THE (?) CORRESPONDENCE PRINCIPLE

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1. INTRODUCTION

Most physicists and physics students understand the correspondence principle as the requirement that the results of quantum physics go over to those of classical physics in some appropriate limit, say of high quantum numbers, or of large numbers of quanta, or as Planck's constant h goes to zero. This is what one typically finds in textbooks. One can also find such formulations by participants of the quantum revolution. For example in his 1933 lectures published under the title *Moderne Physik*, Max Born cites Bohr's correspondence principle as the *Grundidee* of the *provisorische Atommechanik* prior to the emergence of matrix mechanics. According to Born, it demands that "die neue Mechanik für den Grenzfall großer Massen oder großer Bahndimensionen in die klassische Mechanik übergeht." [20, p. 67] In 1925, at the very birth of the new quantum mechanics, in "The Fundamental Equations of Quantum Mechanics," P. A. M Dirac characterizes the correspondence principle as the requirement that "the classical theory gives the right results in the limiting case when the action per cycle of the system is large compared to Planck's constant h, and in certain other special cases."[25, p. 642] According to these formulations, the correspondence principle functions as a *constraint* on theorizing.

This leaves us with a kind of puzzle. Bohr's two fundamental postulates, the principle of stationary states and the Bohr frequency condition,

$$(1) \qquad E' - E'' = h\nu,$$

when applied to atomic systems yield only predictions about the frequencies of radiation emitted or absorbed in transitions between stationary states. They do not yield, on the one hand, any predictions about the state of polarization of emitted radiation. On the other, they do not make any predictions about the emission and absorption probabilities. This means they say nothing about which transitions are allowed or forbidden and nothing about the intensity of the radiation emitted from an ensemble of systems. The above formulation of the correspondence principle as a constraint on quantum theorizing provides for empirical content at best constraining the determination of stationary states in the limit (of slow frequencies or large quantum numbers). It cannot milk out of the two fundamental postulates any information about transition probabilities or polarization. Yet in the period prior to matrix mechanics the correspondence principle was widely presumed to do just that. As a "rational" generalization of classical electrodynamics into the quantum realm, it served to give quantum physics predictive muscle, something that can accrue to theory only by the annexation of substantive physical hypotheses. The puzzle then is that if the

correspondence principle was introduced to extend the quantum theory to make predictions not just about frequencies, but about polarization and transition probabilities as well, how did it come to pass that the correspondence principle is now most commonly taken to constitute a constraint on theory construction?

This also means that, in speaking of "the" correspondence principle, we must be careful about what is to be understood. In what follows I first develop the original doctrine, or strategy associated with Bohr's correspondence principle before attempting to identify a definitive formulation of the principle. In then looking for such a formulation, we'll find that Bohr was not uniformly consistent in stating the principle. Nonetheless, formulations can be found that include the major predictive components of that doctrine and thus cast the principle as making substantive assertions. I'll then turn to a brief survey of various "extensions" of the correspondence principle. Apart from interest in their own right, they will lead us back to the reintroduction of the correspondence principle within the Copenhagen school explicitly as a constraint on quantum theorizing.

2. The Core Doctrine: Transition Probabilities and Polarization

That the correspondence principle in the old quantum theory was intended to extend Bohr's postulates in such a way as to provide predictions about transition probabilities and polarizations is certainly indicated by one of the terser formulations that can be found in the literature. According to Back and Landé, in their 1925 monograph on the Zeeman effect and the multiplet structure of spectral lines:

> The correspondence principle asserts that the rate of occurrence (probability) of a quantum jump, which is manifested by the intensity of the spectral emission line, as well as the polarization of the emitted radiation is determined by the intensity and polarization of the "corresponding" component oscillations [*Einzelschwingungen*] of the ersatz classical radiation process.
> ([1], p. 17)

For this to be helpful, though, one needs to know (a) what are "the 'corresponding' component oscillations of the ersatz classical radiation process," and (b) what is the determining relation.

In order to flesh this out, consider Bohr's address "Über die Serienspektra der Elemente" given to the *Deutsche Physikalische Gesellschaft* in Berlin on April 27, 1920 [5]. Although Bohr is supposed to have first developed the correspondence principle two years earlier in parts one and two of *On the Quantum Theory of Line-Spectra* [2, 3, 4] (hereafter Q.o.L, following Bohr), it is on this occasion that he first used the terminology *Korrespondenzprinzip*. He introduces it in the context of treating the hydrogen atom.

Assume for simplicity the mass of the nucleus to be infinite in comparison to that of the electron. Then, as a Kepler problem due to the Coulomb attraction, the latter revolves classically with angular frequency ω around the former in an ellipse with semi-major axis a with

$$(2) \qquad \omega = \sqrt{\frac{2W^3}{\pi^2 e^4 m}}, \; 2a = \frac{e^2}{W},$$

where e is the charge and m the mass of the electron and W is the work required to remove the electron to infinity. Bohr proceeds to use (what he calls) the Balmer formula

(3)
$$\nu = \frac{K}{n''^2} - \frac{K}{n'^2},$$

where K is the Rydberg constant, as a quantization condition. Multiplying both sides by h yields an instance of the Bohr frequency condition (1), suggesting that the energy of the nth state is

$$E_n = -\frac{Kh}{n^2}.$$

Substituting this for W in (2) yields

(4)
$$\omega_n = \frac{1}{n^3}\sqrt{\frac{2h^3K^3}{\pi^2e^4m}}, \quad 2a_n = \frac{n^2e^2}{hK}$$

for the orbital frequency and semi-major axis, respectively, of the nth stationary state. Now assume $n' > n''$ and $\tau = n' - n''$. Then, holding τ fixed, the ratio $\omega_{n''}/\omega_{n'}$ converges to unity as n' goes to infinity. The Balmer formula (3) can be rewritten

$$\nu = \frac{K}{n''^2} - \frac{K}{n'^2} = \tau K \frac{n' + n''}{(n')^2(n'')^2}.$$

When n' and n'' are large in comparison to their difference τ, we get from this and (4) the approximation

(5)
$$\nu \sim \tau\omega\sqrt{\frac{2\pi^2e^4m}{Kh^3}},$$

where ω is the orbital frequency of one or the other of the two states.[1]

Now e, m, h, and K are all independently measurable physical constants. It so happens that, to within experimental error,

$$K = \frac{2\pi^2e^4m}{h^3}.$$

Thus, the radical on the r.h.s. of (5), as a matter of empirical fact, is unity, so that (5) reduces to

$$\nu \sim \tau\omega.$$

Following Ehrenfest [27] and after him van Vleck [47], call this the *frequency theorem*. Although here it follows from the *ad hoc* use of the Balmer formula as a quantization condition and the experimental identity $K = 2\pi e^4m/h^3$, it can be derived generally from

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[1]Bohr's reasoning here appears to be that if $n', n'' >> \tau$, then n' and n'' can be equated in the term

$$\frac{n' + n''}{(n')^2(n'')^2}$$

and

$$\frac{n + n}{n^2n^2} = \frac{2}{n^3}.$$

the usual action-angle quantization conditions for multiply-periodic systems, as Bohr did in Q.o.L.

The physical significance is this. For any periodic motion of frequency ω (and thus for elliptical motion in particular) the displacement x in any given direction can be represented as a function of time by a Fourier series of the form

$$(6) \qquad\qquad x = \sum X_\tau \cos 2\pi(\tau\omega t + c_\tau),$$

where the sum is taken over all positive integer values of τ. Classically, a charged particle undergoing such motion radiates at each of the overtones $\tau\omega$ of the fundamental frequency ω with an amplitude proportional to the square of X_τ. This is the *ersatz radiation process* referred to by Back and Landé above. The *component oscillations* are the overtones $\tau\omega$. The spectral line emitted quantum-mechanically in the transition from the n'th to the n''th state has the same frequency as the τth overtone of the classical ersatz process and hence that quantum transition *corresponds to* the τth component oscillation of the classical ersatz process. Since I want to distinguish between the various components of the doctrine associated with the correspondence principle, let us call this correspondence the *correspondence mapping*.

Before moving on, it should be stressed, as did Bohr, that, although one arrives at this convergence in frequencies, the *mechanisms* of radiation remain utterly distinct even in the limit of slow oscillations. Classically, a single atom radiates at all the overtone frequencies, while quantum theoretically an atom will emit radiation at a single frequency equal to the τth overtone corresponding to the transition from state n' to state n''. An ensemble of atoms is required in the initial state n' in order to replicate the spectrum of the classical radiation process.

We have now given an answer to (a) above, viz., the question, what are "the 'corresponding' component oscillations of the ersatz classical radiation process." It remains to address (b), viz., how is the transition probability of a quantum jump and the polarization of the emitted radiation determined by the intensity and polarization of the radiation classically emitted by the corresponding component of the classical radiation process.

The frequency theorem leads immediately to an expectation concerning intensities. As Bohr puts it in his 1920 Berlin address:

> If we now inquire into a deeper meaning of the correspondence established, we are naturally led to expect first that the correspondence arises not only in an agreement of the frequencies of spectral lines determined by the two methods, but will remain valid also for their *intensities*; an expectation that is equivalent to the proposition that the relative *probability* of a given transition between two stationary states is connected in an easily stated way with the amplitude of the corresponding harmonic component of the motion. ([5], p. 431. Emphasis in original. See also [8], p. 27.)

Assuming $n', n'' >> \tau$, the easily stated connection is that of identity between probability amplitudes and amplitudes of oscillation. In other words, for one dimension the connection is that, the probability of a the transition from state n' to state n'' is proportional to X_τ^2.

For the three dimensional case, we need to consider the Fourier expansions in the y and z directions with amplitudes Y_τ and Z_τ, respectively. Then the probability of the transition is proportional to $X_\tau^2 + Y_\tau^2 + Z_\tau^2$. Call this the *slow-vibration hypothesis for transition-probabilities*.

Given Y_τ and Z_τ in addition to X_τ, the state of polarization of the classical radiation component of frequency $\tau\omega$ is fully determined. Bohr makes the additional assumption that, in the same regime, the state of polarization of the radiation emitted in the quantum transition from state n' to state n'' is the same as for the corresponding classical radiation. Call this *the slow-vibration hypothesis for polarization*.

The slow-vibration hypotheses seem like eminently rational extensions of the frequency theorem. There is, of course, no question of finding proofs of them, as one has for the frequency theorem. For Bohr's theory, as mentioned earlier, simply makes no predictions about intensities and polarizations of emitted light in the vast regime in which n' and n'' are not both large with respect to their difference τ. So there is no place from which to start. Rather, the entire point of the slow-vibration hypothesis is to provide a point of entry to fill that lacuna. Bohr's proposal is to start with the slow-vibration hypothesis and then to attempt to extrapolate downward into the regime in which the frequency theorem fails, i.e., to take the amplitudes of the Fourier components as indicative in some way of the probabilities of the corresponding quantum transitions and the polarization of the light thereby emitted, even if the the quantum numbers lie well below the slow-vibration range. Call this the *downward extrapolation project*. How one should extrapolate downward is not straightforward. As Sommerfeld puts it in a way that might be construed as criticism:

> This determination of the intensity and polarisation is not, however, fully unambiguous, and this fact in itself characterises it as a process of approximation. In calculating the [Fourier expansion] are we to use as our basis the conditions of the initial orbit or those of the final orbit, or, perhaps, an intermediate orbit that is to be defined by taking the mean of both? No answer is vouchsafed to this by the principle of correspondence. It is easy to see in a general way, indeed, that with the asymptotic condition $[n', n'' >> \tau]$, the coefficients $[X_\tau]$ that are obtained from the initial or final orbit, or from an intermediate orbit must come out appreciably equal. In the case of values of $[\tau]$ that are comparable with $[n'$ or $n'']$, however, the $[X_\tau\text{'s}]$ in general become different for the initial and the final orbit and hence a certain arbitrariness remains in applying the principle of correspondence. ([44], pp. 581–582)

Nonetheless various averaging schemes were proposed and calculations performed, most notably by H. A. Kramers in his 1919 doctoral dissertation under Bohr [34] and later by the American physicist Frank Hoyt [30, 31] while visiting Copenhagen as a National Research Fellow.[2] Despite a measure of empirical success, the problem of non-uniqueness remained a thorn. As Max Born explains in his Winter 1925–1926 MIT lectures:

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[2]van Vleck also proposed a general scheme in his 1924 paper. See [47], p. 334. Hoyt was still working along these lines as late as October of 1925. See [32].

> By this method Kramers has succeeded in representing satisfactorily the
> results of observations in certain cases. It is not satisfactory in principle
> that we should not find in quantum theory, in the form here presented, a
> unique determination of the intensities. This is one of the main reasons
> which led us to formulate our new quantum theory, where the difficulty is
> overcome. ([19], pp. 30–31).

There were, however, distinguished cases, in which the non-uniqueness problem could
be overcome and the correspondence principle gained credibility by making correct predic-
tions. These cases are ones in which the Ritz combination principle fails: certain transitions
simply do not occur. The correspondence principle identifies the reason for the "forbidden"
transition with the absence of the corresponding Fourier components. Suppose, now, in
the regime where $n', n'' >> \tau$, the amplitudes X_τ, Y_τ, Z_τ vanish for some particular value
of τ. Then, according to the slow-vibration transition-probability hypothesis, the inten-
sity of the classical radiation emitted is identically zero, and hence the probability of the
transition from state n' to state n'' is identically zero, and so the transition simply cannot
occur. This result can be unambiguously extrapolated downward if the amplitudes of the
τ-th component are zero not only for the initial state, but also for the final state and all
intermediate states. As Sommerfeld explains,

> In this case we shall also have no scruples in inferring the value zero of the
> radiation. The principle of correspondence then becomes specialised and
> condensed into a *principle of selection*; it forbids the occurrence of such
> spectral lines the corresponding partial vibrations of which do not occur in
> the series expansion. ([44], p. 82)

The simplest example is that of a Planckian oscillator. Since for each state no terms occur in
the Fourier expansion other than the term for $\tau = 1$, the selection rule $\Delta \tau = 1$ for emission
and $\Delta \tau = -1$ for absorption follows immediately, and the frequency ν of the emitted
radiation is just the mechanical frequency ω. In Q.o.L Bohr appeals to the absorption
spectra of diatomic gases as an empirical realization of this case. The appearance in some
gases of a faint line at double the fundamental frequency is explained by assuming that
the vibrations along the axis connecting the two atoms is not strictly harmonic. (See [2]
or [4], p. 16)

More sophisticated cases involved the spectra of hydrogen subject to a constant, uniform
external field. These systems are not simply period but conditionally (or, multiply) period.
The Fourier expansions involve multiple fundamental frequencies, in this case two, call them
ω and σ:[3]

$$(7) \qquad\qquad x = \sum X_{\tau,\kappa} \cos 2\pi[(\tau\omega + \kappa\sigma)t + c_{\tau,\kappa}],$$

and similarly for y and z, where τ and κ are summed over all integral values, positive and
negative. For the case of a magnetic field H (Zeeman effect), the effect is the superposition
of a uniform rotation of frequency $\sigma = eH/4\pi mc$ on the original unperturbed orbit with κ

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[3]Think of ω as governing the original elliptical motion and σ the rate of precession of the perturbed
orbit about the axis of symmetry of the field.

restricted to $-1 \leq \kappa \leq 1$. The selection rule for κ is then $\Delta\kappa = -1, 0, 1$ corresponding to the splitting of the unperturbed line into the classical triplet. Moreover, the correct polarizations are predicted. For $\kappa = 0$, the oscillations are rectilinear parallel to the magnetic field with frequencies $\tau\omega$, while for $\kappa = \pm1$, the oscillations are in a plane perpendicular to the field with frequencies $\tau\omega \pm \sigma$.

For the case of a uniform electric field F (Stark effect), $\sigma = 3eF/8\pi^2 ma\omega$ and κ is less restricted, corresponding to the proliferation of new spectral lines. Again, the correct polarizations are given. When $\tau + \kappa$ is even, Bohr argues, the oscillations are linear parallel to the direction of the field, and when odd, elliptical perpendicular to the field. (See [5] or [8] for details.)

3. The Scope of the Principle

Thus, the correspondence principle, or should we say, the doctrine associated with it, was not without its successes. That doctrine, we found, has a number of distinctly identifiable components:

- the frequency theorem
- the correspondence mapping
- the slow-vibration transition-probability hypothesis
- the slow-vibration polarization hypothesis
- the downward extrapolation project
- the selection principle(s)

Which among these, or rather, which subset of these, constitutes the correspondence principle?

Keep in mind that I judiciously selected the Back and Landé quote so as to motivate coverage of all these components of the doctrine. But what right do we have to take Back and Landé to be any more authoritative than Born or Dirac? Can we not find a concise and definitive formulation in Bohr's writings? Unfortunately, Bohr was not always clear nor necessarily consistent. There are passages in which the correspondence principle appears to be no more than a selection principle. For example,

> . . . the possibility of the occurrence of a transition accompanied by radiation, between two states . . . is conditioned by the presence of certain harmonic components in the expression [for] the electric moment of the atom. . . . We, therefore, call these the "corresponding" harmonic components in the motion, and the substance of the above statement we designate as the "Correspondence Principle" for multiply periodic systems. ([11], p. 22; [13], 479)

This has lead, for instance, A. Bokulich to insist that Bohr intended it to be no more than a selection principle. (See [16], pp. 81–94 and [17]) But such a selection principle is just a special case of zero probability, and elsewhere Bohr connects the correspondence principle explicitly with both probabilities and polarizations. In the 7th Guthrie lecture, given March 24, 1922, he says:

> This law, which has been called the "correspondence principle," states that the occurrence of each transition between two stationary states accompanied by emission of radiation is correlated to one of the constituent harmonic oscillations into which the electric moment of the atom considered as a function of time can be resolved, to the extent that the probability of the occurrence of a transition shall depend on the amplitude of the corresponding harmonic oscillation of the atom, in such a way that in the limit when the quantum-number is large, the intensity of the emitted radiation in unit time in the mean shall be the same as that which would follow from the classical laws of electrodynamics. A similar connection with the classical theory will be exhibited by the polarisation of the emitted radiation. If, for instance, the corresponding harmonic oscillation in all states of the atom is a linear vibration or a circular rotation, the radiation will have the same constitution as that which on the classical theory would be emitted by an electron executing harmonic motion of that type. ([9], p. 284; [13], p. 428)

Moreover, Kramers' contribution to the 1923 *Die Naturwissenschaften* festschrift for Bohr claims to speak on Bohr's behalf as follows:

> The occurrence of a transition between two stationary states accompanied by the emission of radiation is uniquely correlated with one of the harmonic oscillations into which the motion of the particle (or more precisely: the electric moment of the atom) can be decomposed. This correlation requires that the probability for the occurrence of a transition depends on the amplitude of the corresponding harmonic component, and in fact so that in the region of large quantum numbers the intensity per unit time of the emitted radiation will be the same on average as would be expected according to classical electrodynamics. The polarization of the emitted radiation will display a similar analogy with classical electrodynamics. Thus, if the corresponding harmonic oscillator in all states is a linear oscillation parallel or a circular rotation perpendicular to a fixed line, then the radiation will possess the same properties as that which is emitted by an electron which executes an oscillation of this type. ([35], p. 552)

Kramers adds to this a remarkable footnote:

> It seems most appropriate to construe the correspondence principle thus formulated as a postulate (or axiom), in complete analogy with Bohr's two fundamental postulates.

It is pretty clear that the correspondence principle thus formulated is more than just a selection principle. And construed as a postulate that yields substantive predictions beyond what is entailed by Bohr's first two postulates, it does not appear to have the form of a constraint on theory construction.[4]

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[4]According to Bohr and Kramers, the correspondence principle also did work in perturbation theory and the construction of the shell structure for the periodic chart. For the former, see [3] and [5]. For the latter, see [6], [7], and [35].

4. Some "Extensions" of the Correspondence Principle

The correspondence principle thus understood provides a guide to the determination of transition probabilities for spontaneous emission and the polarization of the emitted radiation. But what about transitions in the opposite direction that do not involve radiation, such as inelastic collisions in the bombardment of atoms with electrons, as in the Frank-Hertz experiment?[5] How is one to estimate the transition probabilities?

For such processes, Pauli introduced, in his doctoral dissertation under Sommerfeld [41], what he called a "mechanical" correspondence principle, which he attempted to craft in analogy with Bohr's electrodynamical principle. There Pauli starts with the observation that, just as classical electrodynamics breaks down, so should classical mechanics applied to inelastic collision processes. For classically, the amount of energy ΔE that an electron can give up to its target system varies continuously, while quantum mechanically this is impossible, since the target system can exist only in a discrete manifold of stationary states and consequently ΔE can take on only discrete values. In the limit of large quantum numbers, however, the energy states converge toward a continuum, and thus one should expect the laws of classical mechanics to hold in the limit. This can be taken to be the analogue of the frequency theorem, but it is something assumed, not proven. Nonetheless, it is like the frequency theorem it that the agreement holds only for ensembles, and not for individual transitions. For the analogue of transition probabilities for high quantum numbers Pauli proposes that one take the averages of the classical outcomes when one fixes the initial energy and varies the other initial conditions (direction of the initial velocity, distance of the trajectory asymptote from the center of gravity of the atom, phase of the motion of the electron and of the atom bombarded). These are to be regarded as uncontrollable. For smaller quantum numbers, one "extrapolates" downward: classical mechanics *"must give an (at least approximate) measure for the relative frequency of the various possible quantum energetic inelastic collisions (with a given initial energy for the colliding electron)."* ([41], p. 187. Emphasis in original.) If a particular collision process is classically impossible, then one has a *selection rule* forbidding that transition.

Pauli realizes that given the current state of the art this is predictively pretty weak. Nonetheless he holds out hope that eventually some of the calculational difficulties can be overcome. At least one gets some definite consequences by considering that the law of reversibility must continue to hold. Thus, for every quantum mechanical process, the time-inverse process must also be possible. This leads to so-called "collisions of the second kind" postulated by Klein and Rosseland [33] in Copenhagen on the basis of considerations concerning thermal equilibrium between atoms and free electrons. For example, an electron should be able to experience an increase in velocity when striking an atom in an excited state which decays into a lower energy state.

The existence of such inelastic collisions was confirmed [24] but P. Tartakowsky subsequently took Pauli to task for the manner of his formulation of the "mechanical" correspondence principle [45]. According to Tartakowsky, Pauli's point of departure was the

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[5]Also, what about electron capture under bombardment with the the emission of X-rays? For Kramers' attempt to calculate transition probabilities in this case, see [36].

requirement that the results of quantum theory and classical mechanics must coincide in the case of large quantum numbers. That, however, was not the complaint. It was rather that Pauli did not establish a *correspondence*, in Bohr's sense of the word, which, for Bohr, was specifically between the components of the Fourier expansion of the motion in the stationary state and the radiation emitted in quantum jumps. In other words, Pauli had not established an analogue for what I earlier called the *correspondence mapping*. So Tartakowsky took it upon himself to establish an analogous correspondence relation, in this case, between the Fourier expansion for the motion and the energy given up by one system to another in the absence of emission. Again, the self-described point of departure is the requirement that the results of quantum theory and classical mechanics agree in the realm of high quantum numbers. Tartakowsky then takes up the problem, not of electron scattering, but of a radiationless energy exchange between two quantum systems A and B that does not result in the production of additional kinetic energy, in other words for which we have the energy exchange relation

$$E_A - E'_A = -(E_B - E'_B),$$

where the primed quantities refer to final states. He makes the connection with the Fourier expansions by introducing what he calls a hidden coupling mechanism [*verborgenen Koppelungsmechanismus*], the details of which, however interesting, we do not have time for. The important point here is that, although Tartatowsky begins with the constraint that the results of quantum theory go over to those of classical in the limit of large quantum numbers, he does not take this to be the principle itself, but instead only a preliminary to the formulation of a "mechanical" correspondence theorem.

Contrast this with the tenor of the first part of Van Vleck's two part 1924 paper, [47] which bears the subtitle "Some Extensions of the Correspondence Principle." There he makes it clear that, in the case of emission, whereas the asymptotic connection between quantum and classical physics for frequencies is a mathematical consequence of the quantum conditions,[6] the existence of an analogous relation for intensities is a *hypothesis*. This is what I referred to above as the transition-probability hypothesis for slow vibrations. Van Vleck calls it the correspondence principle for emission and formulates it "analytically" by explicitly calculating the Einstein spontaneous emission probability (A-coefficient) in the classical limit as a function of the amplitudes and frequencies of the Fourier expression for the motion of the particle. From this he derives a correspondence principle for absorption, viz., that in the limit of high quantum numbers the energy extracted from a non-zero radiation field as determined by the Einstein B-coefficients for absorption and induced emission agrees with the classical prediction for absorption. Also demonstrated is a second extension of the correspondence principle to a correspondence principle for orbital distortions: in the limit of large quantum numbers the orbital distortion resulting from a transition between nearby stationary states agrees with that which would follow as a result of classical radiation involving only the frequency component associated with the transition.

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[6]Including, of course, the Bohr frequency condition.

You can see from this that in extending "the" correspondence principle, van Vleck focuses on the behavior of quantum systems in the limit of large quantum numbers. He does consider briefly the downward extrapolation project, but not under the rubric of a correspondence principle. Selection rules do not come under discussion.[7] And there is no overt construction of a correspondence mapping. This is not to say these are beyond his ken. For he spells these out explicitly in his 1926 monograph length report for the National Research Council [48].[8] Rather, they do not serve the development towards the principal result, which is to show that Kramers' generalization of Ladenburg's quantum dispersion formula goes over into the classical formula in the limit of large quantum numbers "not just when the quantized system is a linear oscillator, but also when it is the most general type of non-degenerate multiply periodic orbit." (p. 345) In the 1926 report, he calls this *the correspondence principle for dispersion*.

5. The Correspondence Principle as Constraint

That the formula goes over to the classical formula in the limit is claimed by Kramers in the letter to *Nature*, written in March of 1924, in which he first introduced it.

> It is, however, possible to establish a very simple expression . . . , which fulfills the condition, claimed by the correspondence principle, that, in the region where successive stationary states of an atom differ only comparatively little from each other, the interaction between the atom and the field of radiation tends to coincide with the interaction to be expected on the classical theory of electrons. ([37], p. 673)

But note that here we do not have the framing of the proposition that the quantum formula goes over to the classical in the region of high quantum numbers as an *extension* of the correspondence principle. Rather, that it do so, and presumably any other law about the interaction of radiation and matter, is taken to be a *requirement of the original correspondence principle*. Here we have the implementation from within the Copenhagen school of a new significance for "the" correspondence principle. Whereas just a year earlier Kramers had characterized the correspondence principle as something that could be viewed as a fundamental postulate alongside the the frequency condition and the postulate of stationary states, it is now taken explicitly to function in a different capacity, viz., as a constraint on further theory construction. To my knowledge, this is the first appearance in the literature of "the" correspondence principle as explicitly imposing a constraint.

Its use as such caught on immediately, especially in the literature on dispersion. The exchange of letters in *Nature* between Gregory Breit and Kramers that August is premised on the use of the correspondence principle in this capacity [23, 38]. Born, in his "Über

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[7]He uses the selection rule for the harmonic oscillator as a consequence of "the correspondence principle" but does not pause to consider which, among those in his catalogue of correspondence principles, it comes from.

[8]There he continues with the strategy of cataloguing correspondence principles. The reader should not find it remarkable if some of these map onto components of the doctrine associated with the correspondence principle outlined above.

Quantenmechanik" of 1924, comments concerning his quantum formula for perturbation energy that the first of its two most important properties is that "it goes over for large n_k (large in relation to τ_k) into the corresponding classical formula, thereby satisfying the correspondence principle." ([18], p. 389; [46], p. 192) And again, "the [Kramers dispersion] formula in the limit of large quantum numbers (n_k large compared to τ_k) goes over into the corresponding classical formula, as required by the correspondence principle." ([18], p. 390; [46], p. 193) Not surprisingly in their joint paper on dispersion of 1925 Kramers and Heisenberg write, "In particular the principle requires that in the region of high quantum numbers the actual properties of the atom can be described asymptotically with the help of the classical laws of electrodynamics." ([39], p. 684; [46], p. 226) Even Bohr, in a paper dated November 1, 1924 on a different topic — the polarization of fluorescent light — follows Kramer's lead. Despite the absence in degenerate systems of a direct connection between atomic motion and polarization, which is to be taken as analogous to the lack of a direct relation between atomic motion and spectral frequencies, "the asymptotic agreement, demanded by the correspondence principle, of the consequences of the classical theory and the quantum theory in the limit, where neighboring stationary states deviate relatively little from one another, is kept." ([12], p. 1117)

6. Conclusion

I do not mean to suggest that following Kramers' letter to Nature, which appeared in May 1924, all subsequent uses of the term "correspondence principle" suddenly fell in line with this apparently new usage. One has only to consider the quote from Back and Landé's 1925 monograph above. Nor does there appear to be any hint of a sensed tension between the use of the principle as constraint vs. postulate. Earlier in the letter in which Kramers first invokes the principle as constraint, he cites it in its traditional capacity:

> On Bohr's principle of correspondence, the possibility for such transitions is considered as being directly connected with the periodicity properties of the motion of the atom, in such a way that every possible transition between two stationary states is conjugated with a certain harmonic oscillating component in the motion. ([37], p. 673; [46], p. 178)

W. Kuhn in early 1925 writes concerning his sum rule:

> In the region of high quantum numbers, the above theorem on the p-summation can be directly understood as a requirement of the correspondence principle. In the region of low quantum numbers, in case the initial assumptions made about dispersion should prove correct, our rule represents a strengthening [Verschärfung][9] of the correspondence principle .... ([40], p. 410 [46], p. 255.)

Of course, it takes only a minor adjustment in viewpoint in order to regard, e.g., the slow-vibration transition-probability hypothesis not as "something we are lead to expect" as a

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[9]Heisenberg in his 1925 paper on polarization and fluorescence [28] also proposes a *Verschärfung* of the correspondence principle.

matter of inductive generalization from the frequency theorem, but rather as a constraint to be imposed on any theoretical attempt to calculate transition probabilities; or in order to see the frequency theorem as proof that the Bohr frequency condition satisfies the constraint on predicted frequencies. Thus, the "claim" of the correspondence principle when the theory is extended to other aspects of the interaction between radiation and matter.

It is important, though, not to lose sight of the downward extrapolation project, applied across the board, and thus the introduction of empirical content beyond mere constraint, as part of what "the" correspondence principle involved. We have seen this in averaging schemes, selection principles, and polarization predictions. There was also a tradition dating back to Sommerfeld [42] of exploiting a classical-quantum correlation in the guise of replacing differential quotients with quotients of differences, a procedure referred to by Born and Jordan as a *korrespondenzmäßige Umformung*.[10] ([22], p. 870; [46], p. 290) Born and Jordan also refer to the application of an ingenious correspondence consideration [*geistreiche Korrespondenzbetrachtung*] ([22], p. 859; [46], p. 278) in Heisenberg's derivation of the law of multiplication in his seminal *Umdeutung* paper [29]. Indeed, matrix mechanics can be claimed as the ultimate fulfillment of the downward project. As Born, Heisenberg and Jordan put it in their famous *Dreimännerarbeit*:

> This similarity of the new theory with the classical theory also precludes any question of an independent correspondence principle in addition to the theory; rather the theory itself can be regarded as an exact formulation of Bohr's correspondence ideas [*Korrespondenzgedanken*]. ([21], p. 558; [46], p. 322)

Without the need for any principle *outside* the theory, i.e., any downward extrapolations, what is left of "the" principle is the shell of the constraint.

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[10]Duncan and Janssen [26] have emphasized this as part of the lore pertaining to the correspondence principle.

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