Derivation of the Symmetry Postulates for Identical Particles from Pilot-Wave Theories

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Abstract

The symmetries of the wavefunction for identical particles, including anyons, are given a rigorous non-relativistic derivation within pilot-wave formulations of quantum mechanics. In particular, parastatistics are excluded. The result has a rigorous generalisation to n particles and to spinorial wavefunctions. The relation to other non-relativistic approaches is briefly discussed.

1 Quantum indistinguishability

It is well known that identical particles in standard (non-relativistic) quantum mechanics are characterised by symmetry conditions on the wavefunction. For spinless wavefunctions, these are given by

$$\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_n,t)=e^{i\gamma}\psi(\mathbf{x}_1,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_n,t), \qquad (1)$$

where $\gamma=0\,(\mathrm{mod}\ 2\pi)$ for bosons (symmetry), and $\gamma=\pi\,(\mathrm{mod}\ 2\pi)$ for fermions (antisymmetry). In two dimensions, it is possible for γ to be arbitrary (anyons), where the value depends on the homotopy class of the path along which the particles are exchanged, and the wavefunction is multi-valued. For spinorial wavefunctions (which we discuss only in three dimensions) one requires total symmetry or antisymmetry under simultaneous exchange of both spatial and spinorial indices (bosons or fermions).[1]

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[1]The possibility of anyons was first pointed out by Leinaas and Myrheim (1977). They were described in detail by Goldin *et al.* (1981) and by Wilczek (1982), and appear to be relevant to the explanation of the fractional quantum Hall effect (Arovas *et al.*, 1982, Laughlin, 1983, Tsui *et al.*, 1982).

The derivation of these symmetry conditions from first principles, however, is not self-evident. As stressed in the classic discussion by Messiah and Greenberg (1964), the empirical content of quantum indistinguishability is captured by the requirement that all observables commute with the permutation operators,

$$[A, \Pi_{ij}] = 0, \quad \forall A, i, j. \tag{2}$$

But this merely leads to the conclusion that wavefunctions describing indistinguishable particles should lie in subspaces of Hilbert space that are invariant under all elements of an irreducible representation of the permutation group. Since for $n > 2$ particles the permutation group is non-commutative, one obtains in addition to the usual one-dimensional representations also higher-dimensional ones, connected with the so-called *parastatistics*, and which have never been observed experimentally.

It is the purpose of this paper to give a natural proof that parastatistics are excluded and a derivation of the three known symmetry types in the framework of two pilot-wave theories, namely de Broglie–Bohm theory (de Broglie, 1928, Bohm, 1952), and Nelson's (1966) stochastic mechanics. In non-relativistic quantum theory, this is one of the most rigorous proofs to date, in particular in int extension to spinors. The two pilot-wave theories are sketched in Section 2. The main result follows in Section 3 for two spinless particles, and is generalised to n particles and to spinors in Sections 4 and 5, respectively. Finally, in Section 6, we briefly examine the relation of our approach to the topological approaches of Laidlaw and C. DeWitt (1971) and Leinaas and Myrheim (1977), and to the approach of Goldin *et al.* (1980, 1981).

An independent treatment of this question by Dürr *et al.* (in preparation) is forthcoming, which explicitly includes spinorial wavefunctions, and gives a label-free description of identical particles in the Bohm framework. For other partial or related results in de Broglie–Bohm theory see Schneider (1995),[2] Sjöqvist and Carlsen (1995) and Brown *et al.* (1999), and in stochastic mechanics see Nelson (1985) and Goldstein (1987).

2 Pilot-wave theories

The development of wave mechanics, with both de Broglie and Schrödinger, was based on the optico-mechanical analogy. For de Broglie (1928), the Hamilton–Jacobi action, which is related to the canonical momentum of particle i via

$$\mathbf{p}_i = \boldsymbol{\nabla}_i S, \tag{3}$$

[2]Schneider shows under that under quite general conditions, two wavefunctions satisfying the same Schrödinger equation and generating the same de Broglie velocity field are proportional to each other. Applying this to identical particles, he obtains a result similar to ours (for scalar wavefunctions).

should be interpreted as the geometric limiting case of the phase of a wave, for which a wave equation (on configuration space) had meanwhile been derived by Schrödinger. While in Schrödinger's formulation of wave mechanics, as in wave optics, one does away with light rays or particle trajectories, de Broglie's formulation is a dynamical theory of point masses, a pilot-wave theory, defined by a *guidance equation* of the form (3) and the Schrödinger equation for $\psi = Re^{iS/\hbar}$. In the case of non-vanishing vector potential, the guidance equation reads

$$m_i \dot{\mathbf{x}}_i - q_i \mathbf{A}(\mathbf{x}_i) = \boldsymbol{\nabla}_i S, \tag{4}$$

or

$$\dot{\mathbf{x}}_i = \frac{1}{m_i}(\boldsymbol{\nabla}_i S + q_i \mathbf{A}(\mathbf{x}_i)), \tag{5}$$

and is thus gauge invariant.

From the Schrödinger equation one standardly derives the two coupled equations for R^2 and S:

$$\partial_t R^2 = -\sum_i \boldsymbol{\nabla}_i \cdot (R^2 \frac{1}{m_i} \boldsymbol{\nabla}_i S), \tag{6}$$

$$\partial_t S = -\sum_i \frac{(\boldsymbol{\nabla}_i S)^2}{2m_i} - V + \sum_i \frac{\hbar^2}{2m_i} \frac{\Delta_i R}{R}, \tag{7}$$

and accordingly with vector potential. Equation (6) is a continuity equation for R^2, while equation (7) has the form of a Hamilton–Jacobi equation with an additional term, sometimes called the quantum potential,

$$Q = -\sum_i \frac{\hbar^2}{2m_i} \frac{\Delta_i R}{R}. \tag{8}$$

This explains intuitively results such as electron diffraction and interference, as can be seen qualitatively from the shape of the quantum potential (see e.g. Bohm and Hiley, 1993, p. 34). Notice that the quantum potential factorises if and only if the wavefunction does.

After a lull of twenty-five years, the theory was rediscovered by Bohm (1952) and revitalised through his discovery that it allows a consistent quantum description of measurement and of the phenomenology of collapse. We now sketch the latter description, which we need for Section 5.

During a measurement interaction, eigenstates $\psi_i(x)$ of some self-adjoint operator of a system (call it the particle) become coupled with essentially non-overlapping states $\Phi_i(y)$ of some other system (call it the apparatus):

$$\sum_i \lambda_i \psi_i(x) \Phi_i(y). \tag{9}$$

This means that during the measurement interaction, the configuration space of the composite system is carved up into regions separated by high quantum potential barriers, corresponding to the different components of (9). If the components are not reinterfered, the future trajectory of the particle will be guided, via (5), only by one of the states $\psi_i(x)$, the *effective wavefunction* of the particle, as if the wave had physically collapsed. Given a fixed experimental arrangement, it is the initial configuration of the system that determines which component is the effective wavefunction. In this sense, one can talk of the theory as a hidden variables theory. In general, however, the result depends also on the experimental arrangement, so that the standard arguments against such theories do not apply (Bell, 1987, esp. Chaps 1 and 18).

Application of this model to position and momentum measurements already yields a qualitative understanding of uncertainty, and application to parallel Stern–Gerlach measurements on particles in the singlet state yields strict anticorrelations (see Section 5 for the relevant generalisation of the theory to spinors). The quantitative aspects of quantum mechanics are obtained by assuming that particle positions in an ensemble are distributed according to the quantum distribution $R^2 = |\psi|^2$, which will then be true at all times (under a unitary evolution the particle distribution satisfies the same continuity equation as R^2, and in a measurement the preceding analysis shows why no other subensembles can be selected other than those characterised by one of the distributions $|\psi_i|^2$). This distribution plays a role analogous to that of equilibrium distributions in classical statistical mechanics. Indeed, the analogy between the two theories stretches from justifications of equilibrium down to cousins of Maxwell's demon.

Standard references to de Broglie–Bohm theory, in addition to the original papers, are the textbooks by Bohm and Hiley (1993), Holland (1993), Dürr (2001) and the forthcoming one by Valentini (in preparation). A concise non-technical overview is given by Goldstein (2001). The model of spin to be used in Section 5 is due to Bell (1987, Chap. 19), which contains several other papers providing insights into, extending and generally promoting the approach.

Nelson's (1966, 1985) stochastic mechanics can also be seen as a pilot-wave theory, although this is not his own understanding. One modifies de Broglie–Bohm theory by assuming that the particle undergoes not a deterministic evolution but a *diffusion process*, in which the guidance equation is given by the following Itô equation:

$$d\mathbf{x}_i = \mathbf{b}_i dt + d\boldsymbol{\omega}_i, \tag{10}$$

with

$$\mathbf{b}_i = \frac{1}{m_i}\boldsymbol{\nabla}_i S + \frac{q_i}{m_i}\mathbf{A} + \frac{\hbar}{2m_i}\frac{\boldsymbol{\nabla}_i R^2}{R^2}, \tag{11}$$

and where $d\boldsymbol{\omega}_i$ is a Wiener process with

$$\overline{d\boldsymbol{\omega}_i} = 0, \quad \overline{(d\boldsymbol{\omega}_i)^2} = \frac{\hbar}{m_i}dt. \tag{12}$$

Here, $\mathbf{b}_i$ is the drift velocity of particle i, and $\nu_i := \frac{\hbar}{2m_i}$ its diffusion coefficent.

By inserting the distribution R^2, the corresponding Fokker–Planck equation reduces to the continuity equation (6), so that one sees both that R^2 is the equilibrium distribution also for Nelson's theory, and that the average velocity of the particles at equilibrium is their de Broglie–Bohm velocity (5). Thus, one can consider Nelson's theory as de Broglie–Bohm theory with added noise (see also Bohm and Hiley, 1993, Sects 9.5 ff.).

3 Case of two spinless particles

Since pilot-wave theories include more structure than the standard formulation of quantum mechanics, namely particle trajectories, it is now possible to formulate stronger conditions of indistinguishability than merely requiring (2), and this will allow us to derive all the standard symmetry properties for wavefunctions. We start with the case of two spinless particles. Since any Hamiltonian will be symmetrical, it suffices to establish the result at a single instant in time; it will then hold for all instants. We treat the phase S and amplitude R in succession. We shall allow S in general to be multi-valued, while R is obviously single-valued and positive. Notice, however, that although S may be multi-valued, $\partial_t S$ and all the $\boldsymbol{\nabla}_i S$ need to be uniquely defined, in order for $Re^{iS/\hbar}$ to solve the Schrödinger equation, in particular (6) and (7). (A unique $\boldsymbol{\nabla}_i S$ is also needed for the corresponding guidance equation to be well-defined.) In the following, we shall set $m_i = m = 1$ and $q_i = q = 1$ for all i.

The treatment of S is identical in de Broglie–Bohm theory and in Nelson's mechanics. We impose a symmetry condition on the velocities of the particles (respectively, on the average velocities), as given by (5):

$$\boldsymbol{\nabla}_1 S(\xi_1, \xi_2, t) + \mathbf{A}(\xi_1, t)\Big|_{\xi_1=\mathbf{x}, \xi_2=\mathbf{y}} = \boldsymbol{\nabla}_2 S(\xi_1, \xi_2, t) + \mathbf{A}(\xi_2, t)\Big|_{\xi_1=\mathbf{y}, \xi_2=\mathbf{x}}. \tag{13}$$

That is, the velocity (average velocity) of particle 1 in a given configuration is equal to that of particle 2 in the configuration with the particles exchanged. This is the natural requirement of *indistinguishability* at the level of particle trajectories. By renaming the variables on the right-hand side, we see that the vector potential drops out, and we obtain:

$$\boldsymbol{\nabla}_1 \Big(S(\xi_1, \xi_2, t) - S(\xi_2, \xi_1, t) \Big)\Big|_{\xi_1=\mathbf{x}, \xi_2=\mathbf{y}} = 0. \tag{14}$$

Similarly we obtain:

$$\boldsymbol{\nabla}_2 \Big(S(\xi_1, \xi_2, t) - S(\xi_2, \xi_1, t) \Big)\Big|_{\xi_1=\mathbf{x}, \xi_2=\mathbf{y}} = 0. \tag{15}$$

In other words, we have (with $2D$-dimensional notation for the $\boldsymbol{\nabla}$ operator, D the dimension of space):

$$\boldsymbol{\nabla} \Big(S(\mathbf{x}, \mathbf{y}, t) - S(\mathbf{y}, \mathbf{x}, t) \Big) = 0. \tag{16}$$

Were S everywhere defined, we would conclude that $S(\mathbf{x},\mathbf{y},t) = S(\mathbf{y},\mathbf{x},t) + \gamma\,(\mathrm{mod}\ 2\pi)$, and in fact $\gamma = 0$ for all t (setting $\mathbf{x} = \mathbf{y}$). However, S is undefined where R has a zero. For example, if the two particles are trapped in two disjoint boxes with infinite potential walls, the region $\{R \neq 0\}$ is disconnected, and γ can be arbitrary. Indeed, if ψ and φ are single-particle wavefunctions with support in the two boxes, respectively, such that the symmetric combination $\psi(\xi_1)\varphi(\xi_2) + \varphi(\xi_1)\psi(\xi_2)$ solves the Schrödinger equation, the two terms have disjoint support in the configuration space, and solve the Schrödinger equation separately. Thus any wavefunction

$$\psi(\xi_1)\varphi(\xi_2) + \alpha e^{i\beta}\varphi(\xi_1)\psi(\xi_2) \tag{17}$$

is also a solution. (Notice that not only the phase difference may be arbitrary, but also the symmetry of the amplitude may be destroyed. Incidentally, (17) furnishes an example in which there is more than one equilibrium measure for the particle distribution.)

We must thus require, as is implicit also in the standard requirement (2), that the particles be indistinguishable under *all possible circumstances*, in particular if we open the boxes. (This and similar ways of speaking in the following do not imply reference to actual laboratory procedures, but to the fact that (16) needs to hold at all times even with arbitrary time-dependent symmetric potentials, magnetic fields etc.)

We still may not assume, however, that the spreading of the wave function makes R everywhere different from zero, because we know that the diagonal set $\{\mathbf{x} = \mathbf{y}\}$ may be exceptional, from the example of antisymmetric wavefunctions. (It is also exceptional from the point of view of the pilot-wave dynamics, as discussed below in Section 6.) We omit the case $D = 1$, for which the set $\{\mathbf{x} \neq \mathbf{y}\}$ is disconnected. For $D \geq 2$ instead this set is connected. Given the freedom to vary V etc., we shall thus assume that there is an open time interval in which R is different from zero on the set $\{\mathbf{x} \neq \mathbf{y}\}$ (or at least a connected subset). On this interval we can indeed conclude

$$S(\mathbf{x},\mathbf{y},t) = S(\mathbf{y},\mathbf{x},t) + \gamma\,(\mathrm{mod}\ 2\pi). \tag{18}$$

For $D \geq 3$ the set $\{\mathbf{x} \neq \mathbf{y}\}$ is simply connected (or we may assume that so is the connected subset). Thus S is single-valued (mod 2π). Interchanging $\mathbf{x}$ and $\mathbf{y}$ in (18), we see that γ equals 0 or π (bosons or fermions). When $D = 2$ the set $\{\mathbf{x} \neq \mathbf{y}\}$ is multiply connected. The (equal-time) paths λ from $(\mathbf{x},\mathbf{y})$ to $(\mathbf{y},\mathbf{x})$ fall into different homotopy classes, depending on whether the two particles pass each other on the left or on the right (call these 'simple exchange paths'), or on whether and how often they wind around each other in either sense. Any exchange path is homotopic to a concatenation of simple exchange paths. Thus for $D = 2$ the phase difference along an exchange path is independent of $\mathbf{x}$ and $\mathbf{y}$, and is equal to a whole number n, depending on the homotopy class, times an arbitrary constant γ (anyonic phase). This holds over a whole time interval, and the question arises whether γ may be time-dependent. Were this the case, however, $\partial_t S$ would be ill-defined. Thus γ is independent of t, for all $D \geq 2$.

We now establish that R is symmetric. The second proof given below covers both de Broglie and Bohm's theory and Nelson's, but we note that there is a more direct proof in Nelson's theory. (Both proofs hold for all $D \geq 2$.) Indeed, we require not only that the average velocity (5), but also the drift velocity (11) be symmetric. The two conditions combined yield:

$$\frac{\boldsymbol{\nabla} R^2(\mathbf{x},\mathbf{y},t)}{R^2(\mathbf{x},\mathbf{y},t)} = \frac{\boldsymbol{\nabla} R^2(\mathbf{y},\mathbf{x},t)}{R^2(\mathbf{y},\mathbf{x},t)}. \tag{19}$$

We take any instant from the interval for which $R \neq 0$ on $\{\mathbf{x} \neq \mathbf{y}\}$ (or the connected subset), and thus again consider (19) on this set (R is already symmetric on the complement). From (19) we have:

$$\boldsymbol{\nabla} \log R^2(\mathbf{x},\mathbf{y},t) = \boldsymbol{\nabla} \log R^2(\mathbf{y},\mathbf{x},t). \tag{20}$$

Thus,

$$\boldsymbol{\nabla} \log R(\mathbf{x},\mathbf{y},t) = \boldsymbol{\nabla} \log R(\mathbf{y},\mathbf{x},t), \tag{21}$$

and since for $D \geq 2$ the region is connected,

$$\log R(\mathbf{x},\mathbf{y},t) = \log R(\mathbf{y},\mathbf{x},t) + \delta, \tag{22}$$

and

$$R(\mathbf{x},\mathbf{y},t) = e^{\delta} R(\mathbf{y},\mathbf{x},t). \tag{23}$$

It follows that

$$R(\mathbf{x},\mathbf{y},t) = e^{2\delta} R(\mathbf{x},\mathbf{y},t), \tag{24}$$

and since R is single-valued by definition, $e^{2\delta} = 1$, or

$$R(\mathbf{x},\mathbf{y},t) = \pm R(\mathbf{y},\mathbf{x},t); \tag{25}$$

and since it is positive,

$$R(\mathbf{x},\mathbf{y},t) = R(\mathbf{y},\mathbf{x},t), \tag{26}$$

for any instant of the time interval under consideration (both on $\{\mathbf{x} \neq \mathbf{y}\}$ and on $\{\mathbf{x} = \mathbf{y}\}$). Thus, in both phase and amplitude the wavefunction has the desired symmetries on a certain time interval, and since H is symmetric, it has them at all times.

For the case of de Broglie and Bohm's theory, we do not have the additional condition (19), but we use the fact that on the given time interval S is symmetric (up to a time-independent γ), and R and S are coupled via (6) and (7), with symmetric V. Since we can vary the Hamiltonian, we choose the case of zero magnetic field, so in an appropriate gauge $\mathbf{A} = 0$. (This is not only for ease of calculation, but also because at one point in Section 5 we shall need precisely this case.) We introduce the notation $\tilde{S}(\mathbf{x},\mathbf{y},t) := S(\mathbf{y},\mathbf{x},t)$ and $\tilde{R}(\mathbf{x},\mathbf{y},t) := R(\mathbf{y},\mathbf{x},t)$.

First of all, since $Re^{iS/\hbar}$ and $\tilde{R}e^{i\tilde{S}/\hbar}$ both solve the Schrödinger equation and thus (7), and since all of $\partial_t S$, $\boldsymbol{\nabla} S$ and V are symmetric on the given time interval, it follows that the quantum potential (8) is symmetric:

$$-\frac{\hbar^2}{2}\frac{\Delta R}{R} = -\frac{\hbar^2}{2}\frac{\Delta \tilde{R}}{\tilde{R}}, \tag{27}$$

or

$$\tilde{R}\Delta R - R\Delta\tilde{R} = 0 \tag{28}$$

identically, and thus also

$$\partial_t\left(\tilde{R}\Delta R - R\Delta\tilde{R}\right) = 0. \tag{29}$$

From the continuity equation (6), on the region $R \neq 0$, one has

$$\partial_t R = -\boldsymbol{\nabla} R \boldsymbol{\nabla} S - \frac{R}{2}\Delta S, \tag{30}$$

and similarly for $\tilde{R}$ and $\tilde{S}$, since they also satisfy the continuity equation. Substituting into (29) one has:

$$\begin{aligned}
\partial_t\left(\tilde{R}\Delta R - R\Delta\tilde{R}\right) &= (\partial_t\tilde{R})\Delta R + \tilde{R}\Delta(\partial_t R) - (\partial_t R)\Delta\tilde{R} - R\Delta(\partial_t\tilde{R}) \\
&= \boldsymbol{\nabla} R\boldsymbol{\nabla} S\Delta\tilde{R} + \frac{R}{2}\Delta S\Delta\tilde{R} + R\boldsymbol{\nabla}(\Delta\tilde{R})\boldsymbol{\nabla}\tilde{S} + \\
&\quad 2R(\sum_{m,n}\partial_m\partial_n\tilde{R}\partial_m\partial_n\tilde{S}) + R\boldsymbol{\nabla}\tilde{R}\boldsymbol{\nabla}(\Delta\tilde{S}) + \\
&\quad R\frac{\Delta\tilde{R}}{2}\Delta\tilde{S} + R\boldsymbol{\nabla}\tilde{R}\boldsymbol{\nabla}(\Delta\tilde{S}) + R\frac{\tilde{R}}{2}\Delta\Delta\tilde{S} - \\
&\quad \boldsymbol{\nabla}\tilde{R}\boldsymbol{\nabla}\tilde{S}\Delta R - \frac{\tilde{R}}{2}\Delta\tilde{S}\Delta R - \tilde{R}\boldsymbol{\nabla}(\Delta R)\boldsymbol{\nabla} S - \\
&\quad 2\tilde{R}(\sum_{m,n}\partial_m\partial_n R\partial_m\partial_n S) - \tilde{R}\boldsymbol{\nabla} R\boldsymbol{\nabla}(\Delta S) - \\
&\quad \tilde{R}\frac{\Delta R}{2}\Delta S - \tilde{R}\boldsymbol{\nabla} R\boldsymbol{\nabla}(\Delta S) - \tilde{R}\frac{R}{2}\Delta\Delta S
\end{aligned} \tag{31}$$

(where we continue to use $2D$-dimensional notation for $\boldsymbol{\nabla}$ and Δ, and m and n also run from 1 to $2D$). Simplifying using (28) and the symmetry of $\boldsymbol{\nabla} S$ and ΔS yields

$$\begin{aligned}
&\boldsymbol{\nabla} S\Big(\boldsymbol{\nabla} R\Delta\tilde{R} + R\boldsymbol{\nabla}(\Delta\tilde{R}) - \boldsymbol{\nabla}\tilde{R}\Delta R - \tilde{R}\boldsymbol{\nabla}(\Delta R)\Big) + \\
&\quad 2(\boldsymbol{\nabla}\Delta S)(R\boldsymbol{\nabla}\tilde{R} - \tilde{R}\boldsymbol{\nabla} R) + 2\sum_{m,n}\partial_m\partial_n S(R\partial_m\partial_n\tilde{R} - \tilde{r}\partial_m\partial_n R) \quad = \\
&(\boldsymbol{\nabla} S)\boldsymbol{\nabla}(R\Delta\tilde{R} - \tilde{R}\Delta R) + 2(\boldsymbol{\nabla}\Delta S)(R\boldsymbol{\nabla}\tilde{R} - \tilde{R}\boldsymbol{\nabla} R) + \\
&\quad 2\sum_{m,n}\partial_m\partial_n S(R\partial_m\partial_n\tilde{R} - \tilde{r}\partial_m\partial_n R) \quad = \quad 0.
\end{aligned} \tag{32}$$

Thus, again because of (28),

$$(\boldsymbol{\nabla}\Delta S)(R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R)+\sum_{m,n}\partial_m\partial_n S(R\partial_m\partial_n\tilde{R}-\tilde{r}\partial_m\partial_n R)=0, \tag{33}$$

and again taking the time derivative,

$$\begin{aligned}\partial_t\Big((\boldsymbol{\nabla}\Delta S)(R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R)\Big) =& \\ &(\boldsymbol{\nabla}\Delta\partial_t S)(R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R)+\boldsymbol{\nabla}(\Delta S)\partial_t(R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R)+ \\ &\sum_{m,n}(\partial_m\partial_n\partial_t S)(R\partial_m\partial_n\tilde{R}-\tilde{r}\partial_m\partial_n R)+ \\ &\sum_{m,n}(\partial_m\partial_n S)\partial_t(R\partial_m\partial_n\tilde{R}-\tilde{r}\partial_m\partial_n R) \qquad = \quad 0.\end{aligned} \tag{34}$$

In the limit of instantaneously varying V, we can affect (via (7)) the terms containing $\partial_t S$ without changing the others. Indeed, at any time t, we cann add an arbitrary (symmetric) term U to the potential, thus adding to (34) a term of the form

$$-(\boldsymbol{\nabla}\Delta U)(R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R)-\sum_{m,n}\partial_m\partial_n U(R\partial_m\partial_n\tilde{R}-\tilde{r}\partial_m\partial_n R) \tag{35}$$

which has to be zero. If we further choose U of the form

$$U=u^1(\xi_1^1)+\ldots+u^D(\xi_1^D)+u^1(\xi_2^1)+\ldots+u^D(\xi_2^D), \tag{36}$$

all mixed derivatives $\partial_m\partial_n U$ vanish, and the requirement reduces to

$$\sum_n\partial_n^3 U(R\partial_n\tilde{R}-\tilde{R}\partial_n R)+\sum_n\partial_n^2 U(R\partial_n^2\tilde{R}-\tilde{R}\partial_n^2 R)=0, \tag{37}$$

where all $\partial_n^2 U$ and $\partial_n^3 U$ can be varied independently. Thus, in particular,

$$R\boldsymbol{\nabla}\tilde{R}-\tilde{R}\boldsymbol{\nabla}R=0. \tag{38}$$

From this, on $\{R\neq 0\}$, follows again equation (21), and the rest of the proof proceeds as in the case of Nelson's mechanics.

4 Case of n spinless particles

For n particles, one can take over the above arguments largely unmodified, and conclude that for all i,j,

$$S(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_n,t)=S(\mathbf{x}_1,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_n,t)+\gamma_{\{i,j\}}. \tag{39}$$

As before, $\gamma_{\{i,j\}}$ is independent of t, and also of $\mathbf{x}_k$ $(k \neq i,j)$, because $\partial_t S$ and $\boldsymbol{\nabla} S$ would otherwise be ill-defined. As before, for $D \geq 3$ the configuration space without the set where two or more particles coincide is simply connected, and thus the constant $\gamma_{\{i,j\}}$ can take only the two values 0 or π, while for $D = 2$ it is arbitrary. The only additional step that is required is to show that $\gamma_{\{i,j\}}$ is in fact independent of the pair $\{i,j\}$, and, for $D = 2$, to discuss the dependence on the homotopy structure of the exchange paths, which is rather more complicated for n particles than it is for two.

For $D = 3$, the phase difference between S and $\tilde{S}$ is path-independent, and thus $\gamma_{\{i,j\}}$ is uniquely associated with a permutation of the labels i, j. For permutations Π_{ij} we have, however:

$$\Pi_{ij} = \Pi_{jk}\Pi_{ik}\Pi_{jk}, \tag{40}$$

so that

$$\gamma_{\{i,j\}} = \gamma_{\{j,k\}} + \gamma_{\{i,k\}} + \gamma_{\{j,k\}} \, (\mathrm{mod}\ 2\pi). \tag{41}$$

Since

$$\gamma_{\{j,k\}} + \gamma_{\{j,k\}} = 0 \, (\mathrm{mod}\ 2\pi), \tag{42}$$

we have

$$\gamma_{\{i,j\}} = \gamma_{\{i,k\}} \, (\mathrm{mod}\ 2\pi), \tag{43}$$

and exchange of one particle with any other yields the same phase difference. It follows that exchange of any two particles yields the same phase difference.

Take $D = 2$. For two particles we have said that all exchange paths are homotopic to concatenations of the simple exchange paths along which the two particles do not wind around each other. For n particles, two such paths, even if like-handed, are not generally homotopic to each other. The homotopy class further depends on whether and how many other particles may be enclosed within the path, and on whether and how often any such particles are circled by either of the particles being exchanged. We can, however, redefine simple exchange paths as those along which the particles being exchanged neither wind around each other nor enclose any other particles. One can then easily convince oneself that any exchange path is homotopic to a concatenation of such simple exchange paths. The phase difference along a simple exchange path is $\pm\gamma_{\{i,j\}}$, according to whether the path is left-handed or right-handed. What we wish to establish is that this is independent of the pair $\{i,j\}$. And in fact, we can mimick the above argument based on permutations.

We first exchange particles j and k along a simple exchange path, say a left-handed one; then we exchange i and k along another, say also a left-handed one; finally we exchange j and k again, this time along a *right-handed* simple exchange path. This concatenation is homotopic to a simple path exchanging i and j (a left-handed one). In terms of the phase differences along the paths, we obtain:

$$\gamma_{\{i,j\}} = \gamma_{\{j,k\}} + \gamma_{\{i,k\}} - \gamma_{\{j,k\}} \, (\mathrm{mod}\ 2\pi). \tag{44}$$

That is, even without (42), we obtain the result (43), now for the phase differences along simple exchange paths (not enclosing any other particles) in two dimensions.

5 Generalisation for spinorial wavefunctions

We shall now treat the case of spinorial wavefunctions (in three dimensions). These obey the Pauli equation:

$$i\hbar\frac{\partial\mathbf{\Psi}}{\partial t}=-\sum_i\frac{\hbar^2}{2m_i}(\boldsymbol{\nabla}_i-iq_i\mathbf{A}(\mathbf{x}_i))^2\mathbf{\Psi}+V\mathbf{\Psi}+\mu\sum_i\mathbf{S}_i\mathbf{B}(\mathbf{x}_i)\mathbf{\Psi},\tag{45}$$

where $\mathbf{S}_i$ is the spin matrix vector acting on particle i, and $\mathbf{B}$ is the (external) magnetic field. Writing spinorial wavefunctions as

$$\mathbf{\Psi}=\sum_{s_1,\ldots,s_n}\psi_{s_1,\ldots,s_n}|s_1>\ldots|s_n>,\tag{46}$$

the condition to be derived is

$$\begin{aligned}\psi_{s_1\ldots s_i\ldots s_j\ldots s_n}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_n,t)=&\\ e^{i\gamma}\psi_{s_1\ldots s_j\ldots s_i\ldots s_n}&(\mathbf{x}_1,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_n,t),\end{aligned}\tag{47}$$

with $\gamma=0$ for all i,j or $\gamma=\pi$ for all i,j (total symmetry or antisymmetry of the spinorial wavefunction). Notice also that for $\mathbf{B}=0$, the single components of the spinorial wavefunction decouple and all obey the same (symmetric) Schrödinger equation (in which $\mathbf{A}$ can be gauged to zero).

We first generalise the two pilot-wave theories to the case of spinorial wavefunctions, following the most common treatment, due to Bell (1987, Chap. 19). Notice first of all that de Broglie's guidance equation (5) can be written

$$\dot{\mathbf{x}}_i=\frac{\hbar}{m_i}\mathrm{Im}\frac{\psi^*\boldsymbol{\nabla}_i\psi}{\psi^*\psi}+\frac{q_i}{m_i}\mathbf{A}(\mathbf{x}_i),\tag{48}$$

and similarly, Nelson's can be written

$$d\mathbf{x}_i=\Big(\frac{\hbar}{m_i}\mathrm{Im}\frac{\psi^*\boldsymbol{\nabla}_i\psi}{\psi^*\psi}+\frac{q_i}{m_i}\mathbf{A}(\mathbf{x}_i)+\frac{\hbar}{m_i}\mathrm{Re}\frac{\psi^*\boldsymbol{\nabla}_i\psi}{\psi^*\psi}\Big)dt+d\boldsymbol{\omega}_i.\tag{49}$$

The generalisations of (48) and (49) to spin are given, respectively, by

$$\dot{\mathbf{x}}_i=\frac{\hbar}{m_i}\mathrm{Im}\frac{\sum_{s_1,\ldots,s_n}\psi^*_{s_1\ldots s_n}\boldsymbol{\nabla}_i\psi_{s_1\ldots s_n}}{\sum_{s_1,\ldots,s_n}\psi^*_{s_1\ldots s_n}\psi_{s_1\ldots s_n}}+\frac{q_i}{m_i}\mathbf{A}(\mathbf{x}_i),\tag{50}$$

and

$$
\begin{aligned}
d\mathbf{x}_i &= \Big(\frac{\hbar}{m_i}\mathrm{Im}\frac{\sum\limits_{s_1,\dots,s_n}\psi^*_{s_1\dots s_n}\boldsymbol{\nabla}_i\psi_{s_1\dots s_n}}{\sum\limits_{s_1,\dots,s_n}\psi^*_{s_1\dots s_n}\psi_{s_1\dots s_n}} + \frac{q_i}{m_i}\mathbf{A}(\mathbf{x}_i) + \\
&\qquad \frac{\hbar}{m_i}\mathrm{Re}\frac{\sum\limits_{s_1,\dots,s_n}\psi^*_{s_1\dots s_n}\boldsymbol{\nabla}_i\psi_{s_1\dots s_n}}{\sum\limits_{s_1,\dots,s_n}\psi^*_{s_1\dots s_n}\psi_{s_1\dots s_n}}\Big)dt + d\boldsymbol{\omega}_i, \qquad (51)
\end{aligned}
$$

which are manifestly independent of the spin basis chosen. (These are the equations needed to analyse the Bell experiments.)

Our methods of Section 3 are *not* directly applicable to derive (47). There is, however, an elegant way of generalising the results. We make use of the treatment of measurement theory sketched in Section 2, according to which after a measurement interaction one can identify a specific component of the wavefunction (9) of the total system as uniquely responsible for the further motion of the particles. A measurement interaction, however, just means an appropriate choice of potentials and magnetic fields over a period of time, which we are free to make.

Specifically, we first confine the particles via symmetric potentials to n boxes with infinite walls, each box containing a single particle. We then perform (ideal) measurements of spin on each of the particles, yielding results, say, $s^0_1, \dots, s^0_n$. The effective wavefunction for the n particles is now

$$
\sum_{\Pi}\psi_{s^0_{\Pi(1)},\dots,s^0_{\Pi(n)}}|s^0_{\Pi(1)}>\dots|s^0_{\Pi(n)}>, \qquad (52)
$$

where the sum ranges over all and only permutations that yield distinct sequences. Since with a fixed experimental arrangement, the different sequences of results depend on different initial positions of the particles, the requirement that the original wavefunction leave all possible trajectories invariant under exchange of any two particles translates into requiring the same *separately* of all non-zero components of the form (52). (If one such component is zero, then it is already totally symmetric or antisymmetric.)

For illustration, take the case of three spin-1/2 particles. The general case is analogous. (We need at least three particles in order to cover the possibility of parastatistics.) The easiest case is when all spin measurements yield the same result, say up. Then the effective wavefunction is

$$
\psi_{+++}|+>|+>|+>. \qquad (53)
$$

If we choose the magnetic field $\mathbf{B}=0$, the system is effectively described by one scalar wavefunction obeying the Schrödinger equation. We can open the boxes and treat it by the methods used for scalar wavefunctions in Section 3.

In general, however, we will have different spin values in different boxes, say spin up in two boxes and spin down in the third. We first pull down the potential barriers between boxes containing particles with the same spin, so that in this case we have one box containing two spin-up particles and one containing one spin-down particle. The effective wavefunction is now given by

$$\psi_{++-}|+>|+>|-> +\psi_{+-+}|+>|->|+> +\psi_{-++}|->|+>|+>, \quad (54)$$

each term corresponding to a different particle being in the spin-down box, and one still needs the guidance equations (50) or (51). Notice, however, that *as long as* the particles are confined to the two boxes, for any possible configuration only one of the terms in (54) will be non-zero, e.g. for any configuration in which particle 1 is in the spin-down box, the only non-zero term is $\psi_{-++}|->|+>|+>$. It follows that, as long as the particles are in the boxes,

$$\begin{aligned} \nabla S_{++-}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \nabla S_{++-}(\mathbf{y},\mathbf{x},\mathbf{z}), && (55)\\ \nabla S_{+-+}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \nabla S_{-++}(\mathbf{y},\mathbf{x},\mathbf{z}), && (56)\end{aligned}$$

etc. (where S_{++-} is the phase of ψ_{++-}, etc.). Taking $\mathbf{B}=0$, not only do $\psi_{++-}(\mathbf{x},\mathbf{y},\mathbf{z})$ and $\psi_{++-}(\mathbf{y},\mathbf{x},\mathbf{z})$ obey the same (symmetric) Schrödinger equation, but also $\psi_{+-+}(\mathbf{x},\mathbf{y},\mathbf{z})$ and $\psi_{-++}(\mathbf{y},\mathbf{x},\mathbf{z})$, etc.

For each fixed z, when non-empty, the region $\{R_{++-}(\mathbf{x},\mathbf{y},\mathbf{z})\neq 0\}$ is connected (we exchange the two particles in the spin-up box), so that we can apply our previous methods (for both phase and amplitude) to obtain

$$\psi_{++-}(\mathbf{x},\mathbf{y},\mathbf{z}) = \pm\psi_{++-}(\mathbf{y},\mathbf{x},\mathbf{z}) \quad (57)$$

(and similarly for ψ_{+-+} under exchange of $\mathbf{x}$ and $\mathbf{z}$, and for ψ_{-++} under exchange of $\mathbf{y}$ and $\mathbf{z}$). Instead, when non-empty, the region $\{R_{+-+}(\mathbf{x},\mathbf{y},\mathbf{z})\neq 0\}\cup\{R_{-++}(\mathbf{x},\mathbf{y},\mathbf{z})\neq 0\}$ has two disconnected components, according to whether particle 1 or particle 2 is in the spin-down box. We obtain only

$$\psi_{+-+}(\mathbf{x},\mathbf{y},\mathbf{z}) = \delta e^{i\gamma}\psi_{-++}(\mathbf{y},\mathbf{x},\mathbf{z}), \quad (58)$$

with γ and δ arbitrary (and similarly in the other cases). One sees that the spinorial wavefunction must have the form

$$\begin{aligned}\Psi(\mathbf{x},\mathbf{y},\mathbf{z}) = {} & \psi_{++-}(\mathbf{x},\mathbf{y},\mathbf{z})|+>|+>|-> + \\ & \delta e^{i\gamma}\psi_{+-+}(\mathbf{x},\mathbf{z},\mathbf{y})|+>|->|+> + \\ & \delta' e^{i\gamma'}\psi_{-++}(\mathbf{z},\mathbf{y},\mathbf{x})|->|+>|+>, \quad (59)\end{aligned}$$

with ψ_{++-} symmetric or antisymmetric in the first two variables.

As a penultimate step, we flip the spin in the spin-down box, set the magnetic field back to 0, and pull down the wall between this box and the spin-up box. (If there are more than two boxes with different values of spin, we repeat this operation several times.) The wavefunction in (59) becomes

$$\Big(\psi_{++-}(\mathbf{x},\mathbf{y},\mathbf{z}) + \delta e^{i\gamma}\psi_{+-+}(\mathbf{x},\mathbf{z},\mathbf{y}) + \delta' e^{i\gamma'}\psi_{-++}(\mathbf{z},\mathbf{y},\mathbf{x})\Big)|+>|+>|+>, \tag{60}$$

i.e. has the form (53), and must therefore be symmetric in all pairs of variables (if ψ_{++-} is symmetric in the first two variables) or antisymmetric in all pairs of variables (if ψ_{++-} is antisymmetric in the first two variables). It follows easily that, respectively,

$$\delta e^{i\gamma} = \delta' e^{i\gamma'} = \pm 1. \tag{61}$$

Thus, we have shown that also an effective wavefunction with different spin terms such as (54) is totally symmetric or antisymmetric.

Finally, one convinces oneself that the effective wavefunctions need to be either all symmetric or all antisymmetric (e.g. by considering non-maximal measurements after which the effective wavefunction is a superposition of any two thus far considered). Thus, the spinorial wavefunction is totally symmetric or totally antisymmetric at the time of the spin measurements, and since the Pauli equation is symmetric, for all times.

This completes our derivation of the symmetry conditions in the spinorial case, and shows that parastatistics is excluded. Unfortunately, our proof yields no complete connection between spin and statistics, since for arbitrary spin the velocity field is invariant both when the wavefunction is symmetric and when it is antisymmetric. Incidentally, there is another model of spin in pilot-wave theories, due to Bohm and Hiley (1993, Chap. 10), where the guidance equation includes an additional velocity term. (This term is added so to have the velocity field obtained from the Pauli equation be the non-relativistc limit of the one obtained from the Dirac equation.) However, this term also transforms invariantly under exchange irrespectively of whether the wavefunction is symmetric or antisymmetric, so that Bohm and Hiley's model can also not be used to derive the spin-statistics connection.

6 Alternative approaches

To conclude, we briefly mention the alternative approaches to identical particles in non-relativistic quantum mechanics, one of which also aims to exclude parastatistics and bears an interesting relation to our approach above. This is the *topological approach* developed by Laidlaw and C. DeWitt (1971) and by Leinaas and Myrheim (1977), with an anticipation by Finkelstein and Rubinstein (1968). This approach is based on the use of a *reduced configuration space* for identical particles, defined by excising the points in which two or more particles coincide, and identifying configurations related by a permutation of the particles. Exclusion of the coincidence points makes the resulting space multiply connected. In three dimensions or more, it is doubly connected, corresponding to the simply connected case above (without identifications). In two dimensions (treated only by Leinaas and Myrheim), the reduced configuration space is infinitely connected.

Laidlaw and DeWitt use the Feynman-path formalism to calculate transition amplitudes. From the topology of the paths, they derive the usual amplitudes. (Notice the different point of application of the topological considerations: Feynman paths versus equal-time paths.) Leinaas and Myrheim define wavefunctions (and a Schrödinger equation) directly on the reduced configuration space, and obtain two-valued and multi-valued wavefunctions in three and two dimensions, respectively (they also have a discussion of the one-dimensional case). Laidlaw and DeWitt treat only the spinless case, while Leinaas and Myrheim give an extension to spinor-valued wavefunctions on the reduced configuration space (explicitly only for two particles).

In both cases, the choice of the reduced configuration space is conceptually the central step. As pointed out already in Brown *et al.* (1999), the pilot-wave approach gives a particularly perspicuous justification for this step. Indeed, the set of coincidences is dynamically inaccessible, whether or not the wavefunction is zero on the set. This can be seen as follows. We set $\xi_1 = \xi_2 = \mathbf{x}$ in (13) above:

$$\boldsymbol{\nabla}_1 S(\xi_1, \xi_2, t) + \mathbf{A}(\xi_1, t)\Big|_{\xi_1=\xi_2=\mathbf{x}} = \boldsymbol{\nabla}_2 S(\xi_1, \xi_2, t) + \mathbf{A}(\xi_2, t)\Big|_{\xi_1=\xi_2=\mathbf{x}}, \tag{62}$$

and see that if two particles coincide at one instant t, they coincide for all times. Thus, if they do not coincide at one instant, they cannot come to coincide at any later time. The configuration space for two identical particles decomposes into an unphysical configuration space for one particle of double mass and charge, and the configuration space with coincidence points removed. As regards the identification of points related by permutations, it would seem that this is also perfectly natural, since the motions are perfectly symmetrical, and indeed de Broglie–Bohm theory can be consistently formulated in a label-free way, as described in Dürr *et al.* (in preparation) (see also Sjöqvist and Carlsen, 1995).

Goldin *et al.* (1980, 1981) work within the framework of second-quantised non-relativistic fields $\psi(\mathbf{x})$, or more precisely, the Lie algebra formed by the mass density and momentum density operators:

$$\begin{aligned} \rho(\mathbf{x}) &:= m\psi^*(\mathbf{x})\psi(\mathbf{x}), \\ \mathbf{J}(\mathbf{x}) &:= \tfrac{\hbar}{2i}(\psi^*(\mathbf{x})\boldsymbol{\nabla}\psi(\mathbf{x}) - \boldsymbol{\nabla}\psi^*(\mathbf{x})\psi(\mathbf{x})) \end{aligned} \tag{63}$$

(suitably averaged with Schwartz space functions). The representations of this *current algebra* yield fields satisfying the canonical commutation and anticommutation relations, as well as anyonic commutation relations (see also Goldin and Sharp, 1983, where the relation to Leinaas and Myrheim's work is discussed in more detail). Goldin (1987) points out that also this approach yields a natural justification for excising the coincidence points. As opposed to the justification provided by de Broglie–Bohm theory, however, coincidence points are excised also in the case of distinguishable particles.

The current algebra approach provides a rigorous framework for anyons, but it explicitly allows for parastatistics, unless the wavefunction is scalar-valued (Goldin, 1987). Goldin *et al.* (1985) therefore conjecture that the topological approach may be able to rigorously exclude parastatistics only in the case of scalar wavefunctions (e.g. suitable extensions of Feynman-path methods would also allow for parastatistics). If this is correct, the approach based on pilot-wave theories, as in the present paper and in Dürr *et al.* (in preparation) would provide the first rigorous proof that parastatistics are excluded also in the spinorial case.

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