Choosing the Analytic Component of Theories

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Abstract

I provide a compact reformulation of Carnap's conditions of adequacy for the analytic and the synthetic component of a theory and show that, contrary to arguments by Winnie and Demopoulos, Carnap's conditions of adequacy need not be supplemented by another condition. This has immediate implications for the analytic component of reduction sentences.

Keywords: analyticity; analytic-synthetic distinction; Carnap sentence; Ramsey sentence; reduction sentence; Przełęcki reduction pair; relativization sentence

1 Conditions of adequacy for an analytic-synthetic distinction

Given an arbitrary theory ϑ, it is not always obvious what its synthetic component $\mathrm{Syn}(\vartheta)$ is and what its analytic component $\mathrm{An}(\vartheta)$ is. The most widely known and analyzed technical solution to this problem has been suggested by Carnap (1963, 24.D), who assumes a bipartition of the vocabulary $\mathscr{V}$ of ϑ into observational terms $\mathscr{O} = \{O_1, \ldots, O_m\}$ and theoretical terms $\mathscr{T} = \{T_1, \ldots, T_n\}$. He then suggests as ϑ's synthetic component its Ramsey sentence

$$\mathsf{R}_{\mathscr{O}}(\vartheta) := \exists X_1 \ldots X_n \vartheta(O_1, \ldots, O_m, X_1, \ldots, X_n) \;, \tag{1}$$ [1]

which results from ϑ by existentially generalizing on all $\mathscr{T}$-terms in ϑ, and as ϑ's analytic component the Carnap sentence

$$\mathsf{C}_{\mathscr{O}}(\vartheta) = \mathsf{R}_{\mathscr{O}}(\vartheta) \rightarrow \vartheta \;. \tag{2}$$

According to Carnap, $\mathsf{R}_{\mathscr{O}}(\vartheta)$ and $\mathsf{C}_{\mathscr{O}}(\vartheta)$ provide an adequate distinction between the analytic and synthetic component of ϑ. To spell out explicit conditions of adequacy for such a distinction, Carnap (1963, 963) defines the observational content of any sentence S as follows:

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[1]The subscript stands for the vocabulary that remains after the existential generalization.

Definition 1. "The *observational content* or O-content of $S =_{\mathrm{Df}}$ the class of all non-L-true [not logically true] sentences in L'_{O} which are implied by S."

L'_{O} refers to the logically extended observation language, whose sentences contain only $\mathcal{O}$-terms and logical constants and variables of any order (Carnap 1963, 959). On this basis, Carnap (1963, 963) suggests

Definition 2. "S' is *O-equivalent* (observationally equivalent) to $S =_{\mathrm{Df}} S'$ is a sentence in L'_{O} and S' has the same O-content as S."[2]

Finally, Carnap (1963, 965) states the following conditions of adequacy:

Definition 3. $\mathrm{Syn}(\vartheta)$ is an *adequate synthetic component* and $\mathrm{An}(\vartheta)$ an *adequate analytic component* of ϑ if and only if

"(a) The two components together are L-equivalent [logically equivalent] to $TC[:= \vartheta]$.

(b) The first component is O-equivalent to TC.

(c) The second component contains theoretical terms; but its O-content is null, since its Ramsey-sentence is L-true [logically true] in L'_{O}."

Since 'O-content' is defined relative to the extended observation language, it is straightforward to show

Lemma 1. *The observational content of ϑ is equivalent to* $\mathsf{R}_{\mathcal{O}}(\vartheta)$.

Proof. Lutz (2013, Lemma 1). □

More generally,

Claim 2. $\mathrm{An}(\vartheta)$ *is an adequate analytic component of ϑ and* $\mathrm{Syn}(\vartheta)$ *is an adequate synthetic component of ϑ if and only if*

1. $\mathrm{An}(\vartheta) \wedge \mathrm{Syn}(\vartheta) \models\!\!\!\!\!= \vartheta$,

2. $\mathsf{R}_{\mathcal{O}}(\mathrm{Syn}(\vartheta)) \models\!\!\!\!\!= \mathsf{R}_{\mathcal{O}}(\vartheta)$,

3. $\mathrm{Syn}(\vartheta)$ *contains no theoretical terms, and*

4. $\models \mathsf{R}_{\mathcal{O}}(\mathrm{An}(\vartheta))$.

Proof. Lutz (2013, corollary 2). □

Claim 2 straightforwardly leads to

Corollary 3. $\mathsf{C}_{\mathcal{O}}(\vartheta)$ *is an adequate analytic component and* $\mathsf{R}_{\mathcal{O}}(\vartheta)$ *is an adequate synthetic component of ϑ.*

[2]Note that Carnap's definition is asymmetric: S' but not S has to be in L'_{O}.

Carnap (1963, 965) thus states:

> These results show, in my opinion, that this method [of splitting ϑ into $\mathsf{R}_{\mathscr{O}}(\vartheta)$ and $\mathsf{C}_{\mathscr{O}}(\vartheta)$] supplies an adequate explication for the distinction between those postulates which represent factual relations between completely given meanings, and those which merely represent meaning relations.

Thus it seems that one of the central requirements of Carnap's philosophy of science, and indeed of logical empiricism in general, has been fulfilled: There is a way to distinguish precisely between the analytic and the synthetic component of theories.

2 Winnie and Demopoulos on the uniqueness of the analytic component

For a consistent theory ϑ in a first order language without identity, Winnie (1970, theorem 5) shows that beyond $\mathsf{C}_{\mathscr{O}}(\vartheta)$, $\mathsf{C}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta)$ also fulfills the conditions for $\text{An}(\vartheta)$ (cf. Williams 1973, 404–408).[3] The lack of identity is necessary so that ϑ does not restrict the cardinality of its domain, but as Demopoulos (2008, 376–377) points out, this can also be made an explicit condition. Then Winnie's result is easy to recover in higher order logic:

Claim 4. *If ϑ does not restrict the cardinality of its domain, $\mathsf{C}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta)$ is an adequate analytic component and $\mathsf{R}_{\mathscr{O}}(\vartheta)$ is an adequate synthetic component of ϑ.*

Proof. That $\mathsf{R}_{\mathscr{O}}(\vartheta)$ fulfills conditions 2 and 3 of claim 2 follows from corollary 3. Clearly, $\mathsf{C}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta) \wedge \mathsf{R}_{\mathscr{O}}(\vartheta) \mathrel{\models\!\!\!\dashv} \vartheta$. Furthermore, if ϑ does not restrict the cardinality of its domain, $\models \mathsf{R}_{\varnothing}(\vartheta) \mathrel{\models\!\!\!\dashv} [\neg\mathsf{R}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\varnothing}(\vartheta)] \vee [\mathsf{R}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\varnothing}(\vartheta)] \mathrel{\models\!\!\!\dashv} [\neg\mathsf{R}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\varnothing}(\vartheta)] \vee \mathsf{R}_{\mathscr{O}}(\vartheta) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}([\neg\mathsf{R}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta)] \vee \vartheta) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}([\mathsf{R}_{\mathscr{O}}(\vartheta) \rightarrow \vartheta] \wedge \mathsf{R}_{\mathscr{T}}(\vartheta)) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\mathsf{C}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta))$. □

Winnie (1970, 294–296) and Demopoulos (2007, §V) consider this non-uniqueness result something of a confirmation of the Quinean charge that the distinction between analytic and synthetic sentences is arbitrary (cf. Quine 1951). As a defense of Carnap's approach, Winnie (1970, 296–297) and Demopoulos (2007, V) suggest an additional condition of adequacy for $\text{An}(\vartheta)$ that is based on

Definition 4. σ is *observationally vacuous* in ϑ if and only if $\vartheta \models \sigma$ and for any $\mathscr{O}$-sentence ω and $\mathscr{V}$-sentence τ with $\vartheta \models \tau$, $\tau \wedge \sigma \models \omega$ only if $\tau \models \omega$.

Winnie (1970, 296–297) and Demopoulos (2008, 371) justify definition 4 as similar to the notion of $\mathscr{O}$-conservativeness relative to an empty set in first order

[3] $\mathsf{R}_{\mathscr{T}}(\vartheta) = \exists X_1 \ldots X_m \vartheta(X_1, \ldots, X_m, T_1, \ldots, T_n)$, see n. 1. To be precise, Winnie shows that any $\mathscr{T}$-sentence entailed by $\mathsf{C}_{\mathscr{O}}(\vartheta)$ can be conjoined with ϑ.

logic,[4] and point out that an observationally vacuous sentence can never contribute to the inference of an $\mathcal{O}$-sentence (Winnie 1970, 297; Demopoulos 2007, 259). This, of course, is shorthand for the claim that an observationally vacuous sentence can never contribute to the inference of an $\mathcal{O}$-sentence *from a sentence entailed by* ϑ.[5] In fact, σ is observationally vacuous in ϑ if and only if, first, it is entailed by ϑ, and second, it is $\mathcal{O}$-conservative relative to *every sentence* entailed by ϑ. This is quite obviously a much stronger condition than, for example, $\mathcal{O}$-conservativeness relative to ϑ. Indeed, it is so strong that one might suspect that no consequence of ϑ at all is observationally vacuous in ϑ, since for any $\mathcal{V}$-sentence σ and $\mathcal{O}$-sentence ω entailed by ϑ, $\vartheta \vDash \sigma \rightarrow \omega$. Since $\sigma \wedge (\sigma \rightarrow \omega) \vDash \omega$, σ therefore fails to be observationally vacuous unless $\sigma \rightarrow \omega \vDash \omega$, that is, $\neg\omega \vDash \sigma$. In other words, a sentence σ is observationally vacuous only if it is entailed by ϑ and by the negation of every $\mathcal{O}$-sentence ω entailed by ϑ. Since $\mathsf{R}_{\mathcal{O}}(\vartheta)$ entails the same $\mathcal{O}$-sentence as ϑ, σ is observationally vacuous only if $\neg\mathsf{R}_{\mathcal{O}}(\vartheta) \vDash \sigma$ and $\vartheta \vDash \sigma$, that is, $\neg\mathsf{R}_{\mathcal{O}}(\vartheta) \vee \vartheta \vDash \sigma$, or simply $\mathsf{C}_{\mathcal{O}}(\vartheta) \vDash \sigma$. Winnie (1970, corollary 12) also proves the converse, so that the following holds:

Claim 5. *σ is observationally vacuous in ϑ if and only if $\mathsf{C}_{\mathcal{O}}(\vartheta) \vDash \sigma$.*

Since Winnie demands as an additional condition of adequacy that An(ϑ) be observationally vacuous in ϑ, he thus shows that only the Carnap sentence is an adequate explication of the analytic component of ϑ and the Quinean charge of arbitrariness is met.

3 Against uniqueness

Winnie and Demopoulos justify observational vacuity as a condition of adequacy by pointing out that observationally vacuous sentences cannot contribute to the inference of an $\mathcal{O}$-sentence. But since they do not argue that all observationally non-vacuous sentences do so contribute, they show at best that the condition is not too inclusive. Given the strength of the condition, however, it is much more interesting whether it is too exclusive and, indeed, whether it is needed at all.

Winnie and Demopoulos claim that without the demand for observational vacuity of An(ϑ), the analytic-synthetic dichotomy is arbitrary. But this is a tendentious formulation, for one because at most the analytic component of ϑ may be arbitrary, as Syn(ϑ) is uniquely determined up to equivalence by Carnap's conditions of adequacy:

Claim 6. *A sentence ϱ is an adequate synthetic component of ϑ if and only if* $\varrho \mathrel{\Dashv\vDash} \mathsf{R}_{\mathcal{O}}(\vartheta)$.

[4] Suppes (1957, §8.2) discusses $\mathcal{O}$-conservativeness relative to a set of sentences as "non-creativity" relative to a set of "axioms". Demopoulos (2007, n. 12) calls observational vacuity a "special case" of $\mathcal{O}$-conservativeness; but this is misleading, since the former is stronger than the latter.

[5] As Ayer (1946, 11–12) realized the hard way, any sentence σ can contribute to the inference of almost any $\mathcal{O}$-sentence ω, namely in connection with $\sigma \rightarrow \omega$ when $\sigma \rightarrow \omega \nvDash \omega$ (cf. Lewis 1988).

Proof. Assume that ϱ is an adequate synthetic component of ϑ. Then it is an $\mathscr{O}$-sentence, and hence $\varrho \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\varrho)$. By claim 2, it further holds that $\mathsf{R}_{\mathscr{O}}(\varrho) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\vartheta)$. Hence $\varrho \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\vartheta)$. Given claim 2, the converse is immediate. □

Incidentally, it follows from claim 6 that the (absolute) concept of an adequate analytic component of ϑ is well-defined, even though Carnap's conditions only define the adequacy of an analytic component *relative* to an adequate synthetic one: Since up to equivalence only $\mathsf{R}_{\mathscr{O}}(\vartheta)$ is adequate, the analytic component has to be adequate relative to $\mathsf{R}_{\mathscr{O}}(\vartheta)$.[6]

Winnie and Demopoulos's claim of arbitrariness is also too strong because An(ϑ), while not uniquely determined like Syn(ϑ), is far from being *completely* vague: All sentences with a non-tautological Ramsey sentence are definitely in its anti-extension, and all sentences that are observationally vacuous are definitely in its extension.

On the other hand, if analytic sentences are—with Carnap—considered language conventions, then the set of analytic sentences is even more vague than Winnie and Demopoulos let on. For in this case, any theory ϑ is as good as any other theory τ as long as $\mathsf{R}_{\mathscr{O}}(\vartheta) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\tau)$. Hence even if An($\vartheta$) were uniquely determined by ϑ, the analytic sentences could nonetheless be different, namely An(τ). And once one accepts that it is a matter of choice whether ϑ or τ is true, there is no obvious reason for demanding that if ϑ has been chosen, there must be no more choice with respect to An(ϑ).

But *if*, in spite of these considerations, one were to demand that An(ϑ) has to be fixed by ϑ, $\mathsf{C}_{\mathscr{O}}(\vartheta)$ is the wrong analytic component and thus observational vacuity the wrong condition of adequacy. One reason is the following: Take any theory ϑ with non-tautologous theoretical content that does not restrict the cardinality of its domain. Then, if the theory is extended in any way so that its observational content increases, some of the theory's analytic implications will become non-analytic :

Claim 7. *Let ϑ be such that $\mathsf{R}_{\mathscr{T}}(\vartheta)$ is not a tautology and has models of any cardinality, and let τ be any theory. Then $\mathsf{C}_{\mathscr{O}}(\vartheta \wedge \tau) \models \mathsf{C}_{\mathscr{O}}(\vartheta)$ if and only if the observational content of $\vartheta \wedge \tau$ is equivalent to that of ϑ.*

Proof. '$\Leftarrow$': By lemma 1, the observational content of $\tau \wedge \vartheta$ is equivalent to that of ϑ if and only if $\mathsf{R}_{\mathscr{O}}(\vartheta) \mathrel{\models\!\!\!\dashv} \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau)$. It is to be shown that if $\mathsf{C}_{\mathscr{O}}(\vartheta \wedge \tau)$ holds, $\mathsf{R}_{\mathscr{O}}(\vartheta)$ entails ϑ. Since $\mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau) \rightarrow \vartheta \wedge \tau$ holds and $\mathsf{R}_{\mathscr{O}}(\vartheta)$ entails by assumption $\mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau)$, it also entails $\vartheta \wedge \tau$ and hence ϑ.

'$\Rightarrow$': If $\mathsf{C}_{\mathscr{O}}(\vartheta \wedge \tau) \models \mathsf{C}_{\mathscr{O}}(\vartheta)$, then $\models \mathsf{C}_{\mathscr{O}}(\vartheta \wedge \tau) \rightarrow \mathsf{C}_{\mathscr{O}}(\vartheta)$ and, by propositional logic, $\models \mathsf{R}_{\mathscr{O}}(\vartheta) \rightarrow \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau) \vee \vartheta$. Hence $\mathsf{R}_{\mathscr{O}}(\vartheta) \models \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau) \vee \vartheta$ and thus $\mathsf{R}_{\mathscr{O}}(\vartheta) \models \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau) \vee \mathsf{R}_{\mathscr{T}}(\vartheta)$. Since $\mathsf{R}_{\mathscr{T}}(\vartheta)$ is not a tautology and does not restrict the cardinality of its domain, any $\mathscr{O}$-model of $\mathsf{R}_{\mathscr{O}}(\vartheta)$ can be expanded such that

[6] Przełęcki and Wójcicki (1969) give conditions of adequacy for analytic components and separate conditions for synthetic components of ϑ. Their conditions are up to equivalent reformulation of the synthetic component equivalent to Carnap's (Lutz 2013, claim 10).

$\mathsf{R}_{\mathscr{T}}(\vartheta)$ is false. Hence $\mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau)$ must be true in every $\mathscr{O}$-model of $\mathsf{R}_{\mathscr{O}}(\vartheta)$, so that $\mathsf{R}_{\mathscr{O}}(\vartheta) \vDash \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau)$ and thus $\mathsf{R}_{\mathscr{O}}(\vartheta) \mathrel{\vDash\!\dashv} \mathsf{R}_{\mathscr{O}}(\vartheta \wedge \tau)$ □

If only the Carnap sentence was an adequate analytic component, the introduction of any sentence into the theory that leads to new observational information would thus render some previously analytic sentence non-analytic. One example of such a sentence is given by Carnap (1966, 238), who notes that as long as they avoid inconsistency, physicists "are free to add new correspondence rules", which not necessarily (or even typically) lead to an observationally equivalent theory. In a sense then, assuming the Carnap sentence as the unique analytic component of every theory leads to contradictory results when the extension of theories is taken into consideration.

Put slightly differently, $\mathsf{C}_{\mathscr{O}}(\vartheta)$ seems the wrong unique analytic component because if ϑ contains any mathematical claim α, say, because ϑ is a theory in physics that requires the development of new mathematics, the mathematical claim would not be analytic. This can be inferred from a theorem by Winnie (1970, theorem 4), who shows that for a consistent theory ϑ with observational content in a first order language without identity, only tautological $\mathscr{T}$-sentences follow from $\mathsf{C}_{\mathscr{O}}(\vartheta)$ (cf. Williams 1973, theorem 5). In higher order logic, the result is a corollary of claim 7 if α does not restrict the cardinality of its domain.

Corollary 8. *Let α be a $\mathscr{T}$-sentence with models of any cardinality, and let ϑ have observational content. Then $\mathsf{C}_{\mathscr{O}}(\vartheta) \vDash \alpha$ only if $\vDash \alpha$.*

Proof. Assume that $\mathsf{C}_{\mathscr{O}}(\vartheta) \vDash \alpha$. Then $\neg\mathsf{R}_{\mathscr{O}}(\vartheta) \vDash \alpha$ and $\vartheta \vDash \alpha$, so that $\vartheta \mathrel{\vDash\!\dashv} \vartheta \wedge \alpha$ and thus $\mathsf{C}_{\mathscr{O}}(\vartheta \wedge \alpha) \vDash \alpha$. Hence $\mathsf{C}_{\mathscr{O}}(\vartheta \wedge \alpha) \vDash \neg\mathsf{R}_{\mathscr{O}}(\alpha) \rightarrow \alpha \mathrel{\vDash\!\dashv} \mathsf{C}_{\mathscr{O}}(\alpha)$. Since α has models of any cardinality and contains no $\mathscr{O}$-terms, $\vDash \mathsf{R}_{\varnothing}(\alpha) \mathrel{\vDash\!\dashv} \mathsf{R}_{\mathscr{O}}(\alpha)$. Since α is a $\mathscr{T}$-sentence, $\alpha \mathrel{\vDash\!\dashv} \mathsf{R}_{\mathscr{T}}(\alpha)$; if it now were the case that $\nvDash \alpha$, it would follow that $\nvDash \mathsf{R}_{\mathscr{T}}(\alpha)$ and, by claim 7, that $\vDash \mathsf{R}_{\mathscr{O}}(\alpha) \mathrel{\vDash\!\dashv} \mathsf{R}_{\mathscr{O}}(\alpha \wedge \vartheta)$ and thus $\vDash \mathsf{R}_{\mathscr{O}}(\vartheta)$, which contradicts the assumption that ϑ has observational content. □

Hence, assuming that mathematical claims are expressed in theoretical terms, they would be either cardinality claims or tautologies if only the Carnap sentence was an adequate analytic component. The axioms of group theory, for instance, do not restrict the cardinality of their domain and would thus be tautologies, which they are not.[7]

If there had to be a unique analytic component An(ϑ) for every sentence ϑ, it would have to be at least as strong as $\mathsf{C}_{\mathscr{O}}(\vartheta) \wedge \mathsf{R}_{\mathscr{T}}(\vartheta)$ (assuming ϑ does not restrict the cardinality of its domain). For otherwise, one could always be in the situation that $\mathsf{R}_{\mathscr{T}}(\vartheta)$ is a theoretical sentence, which by claim 7 would become non-analytic with the addition of ϑ to one's theory. However, since there is no

[7] Higher order logic is strong enough to explicitly *define* most or all of mathematics and certainly group theory, but it does not already contain mathematical terms like group theory's relation symbol '∘'; whence the need for (non-tautologous) explicit definitions for arriving at mathematics.

reason to treat all theoretical sentences of all theories as analytic, there cannot be a unique An(ϑ) for every sentence ϑ. Therefore, as the conventionality of the choice between ϑ and τ with $\mathsf{R}_{\mathcal{O}}(\vartheta) \mathrel{\text{⫤⊨}} \mathsf{R}_{\mathcal{O}}(\tau)$ already suggests, it is another conventional choice which of the sentences that fulfill Carnap's conditions of adequacy one chooses as analytic.

4 Przełęcki reduction pairs and relativization sentences

That the analytic component of a theory can be chosen has immediate implications for a theory

$$\vartheta \mathrel{\text{⫤⊨}} \forall x[\varphi(x) \rightarrow T_1 x] \wedge \forall x[\psi(x) \rightarrow \neg T_1 x] \tag{3}$$

consisting of two reduction sentences, where φ and ψ are $\mathcal{O}$-formulas and T_1 is a $\mathcal{T}$-term. Then

$$\mathsf{C}_{\mathcal{O}}(\vartheta) \mathrel{\text{⫤⊨}} \forall x[\varphi(x) \rightarrow \neg\psi(x)] \rightarrow \forall x[\varphi(x) \rightarrow T_1 x] \wedge \forall x[\psi(x) \rightarrow \neg T_1 x] \tag{4}$$

is an adequate analytic component of ϑ (Carnap 1963, 964–966). However, as Przełęcki (1969, §7.III) points out, so is the logically stronger sentence

$$\forall x[\varphi(x) \wedge \neg\psi(x) \rightarrow T_1 x] \wedge \forall x[\psi(x) \wedge \neg\varphi(x) \rightarrow \neg T_1 x]\,. \tag{5}$$

I will call the two conjuncts (5) the 'Przełęcki reduction pair for ϑ'.

A particularly intuitive way of arriving at the Przełęcki reduction pair is to think of it as a relativization of the concepts of ϑ to the domain in which $\mathsf{R}_{\mathcal{O}}(\vartheta)$ is true. Even if $\mathsf{R}_{\mathcal{O}}(\vartheta) = \forall x \neg[\varphi(x) \wedge \psi(x)]$ is false, there may be some objects a in the domain for which $\neg[\varphi(a) \wedge \psi(a)]$ is true. The relativization to these objects, that is, the relativization $\vartheta^{(\xi)}$ of ϑ to $\xi := \lambda x \neg[\varphi(x) \wedge \psi(x)]$ (cf. Hodges 1993, 203)[8] results in

$$\begin{aligned}\vartheta^{(\xi)} \mathrel{\text{⫤⊨}} \forall x\big([\neg\varphi(x) \vee \neg\psi(x)] \rightarrow [\varphi(x) \rightarrow T_1 x]\big) \\ \wedge \forall x\big([\neg\varphi(x) \vee \neg\psi(x) \rightarrow [\psi(x) \rightarrow \neg T_1 x]\big)\,,\end{aligned} \tag{6}$$

which is equivalent to the Przełęcki reduction pair. This path to the Przełęcki reduction pair makes it especially transparent that in contradistinction to $\mathsf{C}_{\mathcal{O}}(\vartheta)$, ϑ's Przełęcki reduction pair allows for the relation T_1 to stay reducible even if $\mathsf{R}_{\mathcal{O}}(\vartheta)$ turns out false. Przełęcki reduction pairs can thus be useful in cases where an empirically false theory contains concepts that are still considered to be at least in part helpful.

The reasoning that led to the Przełęcki reduction pair suggests the following generalization:

[8] A relativization of a sentence to some formula ξ restricts all quantifiers occurring in the sentence to ξ.

Definition 5. A *relativization sentence* for ϑ is any sentence $\exists x\xi(x) \to \vartheta^{(\xi)}$ such that $\forall x\xi(x) \Dashv\vDash \mathsf{R}_{\mathcal{O}}(\vartheta)$.

The antecedent $\exists x\xi(x)$ of the relativization sentence ensures $\mathcal{O}$-conservativeness (see the proof of claim 9). Relativization sentences are in a way analogous to Carnap sentences. For those structures in which ϑ's Ramsey sentence is true, the Carnap sentence stipulates that the whole theory is true, and for those structures in which the Ramsey sentence is false, the Carnap sentence stipulates nothing. Analogously, for sets of objects to which the Ramsey sentence applies, relativization sentences stipulate that ϑ applies to them as well, and for sets of objects to which the Ramsey sentence does not apply, they stipulate nothing. Unlike the Carnap sentence, however, relativization sentences are not uniquely determined for every ϑ, since the only requirement for the relativizing formula ξ is that its universal closure must be equivalent to ϑ's Ramsey sentence. And this can be achieved in different ways, for even Carnap sentences are relativization sentences. This can be shown by choosing $\mathsf{R}_{\mathcal{O}}(\vartheta)$ as the relativizing formula; since it contains no free variable, $\forall x \mathsf{R}_{\mathcal{O}}(\vartheta) \Dashv\vDash \mathsf{R}_{\mathcal{O}}(\vartheta)$ and, as is easily shown, $\exists x \mathsf{R}_{\mathcal{O}}(\vartheta) \to \vartheta^{(\mathsf{R}_{\mathcal{O}}(\vartheta))} \Dashv\vDash \mathsf{R}_{\mathcal{O}}(\vartheta) \to \vartheta$.

For a special case, it can be shown that relativization sentences are adequate analytic components of ϑ.

Claim 9. *A relativization sentence for a relational first order sentence ϑ is an adequate analytic component of ϑ.*

Proof. By claims 2 and 6, it suffices to show that $\mathsf{R}_{\mathcal{O}}(\vartheta) \wedge [\exists x\xi(x) \to \vartheta^{(\xi)}] \Dashv\vDash \vartheta$ and $\vDash \mathsf{R}_{\mathcal{O}}[\exists x\xi(x) \to \vartheta^{(\xi)}]$. The former is straightforward: $\mathsf{R}_{\mathcal{O}}(\vartheta) \wedge [\exists x\xi(x) \to \vartheta^{(\xi)}] \Dashv\vDash \forall x\xi(x) \wedge [\exists x\xi(x) \to \vartheta^{(\xi)}] \Dashv\vDash \forall x\xi(x) \wedge \vartheta^{(\xi)} \Dashv\vDash \vartheta$. For the latter, it has to be shown that every $\mathcal{O}$-structure $\mathfrak{A}$ can be expanded to a model of $\exists x\xi(x) \to \vartheta^{(\xi)}$ (Lutz 2013, lemma 6).

If $\mathfrak{A} \nvDash \exists x\xi(x)$, then any expansion $\mathfrak{B}$ of $\mathfrak{A}$ is such that $\mathfrak{B} \nvDash \exists x\xi(x)$, and hence $\mathfrak{B} \vDash \exists x\xi(x) \to \vartheta^{(\xi)}$. Thus assume that $\mathfrak{A} \vDash \exists x\xi(x)$ and let $\mathfrak{C}$ be the restriction $\mathfrak{A}|\xi$ of $\mathfrak{A}$ to ξ.[9] Since ϑ is relational, $\mathfrak{A}$ can be assumed to be relational without loss of generality, and hence such a $\mathfrak{C}$ exists. Trivially, $\mathfrak{A} \vDash \forall x[\xi(x) \to \xi(x)] \Dashv\vDash [\forall x\xi(x)]^{(\xi)}$, so that $\mathfrak{C} \vDash \forall x\xi(x) \Dashv\vDash \mathsf{R}_{\mathcal{O}}(\vartheta)$ by the relativization theorem (Hodges 1993, 203). Hence there is an expansion $\mathfrak{B}^*$ of $\mathfrak{C}$ such that $\mathfrak{B}^* \vDash \vartheta$ (Lutz 2013, lemma 6). Since $\mathfrak{C} \subseteq \mathfrak{A}$, there is an expansion $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B}^* \subseteq \mathfrak{B}$. Since $\mathfrak{B}^* \vDash \vartheta$ and $\mathfrak{B}^* = \mathfrak{B}|\xi$, $\mathfrak{B} \vDash \vartheta^{(\xi)}$, again by the relativization theorem. □

Thus relativization sentences can often be used instead of Carnap sentences. Of course, relativization sentences are not mandatory analytic components either; rather, they provide more choices besides $\mathsf{C}_{\mathcal{O}}(\vartheta)$ and $\mathsf{C}_{\mathcal{O}}(\vartheta) \wedge \mathsf{R}_{\mathcal{T}}(\vartheta)$.

[9] $\mathfrak{C} = \mathfrak{A}|\xi$ if and only if $\mathfrak{C} \subseteq \mathfrak{A}$ and $|\mathfrak{C}| = \{a : a \text{ satisfies } \xi \text{ in } \mathfrak{A}\}$ (Bell and Slomson 1974, 73).

5 Conclusion

It is no problem at all that, unlike a theory's synthetic component, a theory's analytic component is not uniquely determined by Carnap's conditions of adequacy. Specifically, a theory's Carnap sentence is not its only adequate analytic component, because if it were, an increase of our empirical knowledge would be impossible and theories could not have mathematical content. Reduction sentences provide one particularly vivid illustration of the possibility to choose a theory's analytic component, and specifically Przełęcki reduction pairs and, more generally, relativization sentences turn out to be well-motivated, applicable, and hence anything but inadequate.

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