A Topos Perspective on the Kochen-Specker Theorem:
IV. Interval Valuations

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Abstract

We extend the topos-theoretic treatment given in previous papers [2, 3, 4] of assigning values to quantities in quantum theory. In those papers, the main idea was to assign a sieve as a partial and contextual truth value to a proposition that the value of a quantity lies in a certain set $\Delta \subseteq {\rm I\!R}$. Here we relate such sieve-valued valuations to valuations that assign to quantities subsets, rather than single elements, of their spectra (we call these 'interval' valuations). There are two main results. First, there is a natural correspondence between these two kinds of valuation, which uses the notion of a state's support for a quantity (Section 3). Second, if one starts with a more general notion of interval valuation, one sees that our interval valuations based on the notion of support (and correspondingly, our sieve-valued valuations) are a simple way to secure certain natural properties of valuations, such as monotonicity (Section 4).

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1   Introduction

In three previous papers [2, 3, 4] we have developed a topos-theoretic perspective on the assignment of values to quantities in quantum theory. In particular, it was shown that the Kochen-Specker theorem [5] (which states the impossibility of assigning to each bounded self-adjoint operator on a Hilbert space of dimension greater than 2, a real number such that functional relations are preserved) is equivalent to the non-existence of any global elements of a certain presheaf Σ, called the 'spectral presheaf'. This presheaf is defined in closely analogous ways on the category $\mathcal{O}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ (cf. [2, 3]), and on the category $\mathcal{V}$ of commutative von Neumann subalgebras of the algebra of bounded operators on $\mathcal{H}$ (cf. [4]).[3]

A key result of [2, 3, 4] is that, notwithstanding the Kochen-Specker theorem, it *is* possible to define 'generalised valuations' on all quantities, in which any proposition "$A \in \Delta$" (read as saying that the value of the physical quantity A lies in the Borel set of real numbers Δ) is assigned, in effect, a set of quantities that are coarse-grainings (functions) of A. To be precise, such a proposition is assigned as a truth value a certain set of morphisms in the category $\mathcal{O}$ (or $\mathcal{V}$), this set being required to have the structure of a *sieve*. These generalised valuations can be motivated from various different perspectives (cf. also [1]). In particular, they obey a condition analogous to the *FUNC* condition of the Kochen-Specker theorem, which states that assigned values preserve functional relations between operators, and certain other natural conditions too. Furthermore, each (pure or mixed) quantum state defines such a valuation. In Section 2 we will briefly recall the details of these proposals and results.

In this paper, we shall extend this treatment in two main ways (Sections 3 and 4 respectively). Both involve the relation between sieve-valued valuations and valuations that assign to a quantity A, not an individual member of its spectrum, but rather some subset of it; (which we call 'interval valuations'). Though this idea seems at first sight very different from our generalised valuations—that assign sieves to propositions "$A \in \Delta$"—the two types of valuations turn out to be closely related. In fact, there is a natural

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[3]There is also a closely analogous presheaf—the dual presheaf $\mathbf{D}$—that is defined on the category $\mathcal{W}$ of Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ of projectors on $\mathcal{H}$; and the Kochen-Specker theorem is also equivalent to the non-existence of any global elements of $\mathbf{D}$ (cf. [2, 3]). But we shall not discuss $\mathcal{W}$ further in this paper.

correspondence between them that uses the notion of the *support* of a state for a quantity (Section 3). This correspondence is best expressed for the case of $\mathcal{V}$ than for $\mathcal{O}$ since, by using von Neumann algebras as the base category, various measure-theoretic technicalities can be immediately dealt with. However, we shall also discuss the case of $\mathcal{O}$, as it is heuristically valuable.

In Section 4, we describe how if one starts with a yet more general notion of an interval valuation (*i.e.*, one that does not appeal to the notion of support), one sees that our interval valuations based on the notion of support (and correspondingly, our sieve-valued valuations) are a simple way to secure certain natural properties of valuations, such as monotonicity.

2 Review of Our Framework

2.1 The categories $\mathcal{O}$ and $\mathcal{V}$

We will first summarise the definitions given in the previous papers [2, 3, 4], of the categories $\mathcal{O}$ and $\mathcal{V}$ that are defined in terms of the operators on a Hilbert space, and over which various presheaves may be usefully constructed.

The objects of the category $\mathcal{O}$ are defined to be the bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ of some quantum system. A morphism $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ is defined to exist if $\hat{B} = f(\hat{A})$ for some Borel function f. On the other hand, the objects V of the category $\mathcal{V}$ are defined to be the commutative von Neumann subalgebras of the algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The morphisms in $\mathcal{V}$ are the subset inclusions—so if $V_2 \subseteq V_1$, we have a morphism $i_{V_2 V_1} : V_2 \to V_1$. Thus the objects in the category $\mathcal{V}$ form a poset.

The category $\mathcal{V}$ gives the most satisfactory description of the ordering structure of operators. Some reasons for this were discussed in Section 2.1 of [4]. In particular, each von Neumann algebra contains the spectral projectors of all its self-adjoint members; so in a sense $\mathcal{V}$ subsumes both $\mathcal{O}$ and $\mathcal{W}$ (the category of Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$). More important in this paper is the fact that in $\mathcal{V}$ issues about measure theory and spectral theory are easier to treat than they are in $\mathcal{O}$; though we will for heuristic reasons keep $\mathcal{O}$ in play.

2.2 The spectral presheaf on $\mathcal{O}$ and $\mathcal{V}$

The spectral presheaf $\boldsymbol{\Sigma}$ over $\mathcal{O}$ was introduced in [2]. It assigns to each object $\hat{A}$ in $\mathcal{O}$ its spectrum $\sigma(\hat{A})$ as a self-adjoint operator: thus $\boldsymbol{\Sigma}(\hat{A}) := \sigma(\hat{A}) \subseteq \mathbb{R}$. And it assigns to each morphism $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ (so that $\hat{B} = f(\hat{A})$), the corresponding map from $\sigma(\hat{A})$ to $\sigma(\hat{B})$: $\boldsymbol{\Sigma}(f_{\mathcal{O}}) : \sigma(\hat{A}) \to \sigma(\hat{B})$, with $\lambda \in \sigma(\hat{A}) \mapsto f(\lambda) \in \sigma(\hat{B})$.

We now define the corresponding presheaf over $\mathcal{V}$. We recall (see for example, [6]) that the spectrum $\sigma(V)$ of a commutative von Neumann algebra V is the set of all multiplicative linear functionals $\kappa : V \to \mathbb{C}$; these are also the pure states of V. Such a functional assigns a complex number $\kappa(\hat{A})$ to each operator $\hat{A} \in V$, such that $\kappa(\hat{A})\kappa(\hat{B}) = \kappa(\hat{A}\hat{B})$. If $\hat{A}$ is self-adjoint then $\kappa(\hat{A})$ is real and belongs to the spectrum $\sigma(\hat{A})$ of $\hat{A}$.

Furthermore, $\sigma(V)$ is a compact Hausdorff space when it is equipped with the weak-$*$ topology, which is defined to be the weakest topology such that, for all $\hat{A} \in V$, the map $\tilde{A} : \sigma(V) \to \mathbb{C}$ defined by

$$\tilde{A}(\kappa) := \kappa(\hat{A}), \tag{2.1}$$

is continuous. The function $\tilde{A}$ defined in Eq. (2.1) is known as the *Gelfand transform* of $\hat{A}$, and the spectral theorem for commutative von Neumann algebras asserts that the map $\hat{A} \mapsto \tilde{A}$ is an isomorphism of V with the algebra $C(\sigma(V))$ of complex-valued, continuous functions on $\sigma(V)$. The spectral presheaf is then defined as follows.

Definition 2.1 *The* spectral presheaf *over* $\mathcal{V}$ *is the contravariant functor* $\boldsymbol{\Sigma} : \mathcal{V} \to \text{Set}$ *defined as follows:*

- *On objects: $\boldsymbol{\Sigma}(V) := \sigma(V)$, where $\sigma(V)$ is the spectrum of the commutative von Neumann algebra V, i.e. the set of all multiplicative linear functionals $\kappa : V \to \mathbb{C}$.*

- *On morphisms: If $i_{V_2 V_1} : V_2 \to V_1$, so that $V_2 \subseteq V_1$, then $\boldsymbol{\Sigma}(i_{V_2 V_1}) : \sigma(V_1) \to \sigma(V_2)$ is defined by $\boldsymbol{\Sigma}(i_{V_2 V_1})(\kappa) := \kappa|_{V_2}$, where $\kappa|_{V_2}$ denotes the restriction of the functional $\kappa : V_1 \to \mathbb{C}$ to the subalgebra V_2 of V_1.*

When restricted to the self-adjoint elements of V, a multiplicative linear functional κ satisfies all the conditions of a *valuation*, namely:

1. the (real) value $\kappa(\hat{A})$ of $\hat{A}$ must belong to the spectrum of $\hat{A}$;

2. the functional composition principle (*FUNC*)

$$\kappa(\hat{B}) = f(\kappa(\hat{A})) \tag{2.2}$$

holds for any self-adjoint operators $\hat{A}, \hat{B} \in V$ such that $\hat{B} = f(\hat{A})$.

It follows that the Kochen-Specker theorem can be expressed as the statement that (for $\dim \mathcal{H} > 2$) the presheaf $\boldsymbol{\Sigma}$ over $\mathcal{V}$ has no global elements. Indeed, a global element of $\boldsymbol{\Sigma}$ over $\mathcal{V}$ would assign a multiplicative linear functional $\kappa : V \to \mathbb{C}$ to each commutative von Neumann algebra V in $\mathcal{V}$ in such a way that these functionals 'match up' as they are mapped down the presheaf: *i.e.*, the functional κ on V would be obtained as the restriction to V of the functional $\kappa_1 : V_1 \to \mathbb{C}$ for any $V_1 \supseteq V$. So such a global element would yield a global valuation, obeying *FUNC* Eq. (2.2), on all the self-adjoint operators: a valuation that is forbidden by the Kochen-Specker theorem.

2.3 The coarse-graining presheaf on $\mathcal{O}$ and $\mathcal{V}$

Our chief concern is with *generalised* valuations, which are *not* excluded by the Kochen-Specker theorem. These are given by first introducing another presheaf—the *coarse-graining*—presheaf, which gives us a structured collection of propositions about the values of quantities. We summarise the ideas behind the coarse-graining presheaf in this subsection and the next. Then we use the topos-theoretic idea of the subobject classifier to assign sieves as partial, and contextual, truth values to these propositions (Section 2.5).

We begin by representing the proposition "$A \in \Delta$"—that the value of the physical quantity A lies in the Borel set $\Delta \subseteq \sigma(\hat{A}) \subset \mathbb{R}$—by the corresponding spectral projector for $\hat{A}$, $\hat{E}[A \in \Delta]$ (as we will see below, and as discussed in detail in [4], Section 3, this last statement needs to be qualified as regards $\mathcal{V}$). The coarse-graining presheaf is then defined so as to reflect the behaviour of these propositions as they are mapped between the different stages of the base category. Specifically, the coarse-graining presheaf over $\mathcal{O}$ is defined ([2], Defn. 4.3) as the following contravariant functor $\mathbf{G} : \mathcal{O} \to \text{Set}$:

- *On objects in $\mathcal{O}$:* $\mathbf{G}(\hat{A}) := W_A$, where W_A is the spectral algebra of $\hat{A}$ (*i.e.*, the set of all spectral projectors for $\hat{A}$);

- *On morphisms in $\mathcal{O}$:* If $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ (*i.e.*, $\hat{B} = f(\hat{A})$), then $\mathbf{G}(f_{\mathcal{O}}) : W_A \to W_B$ is defined as

$$\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)]. \tag{2.3}$$

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Note that the action of this presheaf coarsens propositions (and their associated projectors) in the sense that, in the partial-ordering of the lattice of projectors, $\hat{E}[f(A) \in f(\Delta)] \geq \hat{E}[A \in \Delta]$, and where the strict inequality arises when f is not injective.

One subtlety is that for Δ a Borel subset of $\sigma(\hat{A})$, $f(\Delta)$ need not be Borel. This is resolved in [2], theorem (4.1), by using the fact that if $\hat{A}$ has a purely discrete spectrum (so that, in particular, $f(\Delta)$ *is* Borel) then

$$\hat{E}[f(A) \in f(\Delta)] = \inf\{\hat{Q} \in W_{f(A)} \subseteq W_A \mid \hat{E}[A \in \Delta] \leq \hat{Q}\} \qquad (2.4)$$

where the infimum of projectors is taken in the (complete) lattice structure of all projectors on $\mathcal{H}$. This motivated the use in [2] of Eq. (2.4) to *define* the coarse-graining operation for a general self-adjoint operator $\hat{A}$: *i.e.*, the projection operator denoted by $\hat{E}[f(A) \in f(\Delta)]$ is *defined* using the right hand side of Eq. (2.4).

This infimum construction is used again in our definition of $\mathbf{G}$ over $\mathcal{V}$. Specifically, we define:

Definition 2.2 *The* coarse-graining presheaf *over $\mathcal{V}$ is the contravariant functor $\mathbf{G} : \mathcal{V} \to \text{Set}$ defined as follows:*

- *On objects: $\mathbf{G}(V)$ is the lattice $\mathcal{L}(V)$ of projection operators in the commutative von Neumann algebra V.*

- *On morphisms: if $i_{V_2 V_1} : V_2 \to V_1$ then $\mathbf{G}(i_{V_2 V_1}) : \mathcal{L}(V_1) \to \mathcal{L}(V_2)$ is the coarse-graining operation defined on $\hat{P} \in \mathcal{L}(V_1)$ by*

$$\mathbf{G}(i_{V_2 V_1})(\hat{P}) := \inf\{\hat{Q} \in \mathcal{L}(V_2) \mid \hat{P} \leq i_{V_2 V_1}(\hat{Q})\} \qquad (2.5)$$

where the infimum exists because $\mathcal{L}(V_2)$ is complete.

The spectral and coarse-graining presheaves will play a central role in the subsequent discussion. First, however, we recall that the interpretation of the propositions "$A \in \Delta$" is more subtle for the base category $\mathcal{V}$ than for $\mathcal{O}$ (as discussed in [2, 3]). For we interpret a projector $\hat{P} \in \mathcal{L}(V)$ as a proposition about the entire stage V_1. Formally, we can make this precise in terms of the spectrum of the *algebra V*. That is to say, we note that:

- Any projector $\hat{P} \in \mathcal{L}(V)$ corresponds not only to a subset of the spectrum of individual operators $\hat{A} \in V$ (where $\hat{P} \in W_A$ so $\hat{P} = \hat{E}[A \in \Delta]$

for some $\Delta \subset \sigma(\hat{A})$, but also to a subset of the spectrum of the whole algebra V, namely, those multiplicative linear functionals $\kappa : V \to \mathbb{C}$ such that $\kappa(\hat{P}) = 1$. It will be convenient in Section 2.4 *et seq.* to have a notation for this subset, so we define $V(\hat{P}) := \{\kappa \in \sigma(V) : V \to \mathbb{C} \mid \kappa(\hat{P}) = 1\}$.

- Coarse-graining respects this interpretation in the sense that if we interpret $\hat{P} \in \mathcal{L}(V_1)$ as a proposition about the spectrum of the algebra V_1, then the coarse-graining of $\hat{P}$ to some $V_2 \subset V_1$, given by $\inf\{\hat{Q} \in \mathcal{L}(V_2) \mid \hat{P} \leq i_{V_2 V_1}(\hat{Q})\}$ is a member of $\mathcal{L}(V_2)$, and so can be interpreted as a proposition about the spectrum of the algebra V_2.

This treatment of propositions as concerning the spectra of commutative von Neumann algebras, rather than the spectra of individual operators, amounts to the semantic identification of all propositions in the algebra corresponding to the same mathematical projector. Thus when we speak of a proposition "$A \in \Delta$" at some stage V, with $\hat{A} \in V$, we really mean the corresponding proposition about the spectrum of the whole algebra V defined using the projector $\hat{E}[A \in \Delta]$. In terms of operators, the proposition "$A \in \Delta$" is *augmented*: it can be thought of as the family of propositions "$B \in \Delta_B$" about operators $\hat{B} \in V$ such that the projector $\hat{E}[A \in \Delta]$ belongs to the spectral algebra of $\hat{B}$, and $\hat{E}[A \in \Delta] = \hat{E}[B \in \Delta_B]$.

2.4   G as the Power Object of Σ

In any topos, any object X has an associated 'power object' $PX := \Omega^X$, which is the topos analogue of the power set of a set. Accordingly, in a topos of presheaves, any presheaf $\mathbf{X}$ has a 'power presheaf' $P(\mathbf{X})$. This presheaf $P(\mathbf{X})$ assigns to each stage A of the base category the power set $P(\mathbf{X}(A))$ of the set $\mathbf{X}(A)$ assigned by $\mathbf{X}$; and it assigns to each morphism $f : B \to A$ in the base-category, the set-function from $P(\mathbf{X}(A))$ to $P(\mathbf{X}(B))$ induced in the obvious way from $\mathbf{X}(f) : \mathbf{X}(A) \to \mathbf{X}(B)$. That is to say, $P(\mathbf{X})(f) : P(\mathbf{X}(A)) \to P(\mathbf{X}(B))$ is defined by

$$P(\mathbf{X})(f) : \Delta \in P(\mathbf{X}(A)) \mapsto (\mathbf{X}(f))(\Delta) \in P(\mathbf{X}(B)). \qquad (2.6)$$

But one can also consider more restricted power presheaves, whose assignment at a stage A contains only certain subsets of the set $\mathbf{X}(A)$. Indeed, in [2] Section 4.2.3, it was noted that the coarse-graining presheaf $\mathbf{G}$ over

$\mathcal{O}$ was essentially the same as the presheaf $B\mathbf{\Sigma}$ over $\mathcal{O}$, which is defined as assigning to each $\hat{A}$ in $\mathcal{O}$ the Borel subsets of the spectrum of $\hat{A}$. The presheaf $B\mathbf{\Sigma}$ on $\mathcal{O}$ was essentially the Borel power object of $\mathbf{\Sigma}$, containing those subobjects of $\mathbf{\Sigma}$ which are formed of Borel sets of spectral values, with a projector $\hat{E}[A \in \Delta] \in \mathbf{G}(\hat{A})$ corresponding to the Borel subset $\Delta \subset \mathbf{\Sigma}(\hat{A})$.

This connection between projectors and subsets of spectra—*i.e.*, the fact that $\mathbf{G}$ is essentially the same as a restricted power presheaf of $\mathbf{\Sigma}$—also holds in the case of $\mathcal{V}$, as follows.

A projection operator $\hat{P} \in V$ has as its Gelfand transform $\tilde{P}$ the characteristic function of a subset of the spectrum of V, namely the set $V(\hat{P})$ of multiplicative linear functionals κ on V such that $\kappa(\hat{P}) = 1$; in other words, $\tilde{P} = \chi_{V(\hat{P})}$. This set is both closed and open (*clopen*) in the compact Hausdorff topology of $\sigma(V)$. Conversely, each clopen subset of $\sigma(V)$ corresponds to a projection operator $\hat{P}$ whose representative function $\tilde{P}$ on $\sigma(V)$ is the characteristic function of this subset.

So in analogy with $B\mathbf{\Sigma}$ on $\mathcal{O}$, we define a similar presheaf on $\mathcal{V}$, viz. the clopen power object of $\mathbf{\Sigma}$, which we will denote $\mathrm{Clo}\mathbf{\Sigma}$:

- On objects: $\mathrm{Clo}\mathbf{\Sigma}(V)$ is defined to be the set of clopen subsets of the spectrum $\sigma(V)$ of the algebra V. Each such clopen set is the set $V(\hat{P})$ of multiplicative linear functionals κ such that $\kappa(\hat{P}) = 1$ for some projector $\hat{P} \in V$. So $\mathrm{Clo}\mathbf{\Sigma}(V) = \{V(\hat{P}) \mid \hat{P} \in \mathcal{L}(V)\}$.

- On morphisms: for $V_2 \subset V_1$, we define in accordance with Eqn (2.6)

$$\mathrm{Clo}\mathbf{\Sigma}(i_{V_2 V_1})(V_1(\hat{P})) := \mathrm{Clo}\mathbf{\Sigma}(i_{V_2 V_1}) \left(\{\kappa \in \sigma(V_1) \mid \kappa(\hat{P}) = 1\} \right)$$
$$= \{\lambda \in \sigma(V_2) \mid \lambda = \kappa \mid_{V_2} \text{ for some } \kappa \in V_1(\hat{P})\} \quad (2.7)$$

The right hand side of Eq. (2.7) is clearly a subset of $\{\lambda \in \sigma(V_2) \mid \lambda(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1\}$. But the converse inclusion also holds: *i.e.*, any $\lambda \in \sigma(V_2)$ such that $\lambda(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1$ is the restriction to V_2 of some $\kappa \in \sigma(V_1)$ such that $\kappa(\hat{P}) = 1$. This follows from Theorem 4.3.13 (page 266) of [6].[4] This theorem concerns extending states from a subspace of a C^*-algebra to the whole C^*-algebra. But using the fact that the set of pure states of a commutative C^*-algebra is its spectrum, the theorem, especially part (iv), implies that $\lambda \in \sigma(V_2)$ can be extended to a $\kappa \in \sigma(V_1)$, with κ chosen so that

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[4]We thank Hans Halvorson for this reference.

$\kappa(\hat{P}) = c$ for any c such that:

$$c \geq \sup\{\lambda(\hat{E}) \mid \hat{E} \text{ is a projector in } V_2 \text{ and } \hat{E} \leq \hat{P}\} \qquad (2.8)$$

$$c \leq \inf\{\lambda(\hat{E}) \mid \hat{E} \text{ is a projector in } V_2 \text{ and } \hat{P} \leq \hat{E}\} \qquad (2.9)$$

For our case of $c = 1$, the first constraint Eq. (2.8) is trivial. And the infimum in the second constraint, Eq. (2.9), is 1; for the fact that $\mathbf{G}(i_{V_2 V_1})(\hat{P})) \leq \hat{E}$ for all $\hat{E} \in V_2$ such that $\hat{P} \leq \hat{E}$ implies that $\lambda(\hat{E}) = 1$ for all $\hat{E} \in V_2$ such that $\hat{P} \leq \hat{E}$. So λ has an extension κ such that $\kappa(\hat{P}) = 1$.

This result, that $\{\lambda \in \sigma(V_2) \mid \lambda = \kappa|_{V_2} \text{ for some } \kappa \in V_1(\hat{P})\} = \{\lambda \in \sigma(V_2) \mid \lambda(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1\}$, implies that $\mathbf{G}$ and $\mathrm{Clo}\boldsymbol{\Sigma}$ are isomorphic in the topos of presheaves on $\mathcal{V}$. Here isomorphism means as usual that: (i) there is a natural transformation N from $\mathbf{G}$ to $\mathrm{Clo}\boldsymbol{\Sigma}$, $i.e.$, a family of maps $N_V : \mathbf{G}(V) \to \mathrm{Clo}\boldsymbol{\Sigma}(V)$ for each stage V in $\mathcal{V}$ such that the diagram for $V_2 \subseteq V_1$:

$$
\begin{array}{ccc}
\mathrm{Clo}\boldsymbol{\Sigma}(V_1) & \xrightarrow{\text{restriction}} & \mathrm{Clo}\boldsymbol{\Sigma}(V_2) \\
\uparrow{\scriptstyle N_{V_1}} & & \uparrow{\scriptstyle N_{V_2}} \\
\mathbf{G}(V_1) & \xrightarrow{\mathbf{G}(i_{V_2 V_1})} & \mathbf{G}(V_2)
\end{array}
\qquad (2.10)
$$

commutes; and (ii) N is invertible.

Such a natural transformation is provided by $N_V : \hat{P} \in \mathbf{G} \mapsto V(\hat{P}) \in \mathrm{Clo}\boldsymbol{\Sigma}(V)$. With this definition of N, the requirement that the diagram in (2.10) commutes is

$$V_1(\hat{P})|_{V_2} = V_2[\mathbf{G}(i_{V_2 V_1})(\hat{P})] \qquad (2.11)$$

which is just the result that $\{\lambda \in \sigma(V_2) \mid \lambda = \kappa|_{V_2} \text{ for some } \kappa \in V_1(\hat{P})\} = \{\lambda \in \sigma(V_2) \mid \lambda(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1\}$. This natural transformation N is invertible since any clopen set $\in \mathrm{Clo}\boldsymbol{\Sigma}(V)$ is of the form $V(\hat{P})$ for a unique projector $\hat{P} \in \mathbf{G}(V)$.

To sum up, we can think of $\mathbf{G}$ on $\mathcal{V}$ as being the clopen power object of $\boldsymbol{\Sigma}$ on $\mathcal{V}$. This isomorphism will be important in Section 3.

2.5 Sieve-Valued Generalised Valuations

We now describe how to use the topos-theoretic idea of a subobject classifier to assign sieves as partial and contextual truth values to the propositions provided by the coarse-graining presheaf; (propositions that, for $\mathcal{V}$ as base category, are "augmented" in the sense discussed at the end of Section 2.3).

As mentioned in Section 1, these *sieve*-valued valuations have certain properties which strongly suggest that they are appropriate generalisations of the usual idea of a valuation. In particular, they satisfy a functional composition principle analogous to Eq. (2.2). Furthermore, these valuations can be motivated from various different perspectives, discussed in [2, 3, 1]. The main motivation, which applies equally to either $\mathcal{O}$ or $\mathcal{V}$ as base category, lies in the facts that:

1. For any base-category, $\mathcal{C}$ say, the subobject classifier $\boldsymbol{\Omega}$ in the topos of presheaves $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the presheaf that (i) assigns to each A in $\mathcal{C}$ the set $\boldsymbol{\Omega}(A)$ of all sieves on A, where a sieve on A is a set of morphisms in $\mathcal{C}$ with codomain A that is 'closed under composition', *i.e.* if S is sieve on A and $f : B \to A$ is a morphism in S, then for any $g : C \to B$, the composite $f \circ g : C \to A$ is in S; and (ii) assigns to each morphism $f : B \to A$ in $\mathcal{C}$, the pullback map on sieves $\boldsymbol{\Omega}(f) : \boldsymbol{\Omega}(A) \to \boldsymbol{\Omega}(B)$; (cf. [2], Appendix).

   We note that the base-category of most interest for us, $\mathcal{V}$, is a poset; and since in a poset there is at most one morphism between objects, sieves can be identified with lower sets in the poset. Thus on $\mathcal{V}$, the subobject classifier $\boldsymbol{\Omega}$ is as follows:

   - On objects: $\boldsymbol{\Omega}(V)$ is the set of sieves in $\mathcal{V}$ on V. We recall that $\boldsymbol{\Omega}(V)$ has (i) a minimal element, the empty sieve, $0_V = \emptyset$; and (ii) a maximal element, the principal sieve, $\mathrm{true}_V = \downarrow V := \{i_{V'V} : V' \to V \mid V' \subseteq V\} = \{V' \mid V' \subseteq V\}$.

   - On morphisms: $\boldsymbol{\Omega}(i_{V_2V_1}) : \boldsymbol{\Omega}(V_1) \to \boldsymbol{\Omega}(V_2)$ is the pull-back of the sieves in $\boldsymbol{\Omega}(V_1)$ along $i_{V_2V_1}$ defined by:

$$
\begin{aligned}
\boldsymbol{\Omega}(i_{V_2V_1})(S) = i_{V_2V_1}^*(S) \quad &:= \quad \{i_{V_3V_2} : V_3 \to V_2 \mid i_{V_2V_1} \circ i_{V_3V_2} \in S\} \quad (2.12) \\
&= \quad \{V_3 \subset V_2 \mid V_3 \in S\} \quad\quad\quad (2.13)
\end{aligned}
$$

   for all sieves $S \in \boldsymbol{\Omega}(V_1)$.

2. In any topos of presheaves $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, morphisms from an arbitrary presheaf $\mathbf{X}$ to the subobject classifier $\boldsymbol{\Omega}$ generalize the characteristic functions χ in set theory that map an arbitrary set X to the two classical truth values $\{0, 1\}$. In particular, just as the characteristic function $\chi_K : X \to \{0, 1\}$ for a given subset $K \subseteq X$ encodes the answers to the

questions for each $x \in X$, "$x \in K$?", so also the morphism $\chi_{\mathbf{K}} : \mathbf{X} \to \mathbf{\Omega}$ for a given subobject (sub-presheaf) $\mathbf{K}$ of $\mathbf{X}$ encodes the answers to the questions for each stage A in the base-category, and each $x \in \mathbf{X}(A)$, "at what stage does a 'descendant' of x enter $\mathbf{K}$?".

Thus the sieves can be considered as generalized—more precisely, partial and contextual—truth values.

The actual definition of a sieve-valued generalised valuation on $\mathcal{V}$ is as follows. (The definition on $\mathcal{O}$ can be obtained *mutatis mutandis*.)

Definition 2.3 *A sieve-valued generalised valuation on the category $\mathcal{V}$ in a quantum theory is a collection of maps $\nu_V : \mathcal{L}(V) \to \mathbf{\Omega}(V)$, one for each 'stage of truth' V in the category $\mathcal{V}$, with the following properties:*

(i) Functional composition:

For any $\hat{P} \in \mathcal{L}(\mathcal{V})$ and any $V' \subseteq V$, so that $i_{V'V} : V' \to V$, we have

$$\nu_{V'}(\mathbf{G}(i_{V'V}(\hat{P}))) = i_{V'V}^{*}(\nu_V(\hat{P})) \tag{2.14}$$

where $i_{V'V}^{*}$ is the pull-back of the sieves in $\mathbf{\Omega}(V)$ along $i_{V'V}$ defined by

$$\mathbf{\Omega}(i_{V'V})(S) := i_{V'V}^{*}(S) \quad := \quad \{i_{V''V'} : V'' \to V' \mid i_{V'V} \circ i_{V''V'} \in S\} \tag{2.15}$$

for all sieves $S \in \mathbf{\Omega}(V)$.
(ii) Null proposition condition:

$$\nu_V(\hat{0}) = 0_V \tag{2.16}$$

(iii) Monotonicity:

If $\hat{P}, \hat{Q} \in \mathcal{L}(V)$ with $\hat{P} \leq \hat{Q}$, then $\nu_V(\hat{P}) \leq \nu_V(\hat{Q})$. $\tag{2.17}$

We may wish to supplement this list with:
(iv) Exclusivity:

If $\hat{P}, \hat{Q} \in \mathcal{L}(V)$ with $\hat{P}\hat{Q} = \hat{0}$ and $\nu_V(\hat{P}) = \text{true}_V$, then $\nu_V(\hat{Q}) < \text{true}_V$

$$\tag{2.18}$$

and
(v) Unit proposition condition:

$$\nu_V(\hat{1}) = \text{true}_V. \tag{2.19}$$

Note that in writing Eq. (2.14), we have employed Definition 2.2 to specify the coarse-graining operation in terms of an infimum of projectors, as motivated by Theorem 4.1 of [2].

The topos interpretation of these generalised valuations remains as discussed in Section 4.2 of [2] and Section 4 of [3]. Adapting the results and discussion to the category $\mathcal{V}$, we have in particular the result that because of the $FUNC$ condition, Eq. (2.14), the maps $N_V^\nu : \mathcal{L}(V) \to \mathbf{\Omega}(V)$ defined at each stage V by:

$$N_V^\nu(\hat{P}) = \nu_V(\hat{P}) \tag{2.20}$$

define a natural transformation N^ν from $\mathbf{G}$ to $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is the subobject-classifier of the topos of presheaves, $\mathrm{Set}^{\mathcal{V}^{\mathrm{op}}}$, these natural transformations are in one-to-one correspondence with subobjects of $\mathbf{G}$; so that each generalised valuation defines a subobject of $\mathbf{G}$. We will pursue this topic in more detail in Section 3.

2.6 Sieve-Valued Valuations Associated with Quantum States

We recall (for example, [2], Definition 4.5) that each quantum state ρ defines a sieve-valued generalised valuation on $\mathcal{O}$ in a natural way by

$$
\begin{aligned}
\nu^\rho(A \in \Delta) \;\; &:= \;\; \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{Prob}(B \in f(\Delta); \rho) = 1\} \\
&= \;\; \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\Delta)]) = 1\}. \tag{2.21}
\end{aligned}
$$

Thus the generalised valuation associates to the proposition "$A \in \Delta$" at stage $\hat{A}$ all arrows in $\mathcal{O}$ with codomain $\hat{A}$ along which the projector corresponding to the proposition coarse-grains to a projector which is 'true' in the usual sense of having a Born-rule probability equal to 1, which in our framework corresponds to the 'totally true' truth value, the principal sieve $\downarrow_{\hat{B}}$ at stage $\hat{B}$. This construction is easily seen to be a sieve, and satisfies conditions analogous to Eqs. (2.14–2.19) for a generalised valuation on $\mathcal{V}$ ([2], Section 4.4).

We also recall that there is a one-parameter family of extensions of these valuations, defined by relaxing the condition that the proposition coarse-grains along arrows in the sieve to a 'totally true' projector. That is to say, we can define the sieve

$$\nu^{\rho,r}(A \in \Delta) \;\; := \;\; \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{Prob}(B \in f(\Delta); \rho) \geq r\}$$

$$= \ \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \text{tr}(\rho \, \hat{E}[B \in f(\Delta)]) \geq r\} \qquad (2.22)$$

where the proposition "$A \in \Delta$" is only required to coarse-grain to a projector that is true with some probability greater than r, where $0.5 \leq r \leq 1$.

Furthermore, if one drops the exclusivity condition, one can allow probabilities less than 0.5, *i.e.* $0 < r < 0.5$.

Similarly, each quantum state ρ defines a sieve-valued generalised valuation on $\mathcal{V}$ in a natural way. Recall from Section 2.3 that we interpret a projector $\hat{P} \in \mathcal{L}(V)$ as an "augmented" proposition about the spectrum of the commutative subalgebra V, rather than about the value of just one operator. Thus we define a sieve-valued generalised valuation associated with a quantum state ρ as follows:

Definition 2.4 *The sieve-valued valuation $\nu^{\rho}_{V_1}$ of a projector $\hat{P} \in V_1$ associated with a quantum state ρ is defined by:*

$$\nu^{\rho}_{V_1}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid \rho\,[\mathbf{G}(i_{V_2 V_1})(\hat{P})] = 1\}. \qquad (2.23)$$

This assigns as the truth-value at stage V_1 of a projector $\hat{P} \in \mathcal{L}(V_1)$, a sieve on V_1 containing (morphisms to V_1 from) all stages V_2 at which $\hat{P}$ is coarse-grained to a projector which is 'totally true' in the usual sense of having Born-rule probability 1.

One readily verifies that Eq. (2.23) defines a generalised valuation in the sense of Definition 2.3. (The verification is the same, *mutatis mutandis*, as for generalised valuations on $\mathcal{O}$, given in Section 4.4 of [2].)

Again, we can obtain a one-parameter family of such valuations by introducing a probability r:

$$\nu^{\rho,r}_{V_1}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid \rho\,[\mathbf{G}(i_{V_2 V_1})(\hat{P})] \geq r\}. \qquad (2.24)$$

2.7   Interval Valuations

The sieve-valued generalised valuations on $\mathcal{O}$ and $\mathcal{V}$ discussed in Sections 2.5 and 2.6 (and their analogues on $\mathcal{W}$, discussed in [2, 3]) are one way of assigning a generalised truth value to propositions in a way that is not prevented by the Kochen-Specker theorem. We now turn to relating these to another notion of 'generalised valuation', which we call 'interval valuations' since the intuitive idea is to assign some interval of real numbers to each operator. Note that here 'interval' is used loosely: it means just some (Borel)

subset of $\mathbb{R}$, not necessarily a connected subset; and more generally, it means just some (Borel) subset of the spectrum one is concerned with (at a given stage of the base-category).

In Section 4 of [4], we showed how this intuitive idea can be developed in various ways, even for a single base-category. That discussion focussed on $\mathcal{V}$, and described how a sieve-valued generalised valuation on $\mathcal{V}$—in particular one associated with a quantum state—induces an 'interval valuation' in various senses of the phrase. These various senses differ about whether to take the assigned intervals at the various stages to define:

(i) a subobject of $\mathbf{\Sigma}$, or

(ii) a global element of $\mathbf{G}$, or

(iii) a subobject of $\mathbf{G}$.

But these different senses of 'interval valuation' are similar in that all are defined in terms of the set of 'totally true' projectors at each stage V of the base-category $\mathcal{V}$. Thus for any sieve-valued valuation ν, we defined the *truth set*

$$T^{\nu}(V) := \{\hat{P} \in \mathcal{L}(V) \mid \nu_V(\hat{P}) = \text{true}_V\} \tag{2.25}$$

so that, in particular, for the valuation ν^ρ associated with the quantum state ρ we have

$$T^{\rho}(V) := \{\hat{P} \in \mathcal{L}(V) \mid \rho(\hat{P}) = 1\}. \tag{2.26}$$

We used these truth sets in two ways. First, we defined interval valuations that are subobjects of $\mathbf{\Sigma}$ by assigning to each stage V, the subset of the spectrum $\sigma(V)$ consisting of all functionals that 'make certain' all members of the truth set $T^{\nu}(V)$. That is, for any sieve-valued valuation ν, we assign to V the set:

$$\mathbf{I}^{\nu}(V) = \{\kappa \in \sigma(V) \mid \kappa(\hat{P}) = 1, \ \forall \hat{P} \in T^{\nu}(V)\} = \bigcap_{\hat{P} \in T^{\nu}(V)} V(\hat{P}). \tag{2.27}$$

This assignment gives a subobject of $\mathbf{\Sigma}$ (meaning that if $V_2 \subseteq V_1$, then $\mathbf{I}^{\nu}(V_2) \supseteq \mathbf{I}^{\nu}(V_1) \mid_{V_2}$) provided the following condition (eqn (4.9) of [4])

$$\text{If } V_2 \subseteq V_1 \text{ then inf } T^{\nu}(V_2) \geq \ \text{inf } T^{\nu}(V_1) \tag{2.28}$$

is satisfied—which it always is for the valuations ν^ρ associated with quantum states ρ because for these valuations $T^{\nu^\rho}(V_2) \subseteq T^{\nu^\rho}(V_1)$. It is also satisfied

for the 'probability r' quantum valuations $\nu^{\rho,r}$, *i.e.* with truth sets defined using Eq. (2.24).[5]

Second, we defined interval valuations that are global elements of $\mathbf{G}$ by taking the infima of these truth sets (using the fact that $\mathcal{L}(V)$ is a complete lattice) to define what we called the *support* (of the valuation, or the quantum state) at each stage V:

$$s(\nu, V) := \inf T^\nu(V) = \inf\{\hat{P} \in \mathcal{L}(V) \mid \nu_V(\hat{P}) = \text{true}_V\} \qquad (2.29)$$

so that in particular, for the valuation ν^ρ

$$s(\rho, V) := \inf T^\rho(V) = \inf\{\hat{P} \in \mathcal{L}(V) \mid \rho(\hat{P}) = 1\}. \qquad (2.30)$$

An example of an interval valuation that is a global element of $\mathbf{G}$ is given by assigning to each stage V, the support at that stage, $s(\nu, V)$ or $s(\rho, V)$. This assignment gives a global element of $\mathbf{G}$ provided that supports (infima of truth sets) 'match up' under coarse-graining in the usual sense that

$$\text{If } V_2 \subseteq V_1, \text{ then } \inf\{\hat{P} \in T^\nu(V_2)\} =: s(\nu, V_2) \;\; = \;\; \mathbf{G}(i_{V_2 V_1})\left(\inf\{\hat{P} \in T^\nu(V_1)\}\right)$$
$$= \;\; \mathbf{G}(i_{V_2 V_1})(s(\nu, V_1)). \qquad (2.31)$$

This condition is satisfied for the valuations ν^ρ associated with quantum states ρ (but *not* for the 'probability r' quantum valuations $\nu^{\rho,r}$, *i.e.* with supports $s(\nu^{\rho,r}, V)$ defined on analogy with Eq. (2.29) but using Eq. (2.24)).

We note that the notion of an interval valuation that is a global element of $\mathbf{G}$ is stronger than the notion of a subobject of $\mathbf{\Sigma}$ (treated in case (i) above) in the sense that any global element of $\mathbf{G}$ defines a subobject of $\mathbf{\Sigma}$ but not vice versa. Thus any global element γ of $\mathbf{G}$—*i.e.* an assignment γ such that if $V_2 \subset V_1$ then $\gamma(V_2) = \mathbf{G}(i_{V_2 V_1})(\gamma(V_1))$—defines a subobject $\mathbf{I}^\gamma$ of $\mathbf{\Sigma}$ by

$$\mathbf{I}^\gamma(V_1) := V_1(\gamma(V_1)) = \{\kappa \in \sigma(V_1) \mid \kappa(\gamma(V_1)) = 1\} \qquad (2.32)$$

since $\kappa(\hat{P}) = 1$ implies $\kappa(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1$, so that $\mathbf{I}^\gamma(V_1)|_{V_2} \subseteq \mathbf{I}^\gamma(V_2)$. We can also put this in terms of the isomorphism N in Section 2.4 between $\mathbf{G}$ and $\text{Clo}\mathbf{\Sigma}$ whose component maps $N_V : \hat{P} \in \mathbf{G}(V) \mapsto V(\hat{P}) \in \text{Clo}\mathbf{\Sigma}$ carry the global element γ of $\mathbf{G}$ into a global element of $\text{Clo}\mathbf{\Sigma}$, *i.e.* a subobject of $\mathbf{\Sigma}$.

---

[5]Eqn. (2.27) shows how 'interval' is here used abstractly: an algebra V is assigned a subset of its spectrum, *i.e.* a set of multiplicative linear functionals on V which corresponds to a subset of the spectrum of each operator in the algebra.

3 The Correspondence between Intervals and Sieves

3.1 Prospectus

So much by way of review. In the rest of this paper we shall report some new results about the relation between sieve-valued valuations and interval valuations; where both of these notions will be understood more generally than in Sections 2.5 to 2.7. However, in this Section (though not Section 4) all the interval valuations to be discussed will be like those in Section 2.7, in the sense that they will be based on the notion of truth sets and associated ideas (especially the infima of truth sets, *i.e.*, supports).[6]

In this Section, we will discuss a kind of correspondence between sieve-valued valuations and interval valuations. So despite the marked differences between sieve-valued valuations and interval valuations—for example, in $\mathcal{V}$ we see projectors or propositions *versus* algebras as arguments, and sieves *versus* sets of linear functionals as values—it turns out that they correspond. Indeed, in a sense they mutually determine each other. We have already seen in Section 2.7 how sieve-valued valuations determine interval valuations, via the idea of truth sets. The converse determination, of sieve-valued valuations by interval valuations, is simplest for the case where the interval valuations are global elements of **G**; *i.e.* for case (ii) of Section 2.7, where we use not just truth sets, but their infima, *supports*. We shall present this in Section 3.2.1.

Then in Section 3.2.2 we shall discuss case (i) in Section 2.7, where the interval valuations are subobjects of $\mathbf{\Sigma}$.

In both this Section and the next, our discussion will again concentrate on $\mathcal{V}$ since, as mentioned in the Introduction, using $\mathcal{V}$ avoids measure-theoretic difficulties about the spectra of operators (and functions of them) which arise in $\mathcal{O}$. But since our results about $\mathcal{V}$ are rather abstract, it will be heuristically helpful to report the corresponding claims about $\mathcal{O}$; *i.e.* to state what our results imply about presheaves over $\mathcal{O}$ at those stages (*i.e.* operators) of $\mathcal{O}$ that do not have these measure-theoretic difficulties. (These stages will

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[6]So as in [4], we are not concerned here to appeal to interval valuations to solve the measurement problem, namely by assigning intervals to some or all quantities that are 'narrow' enough to give definite results to quantum measurements and yet 'wide' enough to avoid the Kochen-Specker and other 'no-go' theorems. For a recent discussion of this strategy, as it occurs within the modal interpretation, cf. [7].

include all operators with a pure discrete spectrum.) So we report these corresponding claims about $\mathcal{O}$ in Section 3.3.

3.2 The Correspondence in $\mathcal{V}$

3.2.1 The case of global elements of G

The correspondence between sieve-valued valuations and interval valuations is simplest for the case where the interval valuations are global elements of $\mathbf{G}$. In each 'direction', there is a natural and simple sufficient condition for correspondence, satisfied by 'most' of the valuations discussed in Sections 2.5 to 2.7. More precisely: given a sieve-valued valuation α, and the corresponding interval valuation, s^α say, that α defines in terms of supports, then there is a simple sufficient condition ((i) below) for α to equal a valuation naturally defined by s^α which takes sets of morphisms as values. And in the other direction: given an interval valuation a and the corresponding valuation, α^a say, that a naturally defines and which takes sets of morphisms as values, then a equals the interval valuation defined in terms of the supports of α^a; and there is a simple sufficient condition ((ii) below) for α^a to be sieve-valued. In fact, condition (ii) is that supports should form a global element of $\mathbf{G}$.

But before stating these results it is illuminating to show that, taken together, conditions (i) and (ii) are also sufficient to imply that a valuation is an assignment of sieves, and also obeys $FUNC$. This claim is made precise in Theorem 3.1. It shows that conditions (i) and (ii) taken together are sufficient for an assignment α to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a set of morphisms $\alpha_V(\hat{P}) \subseteq \{i_{V'V} : V' \to V\}$, to satisfy three conditions. Namely, the conditions: (a) that α is sieve-valued (*i.e.* each $\alpha(V)$ is a sieve on V); (b) that α obeys $FUNC$; (c) that α obeys a characterization that encapsulates the correspondence between sieve-valued valuations and interval valuations. (This characterization will also lead in to the discussion in Section 4.)

We begin by noting that for any such α, *i.e.* any such assignments $\alpha_V(\hat{P}) \subseteq \{i_{V'V} : V' \to V\}$, we can define truth sets and (since $\mathcal{L}(V)$ is complete) supports, just as in Eqs. (2.25) and (2.29). So we write these as $T^\alpha(V)$ and $s(\alpha, V)$ respectively. Similarly, for any such α, the condition that supports 'match up' under coarse-graining, makes sense; cf. Eq. (2.31), substituting α for ν.

Theorem 3.1 *Let α be an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a set of morphisms $\alpha_V(\hat{P}) \subseteq \{i_{V'V} : V' \to V\}$ with*

codomain V. Let $T^\alpha(V)$ and $s(\alpha, V)$ be defined as in Eqs. (2.25) and (2.29) respectively (just substituting α for ν). Suppose that α obeys:

(i) If $V_2 \subseteq V_1$ and $\hat{P} \in \mathcal{L}(V_1)$, then $i_{V_2 V_1} : V_2 \to V_1 \in \alpha_{V_1}(\hat{P})$ iff $s(\alpha, V_2) \le \mathbf{G}(i_{V_2 V_1})(\hat{P})$;

(ii) supports give a global element of $\mathbf{G}$, i.e. they match up under coarse-graining, in the sense of Eq. (2.31), i.e.

$$\text{If } V_2 \subseteq V_1, \ s(\alpha, V_2) = \mathbf{G}(i_{V_2 V_1})(s(\alpha, V_1)). \tag{3.1}$$

Then:

(a) each $\alpha_V(\hat{P})$ is a sieve;

(b) α obeys $FUNC$, as in Eq. (2.14), i.e.,

$$\alpha_{V_2}(\mathbf{G}(i_{V_2 V_1}(\hat{P})) = i^*_{V_2 V_1}(\alpha_{V_1}(\hat{P})) \tag{3.2}$$

(c) α obeys

$$\alpha_{V_1}(\hat{P}) = \{i_{V_2 V_1} : V_2 \to V_1 \mid \mathbf{G}(i_{V_2 V_1})(s(\alpha, V_1)) \le \mathbf{G}(i_{V_2 V_1})(\hat{P})\}. \tag{3.3}$$

Proof: (a): Given $i_{V_2 V_1} \in \alpha_{V_1}(\hat{P})$, the condition (i) and the monotonicity of $\mathbf{G}(i_{V_2 V_1})$ imply that for any $V_3 \subseteq V_2$, $\mathbf{G}(i_{V_3 V_2})(s(\alpha, V_2)) \le \mathbf{G}(i_{V_3 V_1})(\hat{P})$. But by (ii), $\mathbf{G}(i_{V_3 V_2})(s(\alpha, V_2)) = s(\alpha, V_3)$; so that by (i), $i_{V_3 V_1} \in \alpha_{V_1}(\hat{P})$.

(b): condition (i) implies that $\alpha_{V_2}(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = \{i_{V_3 V_2} : V_3 \to V_2 \mid s(\alpha, V_3) \le \mathbf{G}(i_{V_3 V_1})(\hat{P})\}$ and that $i^*_{V_2 V_1}(\alpha_{V_1}(\hat{P})) := \{i_{V_3 V_2} : V_3 \to V_2 \mid i_{V_2 V_1} \circ i_{V_3 V_2} \in \alpha_{V_1}(\hat{P})\} = \{i_{V_3 V_2} : V_3 \to V_2 \mid s(\alpha, V_3) \le \mathbf{G}(i_{V_3 V_1})(\hat{P})\}$. (So result (b) depends only on the condition (i)).

(c): Immediate: apply (ii) *i.e.*, Eq. (3.1) to the condition in (i) that $s(\alpha, V_2) \le \mathbf{G}(i_{V_2 V_1})(\hat{P})$. QED.

In particular, the sieve-valued valuations associated with quantum states (for probability 1, but not r with $0 \le r < 1$) obey the conditions of Theorem 3.1. For we noted in Section 4.3 of [4] that (ii) *i.e.* Eq. (2.31), holds for these valuations; and (i) holds trivially for them, since $\rho(\mathbf{G}(i_{V_2 V_1})(\hat{P})) = 1$ iff $s(\rho, V_2) \le \mathbf{G}(i_{V_2 V_1})(\hat{P})$.

We turn to describing how conditions (i) and (ii) are, respectively, natural sufficient conditions for: (a) a sieve-valued valuation to be determined by an interval-valued valuation that it itself determines; and (b) an interval valuation to be determined by a sieve-valued valuation that it itself determines.

First, suppose α is an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a sieve on V. We can define truth sets and (since $\mathcal{L}(V)$ is

complete) supports, as in Eqs. (2.25) and (2.29). Let us write these as $T^\alpha(V)$ and $s^\alpha(V)$ respectively. Then we define a valuation with sets of morphisms as values, in terms of the $s^\alpha(V)$, by

$$\alpha_{V_1}^{s^\alpha}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid s^\alpha(V_2) \leq \mathbf{G}(i_{V_2 V_1})(\hat{P})\} \qquad (3.4)$$

and ask: does $\alpha_{V_1}^{s^\alpha}(\hat{P}) = \alpha_{V_1}(\hat{P})$? The answer is trivially: 'Yes' if and only if α obeys condition (i) of Theorem 3.1. (We note incidentally that this argument, including its definition of truth sets and supports, does not require that the given α assign sieves. It is enough, as in Theorem 3.1, that α be an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a set of morphisms with codomain V: the conclusion, that $\alpha^{s^\alpha} = \alpha$ iff α obeys condition (i), is unaffected.)

Second, suppose a is an assignment at each stage V in $\mathcal{V}$ of an element of $\mathbf{G}(V) := \mathcal{L}(V)$. (We do not for the moment require that a define a global element of $\mathbf{G}$.) Then we define a valuation α^a, with sets of morphisms as values, on all $\hat{P}$ in each $\mathbf{G}(V) := \mathcal{L}(V)$, in terms of a, as follows:

$$\alpha_{V_1}^{a}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid a(V_2) \leq \mathbf{G}(i_{V_2 V_1})(\hat{P})\}. \qquad (3.5)$$

It follows that the support $s(\alpha^a, V)$, defined in the usual way (cf. Eq. (2.29)) is equal to $a(V)$. That is, suppose we define truth sets and supports for α^a in the usual way; cf. Eqs. (2.25) and (2.29). Then note that

$$\hat{P} \in T^{\alpha^a}(V) \text{ iff } a(V) \leq \mathbf{G}(i_{VV})(\hat{P}) = \hat{P}, \qquad (3.6)$$

so that $s(\alpha^a, V) := \inf T^{\alpha^a}(V) = a(V)$.

But under what conditions is α^a sieve-valued (i.e. $\alpha_{V_1}^{a}(\hat{P})$ is always a sieve)? In fact, the condition (ii) in Theorem 3.1—i.e. the condition that a defines a global element of $\mathbf{G}$—is a natural sufficient condition for this. For suppose that $i_{V_2 V_1} \in \alpha_{V_1}^{a}(\hat{P})$, i.e. $a(V_2) \leq \mathbf{G}(i_{V_2 V_1})(\hat{P})$, and pick any $i_{V_3 V_2} : V_3 \to V_2$. Since $\mathbf{G}(i_{V_3 V_2})$ is monotonic, we get $\mathbf{G}(i_{V_3 V_2})a(V_2) \leq \mathbf{G}(i_{V_3 V_1})(\hat{P})$. Assuming (ii), i.e. $\mathbf{G}(i_{V_3 V_2})a(V_2) = a(V_3)$, it follows that $i_{V_3 V_1} \in \alpha_{V_1}^{a}(\hat{P})$, i.e. $\alpha_{V_1}^{a}(\hat{P})$ is a sieve.

3.2.2 The case of subobjects of Σ

We return to case (i) of Section 2.7. We recall that for any sieve-valued valuation ν, the interval valuation that assigns to each stage V the subset

of the spectrum $\sigma(V)$ consisting of all functionals that 'make certain' all members of the truth set $T^\nu(V)$, *i.e.* the interval valuation of Eq. (2.27):

$$\mathbf{I}^\nu(V) := \{\kappa \in \sigma(V) \mid \kappa(\hat{P}) = 1, \ \forall \hat{P} \in T^\nu(V)\} = \bigcap_{\hat{P} \in T^\nu(V)} V(\hat{P}) \qquad (3.7)$$

defines a subobject of $\mathbf{\Sigma}$ provided Eq. (2.28) is satisfied:

$$\text{If } V_2 \subseteq V_1 \text{ then } \inf T^\nu(V_2) \geq \ \inf T^\nu(V_1), \ i.e. \ s(\nu, V_2) \geq s(\nu, V_1). \qquad (3.8)$$

(Incidentally, this argument does not require that ν be a sieve-valued valuation in the strong sense of Definition 2.3 (Section 2.5); it works for any assignment, to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$, of a sieve on V.)

To obtain the analogue of Theorem 3.1 for the case of subobjects of $\mathbf{\Sigma}$, we note that condition (i) of Theorem 3.1 says that $i_{V_2V_1} : V_2 \to V_1 \in \alpha_{V_1}(\hat{P})$ if and only if $s(\alpha, V_2) \leq \mathbf{G}(i_{V_2V_1})(\hat{P})$, *i.e.* iff $\mathbf{G}(i_{V_2V_1})(\hat{P})$ is certain at V_2 according to α. So we expect the corresponding condition, for a subobject $\mathbf{I}^\alpha$ of $\mathbf{\Sigma}$, to be that $\mathbf{I}^\alpha(V_2) \subseteq V_2(\mathbf{G}(i_{V_2V_1})(\hat{P}))$ ($\equiv V_1(\hat{P})|_{V_2}$ by the isomorphism Eq. (2.11) in Section 2.4). Indeed we have:

Theorem 3.2 *Let α be an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a set of morphisms $\alpha_V(\hat{P}) \subseteq \{i_{V'V} : V' \to V\}$ with codomain V. Let $T^\alpha(V)$ and $\mathbf{I}^\alpha(V)$ be defined as in Eqs. (2.25) and (2.27) respectively (just substituting α for ν). Suppose that α obeys:*
(i) If $V_2 \subseteq V_1$ and $\hat{P} \in \mathcal{L}(V_1)$, then $i_{V_2V_1} : V_2 \to V_1 \in \alpha_{V_1}(\hat{P})$ iff $\mathbf{I}^\alpha(V_2) \subseteq V_1(\hat{P}) |_{V_2}$;
(ii) the intervals $\mathbf{I}^\alpha$ give a 'tight' subobject of $\mathbf{\Sigma}$ in the sense that they match up exactly under restriction, i.e.

$$\text{If } V_2 \subseteq V_1, \text{ then } \mathbf{I}^\alpha(V_2) = \mathbf{I}^\alpha(V_1) |_{V_2} : \textit{not merely } \mathbf{I}^\alpha(V_2) \supseteq \mathbf{I}^\alpha(V_1) |_{V_2}; \quad (3.9)$$

Then:
(a) each $\alpha_V(\hat{P})$ is a sieve;
(b) α obeys FUNC, just as in Eqs. (2.14) and (3.2), i.e.

$$\alpha_{V_2}(\mathbf{G}(i_{V_2V_1}(\hat{P})) = i^*_{V_2V_1}(\alpha_{V_1}(\hat{P})) \qquad (3.10)$$

(c) α obeys

$$\alpha_{V_1}(\hat{P}) = \{i_{V_2V_1} : V_2 \to V_1 \mid \mathbf{I}^\alpha(V_1) |_{V_2} \subseteq V_1(\hat{P}) |_{V_2}\} \qquad (3.11)$$

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Proof: (a): Given $i_{V_2V_1} \in \alpha_{V_1}(\hat{P})$, the condition (i) and the monotonicity of taking restrictions (*i.e.* if X and Y are sets of functions on a common domain of which Z is a subset, then $X \subseteq Y$ implies $X|_Z \subseteq Y|_Z$) imply that for any $V_3 \subseteq V_2$, $\mathbf{I}^\alpha(V_2)|_{V_3} \subseteq V_1(\hat{P})|_{V_3}$. But (ii) implies $\mathbf{I}^\alpha(V_3) \subseteq \mathbf{I}^\alpha(V_2)|_{V_3}$; so that $\mathbf{I}^\alpha(V_3) \subseteq V_1(\hat{P})|_{V_3}$ and by (i), $i_{V_3V_1} \in \alpha_{V_1}(\hat{P})$.

(b): condition (i) implies that $\alpha_{V_2}(\mathbf{G}(i_{V_2V_1})(\hat{P})) = \{i_{V_3V_2} : V_3 \to V_2 \mid \mathbf{I}^\alpha(V_3) \subseteq V_2(\mathbf{G}(i_{V_2V_1})(\hat{P}))|_{V_3}\}$. But the isomorphism of $\mathbf{G}$ and $\mathrm{Clo}\boldsymbol{\Sigma}$ (cf. diagram 2.10 and Eq. (2.11)) means that $V_2(\mathbf{G}(i_{V_2V_1})(\hat{P})) = V_1(\hat{P})|_{V_2}$; restricting both sides of this equation to any $V_3 \subseteq V_2$, we get $V_2(\mathbf{G}(i_{V_2V_1})(\hat{P}))|_{V_3} = V_1(\hat{P})|_{V_3}$. On the other hand, condition (i) also implies that $i^*_{V_2V_1}(\alpha_{V_1}(\hat{P})) := \{i_{V_3V_2} : V_3 \to V_2 \mid i_{V_2V_1} \circ i_{V_3V_2} \in \alpha_{V_1}(\hat{P})\} = \{i_{V_3V_2} : V_3 \to V_2 \mid \mathbf{I}^\alpha(V_3) \subseteq V_1(\hat{P})|_{V_3}\}$. (So result (b) depends only on the condition (i)).

(c): Immediate: apply (ii) *i.e.* Eq. (3.9) to the condition (i) that $\mathbf{I}^\alpha(V_2) \subseteq V_1(\hat{P})|_{V_2}$. $\hfill$ QED.

We remark that the analogy with Theorem 3.1 is very close, but we could equally well have proven Theorem 3.2 first. Indeed, much of Theorem 3.2 can be stated and proved without mention of $\mathbf{G}$. More precisely, the coarse-graining map $\mathbf{G}(i_{V_2V_1})(\cdot)$, taking projectors $\hat{P} \in \mathbf{G}(V_1)$ to projectors in $\mathbf{G}(V_2)$, is for the most part replaced by the map $V_1(\cdot)|_{V_2}$, taking projectors $\hat{P} \in \mathbf{G}(V_1)$ to subsets of $\boldsymbol{\Sigma}(V_2)$. In particular, only part (b) needs to mention $\mathbf{G}$ and to make use of the isomorphism in Section 2.4 between $\mathbf{G}$ and $\mathrm{Clo}\boldsymbol{\Sigma}$.

In particular, the sieve-valued valuations associated with quantum states (for probability 1, but not r with $0 \le r < 1$) obey the conditions of Theorem 3.2. For we noted in Section 4.4.1 of [4] that (ii) *i.e.*, Eq. (3.9), holds for these valuations. Besides, (i) holds for these valuations because of the isomorphism between $\mathbf{G}$ and $\mathrm{Clo}\boldsymbol{\Sigma}$, specifically Eq. (2.11), $V_2(\mathbf{G}(i_{V_2V_1})(\hat{P})) = V_1(\hat{P})|_{V_2}$; as follows. By the definition of ν^ρ (cf. Section 2.6), $i_{V_2V_1} : V_2 \to V_1 \in \nu^\rho_{V_1}(\hat{P})$ iff $\mathbf{G}(i_{V_2V_1})(\hat{P}) \in T^\rho(V_2)$. On the other hand, condition (i) for ν^ρ is that $\mathbf{I}^\rho(V_2) \subseteq V_1(\hat{P})|_{V_2} \equiv V_2(\mathbf{G}(i_{V_2V_1})(\hat{P}))$, *i.e.* that if $\kappa \in \sigma(V_2)$ and $\kappa(\hat{P}) = 1, \forall \hat{P} \in T^\rho(V_2)$, then $\kappa(\mathbf{G}(i_{V_2V_1})(\hat{P})) = 1$; which is just that $\mathbf{G}(i_{V_2V_1})(\hat{P}) \in T^\rho(V_2)$.

Furthermore, the discussion of the second half of Section 3.2.1 also carries over *mutatis mutandis*; (though there is one difference). That is: conditions (i) and (ii) are again, respectively, natural sufficient conditions for: (a) a sieve-valued valuation to be determined by an interval-valued valuation that it itself determines; and (b) an interval valuation to be determined by a sieve-valued valuation that it itself determines.

First, suppose α is an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a sieve on V. We can define truth sets as in Eq. (2.25), *i.e.,*

$$T^\alpha(V) := \{\hat{P} \in \mathcal{L}(V) \mid \alpha_V(\hat{P}) = \text{true}_V\} \tag{3.12}$$

and intervals as in Eq. (2.27), *i.e.*

$$\mathbf{I}^\alpha(V) := \{\kappa \in \sigma(V) \mid \kappa(\hat{P}) = 1, \forall \hat{P} \in T^\alpha(V)\} \tag{3.13}$$

Then we define a valuation with sets of morphisms as values, in terms of the $\mathbf{I}^\alpha(V)$, by:

$$\alpha_{V_1}^{\mathbf{I}^\alpha}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid \mathbf{I}^\alpha(V_2) \subseteq V_1(\hat{P})\mid_{V_2}\}; \tag{3.14}$$

and ask: does $\alpha_{V_1}^{\mathbf{I}^\alpha}(\hat{P}) = \alpha_{V_1}(\hat{P})$? The answer is trivially: 'Yes' if and only if α obeys condition (i) of Theorem 3.2. (As in Section 3.2.1, we note incidentally that this argument, including its definition of truth sets and intervals, does not require that the given α assign sieves. It is enough, as in Theorem 3.2, that α be an assignment to each stage V in $\mathcal{V}$ and each $\hat{P} \in \mathbf{G}(V) := \mathcal{L}(V)$ of a set of morphisms with codomain V: the conclusion, that $\alpha^{\mathbf{I}^\alpha} = \alpha$ if and only if α obeys condition (i), is unaffected.)

Second, suppose a is an assignment at each stage V in $\mathcal{V}$ of a subset $a(V)$ of $\mathbf{\Sigma}(V) := \sigma(V)$. (We do not for the moment require that a define a subobject of $\mathbf{\Sigma}$.) Then we define a valuation α^a, with sets of morphisms as values, on all $\hat{P}$ in each $\mathbf{G}(V) := \mathcal{L}(V)$, in terms of a, as follows:

$$\alpha_{V_1}^a(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid a(V_2) \subseteq V_1(\hat{P})\mid_{V_2}\}. \tag{3.15}$$

As in Section 3.2.1 (after Eq. (3.5)), we ask whether the interval $\mathbf{I}^{\alpha^a}(V)$, defined in the usual way (cf. Eqs. (2.27), (3.12) and (3.13)) is equal to $a(V)$. But in Section 3.2.1, the answer was automatically 'Yes'; now it is not. For defining truth sets and intervals in this way, we get:

$$\hat{P} \in T^{\alpha^a}(V) \text{ iff } a(V) \subseteq V(\hat{P})\mid_V = V(\hat{P}), \tag{3.16}$$

so that $\kappa \in \sigma(V)$ is in $\mathbf{I}^{\alpha^a}(V)$ if and only if for all $\hat{P}$ with $a(V) \subseteq V(\hat{P})$, we have $\kappa(\hat{P}) = 1$. All elements of $a(V)$ fulfill this condition so that $a(V) \subseteq \mathbf{I}^{\alpha^a}(V)$. But the converse inclusion requires that if $\kappa \notin a(V)$ then there is $\hat{Q}$ with $a(V) \subseteq V(\hat{Q})$ and $\kappa(\hat{Q}) \neq 1$. And in general this will not hold: if κ is in the closure of $a(V)$, but $\kappa \notin a(V)$, then any clopen (so closed) superset

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Y of $a(V)$ must contain κ; and any such Y is $V(\hat{Q})$ for some $\hat{Q}$. To get this converse, and so $a(V) = \mathbf{I}^{\alpha^a}(V)$, the natural sufficient condition is that the given sets $a(V)$ should be clopen. (Recall from Section 2.4 that every clopen subset of $\sigma(V)$ corresponds to a projector whose Gelfand transform on $\sigma(V)$ is the characteristic function of the subset.)

But under what conditions is α^a sieve-valued (*i.e.* $\alpha^a_{V_1}(\hat{P})$ is always a sieve)? In fact, the condition (ii) in Theorem 3.2—*i.e.* the condition that a defines a 'tight' subobject of $\mathbf{\Sigma}$—is a natural sufficient condition for this. For suppose that $i_{V_2 V_1} \in \alpha^a_{V_1}(\hat{P})$, so that $a(V_2) \subseteq V_1(\hat{P})|_{V_2}$, and pick any $i_{V_3 V_2} : V_3 \to V_2$. Since restriction is monotonic, $a(V_2)|_{V_3} \subseteq V_1(\hat{P})|_{V_3}$. Assuming (ii), *i.e.* $a(V_2)|_{V_3} = a(V_3)$, it follows that $i_{V_3 V_1} \in \alpha^a_{V_1}(\hat{P})$, thus $\alpha^a_{V_1}(\hat{P})$ is a sieve.

3.3 The Correspondence in $\mathcal{O}$

The discussion in Section 3.2 is quite abstract. So it is illuminating to present the same ideas in a more concrete setting: namely, the valuations on $\mathcal{O}$ associated with a quantum state $\psi \in \mathcal{H}$, or more generally a density matrix ρ, which (cf. Section 2.6, especially Eq. (2.21)) are defined by

$$\nu^\psi(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{E}[B \in f(\Delta)]\psi = \psi\} \qquad (3.17)$$

and

$$\nu^\rho(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \operatorname{tr}(\rho \hat{E}[B \in f(\Delta)]) = 1\}. \qquad (3.18)$$

However, as mentioned in Section 3, various measure-theoretic difficulties about the spectra of operators (and functions of them) arise in $\mathcal{O}$. These centre around the fact that if $\hat{B} = f(\hat{A})$ (so that there is a morphism $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ in $\mathcal{O}$) then in general, the corresponding spectra (now consisting of elements of $\mathbb{R}$, not of linear functionals on operators!) have only a subset inclusion

$$f(\sigma(\hat{A})) \subseteq \sigma(f(\hat{A})) \qquad (3.19)$$

not necessarily an equality; though of course, if $\hat{A}$ has a pure discrete spectrum, then $f(\sigma(\hat{A})) = \sigma(f(\hat{A}))$.

This situation prompts three further remarks:

1. For the role of Eq. (3.19) in defining the spectral presheaf on $\mathcal{O}$, cf. Section 2 of [2].

2. As noted in Section 2.1 of [2], the set of self-adjoint operators on $\mathcal{H}$ that have a pure discrete spectrum is closed under taking functions of its members, and so forms a base-category $\mathcal{O}_d$ on which we can define a spectral presheaf and a coarse-graining presheaf in a manner exactly parallel to the definitions over $\mathcal{O}$.

3. For a more precise statement of the relation of $f(\sigma(\hat{A}))$ and $\sigma(f(\hat{A}))$, cf. Eq. (2.9) of [2].

To sum up: it will be heuristically helpful to report what the results in Section 3.2 imply about presheaves over $\mathcal{O}$ for those operators for which these measure-theoretic difficulties do *not* arise. As just mentioned, this will include all operators with a pure discrete spectrum; and the rest of this Section can be read as strictly true for the spectral presheaf and coarse-graining presheaf defined on $\mathcal{O}_d$.

We will do this in two stages, in the next two subsections. Both depend on the following 'definition' of what we will call the *elementary support* of a quantum state, relative to a stage (*i.e.*, relative to an operator $\hat{A}$ in $\mathcal{O}$). We say 'definition' since the infimum of a family of Borel sets is not in general Borel, so that the definition applies only in special cases, in particular in $\mathcal{O}_d$. (And we say 'elementary', since these supports are, as usual, subsets of $\mathbb{R}$, and we want to emphasise the distinction from the rigorously defined supports discussed in Section 3.2.)

Definition 3.1 *The* elementary support, $s(\psi, \hat{A})$, *of a vector state* $\psi \in \mathcal{H}$ *for a quantity* $\hat{A}$, *is the smallest set (measure-theoretic niceties apart!) of real numbers for which* ψ *prescribes probability 1 of getting a result in the set, on measurement of the physical quantity A. And similarly for a density matrix* ρ. *More precisely:*

$$s(\psi, \hat{A}) := \inf\nolimits_{\text{Borel}}\{\Delta \subseteq \mathbb{R} \mid \hat{E}[A \in \Delta]\psi = \psi\}. \qquad (3.20)$$
$$s(\rho, \hat{A}) := \inf\nolimits_{\text{Borel}}\{\Delta \subseteq \mathbb{R} \mid \text{tr}[\rho\hat{E}[A \in \Delta]] = 1\}.$$

3.3.1 Characterizing quantum valuations with elementary supports

Given this definition of supports, we can deduce a characterization of the sieve-valued valuations on $\mathcal{O}$ associated with quantum states as defined in Eqs. (3.17) and (3.18). This characterization is the analogue of those in

Theorem 3.1, *i.e.*, Eq. (3.3) of part (c), and in Theorem 3.2, *i.e.*, Eq. (3.11); and this characterization, being more concrete, is heuristically valuable.

For any $\hat{A}$, Δ, and $f : \Delta \subseteq f^{-1}(f(\Delta))$, we have $\hat{E}[f(A) \in f(\Delta)] = \hat{E}[A \in f^{-1}(f(\Delta))]$, and hence $\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in f(\Delta)]$. This gives as a sufficient condition for an arrow $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ to be in $\nu^{\psi}(A \in \Delta)$, that $f(s(\psi, \hat{A})) \subseteq f(\Delta)$. For suppose that $f(s(\psi, \hat{A})) \subseteq f(\Delta)$. Then $\hat{E}[A \in s(\psi, \hat{A})] \leq \hat{E}[f(A) \in f(s(\psi, \hat{A}))] \leq \hat{E}[f(A) \in f(\Delta)]$. So since $\hat{E}[A \in s(\psi, \hat{A})]\psi = \psi$, we have $\hat{E}[f(A) \in f(\Delta)]\psi = \psi$.

This condition, that $f(s(\psi, \hat{A})) \subseteq f(\Delta)$, is also necessary. For since $\hat{E}[f(A) \in f(\Delta)] = \hat{E}[A \in f^{-1}(f(\Delta))]$, we have that an arrow $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ is in $\nu^{\psi}(A \in \Delta)$ if and only if $f^{-1}(f(\Delta)) \supseteq s(\psi, \hat{A})$. But applying f to this last we get: $f(f^{-1}(f(\Delta))) = f(\Delta) \supseteq f(s(\psi, \hat{A}))$.

Thus we have the result (strictly in $\mathcal{O}_d$, and in $\mathcal{O}$, for those $\hat{A}, \hat{B}, \Delta$ for which measure-theoretic difficulties do not arise):

$$\nu^{\psi}(A \in \Delta) = \{f_{\mathcal{O}} : \hat{B} \to \hat{A} : f(\Delta) \supseteq f(s(\psi, \hat{A}))\}. \qquad (3.21)$$

This argument can be adapted to ν^{ρ} and $s(\rho, \hat{A})$. We use the fact that $f_{\mathcal{O}} : \hat{B} \to \hat{A} \in \nu^{\rho}(A \in \Delta)$ if and only if $\mathrm{tr}[\rho \hat{E}[A \in f^{-1}(f(\Delta))]] = 1$ if and only if $f^{-1}(f(\Delta)) \supseteq s(\rho, \hat{A})$; which, applying f, implies that $f(\Delta) \supseteq f(s(\rho, \hat{A}))$. So we get the result (again, strictly in $\mathcal{O}_d$; and in $\mathcal{O}$, measure-theoretic difficulties apart):

$$\nu^{\rho}(A \in \Delta) = \{f_{\mathcal{O}} : \hat{B} \to \hat{A} : f(\Delta) \supseteq f(s(\rho, \hat{A}))\}. \qquad (3.22)$$

Each of Eqs. (3.21) and (3.22) is clearly an analogue of part (c) of Theorem 3.1, which was

$$\alpha_{V_1}(\hat{P}) = \{i_{V_2 V_1} : V_2 \to V_1 \mid \mathbf{G}(i_{V_2 V_1})(s(\alpha, V_1)) \leq \mathbf{G}(i_{V_2 V_1})(\hat{P})\}. \qquad (3.23)$$

and of part (c) of Theorem 3.2, which was

$$\alpha_{V_1}(\hat{P}) = \{i_{V_2 V_1} : V_2 \to V_1 \mid \mathbf{I}^{\alpha}(V_1)|_{V_2} \subseteq V_1(\hat{P})|_{V_2}\} \qquad (3.24)$$

In short we see that (i) $\hat{A}$ corresponds to V_1; (ii) Δ corresponds to $\hat{P}$; (iii) f corresponds to coarse-graining by $\mathbf{G}(i_{V_2 V_1})$ in Theorem 3.1, and by restriction to V_2 in Theorem 3.2; and (iv) elementary supports correspond to the rigorous supports in Theorem 3.1 and to the intervals in Theorem 3.2.

We note incidentally that the fact that $\psi \in \mathcal{H}$ is determined by the set of 'certainly true' pairs $\langle \hat{A}, \Delta \rangle$ (*i.e.* the pairs for which $\nu(A \in \Delta) = \downarrow \hat{A}$),

together with the fact that ψ itself determines $\nu = \nu^\psi$ by Eq. (3.17), implies that $\nu = \nu^\psi$ is determined by the set of 'certainly true' pairs $\langle \hat{A}, \Delta \rangle$. This 'two-step-determination' argument (going via ψ) shows that for pure quantum states ψ, one of the sieve-valued valuations ν^ψ (a sieve-valued valuation that is induced by some or other ψ according to Eq. (3.17) is determined by the 'certainly true' *i.e.* true$_A$ assignments that it makes.

3.3.2   Supports give subobjects of Σ on $\mathcal{O}$

We recall that assigning to each $\hat{A} \in \mathcal{O}$ a subset $a(\hat{A})$ of its spectrum $\sigma(\hat{A})$ gives a subobject of Σ (rather than the global elements prohibited by the Kochen-Specker theorem) provided the assignment obeys the 'subset' version of $FUNC$: viz.,

$$f(a(\hat{A})) \subseteq a(f(\hat{A})). \tag{3.25}$$

In particular, elementary supports, as defined in Definition 3.1, induce subobjects of Σ—*i.e.* interval valuations obeying Eq. (3.25). For even if $\hat{A}$ has in part a continuous spectrum, the subset conditions:

$$f(s(\psi, \hat{A})) \subseteq s(\psi, f(\hat{A})); f(s(\rho, \hat{A})) \subseteq s(\rho, f(\hat{A})) \tag{3.26}$$

hold. So each of the interval valuations defined by

$$a^\psi(\hat{A}) := s(\psi, \hat{A}); a^\rho(\hat{A}) := s(\rho, \hat{A})) \tag{3.27}$$

is indeed a subobject of Σ. If $\hat{A}$ has pure discrete spectrum, Eq. (3.26) becomes an equality, both for a vector state and a density matrix:

$$f(s(\psi, \hat{A})) = s(\psi, f(\hat{A})); f(s(\rho, \hat{A})) = s(\rho, f(\hat{A})). \tag{3.28}$$

4   Defining sieve-valued valuations in terms of subobjects of Σ

As we have seen, the correspondence, indeed mutual determination, in Section 3 between sieve-valued and interval-valued valuations holds for a wider class of valuations than just those discussed in Sections 2.5 to 2.7. In particular, Theorems 3.1 and 3.2 used only the first clause of the definition in Section 2.5 of a sieve-valued valuation (Definition 2.3), viz. the requirement that a sieve-valued valuation obey $FUNC$. This situation suggests that it

would be worth surveying different ways of defining sieve-valued and interval-valued valuations—and the properties that ensue from these definitions. In this Section we undertake a part of such a survey. It will show in particular that the valuations we have considered are a very natural way to secure the properties listed in the other clauses of Definition 2.3.

To be precise: we will focus on the role played in the results of Section 3 by our having defined generalised valuations (with sets of morphisms as values) in terms of the partial order relation at each stage. That is, we note that in the discussion of global elements of $\mathbf{G}$ in Section 3.2.1, the results about such valuations repeatedly invoked the partial order $\leq$ in $\mathbf{G}(V) := \mathcal{L}(V)$ (cf. in particular, (i) of Theorem 3.1); and in the discussion in Section 3.2.2 of subobjects of $\mathbf{\Sigma}$, the results repeatedly invoked subsethood $\subseteq$ among subsets of the spectrum $\sigma(V)$ (cf. (i) of Theorem 3.2). In both cases, the relation is used at V_2, the coarser of two stages $V_2 \subseteq V_1$, to connect a notion 'intrinsic' to V_2 (i.e., $s(\alpha, V_2)$) and $\mathbf{I}^\alpha(V_2)$ respectively) to a notion got by coarse-graining from V_1 (i.e., $\mathbf{G}(i_{V_2 V_1})(\hat{P})$ and $V_1(\hat{P})|_{V_2} = V_2(\mathbf{G}(i_{V_2 V_1})(\hat{P}))$ respectively).

So we will now ask how the properties of valuations taking sets of morphisms as values that are defined in terms of interval valuations by using a relation R to connect a notion intrinsic to a stage V_2 to another notion got by coarse-graining from a finer stage V_1, depend upon the choice of the relation R. That is to say, we will now consider the following schema for defining from a given global element a of $\mathbf{G}$, a valuation taking sets of morphisms as values, in terms of an arbitrary binary relation R:

$$\alpha_{V_1}^{a,R}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid a(V_2) \ R \ \mathbf{G}(i_{V_2 V_1})(\hat{P})\}. \qquad (4.1)$$

The analogous general schema starting from an interval valuation a that is a subobject of $\mathbf{\Sigma}$ is

$$\alpha_{V_1}^{a,R}(\hat{P}) := \{i_{V_2 V_1} : V_2 \to V_1 \mid a(V_2) \ R \ V_1(\hat{P}) \mid_{V_2}\}. \qquad (4.2)$$

Similarly, for the case of $\mathcal{O}$ (cf. Section 3.3), the general schema is

$$\alpha_{\hat{A}}^{a,R}(\Delta) := \alpha^{a,R}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid a(\hat{B}) \ R \ f(\Delta)\}. \qquad (4.3)$$

But we will discuss only Eq. (4.1); our results carry over *mutatis mutandis* to the cases of Eqs. (4.2) and (4.3).

This leads us to ask what conditions on R in Eq. (4.1) correspond, as either necessary or sufficient conditions, to various properties of the valuation

$\alpha^{a,R}$? The following results can be immediately verified. We give them in the same order as the conditions listed in our original definition of a sieve-valued valuation (Defn. 2.3. in Section 2.5).

(i) $\alpha_{V_1}^{a,R}(\hat{P})$ is a sieve if and only if R is *stable under coarse-graining* in the sense that if $a(V_2)\,R\,\mathbf{G}(i_{V_2V_1})(\hat{P})$, then for all $V_3 \subseteq V_2$, $a(V_3)\,R\,\mathbf{G}(i_{V_3V_1})(\hat{P})$.

Since a is assumed to be a global element of $\mathbf{G}$, so that for all $V' \subseteq V$, $a(V') = \mathbf{G}(i_{V'V})a(V)$, the consequent in this condition becomes

$$a(V_3) = \mathbf{G}(i_{V_3V_2})a(V_2)\,R\,\mathbf{G}(i_{V_3V_1})(\hat{P}) = \mathbf{G}(i_{V_3V_2})\mathbf{G}(i_{V_2V_1})(\hat{P}). \qquad (4.4)$$

Since $\mathbf{G}$ is monotonic with respect to $\leq$, choosing R to be $\leq$, as we have done (cf. Eqs. (3.1) and (3.3)) is a very natural way to secure sievehood.

(ii) For any relation R whatsoever, $\alpha^{a,R}$ obeys functional composition in the form of Eq. (3.2), *i.e.*

$$\alpha_{V_2}^{a,R}(\mathbf{G}(i_{V_2V_1}(\hat{P})) = i_{V_2V_1}^*(\alpha_{V_1}^{a,R}(\hat{P})). \qquad (4.5)$$

To see this, note that in the argument of part (b) of Theorem 3.1, any relation R could be substituted for $\leq$.

(iii) $\alpha^{a,R}$ obeys the null proposition condition, *i.e.* $\alpha_{V_1}^{a,R}(\hat{0}) = \emptyset$, if and only if there is no $V_2 \subseteq V_1$ with $a(V_2)\,R\,\hat{0}$ (since $\hat{0}$ coarse-grains to $\hat{0}$). Provided a always assigns a non-zero projector, this condition is satisfied by our choice of R as $\leq$.

(iv) $\alpha^{a,R}$ obeys the monotonicity condition, i.e. if $\hat{P} \leq \hat{Q} \in \mathcal{L}(V_1)$ then $\alpha_{V_1}^{a,R}(\hat{P}) \leq \alpha_{V_1}^{a,R}(\hat{Q})$, if and only if R is *isotone under coarse-graining* in the sense that

$$[\hat{P} \leq \hat{Q} \text{ and } a(V_2)\,R\,\mathbf{G}(i_{V_2V_1})(\hat{P})] \Rightarrow a(V_2)\,R\,\mathbf{G}(i_{V_2V_1})(\hat{Q}). \qquad (4.6)$$

Since $\mathbf{G}$ is monotonic with respect to $\leq$, the natural sufficient condition for this is that the relation R is stable under taking larger elements on its left-hand side, i.e.

$$[\hat{S} \leq \hat{T} \text{ and } a(V_2)\,R\,\hat{S}] \Rightarrow a(V_2)\,R\,\hat{T}. \qquad (4.7)$$

Again, the natural choice for satisfying this is that R is taken to be $\leq$.

(v) $\alpha^{a,R}$ obeys the exclusivity condition, that if $\hat{P}\hat{Q} = 0$ and $\alpha_{V_1}^{a,R}(\hat{P}) = \downarrow_{V_1} = \text{true}_{V_1}$, then $\alpha_{V_1}^{a,R}(\hat{Q}) < \text{true}_{V_1}$, if and only if

If $\hat{P}\hat{Q} = 0$ and $\forall V_2 \subseteq V_1, a(V_2)\,R\,\mathbf{G}(i_{V_2V_1})(\hat{P})$, then $\exists V_3 \subseteq V_1$

such that not $a(V_3)\,R\,\mathbf{G}(i_{V_3V_1})(\hat{Q})$. (4.8)

Here, the condition in terms of R is not very different from exclusivity in the original form; and so seems not very illuminating. But provided a always assigns a non-zero projector, this condition is satisfied by our choice of R as $\leq$. For if $\hat{P}\hat{Q} = 0$ and, for all $V_2 \subseteq V_1$ we have $a(V_2) \leq \mathbf{G}(i_{V_2V_1})(\hat{P})$, then $a(V_1) \leq \hat{P}$, so that $\neg a(V_1) \leq \hat{Q}$ (since $a(V_1) \neq \hat{0}$), and hence $\alpha_{V_1}^{a;\leq}(\hat{Q}) \neq \text{true}_{V_1}$.

(vi) $\alpha^{a,R}$ obeys the unit proposition condition, that $\alpha_{V_1}^{a,R}(\hat{1}) = \text{true}_{V_1}$, if and only if for all $V_2 \subseteq V_1$ we have $a(V_2)\,R\,\mathbf{G}(i_{V_2V_1})(\hat{1}) = \hat{1}_{V_2}$ (since $\hat{1}$ coarse-grains to $\hat{1}$). Again, the natural way for satisfying this is to choose the relation R to be $\leq$.

These results show that there is a natural choice of the relation R, viz. $R := \leq$, which is sufficient to yield *all* of the properties (i.e. clauses (i)–(v) of Definition 2.3), provided a always assigns a non-zero projector. And again this conclusion reflects the theme of Section 3, viz. the correspondence between sieve-valued and interval-valued valuations, and in particular the characterization Eq. (3.3) (part (c) of Theorem 3.1).

Furthermore, analogous results are easily verified for the schemas in Eqs. (4.2) and (4.3). More precisely: taking the relation R as subsethood, $\subseteq$ in these schemas is sufficient for these properties, provided some 'regularity conditions' hold. These conditions include:

1. The analogue of the proviso above, that a always assigns a non-zero projector (*i.e.* that a always assigns a non-empty subset).

2. The requirement that a defines a 'tight' subobject of $\mathbf{\Sigma}$, in the sense of Eq. (3.9).

3. For the case of $\mathcal{O}$, (*i.e.* schema 4.3) for all bounded Borel functions f and all $\hat{A}$, we have the equality $f(\sigma(\hat{A})) = \sigma(f(\hat{A}))$ (as always occurs if $\hat{A}$ has pure discrete spectrum), not merely $f(\sigma(\hat{A})) \subseteq \sigma(f(\hat{A}))$ as in Eq. (3.19).

But we will not go into details of just how these regularity conditions make choosing R as subsethood $\subseteq$ sufficient for the various properties (i)–(vi) above, for the schemas of Eqs. (4.2) and (4.3). But again the conclusion—that taking R as subsethood in these schemas is sufficient for these properties—reflects the correspondence in Section 3 between sieve-valued and interval-valued valuations; and in particular the characterizations, Eq. (3.11) (for $\mathcal{V}$: part (c) of Theorem 3.2), and Eqs. (3.21) and (3.22) (for $\mathcal{O}$).

5   Conclusion

In this paper, we have extended our topos-theoretic perspective on the assignment of values to quantities in quantum theory; principally using the base category $\mathcal{V}$ of commutative von Neumann algebras introduced in [4]. In Section 3, we compared our sieve-valued valuations with interval valuations based on the notion of supports. This discussion (adding to some results reported in Section 4 of [4]) had as its main theme a correspondence (mutual determination) between certain sieve-valued valuations and corresponding interval valuations. This correspondence was summed up (for $\mathcal{V}$) in the characterizations given in parts (c) of Theorems 3.1 and 3.2, Eqs. (3.3) and (3.11); and summed up more heuristically for $\mathcal{O}$, in Eqs. (3.21) and (3.22).

In Section 4, we generalized this discussion: we gave a partial survey of how in defining sieve-valued valuations in terms of interval valuations, certain properties of the sieve-valued valuations derive from the properties of the binary relation R used in the definition. This survey again showed the naturalness of our previous definitions. For taking R to be the partial order $\leq$ among projectors, or to be subsethood $\subseteq$ among subsets of spectra, was a natural and simple sufficient condition for the defined valuations to obey the clauses of our original definition of sieve-valued valuations (Definition 2.3).

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