

Response to:
Three Merry Roads to T-Violation by Bryan W. Roberts

Abhay Ashtekar*

*Institute for Gravitation and the Cosmos & Physics Department,
Penn State, University Park, PA 16802, U.S.A.*

I. PREAMBLE

Dr. Roberts has provided a lucid account of the analytical arguments that underlie the study of T -violation [1]. His clear presentation makes my task rather easy. I will discuss T violation in a very general setting that incorporates quantum mechanics and quantum field theory, but is not tied to them. The focus will be on the conceptual aspects of the two approaches that have led to experimental proofs of T -violation in weak interactions.

Since Dr. Roberts mentioned the macro-world only in passing, let me begin a brief discussion of the manifest arrow of time we perceive in our everyday life and, more generally, in the physics of large or macroscopic systems. For simplicity, let me discuss this issue in the framework of classical physics because the core of the argument is not sensitive to the distinction between classical and quantum mechanics. Consider a large box with a partition that divides it into two parts, say, the right and the left halves. Suppose there is some gas in the left half and vacuum in the right. Once equilibrium is reached, the macroscopic state of this gas is described by the volume it occupies, V_i ; the pressure it exerts on the wall, P_i and its temperature T_i , where i stands for ‘initial’. If we open the partition slowly, the gas will fill the whole box and its macro-state in equilibrium will be described by new parameters, V_f, P_f, T_f . Thus, there has been a transition from the initial macro-state (V_i, P_i, T_i) to a final state (V_f, P_f, T_f) . Our common experience tells us that the time reverse of this process is *extremely* unlikely.

However, we also know that the microscopic variables for the system are the positions and momenta of some 10^{23} molecules in the box. These are subject just to Newton’s laws which are manifestly invariant under the time reversal operation T ! Therefore, if we were to reverse the momenta $\vec{p}_{(\alpha)}(t)$ of each of the molecules (labeled by α) at a late time t , keeping the final positions $\vec{x}_{(\alpha)}(t)$ the same, time evolution would indeed move the gas from its final macroscopic state to the initial one. But it is very difficult to construct this time-reversed initial state. Thus there is indeed a macroscopic arrow of time but its origin is *not in the failure of the microscopic laws to be invariant under T* but rather in the fact that the initial conditions we normally encounter are very special. This is reflected in the fact that there are vastly fewer micro-states compatible with the initial macro-state (V_i, P_i, T_i) than there are compatible with the final macro-state (V_f, P_f, T_f) .¹ Put differently, the entropy of the initial macro-state is much lower than that in the final macro-state.

*Electronic address: ashtekar@gravity.psu.edu

¹ This is primarily because the volume V_f allowed for *each* molecule in the final macro-state is twice as large as V_i , allowed in the initial macro-state.

To summarize the fact that there is a clear arrow of time in the macro-world does *not* imply that the microscopic or fundamental laws have to break T -invariance. Indeed, as Dr. Roberts emphasized in the beginning of his article [1], it was common to assume that the fundamental laws *are* invariant under the time reversal operation T . It was a major surprise that the weak interaction violates this premise.

II. WEAK INTERACTIONS AND THE CURIE PRINCIPLE

As Dr. Roberts has explained clearly, what the Cronin-Fitch experiment establishes directly is that the weak interactions are not invariant under CP , i.e., under the simultaneous operation of charge conjugation and parity. The parity operation, as normally formulated, is meaningful only if the underlying space-time is flat, i.e., represented by Minkowski space-time. This means one ignores curvature and therefore gravity. One further assumes that physics is described by a local quantum field theory on this Minkowski space, for which individual physical fields transform covariantly under the action of the Lorentz group and dynamics is generated by a self-adjoint Hamiltonian obtained by integration of a scalar density constructed locally from the physical fields. Then, one has the *CPT theorem* that guarantees that the product CPT of charge conjugation, parity and time reversal is an exact *dynamical* symmetry.² Therefore, as Dr. Roberts explained, if we assume that weak interactions are described by such a theory, then the observed breakdown of CP invariance implies that they violate T invariance as well.

Dr. Roberts describes the mathematical underpinning of the ‘Curie Principle’ in his section 2.4 using a linear transformation R on the Hilbert space of states, which is to play the role of a symmetry of interest. This formulation can be significantly generalized. The main point of my ‘response’ is to provide this formulation.

As Dr. Roberts emphasizes, his analysis has the advantage that it does not assume a specific Hamiltonian. Let us go a step further and consider a formulation that does not use even the mathematical structure normally used in quantum (or classical) *kinematics*. Both frameworks will be special cases of this *general mechanics*. What it assumes is:

i) We have a set \mathcal{S} of states;

ii) There is a 1-1, onto dynamical mapping S —the ‘ S -matrix’— from \mathcal{S} to itself. In practice is it convenient to consider two copies \mathcal{S}_i and \mathcal{S}_f of \mathcal{S} , representing initial and final states, and regard S as a map from \mathcal{S}_i to \mathcal{S}_f ;

$$S : \mathcal{S}_i \rightarrow \mathcal{S}_f; \quad S(\sigma_i) = \sigma_f \quad \forall \sigma_i \in \mathcal{S}_i \quad (2.1)$$

and,

iii) A 1-1, onto map R from \mathcal{S} to itself, to be thought of a potential symmetry. We will first consider the case in which R maps \mathcal{S}_i to itself and \mathcal{S}_f to itself. This is the case if R is, for example, the discrete symmetry represented by C , or P or CP .

² This is a heuristic version one finds in quantum field theory text books (see, e.g. [2]). More rigorous versions based on Wightmann axioms [3] and the algebraic approach [4] are also available in the literature. However, one should note that we do not have a single example of a 4-dimensional, interacting quantum field theory satisfying either the Wightmann axioms or the axioms of the algebraic quantum field theory.

Now, in the spirit of the Curie principle, suppose that there exists some $\sigma_i \in \mathcal{S}$ such that

$$R\sigma_i = \sigma_i \quad \text{but} \quad R\sigma_f \neq \sigma_f \quad (2.2)$$

Then,

$$R(S\sigma_i) = R\sigma_f \neq \sigma_f = S(R\sigma_i) \quad (2.3)$$

whence $SR \neq RS$. Thus, the dynamical map S does not commute with the candidate symmetry R : It is *not a dynamical symmetry*. We therefore conclude: *If there exists a state σ_i such that $R\sigma_i = \sigma_i$ and $S\sigma_i = \sigma_f$ but $R\sigma_f \neq \sigma_f$ then R is not a dynamical symmetry of the system.* Thus the Curie principle naturally extends to *general mechanics*. (It is straightforward to alter the argument to obtain the other desired conclusion of fact 1 in section 2.4 of [1].)

Since we assumed \mathcal{S} to be only a set, we cannot speak of continuous evolution. But one could achieve this trivially by endowing \mathcal{S} with a topology and replacing S with a continuous evolution map $E(t)$, where t is to be thought of as a time parameter. The argument given above will then imply that $E(t)$ will not commute with R .

Note that in this more general formulation one does not even assume that the space of states has a Hilbert space structure, whence, when applied to the quantum mechanical system, the argument does not require R or S to be linear mappings. In particular, they need not be unitary. Note also that this general framework enables one to discuss in one go all symmetries in classical mechanics and linear symmetries in quantum mechanics. More importantly, it should be useful also in non-linear generalizations of quantum mechanics (discussed, e.g., in [5]).

However, we did assume that R maps the space \mathcal{S}_i of initial states to itself and the space \mathcal{S}_f of final states to itself. This assumption is *not* satisfied by the time reversal operation T which maps initial states to final states (and vice versa): T is a 1-1 onto map from \mathcal{S}_i to \mathcal{S}_f . Therefore, in this case, T is a dynamical symmetry if and only if

$$S\sigma_i = \sigma_f \quad \implies \quad S^{-1}(T\sigma_i) = T^{-1}(\sigma_f) \quad (2.4)$$

i.e., the time reverse of σ_i (which is in \mathcal{S}_f) is mapped by dynamics to the time reverse of σ_f . The generalization of the Curie principle discussed above does not have any implication in this case. In this respect, the situation is the same as in section 2 of [1].

Remark: While R invariance of dynamics is captured by the property $RS = SR$ of the S-matrix S while the T invariance is captured by $S^{-1}T = T^{-1}S$. The above argument shows that the difference arises simply because while R preserves each of \mathcal{S}_i and \mathcal{S}_f , T maps one to the other. At a fundamental level, then, the difference is not because R is linear while T is anti-linear although, in standard quantum mechanics, one can use linearity of R and anti-linearity of T to arrive at the difference.

III. THE KABIR PRINCIPLE

Can we extend the arguments from *general mechanics* to encompass time reversal in the spirit of Kabir's argument discussed in section 3 of [1]? The answer is in the affirmative. However, to state Kabir's formulation, one needs to introduce additional structure in *general mechanics* which does not have a natural analog in classical mechanics. This is because Kabir's formulation refers to probabilities.

Let us then introduce an *overlap map* \mathcal{O} on the space of states \mathcal{S} (and therefore on each of \mathcal{S}_i and \mathcal{S}_f): $\mathcal{O} : \mathcal{S} \times \mathcal{S} \rightarrow [0, 1] \in R$, such that

$$\mathcal{O}(\sigma, \sigma') = \mathcal{O}(\sigma', \sigma), \quad \forall \sigma, \sigma' \in \mathcal{S}. \quad (3.1)$$

$\mathcal{O}(\sigma, \sigma')$ is to be thought of as the overlap between states σ and σ' , the generalization of the absolute value of the quantum mechanical inner product between normalized states $|\langle \Psi, \Psi' \rangle|$. The overlap map is part of kinematics. Therefore, to qualify as symmetry, the map R we discussed in section II has to satisfy

$$\mathcal{O}_i(R\sigma_i, R\sigma'_i) = \mathcal{O}_i(\sigma, \sigma') \quad (3.2)$$

on \mathcal{S}_i (and similarly on \mathcal{S}_f). Similarly to qualify as symmetry, the Time reversal map which maps \mathcal{S}_i to \mathcal{S}_f must satisfy

$$\mathcal{O}_f(T\sigma_i, T\sigma'_i) = \mathcal{O}_i(\sigma_i, \sigma'_i) \quad (3.3)$$

for all $\sigma_i \in \mathcal{S}_i$.

The dynamical map $S : \mathcal{S}_i \rightarrow \mathcal{S}_f$ should be compatible with this kinematical structure, i.e., satisfy

$$\mathcal{O}_i(\sigma_i, \sigma'_i) = \mathcal{O}_f(S\sigma_i, S\sigma'_i) \equiv \mathcal{O}_f(\sigma_f, \sigma'_f). \quad (3.4)$$

Given a dynamical map S , the *transition probability* between an initial state $\sigma_i \in \mathcal{S}_i$ and *any* given final state σ'_f is defined to be

$$P(\sigma'_f, \sigma_i) := [\mathcal{O}(\sigma'_f, S\sigma_i)]^2 \equiv [\mathcal{O}(\sigma'_f, \sigma_f)]^2. \quad (3.5)$$

This is the additional kinematical structure we need on our *general mechanics* to formulate the Kabir principle.

Recall that T is a dynamical symmetry if and only if $S^{-1}T = T^{-1}S$. Suppose a dynamical map S satisfies this condition. Then,

$$\mathcal{O}_f(T\sigma_i, S(T^{-1}\sigma'_f)) = \mathcal{O}_f(T\sigma_i, T(S^{-1}\sigma'_f)) = \mathcal{O}_f(T\sigma_i, T\sigma'_i) = \mathcal{O}_i(\sigma_i, \sigma'_i) \quad (3.6)$$

where in the second step we have set $\sigma'_i = S^{-1}\sigma'_f$ and in the last step we used (3.3). On the other hand, (3.4) and (3.1) imply

$$\mathcal{O}_i(\sigma_i, \sigma'_i) = \mathcal{O}_f(S\sigma_i, S\sigma'_i) = \mathcal{O}_f(\sigma'_f, S\sigma_i). \quad (3.7)$$

The last two equations and the definition (3.5) of transition probability implies

$$P(\sigma'_i, \sigma_i) = P(T\sigma'_i, T^{-1}\sigma'_f). \quad (3.8)$$

Thus, we have shown that if T is a symmetry of the dynamical map S then the transition probability between the states σ_i and σ'_f must equal that between the two states obtained by a time reversal. Therefore if the transition probability between *any* two states and their time reversed versions differ observationally, then the time reversal symmetry is broken by dynamics.³

³ It is worth noting that the actual transition rate is not determined solely by the transition probability. In the leading order approximation (“Fermi’s golden rule”) the transition probability has to be multiplied by the density of final states. But in practice one can easily take care of this issue and verify whether or not the transition probability is symmetric under time reversal.

As with the discussion of the generalized Curie principle of section II, this generalization of the Kabir criterion does not refer to the Hilbert space structure of the space of states or linearity (or anti-linearity) of various maps. In particular, in the case of quantum mechanics, while it incorporates the standard treatment neatly summarized in section 3 of [1], the results would hold even if, say, the S-matrix were anti-unitary. As with the Curie principle, this generalization may be useful to non-linear generalizations of quantum mechanics. However, in classical mechanics, there are no obvious structures corresponding to the overlap map and the subsequent notion of transition probability. Therefore, unlike our discussion of section II, the present discussion has no implications to classical mechanics.

IV. DISCUSSION

Apart from obvious advantages of inherent to a generalization, already in the context of quantum mechanics, the setting of *general mechanics* presented here serves to bring out the elements and structures that are essential in the discussion of CP and T violation. In particular, neither the linear structure nor the details of the Hermitian inner product of the space of quantum mechanical states is essential. Secondly, the primary distinction between C , P , CP and T lies in the fact that while C , P and CP leave the space of ‘in’ and ‘out’ states invariant, T maps the incoming states to the outgoing ones. In standard quantum mechanics, this has the implication that while C , P and CP are represented by linear, unitary maps, T is represented by an anti-linear, anti-unitary map. However, from the perspective of *general mechanics* this difference is not primary to the distinction between the Curie and Kabir criteria.

Finally, Ref. [1] also provides a succinct summary of ideas for testing T violations through the measurement of the dipole moment of elementary particles, such as a neutron. I will just add a phenomenological remark. The electric dipole moment is not invariant also under the parity operation P . Therefore, even if one were to observe an electric dipole moment, one cannot *directly* conclude that there is T violation.

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- [1] B.̃. Roberts, Three merry roads to T-violation, talk at the Worksop on Cosmology and Time, held at Penn State, April 2013 (pre-print).
 - [2] S. Weinberg, *Quantum Theory of Fields* Vol 1, pp 244-246 (Cambridge University Press, Cambridge, 1995).
 - [3] R. F. Streater and A. Wightman, *PCT, Spin, Statistics and All That*, Chapter 4 (Benjamin, New York, 1964).
 - [4] J. Yngvason and H. J. Borchers, H. J. On the PCT-theorem in the theory of local observables, [arXiv:math-ph/0012020](https://arxiv.org/abs/math-ph/0012020).
 - [5] A. Ashtekar and T. Schilling, Geometrical formulation of quantum mechanics, in: *On Einsteins Path*, edited by A. Harvey (Springer-Verlag, Berlin, 1998), 23-66; also available at [arXiv:gr-qc/9706069](https://arxiv.org/abs/gr-qc/9706069).