential presuppositions, arguing that the very idea of a syntactic rule presupposes that of a finite sequence of symbols. The theory of manifolds, in turn, Gödel explicitly equates with the theory of integers.[20] These characteristics are an ineliminable part of any attempt to build up mathematics as a theory of syntax alone, but this is already a good deal of mathematics.

There is, in this, a clear link to Bernays' discussion. As we saw, Bernays is somewhat vague about what sorts of mathematical analysis of the logical system there will be. But it seems clear that an analysis of the *combinatorial* properties of the 'finite manifolds', which the elements of the formal logical system give rise to, is one clear way in which the mathematical properties of the logic might be explored. In the following passage, for example, Bernays contrasts the logical and the mathematical elements ('moments') at work in a demonstration, saying that the mathematical aspect of it is combinatorial:

> In the process of demonstration, there are two significant moments that work together: the clarification of the concepts, that is, the moment of *reflection*, and the mathematical moment of *combination*. (Bernays, 1930, p. 240)

In a later paper, explaining the fundamental idea of Hilbert's Programme, Bernays makes it clear that Hilbert's notion of finitary arithmetic was combinatorial:

> Hilbert has sketched a detailed program of a theory of proof, indicating the leading ideas of the arguments (for the main consistency proofs). His intention was to confine himself to intuitive and combinatorial considerations; his "finitary point of view" was restricted to these methods. (Bernays, 1935, p. 270)

Bernays goes on to note, there, that Gödel proved such elementary combinatorial methods insufficient; but the crucial point is that combinatorial methods are at the very core of mathematics, i.e., finitary arithmetic. There is, in Bernays' and Gödel's appeal to the combinatorial notion of a finite manifold, a direct link also to Hilbert's Programme, explored further in the next subsection.

[20] See Gödel (1951), p. 320, especially footnote 29.

6.2 Logic's extralogical presupposition

Consider the following passage from Hilbert's "On the Infinite", repeated in very similar form in various other of his writings:

> Rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation [*in der Vorstellung*], certain extralogical concrete objects that are intuitively [*anschaulich*] present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding and communication. (Hilbert, 1926, p. 376)

Hilbert never makes quite clear what the 'extralogical' is that he has in mind. From a similar statement in his (1922) and scattered comments in other writings, one can reconstruct what Hilbert had in mind as two kinds of 'extralogical' presuppositions. One has to do with our physical, or one could say biological, ability to perceive and work with concrete objects: recognizing that the same symbol or string of symbols appears in more than one place, for example, and similar basic perceptual capacities. This might be called the 'engineering' part of what is presupposed for logic and mathematics to be possible. Of greater interest, however, is something else which is arguably presupposed: certain basic cognitive capacities, which can be called the 'computational' part of what must be presupposed. Namely, we must have the ability to follow simple instructions or algorithms, to carry out certain procedures, for example: examining two sequences of strokes such as '||||' and '|||' and being able to compare the two to discern that one is a longer string than the other. Such procedures are clearly what Hilbert is referring to when he speaks of concatenation and comparison, and the claim is clearly that we must be able to carry these out, not just for short sequence of strokes, but in principle for any finite sequence of symbols. This is an 'extralogical' capacity to carry out algorithms and basic combinatorial operations such as concatenation. This, too, is the use of the mathematical in the analysis of the logical, something which belongs to the 'extra-logical'. It is usually accepted that this statement of Hilbert's gives a statement, albeit imprecise, of what underlies what is usually called finitary mathematics.

Gödel, too, recognizes finitary mathematics with what is concrete, in the combinatorial sense. Here is a passage from his (1958), which appeared in a special volume of the journal *Dialectica* dedicated to Bernays:

> [S]ince finitary mathematics is defined as the mathematics in which evidence rests on what is *intuitive*, certain *abstract* notions are required for the proof of the consistency of number theory. ... Here, by abstract (or nonintuitive) notions we must understand those that are essentially of second or higher orders, that is, notions that do not involve properties or relations of *concrete objects* (for example, of combinations of signs), but that relate to *mental constructions* (for example, proofs, meaningful statements, and so on); and in the proofs we make use of insights, into these mental constructs, that spring not from the combinatorial (spatiotemporal) properties of the sign combinations representing the proofs, but only from their *meaning*. (Gödel, 1958, p. 241)

The finitary is identified as the combinatorial manipulation of finite objects, and that is also the 'intuitive'. Referring to the failure of Hilbert's Programme, Gödel explains that it is (demonstrably) impossible to prove the consistency of number theory on such a basis, and notions that go beyond the concrete-combinatorial finitary are required. Gödel's comments are connected with his critique of Carnap, too: he remarks, in the unpublished revised version of his (1958) paper, that Gentzen's consistency proof for arithmetic by means of recursion up to ϵ_0 shows that no consistency proof for arithmetic can be made 'immediately evident' (i.e., 'intuitive' in the sense just discussed).[21] That is, that sticking merely with the 'concrete', as Carnap's syntactic approach tries to do, is insufficient, and that the mathemtical analysis of the consistency of syntactically presented logical frameworks already requires substantial abstract mathematics. In sum, the agreement between all of Bernays, Hilbert and Gödel is on the characteristic of mathematics as the study of the combinatorial and structural. The disagreement is on what precisely is required for the analysis of formal consistency, especially in the case of arithmetic.

[21] See Gödel (1972).

6.3 Intensional and extensional formal theories

Let me conclude these reflections with discussion of another division Gödel makes between logic and mathematics, which aligns in an interesting way with Bernays' analysis of logic and mathematics as abstracting in two different directions. What I have in mind is Gödel's characterisation of logic as *intensional*, and mathematics, construed here as set theory, as *extensional*. Gödel first makes this distinction, albeit unsystematically and without much detail, in his paper on Russell (Gödel (1944)). There, Gödel stresses that paradoxes having to do with sets and classes admit of different solutions, depending on whether one thinks of classes in terms of content and meaning (i.e., intensionally) , or in terms of extension alone.

Gödel returned to the division between the intensional and extensional later in his career, as reported in Hao Wang's account of conversations and correspondence with him (Wang (1996)), where he also explicitly separates the distinction intensional/extensional from that between semantic and syntactic, now viewing the two as mutually independent.[22] In his latter-day views, reported by Wang, Gödel connects the division between intensional and extensional with that between logic and mathematics:

> The subject matter of logic is intensions (concepts); that of mathematics is extensions (sets). Predicate logic can be taken either as logic or as mathematics: it is usually taken as logic. The general concepts of logic occur in every subject. A formal science applies to every concept and every object. There are extensional and intensional formal theories. (Wang, 1996, p. 274)

This passage is rather dense and merits careful reading. Mathematics is assumed to be [reducible to] set theory; a view which I accept for the purpose of the present discussion. In saying that set theory is extensional, Gödel presumably means the fact that sets are individuated by their extensions alone. This makes the contrast with logic clearer: logic deals with concepts, which are individuated intensionally, rather than extensionally.[23]

[22] For example, Gödel considers the Liar Paradox to be semantic, and solvable by means of relativising the truth-predicate to a language. In contrast, he considers paradoxes having to do with logical self-reference (e.g., the paradox of whether the concept of heterologicality is itself heterological) to be *intensional* and not solvable by relativisation to a language. See (Wang, 1996, p. 272).

[23] See (Wang, 1987, p. 297), where Gödel is also reported as drawing the intensional/extensional division in the same way.

However, the picture with respect to logic is far less clear, since, as Gödel points out in the very same passage, predicate logic (for example) can be construed both 'as logic' and 'as mathematics'. Here, Bernays' remarks on the mathematical in logic are very helpful in understanding what would otherwise be a rather obscure statement by Gödel: to treat predicate logic as logic is to consider it a study of inference and validity, and to treat it as mathematics is to focus on the mathematical in logic, e.g., to treat it as a formal calculus. The rest of the passage refers to the equal generality of both intensional and extensional formal theories: this would correspond to what Bernays referred to as 'the logical' and 'the mathematical' [or, 'the formal']: both logic and mathematics hold fully generally, but this creates no conflict.

However, Gödel's philosophical views in later years were unsystematic and in flux, which was apparently the main reason why he was reluctant to publish his philosophical papers and lectures during the 1950's and 1960's.[24] The same is true for his ideas regarding a 'theory of concepts', ideas which complicate Gödel's division of labour between the intensional and extensional. Gödel had a vision for a rigorous theory of concepts, a theory which would be parallel to our rigorous theory of sets.[25] There is a sense in which this envisaged intensional theory of concepts is more general than mathematics (set theory), for Wang explicitly reports that Gödel took logic, construed as the study of concepts, to have 'a more inclusive domain' than that of mathematics (Wang, 1987, p. 189). This is elaborated in the following quote from near the end of Gödel's life, circa 1976:

> Logic is the theory of the formal. It consists of set theory and the theory of concepts. ... Set is a formal concept. If we replace the concept of set by the concept of concept, we get logic. The concept of concept is certainly formal and, therefore, a logical concept. ... A plausible conjecture is: Every set is the extension of some concept. ... The subject matter of logic is intensions (concepts); that of mathematics is extensions (sets). (Wang, 1996, p. 247)

To dispel the appearance of a contradiction between these quotes and Gödel's views discussed earlier in this section, it should be noted that 'logic' is used here in a broader sense than previously, for it embraces both kinds of formal theory: the intensional formal

[24] See, for example, Goldfarb's introductory note to Gödel's (1951) (Gödel, 1995, pp. 324–334).
[25] Wang discusses this in several of his writings. See, e.g., (Wang, 1987, pp. 309–313).

theory, i.e., logic in the narrower sense, as well as the extensional formal theory, i.e., set theory. Moreover, Gödel conjectures that the intensional is more general, that it *subsumes* the extensional, in that every set is the extension of some concept, though not every concept has a set as its extension.[26] It should nevertheless be noted that Gödel's overarching 'theory of concepts' was almost entirely a hypothetical future creation. Gödel does give a few examples of concepts analyzed in the manner he envisaged such a theory to do, such as 'computation' and 'set', but there is little reason to think this can be generalised to all logical and mathematical concepts, let alone to our concepts generally.[27] Nevertheless, Gödel's views on the theory of concepts suggest what Bernays might mean by the view, which he rejects, that logic is more general than mathematics: an all-pervasive conception of 'logic in the broad sense'.

7 Concluding remarks

The primary purpose of this paper has been twofold. First, to explore the relationship between logic and mathematics, and the two respective sorts of formality that are typical of each, via Bernays's reflections on logic and mathematics. In its first part, I argued that the relationship between logic and mathematics is a two-way relationship (of essential presupposition), with each falling under the other's purview. This brought us to consider what this purview is: what essential characteristic of each discipline these considerations reveal. Logic and mathematics, it was argued, abstract from the same content but along different lines. Logic abstracts away from the content (e.g., of names and predicates) to arrive at general rules for valid inference. Mathematics abstracts away from content to arrive at a general study of rule-governed structures of objects.

This division was then compared, in the second part, with Gödel's views, particularly in his criticism of Carnap. While Bernays appears to be arguing that one can undertake mathematical analyses of logic, because of the (presumably variously described) structural

[26] If every concept did have an extension, we would get a version of Russell's paradox.

[27] Wang, ordinarily a very sympathetic commentator on Gödel's views, does note with respect to Gödel's notion of a theory of concepts that it is an instance of Gödel's tendency to be excessively optimistic with respect to how far success in particular instances can be generally repeated; see (Wang, 1987, 191).

elements that a formal logic reveals, Gödel argued (in the light of his studies of incompleteness, and contra Hilbert's conjecture) that a proof of consistency for formal arithmetic already requires appeal to abstract mathematics. I argued further that Bernays's scheme maps quite well onto Gödel's later division of formal theories into intensional (logical) and extensional (mathematical) varieties, which illuminates Gödel's otherwise rather obscure views on this matter. This study is thus meant both as a philosophical argument in its own right, and as an historical-exegetical elucidation.

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