Spontaneous Symmetry Breaking in Quantum Systems: Emergence or Reduction?

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Dedicated to the memory of Gérard Emch (1936–2013)

Abstract

Beginning with Anderson (1972), spontaneous symmetry breaking (SSB) in infinite quantum systems is often put forward as an example of (asymptotic) emergence in physics, since in theory no finite system should display it. Even the correspondence between theory and reality is at stake here, since numerous real materials show SSB in their ground states (or equilibrium states at low temperature), although they are finite. Thus against what is sometimes called 'Earman's Principle', a genuine physical effect (viz. SSB) seems theoretically recovered only in some idealization (namely the thermodynamic limit), disappearing as soon as the the idealization is removed.

We review the well-known arguments that (at first sight) no finite system can exhibit SSB, using the formalism of algebraic quantum theory in order to control the thermodynamic limit and unify the description of finite- and infinite-volume systems. Using the striking mathematical analogy between the thermodynamic limit and the classical limit, we show that a similar situation obtains in quantum mechanics (which typically forbids SSB) versus classical mechanics (which allows it). This discrepancy between formalism and reality is quite similar to the measurement problem, and hence we address it in the same way, adapting an argument of the author and Reuvers (2013) that was originally intended to explain the collapse of the wave-function within conventional quantum mechanics. Namely, exponential sensitivity to (asymmetric) perturbations of the (symmetric) dynamics as the system size increases causes symmetry breaking already in finite but very large quantum systems. This provides continuity between finite- and infinite-volume descriptions of quantum systems featuring SSB and hence restores Earman's Principle (at least in this particularly threatening case).

Motto

"The characteristic behaviour of the whole *could* not, even in theory, be deduced from the most complete knowledge of the behaviour of its components, taken separately or in other combinations, and of their proportions and arrangements in this whole. This (...) is what I understand by the 'Theory of Emergence'. I cannot give a conclusive example of it, since it is a matter of controversy whether it actually applies to anything.

(Broad, 1925, p. 59)

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1 Introduction

In a philosophical context, the notion of 'emergence' is usually ascribed to J.S. Mill, who did not actually use this terminology himself but drew attention to "a distinction so radical, and of so much importance, as to require a chapter to itself." (wow!) The distinction in question is the one between the principle of the "Composition of Causes", according to which the joint effect of several causes is identical with the sum of their separate effects, and the negation of this principle. For example, in the context of his overall materialism, Mill believed that although all "organised bodies" are composed of material parts,

> "the phenomena of life, which result from the juxtaposition of those parts in a certain manner, bear no analogy to any of the effects which would be produced by the action of the component substances considered as mere physical agents. To whatever degree we might imagine our knowledge of the properties of the several ingredients of a living body to be extended and perfected, it is certain that no mere summing up of the separate actions of those elements will ever amount to the action of the living body itself."
>
> Mill (1952 [1843], p. 243).[1]

This kind of thinking initiated what is now called 'British Emergentism' (cf. Stephan, 1992; McLaughlin, 2008; O'Connor and Wong, 2012), a school of thought which included A. Bain, and G.H. Lewes in the 19th century and more or less ended with C.D. Broad (who has our sympathy against Mill because of the doubt he expresses in our motto on the title page). Of this group, the most modern views seem to have been those of S. Alexander, who was committed to a view of emergence as

> "the appearance of novel qualities and associated, high-level causal patterns which cannot be directly expressed in terms of the more fundamental entities and principles. But these patterns do not supplement, much less supersede, the fundamental interactions. Rather, they are macroscopic patterns running through those very microscopic interactions. Emergent qualities are something truly new (...), but the world's fundamental dynamics remain unchanged."
>
> (O'Connor & Wong, 2012)

Alexander's idea that emergent qualities "admit no explanation" and had "to be accepted with the 'natural piety' of the investigator" seems to foreshadow the contemporary notion of explanatory emergence.[2] More precisely, the authorities wrote:

> "The concept of *emergence* has been used to characterize certain phenomena as 'novel', and this not merely in the psychological sense of being unexpected, but in the theoretical sense of being unexplainable, or unpredictable, on the basis of information concerning the spatial parts or other constituents of the systems in which the phenomena occur, and which in this context are often referred to as 'wholes'." (Hempel & Oppenheim, 2008 [1965], p. 62).

[1] Quotation taken from O'Connor and Wong (2012).

[2] Seen as a branch of 'epistemological' emergence. Philosophers distinguish between *ontological* and *epistemological* reduction or emergence, but ontological emergence seems a relic from the days of vitalism and other immature understandings of physics and (bio)chemistry (including the formation of chemical compounds, which Broad and some of his contemporaries still saw as an example of emergence in the strongest possible sense, i.e., falling outside the scope of the the laws of physics). Recent literature, including the present paper, is concerned with epistemological (or rather *explanatory*) emergence.

More recently, Silberstein states (slightly paraphrased) that some higher-level theory X

> "bears predictive/explanatory emergence with respect to Y if Y cannot *replace* X, if X cannot be *derived* from Y [i.e., Y cannot reductively explain X], or if Y cannot be shown to be *isomorphic* to X."
>
> Silberstein (2002, p. 92)

In similar vein, Batterman (2002, p. 20) paraphrases Kim as stating that emergent properties are neither explainable nor predictable even from "exhaustive information concerning their basal [i.e., lower-level] conditions". Wayne & Arciszewski (2009, p. 852) propose that "the failure of reductive explanation is constitutive of emergence in physics". Last but not least, the John Templeton Foundation defines (strongly) emergent phenomena as those which "are, in principle, not derivable from the laws or organizing principles for, or even from an exhaustive knowledge about, their constituents".[3]

In giving quotations like this, we are not so much interested in giving a precise definition of emergence (which may well be impossible and arguably also useless for such a broad concept), but rather in stressing that it is usually meant to be the very opposite of the idea of 'reduction', or 'mechanicism', as Broad (1925) calls it. Indeed, for many authors this very opposition seems to be the principal attraction of emergence. In principle, two rather different notions of reduction then (contrapositively) lead to two different kinds of emergence, which are easily mixed up but should be distinguished (Norton, 2012):

- the reduction of a whole (i.e, a composite system) to its parts;
- the reduction of a 'higher-level' theory (also known in physics as a 'phenomenological' theory, or in philosophy as a 'reduced' theory) to a lower-level one (called a 'fundamental' or a 'reducing' theory in physics and philosophy, respectively).

In older literature, e.g. concerned with the reduction of chemistry to physics (still challenged by Broad (1925)), the first notion also referred to 'wholes' consisting of a *small* number of particles, but in this case the possibility of whole-part emergence seems a lost cause.[4] Thus Anderson (1972), who initiated the modern discussion on emergence in physics,[5] put the emphasis on the possibility of emergence in *large* systems. In particular, he claimed SSB to be an example of emergence (if not *the* example), and duly added that one really had to take the $N \to \infty$ limit. Thus the interesting case for emergence in the first (i.e. whole-part) sense arises if the 'whole' is strictly infinite, as in the thermodynamic limit of (classical or quantum) statistical mechanics. The latter plays the role of the fundamental theory, and, as we shall see, there are two choices for the corresponding phenomenological theory describing the 'infinite whole', namely thermodynamics as a theory of *macroscopic* observables, and infinite-volume statistical mechanics as a theory of (quasi-) *local* observables. But such consideration take us straight into the realm of the second kind of emergence, especially if the inter-theory relation is asymptotic in the sense that the phenomenological theory is a limiting case of the fundamental one.

[3] From `www.templeton.org/what-we-fund/funding-competitions/the-physics-of-emergence`, online at least from 2011–2013. The JTF specifically funds research on emergence in this 'strong' sense.

[4] "If a characteristic of a whole is counted as emergent simply if its occurrence cannot be inferred from a knowledge of all the properties of its parts, then (...) no whole can have any emergent characteristics. Thus (...) the properties of hydrogen include that of forming, if suitably combined with oxygen, a compound which is liquid, transparent, etc." (Hempel & Oppenheim, 2008 [1965], p. 62).

[5] Surprisingly, Anderson actually avoids the term 'emergence' but speaks about new laws and even "a whole new conceptual structure" that each (higher) level can acquire.

The conclusion, then, is that in the interesting case of whole-part reduction, namely when the whole is infinite, one is actually dealing with asymptotic inter-theory reduction, so that the distinction between the two kinds of emergence is blurred. The other case we will consider, viz. the classical limit of quantum mechanics (as the fundamental theory, where the phenomenological theory is classical mechanics) does not fall under the umbrella of the whole-part dichotomy and is purely a case of asymptotic inter-theory reduction.

As we shall see, it is the inter-theoretic reduction perspective (rather than the whole-part notion of reduction) that will be the most useful for us, because we will display striking analogies between SSB in classical mechanics (as a limiting theory) and SSB in thermodynamics (*idem*). Fortunately, it is exactly this 'asymptotic' situation that contemporary research on emergence in physics is mostly concerned with. In particular, it has been claimed that in certain situations some properties of the phenomenological theory (like SSB) could in principle 'emerge' asymptotically, in that *nothing* in the fundamental theory corresponds or gives rise to those properties, not even if one is close to the limit. In this case, the phenomenological theory is claimed to be 'ineliminable', or, in other words, the fundamental theory is said to be 'explanatory insufficient'. This scenario has been developed most notably by Batterman (2002, 2011) and Rueger (2000, 2006), who gave a number of beautiful examples illustrating their point; for criticism of a different kind from the present paper see Hooker (2004), Belot (2005) with a reply by Batterman (2005), Wayne & Arciszewski (2009), Butterfield (2011), and Menon & Callender (2013).

We assume that we understand the limiting phenomenological theory $\mathcal{T}_\infty$; historically it typically preceded the fundamental theory $\mathcal{T}_N$ for $N < \infty$, as in the cases of thermodynamics vis-à-vis statistical mechanics, and classical mechanics vs quantum mechanics (where we use the same notation, putting $N = 1/\hbar$, see below). If not, we simply assume that we have some widely accepted construction of it (as in the case of infinite-volume statistical mechanics with quasi-local observables, see below). Note that the fundamental 'theory' is really a family of theories, parametrized (in this case) by N or $\hbar$. A distinction then needs to be made in principle between two different questions about asymptotic emergence or reduction, which may easily be confused because in practice they often coalesce:

1. Can the phenomenological theory be derived as a limiting case of the fundamental one(s)?

2. Do the fundamental theories $\mathcal{T}_N$ for sufficiently large N approximately behave like the phenomenological one $\mathcal{T}_\infty$?

At first sight these questions appear to be virtually identical, and indeed they are very closely related, but conceptually they are different: the first is a question about the phenomenological theory (and its possible origin in the fundamental one), whereas the second concerns the fundamental theory (in which the phenomenological theory plays an ancillary role). For example, the examples of the rainbow and the WKB approximation in Batterman (2002) deal with the *second* question, as does the research field of quantum chaos, whereas the author's (older) work on the relationship between classical and quantum mechanics tries to answer the *first* (Landsman, 1998).[6] The difference between these questions, as well as their sensitivity to the notions of limit and approximation involved, may also be illustrated by Butterfield's (2011) sequence (g_N) of functions $g_N : \mathbb{R} \to \mathbb{R}$, defined by

$$g_N(x) = -1\,(x \leq -1/N), g_N(x) = Nx\,(-1/N \leq x \leq 1/N), g_N(x) = 1\,(x \geq 1/N).$$

[6]The review Landsman (2007) covers *both*.

Each g_N is continuous, but as $N \to \infty$ this sequence converges pointwise to the discontinuous function $g_\infty(x) = -1$ if $x < 0$ and $g_\infty(x) = 1$ if $x \geq 0$.

- If the theories $\mathcal{T}_N$ are simply the functions g_N themselves and the notions of limit and approximation are pointwise, then the answer to both questions would be "yes".
- If $\mathcal{T}'_N = 0$ if g_N is continuous and $\mathcal{T}'_N = 1$ if it isn't, then with the same notion of limit etc. the answer to the first question remains "yes", yet the second takes "no".

In this case, a reductionist would prefer $\mathcal{T}$, whereas an emergentist would probably go for $\mathcal{T}'$. It could be argued, however, that the latter is being unreasonably strict here, since his/her predicate depends on the behaviour of some function in just one point,[7] and even within this limited scope, the discontinuity of the limit function g_∞ at zero is well approximated by the derivatives $g'_N(0) = N$ converging to infinity.

Another instructive mathematical example, moving from single functions to function spaces but otherwise in the same spirit, is given by the following theory:

1. $\mathcal{T}_N = \ell(\underline{N})$ for $N < \infty$, where $\underline{N}$ is the finite set $\{0, 1, 2, \ldots, N-1\}$ and $\ell(\underline{N})$ consists of all functions $f : \underline{N} \to \mathbb{C}$;
2. $\mathcal{T}_\infty = \ell_0(\mathbb{N})$, i.e., the space of all functions $f : \mathbb{N} \to \mathbb{C}$ that vanish at infinity (in other words, $\lim_{n\to\infty} f(n) = 0$), seen as a Banach space in the supremum-norm $\| \cdot \|_\infty$.[8]

As we shall see, the fundamental theories $\mathcal{T}_N$ model certain primitive aspects of a theory with N degrees of freedom, of which the phenomenological $\mathcal{T}_\infty$ is the 'thermodynamic limit', which describes a theory of quasi-local observables in the corresponding infinite system. For later use, we define the maps $\iota_{M,N} : \ell(\underline{M}) \to \ell(\underline{N})$ for $N \geq M$ by $\iota_{M,N}(f)(n) = f(n)$ for all $n = 0, \ldots, M-1$ and $f(n) = 0$ for $n \geq M$. Similarly, one has maps $\iota_{M,\infty} : \ell(\underline{M}) \to \ell_0(\mathbb{N})$. We now answer the first question in the positive (whilst also making it precise for the case at hand), arguing that symbolically $\mathcal{T}_\infty = \lim_{N\to\infty} \mathcal{T}_N$ in the following sense. For each $f \in \ell_0(\mathbb{N})$ the sequence (f_N), where $f_N \in \ell(\underline{N})$ is given by $f_N = f_{|\underline{N}}$, has the property that $\iota_{N,\infty}(f_N)$ uniformly converges to f_N as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} \|f - \iota_{N,\infty}(f_N)\|_\infty = 0. \tag{1.1}$$

The second question also has a positive answer, which for later use we state in a rather more abstract way than strictly needed at this point. We say that a sequence of functions (f_N), where once again $f_N \in \ell(\underline{N})$,[9] is *local* if there exists some $M \in \mathbb{N}$ such that $f_N = \iota_{M,N}(f_M)$ for all $N \geq M$ (this implies $f_N(n) = 0$ for all $N \geq M$ and $n \geq M$). If $f \in \ell_0(\mathbb{N})$ has finite support, then the restrictions $f_N = f_{|\underline{N}}$ form a local sequence. but this is not necessarily the case for arbitrary $f \in \ell_0(\mathbb{N})$ (since the condition just given in brackets may not be satisfied). Thus we say that a sequence (f_N) with $f_N \in \ell(\underline{N})$ is *quasi-local* if for each $\varepsilon > 0$ there is some $M = M(\varepsilon)$ and some local sequence (f'_N) such that $\|f_N - f'_N\|_\infty < \varepsilon$ for all $N > M$. As intended, sequences like $(f_{|\underline{N}})$ are indeed quasi-local for any $f \in \ell_0(\mathbb{N})$. Conversely, a quasi-local sequence (f_N) has a limit $f \in \ell_0(\mathbb{N})$, given pointwise by $f(n) = \lim_{N\to\infty} f_N(n)$; an elementary $\varepsilon/3$ argument shows that this limit exists and that the ensuing function f satisfies (1.1).

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[7]More generally, by Lusin's theorem from measure theory the difference between continuity and discontinuity is just a matter of epsilonics; see e.g., Stein and Shakarchi (2005), p. 34.

[8]This norm is defined by $\|f\|_\infty = \sup\{\|f(n)\|, n \in \mathbb{N}\}$.

[9]To avoid any confusion: this notation does *not* imply that $f_N = f_{|\underline{N}}$ for some $f \in \ell_0(\mathbb{N})$.

The precise sense, then, in which the $\mathcal{T}_N$ approximately behave like $\mathcal{T}_\infty$, is that any $f \in \ell_0(\mathbb{N})$ may be approximated in the sense of (1.1) by the quasi-local sequence (f_N) with $f_N = f_{|\underline{N}}$, and that such f may be reconstructed from the approximating sequence.

An asymptotic emergentist could challenge this reasoning by adding the following predicate to $\mathcal{T}_\infty$: any $f \in \ell_0(\mathbb{N})$ must have infinite support. This condition is 'emergent' in the limit $N \to \infty$, since none of the approximants $f_{|\underline{N}}$ satisfy it. However, the latter do satisfy it 'up to epsilon', since up to arbitrary precision (in the supremum-norm, which is the sharpest available) any $f \in \ell_0(\mathbb{N})$ can be approximated by functions with finite support. Hence our emergentist should admit that for all practical purposes his/her limiting theory is indistinguishable from $\mathcal{T}_\infty$ and hence is 'almost' obtained from the $\mathcal{T}_N$ by reduction.

In examples from physics one would naturally expect our two questions to practically coincide, or at least, to have the same answers. To be specific, let us move to the context of the type of models from solid state physics considered by Anderson (1972) and others, in which spontaneous symmetry breakdown (SSB) is the (alleged) 'emergent' property. In that context, all systems in reality are finite, so that we regard mathematical models of infinite systems as approximations or idealizations of finite ones.[10] The above expectation then corresponds to what Jones (2006) calls *Earman's Principle*:[11]

> "While idealizations are useful and, perhaps, even essential to progress in physics, a sound principle of interpretation would seem to be that no effect can be counted as a genuine physical effect if it disappears when the idealizations are removed." (Earman, 2004, p. 191)

Similarly, *Butterfield's Principle* is the claim that in this and similar situations, where it has been argued (by other authors) that certain properties emerge strictly in some idealization (and hence have no counterpart in any part of the fundamental theory),

> "there is a weaker, yet still vivid, novel and robust behaviour [compared to Butterfield's own definition of emergence as 'behaviour that is novel and robust relative to some comparison class', which removes the reduction-emergence opposition] that occurs before we get to the limit, i.e. for finite N. And it is this weaker behaviour which is physically real." (Butterfield, 2011)

Both principles are undeniably reductionist in spirit, but they just appear to be common sense and it would be provocative to deny them. Now the problem for reductionists, and hence a potential trump card for asymptotic emergentists, is that both principles appear to be violated in two important examples, namely the thermodynamic limit of large quantum systems displaying SSB, and the classical limit of certain models of Schrödinger's Cat, such as single-particle quantum mechanics with a symmetric double-well potential (Landsman & Reuvers, 2013). Moreover, in both examples their apparent violation is not a matter of mathematical epsilonics or philosophical hairsplitting, but occurs rather dramatically, by a wide margin, and for essentially the same reason (as the second example also involves SSB in a suitable sense). Given our limited expertise, we will just restrict ourselves to these examples of asymptotic theory reduction, which we feel reasonably competent to write about, and refrain from a general analysis. However, since these examples appear to be extremely favorable to the asymptotic emergence scenario, we actually (perhaps too immodestly) suggest that the whole idea stands or falls with its validity in these cases.

[10] See Norton (2012) for a fine distinction between approximations and idealizations, which does not seem to matter here.

[11] Jones in fact quotes a slightly different version of the same idea from Earman (2003).

We start with spontaneous symmetry breakdown in large systems. Here the emergentist case is much stronger in quantum physics than in classical physics, which in the context of phase transition has adequately been dealt with (Butterfield and Bouatta, 2011; Melon and Callender, 2013). In quantum theory, the crucial point is that for finite systems the ground state (or the equilibrium state at sufficiently low temperature) of almost any physically relevant Hamiltonian is unique and hence invariant under whatever symmetry group G it may have. Hence, mathematically speaking, the possibility of SSB, in the sense of having a family of ground states (etc.) related by the action of G, seems to be reserved for infinite systems (for which the arguments proving uniqueness break down).

This leads to two closely related puzzles, one seemingly devastating to the formalism, the other casting serious doubt on the link between theory and reality:

- The formal problem is the discontinuity between a large system and a strictly infinite one: with appropriate dynamics, the former has a unique and invariant ground state, whereas the latter has a family of ground states, each asymmetric. Of course, this is exactly the place where the asymptotic emergentist scores its goal in claiming that the thermodynamic limit is 'singular' and hence SSB is an 'emergent' property.

- The physics problem seems even more serious: nature displays SSB in finite samples, yet the theory is unable to reproduce this and seems to need the infinite idealization.

Thus at least in case of SSB, quantum theory appears to violate Earman's and Butterfield's Principles. In other words, any description of a large quantum system (i.e., large enough so that its physical manifestation displays SSB) that models such a system as a finite system (which it is!) is empirically false (at least as far as SSB is concerned), whereas a description that models it as an infinite system (which it is not!) is empirically adequate for SSB.

The situation involving the classical limit $\hbar \to 0$ of quantum mechanics is analogous (Landsman & Reuvers, 2013). Take a particle moving in some G-invariant potential whose absolute minima from a nontrivial G-space (in particular, each minimum fails to be G-invariant); the simplest example is the symmetric double well in dimension one, where $G = \mathbb{Z}_2$. For any $\hbar > 0$ the quantum theory typically has a unique ground state, peaked above the family of classical minima. In the limit $\hbar \to 0$ the probability distribution on phase space canonically defined by this state does not converge to any one of the classical ground states, but converges to a symmetric convex sum or integral thereof. Nature, however, displays *one* of the localized classical states. Yet for any positive $\hbar$, however small, the quantum ground state is delocalized and shows no tendency towards one peak or the other. For $G = \mathbb{Z}_2$ this is the Schrödinger Cat problem in disguise, which therefore seems to block any asymptotic derivation of classical physics from quantum physics.

As a less familiar intermediate case residing between the thermodynamic limit of quantum statistical mechanics and the classical limit of quantum mechanics, let us recall (Landsman, 2007) that there are two very different different ways of taking the former as far as the choice of observables of the corresponding infinite system is concerned:

- the *local* observables describe finite (but arbitrarily large) subsystems;

- the *global* (or *macroscopic*) observables describe thermodynamic averages.

The former fully retain the quantum-mechanical (i.e., noncommutative) character of the fundamental theory, whereas the latter form a commutative algebra, so that the phenomenological theory acting as the limit of quantum statistical mechanics is classical.

As we shall see, in the latter case the problem of SSB is virtually the same as in the previous two. Summarizing these three cases, the fundamental problem of SSB is as follows:

1. One has a fundamental theory formulated in terms of a parameter $N < \infty$ (quantum statistical mechanics) or $\hbar > 0$ (quantum mechanics);

2. The ground state (or equilibrium state) of some Hamiltonian with symmetry group G is unique and hence G-invariant for any value of $N < \infty$ or $\hbar > 0$;

3. The $N \to \infty$ or $\hbar \to 0$ limit of the ground state (etc.) exists, is still G-invariant, but is now mixed (non-extremal i.e. not a pure thermodynamic phase);

4. The limit theories at $N = \infty$ (being either infinite-volume quantum statistical mechanics or thermodynamics) or $\hbar = 0$ (classical mechanics) exist on their own terms (i.e. without taking any limit) and are completely understood;

5. These limit theories may display SSB (depending on the model): they may have a family of G-variant pure ground states (extremal KMS states), forming a G-space;

6. Nature may display SSB, in which case physical samples modelled by such Hamiltonians behave like the limit theory (although in reality $N < \infty$ or $\hbar > 0$);

7. Thus for any $N < \infty$ or $\hbar > 0$ the theory neither approximates the limit theory nor models reality correctly: indeed, it spectacularly and totally fails to do so.

Perhaps not exactly in those terms, these and similar problems with the thermodynamic limit have been noted, but as far as we know they have by no means been resolved.[12] For example, in response to Earman, Liu and Emch (2005, p. 155) first write that it is a mistake to regard idealizations as acts of "*neglecting the negligible*", which already appears to deny both Earman's and Butterfield's Principles, and continue by:

> "The broken symmetry in question is *not reducible* to the configurations of the microscopic parts of any *finite* systems; but it should *supervene* on them in the sense that for any two systems that have the exactly (sic) duplicates of parts and configurations, both will have the same spontaneous symmetry breaking in them because both will behave identically in the limit. In other words, the result of the macroscopic limit is determined by the non-relational properties of parts of the finite system in question." Liu & Emch (2005, p. 156)

With due respect to especially our posthumous dedicatee, who was a master of mathematical aspects of infinite-volume idealizations, it would be hard to explain even to philosophers (not to speak of physicists) how these comments solve the problem. Also in reply to Earman (2004), Ruetsche (2011) proposes to revise Earman's Principle as follows:

> No effect predicted by a non-final theory can be counted as a genuine physical effect if it disappears from that theory's successors." Ruetsche (2011, p. 336)

Fortunately, both of these defensive manouvres are unnecessary.[13]

[12] The problem of SSB in the thermodynamic limit is rarely if ever described in conjunction with the analogous situation in the classical limit, not even in Landsman (2007), to which this paper is a successor.

[13] As Ruetsche's notes herself, her Principle "has the pragmatic shortcoming that we can't apply it until we know what (all) successors to our present theories are." (Ruetsche 2011, p. 336). A more pragmatic suggestion she makes, which is by no means inconsistent with her revision of Earman's Principle, is to realize that even quantum statistical mechanics in finite volume has an infinite number of degrees of freedom, as the underlying theory should ultimately be quantum field theory.

Namely, using the same idea in all cases, we shall establish the conceptual and mathematical continuity of SSB in both the thermodynamic limit and the classical limit, and hence rescue Earman's and Butterfield's Principles. This idea will be presented in detail in some models with a $\mathbb{Z}_2$-symmetry, viz. the *quantum Ising chain* in the thermodynamic limit described using local observables, the closely related *quantum Curie–Weisz model* in the same limit but now using global observables, and thirdly, the quantum particle moving in a *symmetric double-well* potential in the classical limit. Although these models are somewhat special (we have adopted them because almost everything is known about them), the conceptual lesson from these examples is clearly general, whilst the generalization to different models and larger symmetry groups seems a matter of technique.[14]

Leaving the mathematical details to the main body of this paper, we now sketch this idea, which in the restricted context of the classical limit of quantum mechanics has already been proposed as a possible solution to the measurement problem (Landsman & Reuvers, 2013). Our present work strengthens this proposal, which we now extend from the traditional classical limit $\hbar \to 0$ to the thermodynamic limit $N \to \infty$ and its associated classical realm. Indeed, in our view the measurement problem is just a special case of the problem of emergence (Landsman, 2007). For at first sight the fundamental theory (i.e. quantum mechanics) fails to reproduce a distinct feature of the phenomenological theory (i.e. classical physics), namely the absence of Schrödinger Cat states.[15] Moreover, the stability property of real post-measurement states (that is, 'collapsed' states representing single outcomes) makes it very natural to model them as ground states, as we do here.[16]

For any $N < \infty$, let $\Psi_N^{(0)}$ be the unique and hence $\mathbb{Z}_2$-invariant ground state of the quantum Ising chain. This state, traditionally seen as a unit vector in Hilbert space, define a state $\psi_N^{(0)} : \mathcal{B}_N \to \mathbb{C}$ in the algebraic sense by taking expectation values, where $\mathcal{B}_N = \otimes^N M_2(\mathbb{C})$ (i.e., the N-fold tensor product of the 2×2 matrices) is the pertinent algebra of observables. For $N \to \infty$, there is a satisfactory notion of convergence to the sequence to a state $\psi_\infty^{(0)}$ on $\mathcal{B}_\infty^l$, the algebra of (quasi-) local observables the corresponding infinite system.[17] The point is that $\psi_\infty^{(0)}$ is mixed, whereas the two physically realized ground states would be the states ψ_∞^+ or ψ_∞^- with all spins up or down, respectively. By $\mathbb{Z}_2$-invariance, one has the Schrödinger Cat state

$$\psi_\infty^{(0)} = \tfrac{1}{2}(\psi_\infty^+ + \psi_\infty^-). \tag{1.2}$$

The situation for the Curie–Weisz model is analogous: the local algebras $\mathcal{B}_N$ are the same as for the quantum Ising chain, but the algebra of global observables $\mathcal{B}_\infty^g$ is now commutative: it is isomorphic to the algebra $C(\mathsf{B}^3)$ of continuous functions on the three-ball B^3 in $\mathbb{R}^3$, playing the role of the state space of a quantum-mechanical two-level system, so that its boundary is the familiar Bloch sphere. Once again, the quantum-mechanical ground state for finite N is unique, and its limit as $N \to \infty$ takes the form (1.2), too. But this time the states $\psi_\infty^\pm$ are simply points on the Bloch sphere, reinterpreted as probability measures on B^3, and the left-hand side is their symmetric convex sum, seen as a probability measure on the same space. Being convex-decomposable, the left-hand side is again mixed.

[14] Thus we prophylactically plead not guilty to the 'case-study gambit' warned against by Butterfield (2011): "trying to support a general conclusion by describing examples that have the required features, though in fact the examples are not typical, so that the attempt fails, i.e. the general conclusion, that all or most examples have the features, does not follow." Our examples *are* typical, or so we believe.

[15] See the above-mentioned paper, but our point will also become obvious on reading the present paper.

[16] The following scenario combines the technical work of Jona-Lasinio, Martinelli, & Scoppola (1981a,b) and Koma & Tasaki (1994) on the classical limit and the thermodynamic limit, respectively.

[17] The quasi-local observables are local 'up to ε', see Landsman (2007) or below.

Similarly, for $\hbar > 0$, the unique and hence $\mathbb{Z}_2$-invariant ground state $\Psi_\hbar^{(0)}$ of the double-well potential defines an algebraic state $\psi_\hbar^{(0)} : \mathcal{A}_\hbar \to \mathbb{C}$, where $\mathcal{A}_\hbar \cong \mathcal{A}_1 = K(L^2(\mathbb{R}))$ is the algebra of compact operators on the Hilbert space $L^2(\mathbb{R})$ of square-integrable wave-functions, independent of $\hbar > 0$. The family $(\psi_\hbar^{(0)})$ has a well-defined limit $\psi_0^{(0)}$ as $\hbar \to 0$, which limit is a state on the classical algebra $\mathcal{A}_0 = C_0(\mathbb{R}^2)$ of continuous functions on phase space (taken to vanish at infinity for simplicity),[18] i.e., a probability measure on phase space. Once again, this limit is the familiar symmetric convex combination

$$\psi_0^{(0)} = \tfrac{1}{2}(\psi_0^+ + \psi_0^-), \tag{1.3}$$

where the Dirac (or 'point') measure $\psi_0^{(\pm)}$ is concentrated in $(p = 0, q = \pm a)$.[19]

The key to get rid of these Schrödinger Cat states à la (1.2) and (1.3) is to broaden one's view and also take the first excited state $\Psi_\bullet^{(1)}$ into account, where $\bullet$ is either N or $\hbar$ as appropriate.[20] The point is that in our models, the energy difference $\Delta E_\bullet = E_\bullet^{(1)} - E_\bullet^{(0)}$ between $\Psi_\bullet^{(1)}$ and $\Psi_\bullet^{(0)}$ vanishes exponentially as

$$\Delta E_N \sim \exp(-C \cdot N), \quad \Delta E_\hbar \sim \exp(-C'/\hbar)$$

as $N \to \infty$ and $\hbar \to 0$, respectively. This implies that almost any asymmetric perturbation (that does not vanish as quickly in the limit, e.g., by being independent of N or $\hbar$ or having at most a power-law decay) will eventually dominate the effective Hamiltonian governing the two lowest energy states (which is just a 2×2 matrix). The ground state of the perturbed Hamiltonian will then be one of the states

$$\Psi_\bullet^\pm = (\Psi_\bullet^{(0)} \pm \Psi_\bullet^{(1)})/\sqrt{2}, \tag{1.4}$$

the sign being determined by the localization of the perturbation (in configuration space). For either sign $\pm$ the associated family of algebraic states $\psi_{N/\hbar}^\pm$ then converges to $\psi_{\infty/0}^\pm$.

For physics this means that the behaviour of real samples is well approximated by the theory at very large N or very small $\hbar$. For mathematics this mechanism implies that the limit states are continuously approached.[21] Either way, the job is done by the exponential instability of the ground state (etc.) under asymmetric perturbations for large N or small $\hbar$, which should cause the system to pick a specific symmetry-breaking ground (or KMS) state $\Psi_\bullet^\pm$ already for some *finite* value of N or *positive* value of $\hbar$ (as opposed to the limiting values $N = \infty$ or $\hbar = 0$, which are physically irrelevant).

Such perturbations may be induced either by the environment of the system, as in the 'decoherence' (non) solution to the measurement problem (see Landsman & Reuvers (2013) for more on this), or possibly by material defects. Consequently, in the real world (of finite objects) totally isolated quantum system (at least with perfectly symmetric Hamiltonians) should display SSB, as the naive theory indeed predicts. In other words, truly macroscopic Schrödinger Cat states are only possible if asymmetric perturbations can be totally suppressed (so in theory Schrödinger's Cat can indeed exist!)

[18] In principle there would be a similar ambiguity about the choice of the algebra of observables in the $\hbar \to 0$ limit. We here take the (quasi-)local observables, but in this case the delocalized observables $C_b(\mathbb{R}^2)/C_0(\mathbb{R}^2)$ would be irrelevant, because unlike the previous case the SSB problem has a local nature.

[19] Here $\pm a$ are the the minima of the double-well potential.

[20] *Mutatis mutandis* the same arguments apply to thermal equilibrium states (i.e., KMS states) instead of ground states, at least at low temperature. This also gives continuity of the $T \to 0$ limit.

[21] We will make this precise using the formalism of continuous fields of C*-algebras (Dixmier, 1977) and continuous fields of states thereon (Landsman, 2007).

Where does this leave us? What seems dubious now about the idea of emergence (at least in the case at hand of SSB) is, in our view, its opposition to reduction. So although for many this juxtaposition was the essence of the idea, it would seem wise to give it up. Thus we largely side with Butterfield (2011) in rescuing the side of emergence that talks about 'novel and robust behaviour', but taking this novelty (with reference to our earlier quotation of Hempel and Oppenheim) solely in the psychological sense of being unexpected, rather than in the theoretical sense of being unexplainable or unpredictable (i.e., on the basis of the underlying fundamental theory).

Even so, one aspect of the idea that the phenomenological theory is 'ineliminable' survives, namely the particular choice of observables within the fundamental theory that guarantees the correct limiting behaviour. In quantum statistical mechanics, it is either the (quasi-)local or the macroscopic observables, each with their very peculiar N-dependence, that survive the limit $N \to \infty$. Analogously, in quantum mechanics it is the semiclassical observables with their very peculiar $\hbar$-dependence that survive the limit $h \to 0$.

On the one hand, the choice of these observables is dictated by the limit theories *taken by themselves* (which therefore have to be known in advance) and is *not* intrinsic to the underlying fundamental theories. On the other hand, these specific observables are defined *within* the latter, whose (mathematical) structure gives rise to the very possibility of singling them out. In other words, it is in the nature of (quantum) statistical mechanics that one is able to define the small family of observables that in the thermodynamic limit behave according to the laws of thermodynamics, and it is quantum mechanics itself that allows the selection of the observables with the 'right' $\hbar$-dependence (such as the usual position operator x and the momentum operator $-i\hbar d/dx$).[22] Hence in our view it merely seems a semantic issue whether the act of picking precisely the observables with appropriate limiting behaviour (or just the possibility thereof, or even merely their existence) falls within the scope of reduction, or instead is the hallmark of emergence. Either way, novel and robust behaviour is predicated on this act of choice.

In the remainder of this paper we explain the technical details of this scenario.[23] Section 2 gives a description of our models and their symmetries. Section 3 describes some remarkable continuity properties of these models as $\hbar \to 0$ or $N \to \infty$, putting the folklore that such limits are "singular" in perspective.[24] In section 4 we describe the ground states of our models, proving their discontinuity at $\hbar = 0$ or $N = \infty$. But in the final section 5 we put the record straight by invoking the first excited states, which save the day and restore Earman's Principle and Butterfield's Principle, as outlined above.

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The author is indebted to Rob Batterman for sending him the book (Batterman, 2002) a decade ago, which triggered the his interest in Emergence as a topic amenable to mathematical analysis. This interest was subsequently kept alive by Jeremy Butterfield. Thanks also to Aernout van Enter, Bruno Nachtergaele, and Hal Tasaki for answering some technical questions on quantum spin systems and help in finding a number of references.

[22] Or, more generally, the quantizations $Q_\hbar(f)$ of functions f on classical phase space (Landsman, 2007).

[23] Given the complexity of the underlying material from mathematical physics and the spatial constraints of a journal paper, our treatment cannot possibly be self-contained, however. For *really* technical details or sub-details we briefly summarize the relevant arguments in footnotes, including references to the literature for those who wish to study the background (or check our reasoning). We hope to give a full account in a planned textbook on the foundations of quantum mechanics called *Bohrification*, scheduled for 2015.

[24] This section is quite technical and may be skipped at a first reading.

2 Models

To make our point, it suffices to treat the *quantum Ising chain* in the thermodynamic limit with (quasi-) local observables, the *quantum Curie–Weisz model* in the same limit with global observables, and the *symmetric double well* potential in the classical limit. For pedagogical reasons we start with the latter, which is much easier to understand; cf. Landsman & Reuvers (2013) for a more detailed treatment from our current perspective.

2.1 Double well

The quantum double-well Hamiltonian on the real axis is given by

$$H_\hbar^{DW} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{4}\lambda(x^2 - a^2)^2, \tag{2.5}$$

defined as an unbounded operator on the Hilbert space

$$\mathcal{H}_\hbar^{DW} = L^2(\mathbb{R}); \tag{2.6}$$

more precisely, on a domain like $C_c(\mathbb{R})$ or the Schwartz space of test functions $\mathcal{S}(\mathbb{R})$, where it is essentially self-adjoint (Reed & Simon, 1978). We assume $\lambda > 0$, and $a > 0$.

Reflection in the origin of the position coordinate endows this model with a $\mathbb{Z}_2$ symmetry, which is implemented by the unitary operator $u : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$u\Psi(x) = \Psi(-x). \tag{2.7}$$

The $\mathbb{Z}_2$-symmetry of the Hamiltonian then reads $[H_\hbar^{DW}, u] = 0$, or, equivalently,

$$uH_\hbar^{DW}u^* = H_\hbar^{DW}. \tag{2.8}$$

Although this is not necessary for a mathematically correct treatment of this quantum system, in order to better understand the classical limit $\hbar \to 0$ as well as to see the analogy with our other two models from quantum statistical mechanics, it is convenient to model the double-well system through the (admittedly idealized[25]) algebra of observables

$$\mathcal{A}_\hbar = K(L^2(\mathbb{R})),\ \hbar > 0, \tag{2.9}$$

i.e., the C*-algebra of compact operators on $L^2(\mathbb{R})$. Algebraically, time-evolution is given by a (strongly continuous) group homomorphism $\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_\hbar)$ from the time-axis $\mathbb{R}$ (as an additive group) to the group of all automorphism of $\mathcal{A}_\hbar$,[26] written $t \mapsto \tau_t$; we sometimes abbreviate $a(t) \equiv \tau_t(a)$. In the model at hand, for any $\hbar > 0$ we have

$$\tau_t^{(\hbar)}(a) = u_t^{(\hbar)} a (u_t^{(\hbar)})^*, \tag{2.10}$$

where the unitary operators $u_t^{(\hbar)}$ are given by

$$u_t^{(\hbar)} = e^{itH_\hbar^{DW}/\hbar}. \tag{2.11}$$

[25] Any bounded operator may be obtained as a weak or strong limit of some sequence of compact operators, whereas any possibly unbounded self-adjoint operator resurfaces as a generator of some unitary (hence bounded!) representation of $\mathbb{R}$ as an additive group, as in Stone's Theorem. This sort of idealization by mathematical convenience is unrelated to the problems involved in the idealizations $\hbar = 0$ (i.e., classical physics) or $N = \infty$ (infinite systems) that form the main subject of this paper.

[26] An automorphism of a C*-algebra $\mathcal{A}$ is an invertible linear map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfying $\alpha(ab) = \alpha(a)\alpha(b)$ and $\alpha(a^*) = \alpha(a)^*$. It is *inner* if there is unitary element $u \in \mathcal{A}$ such that $\alpha(a) = uau^*$ for all $a \in \mathcal{A}$.

Similarly, the $\mathbb{Z}_2$-symmetry of the model is algebraically described by a group homomorphism $\tilde{\gamma} : \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{A}_\hbar)$, where $\gamma(1) = \mathrm{id}$, whilst the nontrivial element -1 of $\mathbb{Z}_2$ is mapped to the automorphism $\gamma \equiv \tilde{\gamma}(-1)$ defined in terms of the unitary u in (2.7) by

$$\gamma(a) = uau^*. \tag{2.12}$$

For $\hbar > 0$, the $\mathbb{Z}_2$-invariance of the model is then expressed algebraically by the property

$$\tau_t^{(\hbar)} \circ \gamma = \gamma \circ \tau_t^{(\hbar)},\ t \in \mathbb{R}. \tag{2.13}$$

The classical limit of this system is a particle in phase space $\mathbb{R}^2$, with Hamiltonian

$$h^{DW}(p,q) = \frac{p^2}{2m} + \tfrac{1}{4}\lambda(q^2 - a^2)^2. \tag{2.14}$$

The classical $\mathbb{Z}_2$-symmetry is given by the map $\gamma_* : (p,q) \mapsto (p,-q)$ on phase space $\mathbb{R}^2$, with pullback $f \mapsto \gamma_*^* f \equiv \gamma^{(0)} f$ on functions f on $\mathbb{R}^2$, i.e., $\gamma^{(0)} f(p,q) = f(p,-q)$. Clearly,

$$\gamma^{(0)} h^{DW} = h^{DW}. \tag{2.15}$$

Algebraically, we take the (idealized) classical observables to be

$$\mathcal{A}_0 = C_0(\mathbb{R}^2), \tag{2.16}$$

i.e., the commutative C*-algebra of continuous functions $f : \mathbb{R}^2 \to \mathbb{C}$ that vanish at infinity (with pointwise multiplication). Time-evolution $t \mapsto \tau_t^{(0)}$ on $\mathcal{A}_0$ is defined by

$$\tau_t^{(0)} f(p,q) = f(p(t), q(t)), \tag{2.17}$$

where $(p(t), q(t))$ is the solution of the Hamiltonian equations of motion following from (2.14), with initial conditions $(p(0), q(0)) = (p,q)$. Then $\mathbb{Z}_2$-invariance is expressed by

$$\tau_t^{(0)} \circ \gamma^{(0)} = \gamma^{(0)} \circ \tau_t^{(0)},\ t \in \mathbb{R}. \tag{2.18}$$

2.2 Quantum Ising chain

For $N < \infty$ the Hamiltonian of the quantum Ising chain (with $J = 1$ for simplicity) is

$$H_N^I = -\sum_{i=-\frac{1}{2}N}^{\frac{1}{2}N-1} \sigma_i^z \sigma_{i+1}^z - B \sum_{i=1}^{N} \sigma_i^x, \tag{2.19}$$

where N is even and and we adopt free boundary conditions. This operator is defined on

$$\mathcal{H}_N = \otimes^N \mathbb{C}^2, \tag{2.20}$$

i.e., the N-fold tensor product of $\mathbb{C}^2$, where we label the first copy with $j = -\frac{1}{2}N$, the second with $j = -\frac{1}{2}N + 1$, ..., and the last one with $j = \frac{1}{2}N - 1$.[27] We assume $B > 0$. This model describes a chain of N immobile spin-$\frac{1}{2}$ particles with ferromagnetic coupling in a transverse magnetic field (Pfeuty, 1970; Sachdev, 2011; Suzuki et al, 2013).[28]

[27] The Pauli matrix $\sigma_i^\mu \equiv \otimes_{j=-\frac{1}{2}N}^{i-1} 1_2 \otimes \sigma^\mu \otimes_{k=i+1}^{\frac{1}{2}N-1} 1_2$ $(i = 1, \dots, N,\ \mu = x, y, z)$ acts on the i'th $\mathbb{C}^2$.

[28] The quantum Ising model is a special case of the XY-model, to which the same conclusions apply.

Although the physically relevant operators (like the above Hamiltonian) are most simply given if the Hilbert space is realized in the tensor product form (2.20), the physical interpretation of states tends to be more transparent if we realize $\mathcal{H}_N$ as $\ell^2(S_N)$,[29] where $S_N = \underline{2}^{\underline{N}}$ is the space of classical spin configurations $s : \underline{N} \to \underline{2}$. Here

$$\underline{2} = \{-1, 1\}; \tag{2.21}$$

$$\underline{N} = \{-\tfrac{1}{2}N, \tfrac{1}{2}N - 1\} \ (N > 2). \tag{2.22}$$

In terms of the standard basis $|1\rangle = (1,0)$ and $|-1\rangle = (0,1)$ of $\mathbb{C}^2$, where the label $|\lambda\rangle$ is the corresponding eigenvalue of σ^z, a suitable unitary equivalence $v_N : \ell^2(S_N) \to \otimes^N \mathbb{C}^2$ (each isomorphic to $\mathbb{C}^{2^N}$) is given by linear extension of

$$v_N \delta_s = \otimes_{j=-\frac{1}{2}N}^{\frac{1}{2}N-1} |s(j)\rangle, \tag{2.23}$$

where δ_s is defined by $\delta_s(t) = \delta_{st}$; such functions form an orthonormal basis of $\ell^2(S_N)$.

For example, the state with all spins up, i.e., $\otimes^N |1\rangle$, corresponds to $\delta_{s_\uparrow}$, where $s_\uparrow(j) = 1$ for all j, and analogously $s_\downarrow(j) = -1$ for all j is the state with all spins down. Thus the advantage of this realization is that we may talk of localization of states in spin configuration space, in the sense that some $\Psi \in \ell^2(S_N)$ may be peaked on just a few spins configurations (whilst typically having small but nonzero values elsewhere).

For any B, the quantum Ising chain has a $\mathbb{Z}_2$-symmetry given by a 180-degree rotation around the x-axis. This symmetry is implemented by the unitary operator

$$u_N = \otimes_{i \in \underline{N}} \sigma_i^x \tag{2.24}$$

on $\mathcal{H}_N$, which satisfies $[H_N^I, u_N] = 0$, or, equivalently,[30]

$$u_N H_N^I u_N^* = H_N^I. \tag{2.25}$$

As in the previous subsection, we could take a purely algebraic approach by defining the C*-algebra of observables of the system for $N < \infty$ to be

$$\mathcal{B}_N = \otimes^N M_2(\mathbb{C}), \tag{2.26}$$

i.e., the N-fold tensor product of the 2×2 matrices, labeled as described after (2.20). Time-evolution is given by the analogue of (2.10), which explicitly reads

$$\tau_t^{(N)}(a) = u_t^{(N)} a (u_t^{(N)})^*; \tag{2.27}$$

$$u_t^{(N)} = e^{itH_N^I}. \tag{2.28}$$

Furthermore, the $\mathbb{Z}_2$-symmetry is given by the automorphism γ_N of $\mathcal{B}_N$ defined by

$$\gamma_N(a) = u_N a u_N^*, \ a \in \mathcal{B}_N. \tag{2.29}$$

In this language, $\mathbb{Z}_2$-invariance of the model is expressed as in (2.13), viz.

$$\tau_t^{(N)} \circ \gamma_N = \gamma_N \circ \tau_t^{(N)}, \ t \in \mathbb{R}. \tag{2.30}$$

Without taking the dynamics into account, given (2.26) there are two (interesting) possibilities for the limit algebra at $N = \infty$ (Landsman, 2007):

[29] For any countable set S, the Hilbert space $\ell^2(S)$ consists of all functions $f : S \to \mathbb{C}$ that satisfy $\sum_{s \in S} |f(s)|^2 < \infty$, with inner product $\langle f, g \rangle = \sum_{s \in S} \overline{f(s)} g(s)$. Of course, for $N < \infty$, S_N is a finite set.

[30] Note that $u_N \sigma_i^x u_N^* = \sigma_i^x$, $u_N \sigma_i^y u_N^* = -\sigma_i^y$, $u_N \sigma_i^z u_N^* = -\sigma_i^z$, which implies (2.25).

- The C*-algebra of (quasi-)*local* observables is

$$\mathcal{B}^l_\infty = \otimes^{\mathbb{Z}} M_2(\mathbb{C}), \tag{2.31}$$

the infinite tensor product of $M_2(\mathbb{C})$ as defined in Kadison & Ringrose (1986, §11.4);

- The C*-algebra of *global* observables is

$$\mathcal{B}^g_\infty = C(\mathcal{S}(M_2(\mathbb{C}))), \tag{2.32}$$

the C*-algebra of continuous functions on the state space $\mathcal{S}(M_2(\mathbb{C}))$ of $M_2(\mathbb{C})$.

Thus $\mathcal{B}^l_\infty$ is highly noncommutative, like each of the $\mathcal{B}_N$, which is embedded in $\mathcal{B}^l_\infty$ by tensoring with infinitely many unit matrices in the obvious way, whereas $\mathcal{B}^g_\infty$ is obviously commutative (under pointwise multiplication, that is). Which of these limit algebra is the appropriate one depends on the Hamiltonian: for short-range interactions, as in (2.19), is the the first, because the finite-N Hamiltonians induce a well-defined time-evolution on $\mathcal{B}^l_\infty$. For mean-field models like the quantum Curie–Weisz model, on the other hand, it is the second, for the same reason; see below and the next subsection, respectively.

Indeed, in the first case, for Hamiltonians like (2.19), by Theorem 6.2.4 in Bratteli & Robinson (1997) there exists a unique time-evolution τ on $\mathcal{B}^l_\infty$, again in the sense of a strongly continuous group homomorphism $\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{B}^l_\infty)$, that extends the local dynamics given by the local Hamiltonians H^I_N in that

$$\tau_t(a) = \lim_{N\to\infty} \tau^{(N)}_t(a),\ a \in \mathcal{B}^l_\infty. \tag{2.33}$$

Equivalently, for each $a \in \mathcal{B}^l_\infty$ that is *local* in being contained in some $\mathcal{B}_N \subset \mathcal{B}^l_\infty$,

$$\frac{da(t)}{dt} = i \lim_{N\to\infty} [H^I_N, a(t)], \tag{2.34}$$

where $a(t) \equiv \tau_t(a)$. So the limit theory is the pair $(\mathcal{B}^l_\infty, \tau)$, in which the local Hamiltonians H^I_N have been replaced by the single one-parameter automorphism group τ.

Finally, the infinite-volume relic of the $\mathbb{Z}_2$-symmetry u_N is the automorphism γ of $\mathcal{B}^l_\infty$ that is uniquely defined by the property $\gamma(a) = u_N a u_N^*$ for each $a \in \mathcal{B}_N$, cf. (2.25). The invariance property (2.25) of the local Hamiltonians then becomes the $\mathbb{Z}_2$-symmetry of the time-evolution τ, as expressed by (2.13) (*mutatis mutandis*).

2.3 Quantum Curie-Weisz model

For $N < \infty$, the Hamiltonian of the quantum Curie-Weisz model is

$$H^{CW}_N = -\frac{1}{2N} \sum_{i,j=-\frac{1}{2}N}^{\frac{1}{2}N-1} \sigma^z_i \sigma^z_j - B \sum_{i=1}^{N} \sigma^x_i, \tag{2.35}$$

acting on the same Hilbert space $\mathcal{H}_N$ as the Hamiltonian (2.19), i.e., (2.20), but differing from it by the spin-spin interaction being nonlocal and even of arbitrary range.[31]

[31] The name *Lipkin model* may be found in the nuclear physics literature, and without the B-term, (2.35) is sometimes called the *Weisz model*. It may be treated in any spatial dimension in much the same way.

In terms of the marcoscopically averaged spin operators

$$S_\mu = \frac{1}{2N} \sum_{i=-\frac{1}{2}N}^{\frac{1}{2}N-1} \sigma_i^\mu,\ \mu = 1,2,3, \tag{2.36}$$

this Hamiltonian assumes the perhaps more transparent form

$$H_N^{CW} = -2N(S_z^2 + BS_x). \tag{2.37}$$

This model has exactly the same $\mathbb{Z}_2$-symmetry as its local counterpart (2.19), and for finite N one can simply copy all relevant formulae from the previous subsection.

What is markedly different for long-range forces (as opposed to local ones), however, is the correct choice of the limit algebra, which (in the context of similar models, starting from the BCS Hamiltonian for superconductivity) emerged from the work of Bogoliubov, Haag, Thirring, Bona, Duffield, Raggio, Werner, and others; see Landsman (2007) for references, as well as for the reformulation in terms of continuous fields of C*-algebras to be given in section 3 below. To make a long story short, the local Hamiltonians (2.37) do not induce a time-evolution on $\mathcal{B}_\infty^l$, but they do so on the commutative algebra (2.32).

This limiting dynamics turns out to be of Hamiltonian form, as was to be expected for a classical theory, albeit of a generalized form, where the underlying phase space is a Poisson manifold that is not symplectic, see e.g. Landsman (1998). Specifically, we use the fact that the state space $\mathcal{S}(M_2(\mathbb{C}))$ of $M_2(\mathbb{C})$ appearing in (2.32) is isomorphic (as a compact convex set) to the three-ball B^3, which consists of all $(x, y, z) \in \mathbb{R}^3$ satisfying $x^2+y^2+z^2 \leq 1$. The isomorphism in question is given by the well-known parametrization

$$\rho(x, y, z) = \tfrac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \tag{2.38}$$

of an arbitrary density matrix on $\mathbb{C}^2$. This parametrization is such that pure states (i.e., those for which ρ is a one-dimensional projection, so that $\rho^2 = \rho$) are mapped to the boundary S^2 of B^3, whose points satisfy $x^2 + y^2 + z^2 \leq 1$ (this is just the familiar Bloch sphere from physics texts). Now $\mathbb{R}^3$ is equipped with (twice) the so-called Lie–Poisson bracket,[32] which on the coordinate functions $(x_1, x_2, x_3) \equiv (x, y, z)$ is given by

$$\{x_i, x_j\} = -2\varepsilon_{ijk}x_k, \tag{2.39}$$

and is completely defined by this special case through the Leibniz rule. In terms of this bracket, time-evolution is given by the familiar Hamiltonian formula

$$\frac{dx_i}{dt} = \{h^{CW}, x_i\}, \tag{2.40}$$

where the Hamiltonian giving the limiting dynamics of (2.37) is given by

$$h^{CW}(x, y, z) = -(\tfrac{1}{2}z^2 + Bx). \tag{2.41}$$

Thus the equations of motion (2.40) are given by

$$\frac{dx}{dt} = 2yz; \quad \frac{dy}{dt} = 2z(B-x); \quad \frac{dz}{dt} = -2By. \tag{2.42}$$

[32]See e.g. Marsden & Ratiu (1994); our phase space $\mathbb{R}^3$ is the dual of the Lie algebra of $SO(3)$. The factor 2 in (2.39) is caused by the fact that our model has basic spin one-half (or use $x_i' = \frac{1}{2}x_i$ to avoid it).

3 Continuity

It is often stated that limits like $N \to \infty$ and especially $\hbar \to 0$ are "singular", and indeed they are, if one merely looks at the Hamiltonian: putting $\hbar = 0$ in (2.5) yields an operator that has practically nothing to do with the small $\hbar$-behaviour of its originator, letting $\hbar \to 0$ in (2.11) seems to make no sense, and the limit $N \to \infty$ of (2.19) or (2.35) is simply undefined. Nonetheless, *the limits in question are continuous* if treated in the right way.[33]

They key, which may be unexpected at first sight, is that in discussing the $\hbar \to 0$ limit of quantum mechanics it turns out to be possible to "glue" the (highly noncommutative!) algebras of quantum observables $\mathcal{A}_\hbar$ in (2.9) continuously to the (commutative!) algebra of classical observables $\mathcal{A}_0$ in (2.16), whereas for the $N \to \infty$ limit of quantum statistical mechanics there are even two (relevant) possibilities: the (noncommutative) algebras $\mathcal{B}_N$ in (2.26) can be glued continuously to either the algebra of (quasi-) local observables $\mathcal{B}^l_\infty$ in (2.31), which is noncommutative, too (and hence is the more obvious choice), or to the commutative algebra of global observables $\mathcal{B}^g_\infty$ in (2.32). The "glueing" is done using the formalism of *continuous fields of C*-algebras* (of observables); we will just look at the cases of interest for our three models, and refer the reader to Dixmier (1977) and Kirchberg & Wassermann (1995) for the abstract theory.[34] Continuity of the dynamics in the various limits at hand will be a corollary, provided time-evolution is expressed in terms of the one-parameter automorphism groups τ. Finally, the ensuing *continuous fields of states* defined by the continuous fields of observables will provide the right language for the technical elaboration of our solution of the problem of emergence, as described in the Introduction (N.B.: the use of this formalism itself is *not* yet the solution!).

3.1 Continuous fields of C*-algebras

Each of our three continuous fields $\mathfrak{A}^c$, $\mathfrak{A}^l$, and $\mathfrak{A}^g$ of interest is defined over a ("base") space $I_\alpha \subseteq [0,1]$ defined for $\alpha = c, l, g$ as $I_c = [0,1]$ and $I_l = I_g = \{0\} \cup 1/(2\mathbb{N}_*)$,[35] with the topology inherited from $[0,1]$ (in which each I_α is compact). The fibers are as follows:

1. $\mathfrak{A}^c_\hbar = \mathcal{A}_\hbar = K(L^2(\mathbb{R}))$ for $h \in (0,1]$ and $\mathfrak{A}^c_0 = \mathcal{A}_0 = C_0(\mathbb{R}^2)$;
2. $\mathfrak{A}^l_{1/N} = \mathcal{B}_N = \otimes^N M_2(\mathbb{C})$ for even $N < \infty$ and $\mathfrak{A}^l_0 = \mathcal{B}^l_\infty = \otimes^{\mathbb{Z}} M_2(\mathbb{C})$;
3. $\mathfrak{A}^g_{1/N} = \mathcal{B}_N = \otimes^N M_2(\mathbb{C})$ for even $N < \infty$ and $\mathfrak{A}^g_0 = \mathcal{B}^g_\infty = C(\mathcal{S}(M_2(\mathbb{C})))$.

The continuity structure is given by specifying a sufficiently large family of continuous cross-sections, i.e., maps σ assigning some element $a \in \mathfrak{A}^\alpha_x$ to each $x \in I_\alpha$, as follows.[36]

[33] The following discussion amplifies our earlier treatments of the $\hbar \to 0$, $N \to \infty$ limits in Landsman (1998, 2007), respectively, by including dynamics and, in the next few sections, ground states and SSB.

[34] The idea of a continuous fields of C*-algebras goes back to Dixmier (1977), who give a direct definition in terms of glueing conditions between the fibers, and was usefully reformulated by Kirchberg & Wassermann (1995), who stressed the role of the continuous sections of the field already in its definition. Both definitions are reviewed in Landsman (1998). Our informal discussion below uses elements of both.

[35] That is, $x \in I_l$ if either $x = 0$ or $x = 1/N$ for some even $N \in \mathbb{N}\backslash\{0\}$.

[36] Lest this operation may appear to be circular: there are stringent and highly exclusive conditions on admissible continuity structures, namely, for continuous cross-sections σ of a continuous field $\mathfrak{A}^\alpha$ over I_α:

1. The function $x \mapsto \|\sigma(x)\|_x$, where $\|\cdot\|_x$ is the norm in $\mathfrak{A}_x$, is continuous (from I_α to $\mathbb{R}$);
2. For any $f \in C(I_\alpha)$ and σ as above the cross-section $x \mapsto f(x)\sigma(x)$ for all $x \in I_\alpha$ is again continuous;
3. The totality of all σ form a C*-algebra in the norm $\|\sigma\| = \sup_x \|\sigma(x)\|_x$ and pointwise operations.

1. Each $f \in C_0(\mathbb{R}^2$ defines an operator $Q_\hbar(f) \in K(L^2(\mathbb{R}))$ by *Berezin quantization*,
$$Q_\hbar(f) = \int_{\mathbb{R}^2} \frac{dpdq}{2\pi\hbar}\, f(p,q)|\Phi_\hbar^{(p,q)}\rangle\langle\Phi_\hbar^{(p,q)}|, \tag{3.43}$$
where, for any unit vector Ψ, the one-dimensional projection onto $\mathbb{C}\cdot\Psi$ is denoted by $|\Psi\rangle\langle\Psi|$, and the coherent states $\Phi_\hbar^{(p,q)} \in L^2(\mathbb{R})$, $(p,q) \in \mathbb{R}^2$, are defined by
$$\Phi_\hbar^{(p,q)}(x) = (\pi\hbar)^{-1/4} e^{-ipq/2\hbar} e^{ipx/\hbar} e^{-(x-q)^2/2\hbar}. \tag{3.44}$$
Using this quantization map, for each f we then define a cross-section σ_f of $\mathfrak{A}^c$ by
$$\sigma_f(0) = f; \tag{3.45}$$
$$\sigma_f(\hbar) = Q_\hbar(f),\ \hbar \in (0,1]. \tag{3.46}$$
Thus, although f and $Q_\hbar(f)$ are completely different mathematical objects, for small $\hbar$ they are sufficiently close to each other to be able to say that $\lim_{\hbar\to 0} Q_\hbar(f) = f$, in the sense that if one continuously follows the curve $\hbar \mapsto \sigma_f(\hbar)$ in the total space $\coprod_{\hbar\in[0,1]} \mathfrak{A}_\hbar^c$ of the bundle (equipped with the topology that makes this disjoint union a continuous field of C*-algebras) all the way down to $\hbar = 0$, one ends up with f.

2. In order to describe this case, we have to realize the infinite tensor product $\mathcal{B}_\infty^l$ as equivalence classes of quasi-local sequences (Raggio & Werner, 1989):
   (a) A sequence $(a) \equiv (a_N)_{N\in 2\mathbb{N}_*}$ is *local* if there is an M such that $a_N = \iota_{MN}(a_M)$ for all $N \geq M$, where $\iota_{MN} : \mathcal{B}_M \hookrightarrow \mathcal{B}_N$ is the inclusion map (which takes the tensor product of $a_M \in \mathcal{B}_M$ with as many unit matrices as needed to make it an element of $\mathcal{B}_N$).
   (b) A sequence (a) is *quasi-local* if for any $\varepsilon > 0$ there is an M and a local sequence (a') such that $\|a_N - a'_N\| < \varepsilon$ for all $N > M$ (cf. the Introduction around (1.1)).
   (c) Introduce an equivalence relation on the quasi-local sequences by saying that $(a) \sim (a')$ if $\lim_{N\to\infty} \|a_N - a'_N\| = 0$.
   (d) $\mathcal{B}_\infty^l$ consists of equivalence classes $[a] \equiv a_\infty$ of quasi-local sequences; these form a C*-algebra under pointwise operations (in N) and norm $\|a_\infty\| = \lim_{N\to\infty} \|a_N\|$.

   Continuous cross-section of $\mathfrak{A}^l$ then correspond to quasi-local sequences (a) through
$$\sigma(1/N) = a_N; \tag{3.47}$$
$$\sigma(0) = a_\infty. \tag{3.48}$$

3. Here, the embedding maps ι_{MN} are replaced by the obvious symmetrization maps $j_{MN} : \mathcal{B}_M \hookrightarrow \mathcal{B}_N$, defined for $N > M$ by $j_{MN} = \mathrm{Sym}_N \circ \iota_{MN}$, where the usual symmetrization map $\mathrm{Sym}_N : \mathcal{B}_N \to \mathcal{B}_N$ projects on the completely symmetric tensors.
   (a) A sequence (a) is *symmetric* if there is an M such that $a_N = j_{MN}(a_M)$ for all $N \geq M$.
   (b) A sequence (a) is *quasi-symmetric* if for any $\varepsilon > 0$ there is an M and a symmetric sequence (a') such that $\|a_N - a'_N\| < \varepsilon$ for all $N > M$.

If (a) is a quasi-symmetric sequence, and ω is a state on $M_2(\mathbb{C})$, then the following limit exists:

$$a_0(\omega) = \lim_{N\to\infty} \omega^N(a_N), \tag{3.49}$$

where ω^N is the N-fold tensor product of ω with itself, defining a state on $\mathcal{B}_N$. The ensuing function a_0 on the state space $\mathcal{S}(M_2(\mathbb{C}))$ is continuous, so that a_0 is an element of the limit algebra $\mathcal{B}^g_\infty$. If we identify $\mathcal{S}(M_2(\mathbb{C}))$ with B^3, then from (2.38),

$$a_0(x,y,z) = \lim_{N\to\infty} \mathrm{Tr}\left(\rho(x,y,x)^{\otimes N} a_N\right). \tag{3.50}$$

The continuous cross-sections of $\mathfrak{A}^g$ are then simply given by maps σ of the form

$$\sigma(1/N) = a_N; \tag{3.51}$$
$$\sigma(0) = a_0, \tag{3.52}$$

the latter defined as in (3.49), where (a) is some quasi-symmetric sequence.

One may be surprised by the commutativity of $\mathfrak{A}^g_0$, but a simple example may clarify this (cf. Landsman (2007, §6.1) for the general argument). If we write the left-hand side of (2.36) as $S^{(N)}_\mu$ for clarity, then $a_N = S^{(N)}_\mu = \frac{1}{2} j_{1N}(\sigma^\mu)$ defines a symmetric sequence, and

$$[S^{(N)}_\mu, S^{(N)}_\nu] = \frac{i}{N}\varepsilon_{\mu\nu\rho} S^{(N)}_\rho. \tag{3.53}$$

This commutator evidently vanishes as $N \to \infty$. The limit functions a_0 may be computed from (3.50). Writing $\tilde{S}_\mu$ for the limit of the sequence $a_N = S^{(N)}_\mu$, we obtain

$$\tilde{S}_\mu(x_1, x_2, x_3) = \tfrac{1}{2} x_\mu. \tag{3.54}$$

Similarly, the mean (i.e., averaged) Hamiltonians

$$h^{CW}_N = H^{CW}_N/N = -2(S_z^2 + BS_x) \tag{3.55}$$

of the Curie-Weisz model,[37] cf. (2.37), define a symmetric sequence, with limit (2.41), i.e.,

$$\tilde{h}^{CW} = h^{CW}. \tag{3.56}$$

3.2 Continuous fields of states

Dually, one has *continuous fields of states*, which, given one of our continuous fields of C*-algebras $\mathfrak{A}^\alpha$, are simply defined as families $(\omega) = (\omega_x)_{x\in I_\alpha}$, where ω_x is a state on $\mathfrak{A}^\alpha_x$, such that for each continuous cross-section σ of $\mathfrak{A}^\alpha$, the function $x \mapsto \omega_x(\sigma(x))$ is continuous on I_α. The main purpose of this is to define limits (cf. §4). In our examples:

1. For the classical limit this reproduces the standard notion of convergence of quantum states to classical ones, which is as follows.[38] Let $(\rho_\hbar)$ be a family of density matrices on $L^2(\mathbb{R})$, $\hbar \in (0,1]$. Each $\rho_\hbar$ defines a probability measure $\mu_\hbar$ on phase space $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} d\mu_\hbar\, f = \mathrm{Tr}\,(\rho_\hbar Q_\hbar(f)). \tag{3.57}$$

[37] And similarly for general mean-field models, see Bona (1988) and Duffield & Werner (1992).

[38] Cf. Robert (1987), Paul & Uribe (1996), Landsman & Reuvers (2013), and many other sources.

A state ρ_0 on $C_0(\mathbb{R}^2)$ equally well defines an associated measure μ_0 by Riesz–Markov,

$$\rho_0(f) = \int_{\mathbb{R}^2} d\mu_0\, f. \tag{3.58}$$

The family $(\rho_\hbar)$ then converges to ρ_0 iff $\mu_\hbar \to \mu_0$ weakly, in that for each $f \in C_c(\mathbb{R}^2)$,

$$\lim_{\hbar\to 0} \int_{\mathbb{R}^2} d\mu_\hbar\, f = \int_{\mathbb{R}^2} d\mu_0\, f. \tag{3.59}$$

2. Any state ω_0 on $\mathfrak{A}^l_0 = \mathcal{B}^l_\infty$ defines a state $\omega_{1/N}$ on $\mathfrak{A}^l_{1/N} = \mathcal{B}_N$ by restriction (since $\mathcal{B}_N \subset \mathcal{B}^l_\infty$), and the ensuing field of states (ω) on $\mathfrak{A}^l$ is (tautologically) continuous.

   Conversely, any continuous field of states (ω') on $\mathfrak{A}^l$ is asymptotically equal to one of the above kind, in that the field (ω) defined by $\omega_0 = \omega'_0$ has the property that $\lim_{N\to\infty} |\omega_{1/N}(a_N) - \omega'_{1/N}(a_N)| = 0$ for any fixed quasi-local sequence (a).[39]

3. For $\mathfrak{A}^g$, no independent characterization of continuous fields of states seems available. A nice example, though, comes from permutation-invariant states ω^l on $\mathcal{B}^l_\infty$ (no typo), defined by the property that each restriction $\omega^l_{1/N} = \omega^l_{|\mathcal{B}_N}$ to $\mathcal{B}_N$ is permutation-invariant. These $\omega^l_{1/N}$ define a continuous field of states on $\mathfrak{A}^g$, whose limit state ω^l_0 on $\mathfrak{A}^g_0 = C(\mathsf{B}^3)$ yields a probability measure μ^l_0, which even characterizes the original state by the quantum De Finetti Theorem of Størmer (1969):

$$\omega^l = \int_{\mathsf{B}^3} d\mu^l_0(x,y,z)\, \rho(x,y,z)^\infty. \tag{3.60}$$

   Here we identify the density matrix (2.38) with the corresponding state on $M_2(\mathbb{C})$ via the trace pairing, see (3.50), and $\rho^\infty = \lim_{N\to\infty} \rho^N$, as in (3.49).

   Of course, any continuous fields of states (ω) on $\mathfrak{A}^g$ defines a probability measure μ_0 on B^3 by applying the Riesz–Markov Theorem to the limit state ω_0. We will see various interesting examples of such measures in the remainder of this paper.[40]

3.3 Continuity of time-evolution

Using the continuous field picture developed above, we also obtain a satisfactory notion of convergence of time-evolution in our various limits. We list our three cases of interest.[41] For technical reasons the optimal result, where time-evolved continuous cross-section of $\mathfrak{A}^\alpha$ are again continuous, applies only to $\alpha = l, g$, but for $\alpha = c$ one still has a weaker continuity result after pairing continuous cross-sections with continuous fields of states.

1. Let $(\rho_\hbar)_{\hbar\in[0,1]}$ be a continuous family of states on $\mathfrak{A}^c$, where for $\hbar > 0$ we identify the state with the associated density matrix, with associated probability measures $\mu_\hbar$, defined by (3.57), so that $\lim_{\hbar\to 0} \mu_\hbar = \mu_0$ weakly. Each density matrix $\rho_\hbar$ evolves in time according to the Liouville–von Neumann equation determined by the quantum Hamiltonian (2.5). In other words, $\rho_\hbar(t)$ satisfies

$$\mathrm{Tr}\,(\rho_\hbar(t)a) = \mathrm{Tr}\,(\rho_\hbar \tau^{(\hbar)}_t(a)),\ a \in K(L^2(\mathbb{R})). \tag{3.61}$$

[39] Indeed, for any $\varepsilon > 0$, there is an M such that $|\omega_{1/N}(a_N) - \omega'_{1/N}(a_N)| < \varepsilon$ for all $N > M$.

[40] See also Landsman (2007, §6.2) for a number of abstract general results on such limit measures.

[41] Vast generalizations of the material are possible, but we restrict attention to our three guiding models.

This induces a time-evolution on each $\mu_\hbar$, in that $\mu_\hbar(t)$ is the probability measure determined by $\rho_\hbar(t)$ according to (3.57).

Likewise, μ_0 evolves in time according to the classical Liouville equation given by the classical Hamiltonian (2.14), which is measure-preserving and hence also maps μ_0 into some other probability measure $\mu_0(t)$. For unbounded Hamiltonians of the kind (2.5), the best continuity result is then given by the weak limit[42]

$$\lim_{\hbar\to 0} \mu_\hbar(t) = \mu_0(t),\ t \in \mathbb{R}. \tag{3.62}$$

2. The quasi-local case leads to stronger results, since the operator limit (2.33) implies

$$\tau_t(a_\infty) = \lim_{N\to\infty} \tau_t^{(N)}(a_N),\ t \in \mathbb{R}, \tag{3.63}$$

for any quasi-local sequence (a).[43] In other words, if (a) is a quasi-local sequence with limit a_∞ , then the time-evolved quasi-local sequence $(a(t))$, where each a_N evolves with $\tau_t^{(N)}$, is again quasi-local, with limit $a_\infty(t)$, where a_∞ evolves with the time-evolution τ_t directly defined in infinite volume. Equivalently, time-evolved continuous cross-section of $\mathfrak{A}^l$ are again continuous, where time-evolution is defined separately in each fiber. For continuous fields of states (ω) this implies

$$\lim_{N\to\infty} \omega_{1/N}(t)(a_N) = \omega_0(t)(a_\infty), \tag{3.64}$$

for any quasi-local sequence (a) with limit a_∞, where $\omega_{1/N}(t)$ and $\omega_0(t)$ are defined by "Schrödinger-picturing" (that is, dualizing) $\tau_t^{(N)}$ and τ_t, respectively; cf. (3.62).

3. As in the previous case, time-evolved continuous cross-section of $\mathfrak{A}^g$ are again continuous, but because of the vast difference between the fibers $\mathfrak{A}^g_x$ at $x = 1/N$ and at $x = 0$, this result may be more unexpected. Let (a) be a quasi-symmetric sequence with limit a_0, let $a_N(t) = \tau_t^{(N)}(a_N)$ be defined by the quantum Hamiltonian (2.37) in the usual unitary way (i.e., by (2.27) and (2.28) with H_N^{CW} instead of H_N^I), and finally let $a_0(t)$ be defined by the classical Hamiltonian (2.41) and the Poisson bracket (2.39).[44] Then the time-evolved sequence $(a(t))$ is again quasi-symmetric, with limit $a_0(t)$; see Landsman (2007, §6.5), elaborating on Duffield & Werner (1992). Of course, for continuous fields of states this implies a result analogous to (3.64).

In conclusion, as soon as it has been (re)formulated in the 'right' setting, time-evolution as such turns out to be continuous throughout the limits $N \to \infty$ or $\hbar \to 0$. The precise kind of continuity is even quite strong in our two lattice models, but it is still acceptable for the double well.[45] Given this continuity, the extreme discontinuity in the behaviour of the ground states if one passes from $N < \infty$ to $N = \infty$, or from $\hbar > 0$ to $\hbar = 0$, is remarkable. Displaying this discontinuity in naked form and at the appropriate technical level is the purpose of the next section; resolving it will be the goal of the one after.

[42]This follows from Egorov's Theorem in the form of Theorem II.2.7.2 in Landsman (1998). In the 'academic' case that the quantum Hamiltonian is compact and is given by $H_\hbar = Q_\hbar(h)$, one has operator convergence of the kind $\lim_{\hbar\to 0} \|Q_\hbar(\tau_t^{(0)}(f)) - \tau_t^{(\hbar)}(Q_\hbar(f))\| = 0$, see Prop. 2.7.1 in Landsman (1998). This also implies that time-evolved continuous cross-section of $\mathfrak{A}^c$ are again continuous, which unfortunately does not seem to be the case for the unbounded Hamiltonian (2.5).

[43]This holds within $\mathcal{B}^l_\infty$ in the operator norm, where each $\tau_t^{(N)}(a_N)$ is embedded in $\mathcal{B}^l_\infty$.

[44]That is, if $\vec{x}(t)$ is the solution of (2.40) with initial condition $\vec{x}(0) = \vec{x}$, then $a_0(\vec{x})(t) = a_0(\vec{x}(t))$.

[45]Note that because our three examples display exactly the same anomalies in their ground states (cf. the Introduction as well as the next section), the technical differences between these kinds of continuity seem irrelevant to our analysis of the emergence problem.

4 Ground states

The definition of a ground state of a specific physical system depends on the setting (i.e., classical/quantum, finite/infinite), but is uncontroversial and well understood in all cases.

1. A ground state (in the usual sense) of a quantum Hamiltonian like (2.5) is a unit eigenvector $\Psi_\hbar^{(0)} \in L^2(\mathbb{R})$ of $H_\hbar^{DW}$ for which the corresponding eigenvalue $E_\hbar^{(0)}$ lies at the bottom of the spectrum $\sigma(H_\hbar^{DW})$. Algebraically, such a unit vector $\Psi_\hbar^{(0)}$ defines a state $\psi_\hbar^{(0)}$ on the C*-algebra of observables $\mathcal{A}_\hbar$ given by (2.9), viz.[46]
$$\psi_\hbar^{(0)}(a) = \langle \Psi_\hbar^{(0)}, a\Psi_\hbar^{(0)} \rangle, \; a \in K(L^2(\mathbb{R})), \tag{4.65}$$
where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$. This reformulation is useful already for fixed $\hbar$, where it removes the phase ambiguity in unit vectors, but it is mandatory if we wish to take the limit $\hbar \to 0$. The above definition of a ground state may be reformulated directly in terms of the algebraic state $\psi_\hbar^{(0)}$, see point 2 below.

   A ground state of the corresponding classical Hamiltonian (2.14) is just a point $z_0 \in \mathbb{R}^2$ in phase space where h^{DW} takes an absolute minimum.

2. For $N < \infty$, an analogous discussion applies to the quantum Ising Hamiltonian (2.19): a ground state is simply a unit eigenvector $\Psi_N^{(0)} \in \mathcal{H}_N$ of H_N^I whose eigenvalue $E_N^{(0)}$ is minimal, with the appropriate algebraic reformulation (luxurious for $N < \infty$ but needed as $N \to \infty$) in terms of a state $\psi_N^{(0)}$ on the C*-algebra $\mathcal{B}_N$.

   At $N = \infty$, ground states of the system with C*-algebra $\mathcal{B}_\infty^l$ of quasi-local observables and time-evolution τ, see (2.33), may be defined as pure states ω on $\mathcal{B}_\infty^l$ such that $-i\omega(a^*\delta(a)) \geq 0$ for each $a \in \mathcal{B}_\infty^l$ for which $\delta(a) = \lim_{t\to 0}((\tau_t(a)-a)/t)$ exists.[47]

3. For $N < \infty$, ground states of the quantum Curie–Weisz model (2.35) are defined as in the previous case, whereas at $N = \infty$ a ground state of the classical Hamiltonian (2.41) is a point $\vec{x}_0$ in the "phase space" B^3 that minimizes h^{CW} absolutely.

We now combine these definitions with the notion of a limit of a family of states (in the algebraic sense) following from our discussion of continuous fields in the previous chapter. That is, if $\mathfrak{A}^\alpha$ is our continuous field of C*-algebras, where $\alpha = c, j, g$ (see §3.1), and $(\omega_x)_{x\in I_\alpha}$ is a family of states on this field in that ω_x is a state on $\mathfrak{A}_x^\alpha$, then we say that
$$\omega_0 = \lim_{x\to 0} \omega_x \tag{4.66}$$
if the (ω_x) form a continuous field of states. For $\alpha = c$ this reproduces the notion of convergence $\rho_\hbar \to \rho_0$ discussed after (3.57), whereas for $\alpha = g, l$ this gives meaning to limits like $\lim_{N\to\infty} \omega_{1/N} = \omega_0$, where each $\omega_{1/N}$ is a state on $\mathcal{B}_N$ and ω_0 is a state on $\mathfrak{A}_0^{g,l}$.

[46]A state on a C*-algebra $\mathcal{A}$ is a positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ of norm one, where positivity of ω means that $\omega(a^*a) \geq 0$ for each $a \in \mathcal{A}$. If $\mathcal{A}$ has a unit 1_A, then $\|\omega\| = 1$ iff $\omega(1_A) = 1$ (given positivity). We say that ω is *pure* if $\omega = p\omega_1 + (1-p)\omega_2$ for some $p \in (0,1)$ and certain states ω_1 and ω_2 implies $\omega_1 = \omega_2 = \omega$, and *mixed* otherwise. If $\mathcal{A} = C_0(X)$ is commutative, then by the Riesz–Markov Theorem states on $\mathcal{A}$ bijectively correspond to probability measures on X, the pure states being the Dirac (point) measures δ_x defined by $\delta_x(f) = f(x)$, $x \in X$. For $\mathcal{A} = K(L^2(\mathbb{R}))$, the pure states are just the unit vectors.

[47]This the shortest among many equivalent definitions: see e.g. Bratteli & Robinson (1997, Definition 5.3.18) or Koma & Tasaki (1994, App. A). For finite systems this (nontrivially, see refs.) reproduces the conventional definition of a ground state, and for infinite systems all known examples support its validity.

1. The quantum double-well Hamiltonian (2.5) has a unique ground state $\Psi_\hbar^{(0)}$, which (by a suitable choice of phase) may be chosen to be real and strictly positive.[48] Since the ground state is unique, it is $\mathbb{Z}_2$-invariant (for otherwise its image under u in (2.7) would be another ground state). Seen as a wave-function, $\Psi_\hbar^{(0)}$ has well-separated peaks above a and $-a$, which, as $\hbar \to 0$, become increasingly pronounced.[49]

   However, the classical double-well Hamiltonian (2.14) has *two* ground states

$$\psi_0^\pm = (p = 0, q = \pm a), \tag{4.67}$$

   which are mapped into each other by the $\mathbb{Z}_2$-symmetry $(p, q) \mapsto (p, -q)$. These states may be visualized as the particle being at rest at location $q = \pm a$. Now define

$$\psi_0^{(0)} = \tfrac{1}{2}(\psi_0^+ + \psi_0^-), \tag{4.68}$$

   where we identify $\psi_0^\pm \in \mathbb{R}^2$ with the corresponding (pure) state on $C_0(\mathbb{R}^2)$, so that the right-hand side is a convex sum of states (and hence a state itself).[50] Then the family $(\psi_\hbar^{(0)})_{\hbar\in[0,1]}$ forms a a continuous field of states on the continuous field of C*-algebras $\mathfrak{A}^c$, and we have the result announced in the Introduction, namely

$$\lim_{\hbar\to 0} \psi_\hbar^{(0)} = \psi_0^{(0)}. \tag{4.69}$$

2. Similarly, for any $N < \infty$ and any $B > 0$ the ground state $\Psi_N^{(0)}$ of the quantum Ising model (2.19) is unique and hence $\mathbb{Z}_2$-invariant.[51] The corresponding model at $N = \infty$ with small magnetic field $0 \leq B < 1$, on the other hand, has a doubly degenerate ground state $\psi_\infty^\pm$, in which all spins are either up (+) or down (-).[52] If we relabel the algebraic state corresponding to $\Psi_N^{(0)}$ as $\psi_{1/N}^{(0)}$, and similarly relabel $\psi_\infty^\pm$ as $\psi_0^\pm$, with corresponding mixture (4.68), then the states $(\psi_x)_{x\in I_l}$ form a continuous field of states on the continuous field of C*-algebras $\mathfrak{A}^l$, and,[53]

$$\lim_{N\to\infty} \psi_{1/N}^{(0)} = \psi_0^{(0)}. \tag{4.70}$$

3. The situation for the quantum Curie–Weisz model is the same, *mutatis mutandis*:[54] for $N < \infty$ its ground state is unique and hence $\mathbb{Z}_2$-invariant,[55] but for $N = \infty$ and $0 \leq B < 1$ the classical Hamiltonian (2.41) has two distinct ground states $\psi_0^\pm$, duly related by $\mathbb{Z}_2$ (here realized by a 180-degree rotation around the x-axis, cf. §2.2).[56]

[48] See Reed & Simon (1978, §XIII.12). Uniqueness follows from an infinite-dimensional version of the Perron–Frobenius Theorem of linear algebra, which also yields strict positivity of the wave-function.

[49] See Landsman & Reuvers (2013) for more details as well as some pictures.

[50] If in turn we identify states on $C_0(\mathbb{R}^2)$ with probability measures on $\mathbb{R}^2$, as in (3.58), then the right-hand side of (4.68) is a convex sum of probability measures. Specifically, we have $\psi_0^{(0)}(f) = \frac{1}{2}(f(0,a)+f(0,-a))$.

[51] This was first established in Pfeuty (1970) by explicit calculation, based on Lieb et al (1961). This calculation, which is based on a Jordan–Wigner transformation to a fermonic model, cannot be generalized to higher dimensions d, but uniqueness of the ground state holds in any d, as first shown by Campanino et al (1991) on the basis of Perron–Frobenius type arguments similar to those for Schrödinger operators. The singular case $B = 0$ leads to a violation of the strict positivity conditions necessary to apply the Perron–Frobenius Theorem, and this case indeed features a degenerate ground state even when $N < \infty$.

[52] See Araki & Matsui (1985). For $B \geq 1$ all spins align in the x-direction and the ground state is unique.

[53] This a reformulation in our continuous field language of Corollary B.2 in Koma & Tasaki (1994).

[54] See Rieckers (1981) and Gerisch (1993) for the analogue of (4.70) in the quantum Curie–Weisz model.

[55] This seems well known, but the first rigorous proof we are aware of is very recent (Ioffe & Levit, 2013).

[56] As points of B^3, for $0 \leq B < 1$ these are given by $\psi_0^\pm = (B, 0, \pm\sin(\arccos(B)))$. For $B > 1$ the unique ground state is $(1, 0, 0)$, which for $B = 1$ is a saddle point. These points all lie on the boundary S^2 of B^3.

5 First excited states

In all three cases, the phenomenological theories (i.e., classical mechanics for $\hbar = 0$, (quasi-)local quantum statistical mechanics at $N = \infty$, and global quantum statistical mechanics at $N = \infty$ in the guise of classical mechanics) display SSB of the $\mathbb{Z}_2$-symmetry of the Hamiltonian, whereas the corresponding fundamental theories (that is, quantum mechanics and twice quantum statistical mechanics at $N < \infty$) do not. Consequently, the ($\mathbb{Z}_2$-invariant) ground state of the fundamental theory in question cannot possibly converge to the ground state of the corresponding phenomenological theory, because the latter fails to be $\mathbb{Z}_2$-invariant (and the limiting process preserves $\mathbb{Z}_2$-invariance). Instead, one has (4.68), showing that the 'fundamental' ground state converges to the 'Schrödinger Cat' state (4.68). This leads to the problems discussed at length in the Introduction.

As already mentioned, the solution is to take the first excited state $\Psi_{\bullet}^{(1)}$ into account.

1. For the double-well potential, the eigenvalue splitting $\Delta_\hbar \equiv E_\hbar^{(1)} - E_\hbar^{(0)}$ for small $\hbar$ is well known, see the heuristics in Landau & Lifshitz (1977), backed up by rigor in Hislop & Sigal (1996). The leading term as $\hbar \to 0$ is

$$\Delta_\hbar \cong \frac{\hbar\omega}{\sqrt{\frac{1}{2}e\pi}} \cdot e^{-d_V/\hbar} \;\; (\hbar \to 0), \tag{5.71}$$

where the coefficient in the exponential decay in $-1/\hbar$ is the WKB-factor

$$d_V = \int_{-a}^{a} dx \, \sqrt{V(x)}. \tag{5.72}$$

On the basis of rigorous asymptotic estimates in Harrell (1980) and Simon (1985), which were subsequently verified by numerical computations, Landsman & Reuvers (2013) showed that the algebraic states $\psi_\hbar^\pm$ defined by the linear combinations

$$\Psi_\hbar^\pm = \frac{\Psi_\hbar^{(0)} \pm \Psi_\hbar^{(1)}}{\sqrt{2}}, \tag{5.73}$$

satisfy

$$\lim_{\hbar \to 0} \psi_\hbar^\pm = \psi_0^\pm. \tag{5.74}$$

2. The eigenvalue splitting for the quantum Ising chain (with free boundary conditions) can be determined on the basis of its exact solution by Pfeuty (1970). For the leading term in $\Delta_N \equiv E_N^{(1)} - E_N^{(0)}$ as $N \to \infty$, for $0 < B < 1$, we obtain[57]

$$\Delta_N \cong (1 - B^2)B^N \;\; (N \to \infty), \tag{5.75}$$

showing exponential decay $\Delta_N \sim \exp(-CN)$ with positive coefficient $C = -\ln(B)$.

[57] The first steps in this calculation are given by Karevski (2006), to which we refer for notation and details. To complete it, one has to solve his (1.51) for v also to subleading order as $N \to \infty$, noting first that his leading order solution $v = \ln(h)$ (where his h is our B) has the wrong sign. To subleading order we find $v = -\ln(B) - (1 - B^2)B^{2(N-1)}$. Subsituting $q_0 = \pi + iv$ in the expression for the single-fermion excitation energy $E_N^{(1)} = \varepsilon(q_0) = \sqrt{1 + B^2 + 2B\cos(q_0)}$, one finds $\varepsilon(q_0) = 0$ to leading order and $\varepsilon(q_0) = (1 - B^2)B^N$ to subleasing order. But this is precisely Δ_N, since in the picture of Lieb et al (1961) the ground state is the fermonic Fock space vacuum, which has $E_N^{(0)} = 0$. The energy splitting in higher dimensions does not seem to be known, but Koma & Tasaki (1994, eq. (1.5)) expect similar behaviour.

Furthermore, the analogue of (5.74) holds (Koma & Tasaki, 1994, App. B),[58] viz.

$$\lim_{N\to\infty} \psi_N^{\pm} = \psi_\infty^{\pm}, \tag{5.76}$$

where $\psi_N^{\pm}$ are the algebraic states defined by the unit vectors

$$\Psi_N^{\pm} = \frac{\Psi_N^{(0)} \pm \Psi_N^{(1)}}{\sqrt{2}}. \tag{5.77}$$

3. For the quantum Curie–Weisz model, exponential decay of Δ_N has only been established numerically up to $N \sim 150$ (Botet, Julien & Pfeuty, 1982; Botet & Julien, 1982).[59] Eq. (5.76) may be proved in the same way as in the quantum Ising model.[60]

In summary, in each of our models we may define the unit vectors

$$\Psi_x^{\pm} = \frac{\Psi_x^{(0)} \pm \Psi_x^{(1)}}{\sqrt{2}}, \tag{5.78}$$

where $x = \hbar$ or $x = 1/N$. Given the exponential decay of the eigenvalue splitting, in the asymptotic regime $\hbar \to 0$ or $N \to \infty$ these are 'almost' energy eigenstates. Indeed, the corresponding algebraic states converge to time-independent states of the limit theories:

$$\lim_{x\to 0} \psi_x^{\pm} = \psi_0^{\pm}. \tag{5.79}$$

As explained in the Introduction, this would solve the problem of emergence, provided the system manages to move from its ground state to either Ψ_x^+ or Ψ_x^-, for sufficiently small x (i.e., small $\hbar$ or large N). And so it does, but with another proviso, namely the presence of (asymmetric) perturbations to the (symmetric) dynamics, but: these may be almost arbitrarily small (see below). This, in turn, requires the system to be coupled to something like an environment (which may also be internal to the system at hand).[61]

Given the presence of perturbations, the mechanism inducing this transition is the perturbative instability of the ground state of the double-well potential under arbitrarily small asymmetric perturbations as $\hbar \to 0$ (Jona-Lasinio, Martinelli, & Scoppola, 1981a,b). As this has already been explained at length in Landsman & Reuvers (2013) in the context of the $\hbar \to 0$ limit, all we wish to add here is the universality of the mechanism.

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[58] See especially their Corollary B.2 and subsequent remark. This corollary is formulated in terms of the state $\Psi_N = O_N\Psi_N^{(0)}/\|O_N\Psi_N^{(0)}\|$, where for the quantum Ising model one has $O_N = \sum_{i\in \underline{N}} \sigma_i^z$, but since $\|\Psi_N^{(1)} - \Psi_N\| = O(1/N)$ as $N \to \infty$ by their Lemma B.4, our (5.76) follows. A similar comment applies to the quantum Curie–Weisz model (taking into account the subtlety to be mentioned in the next footnote).

[59] Se also related simulations up to $N = 1000$ in Vidal et al (2004). In the worst case, where for whatever reasons these simulations are misleading, Δ_N would decay as $1/N$, which does not jeopardize our scenario, but would add some constraints on the perturbations destabilizing the ground state, see below. This decay follows from Theorem 2.2 in Koma & Tasaki (1994), where it has to be noted that one of the assumptions in their proof (namely that the support set of H_i, where $H_N = \sum_i H_i$, is bounded in N) is invalid in the quantum Curie–Weisz model; nonetheless, their eq. (2.11) can be proved by direct computation.

[60] It would be interesting to apply the techniques of Ioffe & Levit (2013), designed for $\Psi_N^{(0)}$, to $\Psi_N^{(1)}$.

[61] As explained in Landsman & Reuvers (2013) in connection with the measurement problem, this is *not* the decoherence scenario, which unlike ours fails to predict individual measurement outcomes and hence fails to solve the the measurement problem (it rather reconfirms it). In our present context of SSB, mere decoherence would achieve nothing either, leading to the mixtures $\psi_0^{(0)}$ rather than the pure states $\psi_0^{\pm}$.

Indeed, its subsequent reformulation by Simon (1985), Helffer & Sjöstrand (1986), and Helffer (1988) in terms of the "interaction matrix" immediately shows that the ground state of the quantum Ising chain or the quantum Curie–Weisz model is just as unstable for $N \to \infty$ as is the ground state of the double well Hamiltonian for $\hbar \to 0$.

The point is that in models like the ones discusses in this paper, as $\hbar \to 0$ or $N \to \infty$:

- on the one hand all energy levels merge into a continuum in that the distance between any pair of levels approaches zero (typically as $\sim \hbar$ or some inverse power of N);
- on the other, they split into pairs whose energy difference is exponentially small.

Note that the energy spectrum is discrete for any $\hbar > 0$ or $N < \infty$. In particular, in the asymptotic regime the pair consisting of the ground state and the first excited state entirely controls the low-energy behavior of the system. Now take a basis of the two-level Hilbert space $\mathbb{C}^2$ consisting of $e_1 = \Psi_x^+$ and $e_2 = \Psi_x^-$ (as opposed to $\Psi_x^{(0)}$ and $\Psi_x^{(1)}$). Writing Δ_x for the pertinent eigenvalue splitting, where x is $\hbar$ or $1/N$, and $\Delta_{1/N} \equiv \Delta_N$,

$$H_x = \tfrac{1}{2}\begin{pmatrix} 0 & -\Delta_x \\ -\Delta_x & 0 \end{pmatrix}, \tag{5.80}$$

is the effective Hamiltonian for the low-energy behavior that reproduces the (unperturbed) ground state $(e_1 + e_2)/\sqrt{2} = \Psi_x^{(0)}$. Now add perturbations $\delta_\pm$ that change H_x to

$$H_x^{(\delta)} = \tfrac{1}{2}\begin{pmatrix} \delta_+ & -\Delta_x \\ -\Delta_x & \delta_- \end{pmatrix}. \tag{5.81}$$

Since the Δ_x vanish exponentially fast as $x \to 0$, almost any choice of the $\delta_\pm$ that does not make them vanish as quickly as Δ_x has the effect that as $x \to 0$, the perturbed Hamiltonian (5.81) is dominated by its diagonal, as opposed to the original Hamiltonian (5.80). Hence with the exception of the unlikely symmetric case $\delta_+ = \delta_-$, the ground state of $H_x^{(\delta)}$ will tend to either e_1 or e_2 as $x \to 0$. Consequently, the effect of arbitrary asymmetric perturbations is to change the ground state of the system from a Schrödinger Cat one like $\Psi_x^{(0)}$ to a localized one like $\Psi_x^\pm$ (either in physical space, as for the double well, or in spin configuration space, as in our other two models). The interaction matrix formalism makes this reasoning rigorous, in showing exactly how perturbations of the original Hamiltonians (in the high-dimensional Hilbert spaces where these are defined) may be compressed in just the two numbers $\delta_\pm$, with the effect just described. As we see, the entire argument is just based on linear algebra and the properties of the spectrum described above.

What remains to be done theoretically is to first model the perturbations achieving this dynamically (i.e., in time), and subsequently to study also the dynamical transition from the original, delocalized, unperturbed ground state to the perturbed, localized ground state. For the double-well case this program has been started in Landsman & Reuvers (2013), and for the spin systems this is a matter for future research.[62]

Experimentally, the entire scenario should be put to the test, especially where the envisaged destabilization of the ground state and hence the ensuing collapse of Schrödinger Cat states for semiclassical or large systems are concerned.[63] Such states may indeed be realized in materials amenable to the kind of models studied in this paper.[64]

[62] Relevant literature on this point includes e.g. Caux & Esler (2013).

[63] Such experiments will indeed be designed and performed in the near future in a collaboration with Andrew Briggs and Andrew Steane (Oxford) and Hans Halvorson (Princeton), supported by the TWCF.

[64] See, e.g., Friedenauer (2010) and references therein.

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