Logical and Philosophical Remarks on Quasi-Set Theory

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Abstract

Quasi-set theory is a theory for dealing with collections of indistinguishable objects. In this paper we discuss some logical and philosophical questions involved with such a theory. The analysis of these questions enable us to provide the first grounds of a possible new view of physical reality, founded on an ontology of non-individuals, to which quasi-set theory may constitute the logical basis.

Keywords: quasi-sets, logic of quantum mechanics, identity, non-individuality, quantum ontology, quasi-objects

1 Introduction

"[W]e ought to 'think with the learned, and speak with the vulgar'."
G. Berkeley, *Of the Principles of Human Knowledge*, §51.

Quasi-set theory ($\mathfrak{Q}$) is a theory conceived to handle collections of indistinguishable objects, that is, objects for which the relation of equality (or identity, here employed as a synonym of equality) is, strictly speaking, meaningless. Informally, a quasi-set (qset) may be such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal (not defined by means of ordinals, as usual). From the formal point of view, the theory is constructed so that it extends standard Zermelo-Fraenkel with *Urelemente* (ZFU); thus standard sets (of ZFU) can be viewed as particular qsets (that is, there are qsets that have all the properties of the sets of ZFU; the objects in $\mathfrak{Q}$ corresponding to the *Urelemente* of ZFU are termed M-atoms). But the theory encompasses another kind of *Urelemente*, the m-atoms, to which the standard theory of identity does not apply.

When $\mathfrak{Q}$ is used in connection with quantum physics, these m-atoms are thought of as representing quantum objects (henceforth, q-objects). The m-atoms are viewed as non-individuals in a certain sense, and it is mainly (but not only) to deal with collections of m-atoms that the theory was conceived. So $\mathfrak{Q}$ is a theory of generalized sets, involving non-individuals that, here, will always be q-objects. For details about $\mathfrak{Q}$ and about its historical motivations, see [3, p. 119], [6], [10, Chap. 7], [11], [15].

In $\mathfrak{Q}$, the so called 'pure' qsets have only q-objects as elements, and to them it is assumed that the usual notion of identity cannot be applied (that is, $x = y$, so as its negation, $x \neq y$, is not a well formed formula if x and y stand for q-objects). Notwithstanding, there is a primitive relation $\equiv$ of indistinguishability having the properties of an equivalence relation, and a concept of *extensional identity*, not holding among m-atoms, is defined and has the properties of standard identity of classical set theories. Since the elements of a qset may have properties (and satisfy certain formulas), they can be regarded as *indistinguishable* without turning to be *identical* (that is, being *the same* object), that is, $x \equiv y$ does not entail $x = y$. Since the relation of equality (and the concept of identity) does not apply to m-atoms, they can also be thought of as entities devoid of individuality. We remark further that if the 'property' $x = x$ (to be identical to itself, or *self-identity*) is included as one of the properties of the objects, then the so called Principle of Identity of Indiscernibles in the form $\forall F(F(x) \leftrightarrow F(y)) \rightarrow x = y$ is a theorem of classical second order logic, and hence there cannot be indiscernible but not identical entities (in particular, non-individuals). Thus, if self-identity is linked to the concept of non-individual, and if quantum objects are to be considered as such, then such entities fail to be self-identical, and a logical framework to accommodate them is in order (see [10] for further argumentation, but see the last section).

We have already discussed at length in the references given above (so as in other works) the motivations to build a quasi-set theory, and we shall not return to these points here.[1] We wish just to reinforce some of the claims already given in other works by presenting more arguments that favor the consideration and development of the logical basis of a metaphysics of non-individuals, which we think is in complete agrement with some intuitions regarding quantum objects and quantum physics, at least according to some of their interpretations. Of course we shall provide here only the basic ideas, so this paper can be regarded as a further step in the directions of providing an alternative formulation of a quantum theory which considers the non-individuality of quantum objects *right from the start*, as proposed by Heinz Post [19].

2 The talk of non-individuals

Although we will not recount the whole history here,[2] we shall recall some of the main ideas which have conduced to no-individual quanta, in order to reinforce our claims. In the very beginnings of quantum theory, Planck, when deriving his radiation law, in an 'act of desperation', turned to Boltzmann's combinatorial approach, but instead of being concerned with the distributivity of *individuals* over states (what is typical in Maxwell-Boltzmann statistics, where permutations of indistinguishable particles give raise to distinct physical states), considered the distribution of indistinguishable *non-individual* quanta over oscillators. In other words, if a given total energy is divided into a finite number P of energy elements, according to him the energy can be distributed over N oscillators according to the formula

$$\frac{(N+P-1)!}{(N-1)!P!}.$$

[1] But see [4], [5], [9], [12], [15].

[2] The history of non-individuality is given in [10, Chap. 3].

Later, it was noted by Ehrenfest that the division by $P!$ entails the non-individuality of quanta, since permutations of indistinguishable particles no longer conduce to distinct states.[3] But this basic idea according to which quantum objects would be *non-individuals* came also from other fathers of quantum mechanics. For instance, in 1926, Born, while defending the corpuscular as opposed to wave-like conception, acknowledged that these corpuscles could not be identified as individuals (cf. [10]). In the same year, Heisenberg noted that Einstein's theory of the ideal quantum gas implies that the "individuality of the corpuscle is lost" (ibid, §3.3).[4] Shortly after, Weyl, wrote that "one cannot demand an alibi of an electron" as two identical twins Mike and Ike can in saying "I am Mike", while the another says "I am Ike" when no one can identify them (ibid., p. 105). Schrödinger, when talking about elementary particles, was more specific, providing an idea that can be 'logically' pursued, arguing that the notion of equality (and also of inequality) does not apply to these particles. In fact, he writes that, in the face of quantum physics,

> "... we have ... been compelled to dismiss the idea that ... a particle is an individual entity which retains its 'sameness' forever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no 'sameness' at all". [22, p. 17]

And he continues,

> "I beg to emphasize this and I beg you to believe it: It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning". [22, pp. 17-8]

Still in the twenties, Bose, and later Einstein, yet implicitly, made similar considerations about quanta when proposing their 'statistics', which also presuppose the non-individuality of quanta. Later, the use of symmetric and anti-symmetric state functions reinforced the needs for discussing some questions about the nature of quantum objects, which could no longer be avoided. For instance, in order to capture non-individuality, in the standard formalism (by means of Hilbert spaces) the significant vectors (or state-functions) are to be symmetric or anti-symmetric with respect to the change of variables. This raises a lot of philosophical questions, as those advanced by Redhead and Teller some time ago, including their suggestion of the use of the alternative Fock space formalism [20, 21].

In what respects non-individuals, we may address the following question which we didn't find in the relevant literature: physicists do certain experiments in which an electron (this is also done with other quanta, but electrons suffice here) is trapped by magnetic fields. Then they can say that there is a single electron there and, if they wish, they can even name it, say calling her "Nancy". Is there an *individual* trapped in the apparatus? Apparently not, for in connection with elementary particles, terms like "single", "same", "other", "distinct from", etc., should be plainly meaningless; in effect, they presuppose the relation of identity. Without equality, we cannot say meaningfully that we did catch a *single* electron in a trap, that is, with the help of

[3] Ibid., pp. 85ff.

[4] The history of Planck's formula can also be seen in [10, Chap. 3], where further references and details are provided.

a physical device. However, to deny such assertion, saying that there is no electron (Nancy) in the trap, is also meaningless.

Thus, since physicists *talk* about single particles and since apparently they couldn't, we are faced with a basic problem. To reinforce the point, let us give an example of such a talk. As D. Wick writes,

> "In 1979 in Heidelberg, Germany, Dr. Werner Neuhauser looked up from the eye piece of a low-power microscope and fancied he heard the ghost of Mach whispering: 'Now I believe in atoms'. Neuhauser had just glimpsed what appeared to be a bright blue star floating in the void; it was a single barium ion, caught with an electromagnetic trap and fluorescing in a laser beam. Thus transpired the first observation of an isolated atom using a lens; soon one would be glimpsed with the naked eye as well." [25, p. 137]

Wick adds that

> "Neuhauser collaborated on the 1979 experiment with M. Hohenstatt and P. E. Toschek of the University of Heidelberg and H. Dehmelt of the University of Washington in Seatle. Hans Dehmelt in particular had a lifelong dream of capturing and suspending in space a single atomic particle; he had been promoting the idea for more than a decade, against considerable skepticism from the physicist community. His quest for the perfect atomic measurement was rewarded with the Nobel Prize in 1989. Dehmelt's story is also directly relevant to our concerns, as his work produced the only direct clash between Bohr's philosophy and experimental reality to this date." (ibid., p. 138)

Wick also says that

> "With D. Wineland and P. Ekstrom, Dehmelt first bagged a solitary electron in 1973, trapping it in an invisible cage of electric and magnetic fields. But a few more years were to pass before he and his collaborators in Seatle learned how to cool down and interrogate their prisoner. Once that was accomplished, they could study it at their leisure." (ibid., p. 138)

More generally, are we justified from the logical point of view to assert or deny that in a Bose-Einstein condensate (BEC) there are, say, 1000 atoms? As it is well known, in a BEC, where atoms are cooled till near the absolute zero, all the atoms turn in the same quantum state. Despite they continue to be many, there is no difference among them: they are absolutely indiscernible. How to speak of these entities by using standard languages without the subterfuge of using ad hoc assumptions like symmetric and anti-symmetric vectors? To surmount these logical difficulties, as is clear in the theory of quasi-sets, collections or agglomerations like atoms in a BEC need to have a cardinal number, but not an ordinal.[5]

Thus quasi-set theory, enabling that certain collections of q-objects have a cardinal, but not an associated ordinal, may be useful to accommodate the formal aspects of such conglomerates. But why to do that, if standard languages work fine in present day physics? The answer is that in fact physics works fine from the physical point of view (and perhaps in the mathematical sense too), but of course not from the philosophical or foundational one. In fact, Tian Cao remarks that

[5] G. Domenech and F. Holik are investigating a version of the quasi-set theory in which even a cardinal cannot be associated to certain quasi-sets. Their motivation comes, obviously, from quantum field theory. See [7].

> "Some physicists claim that they feel no need for foundations. This does not mean that their scientific reasoning is detached from the logical structure of concepts in their discipline, which represents the causal structure of the domain under investigation. What it means is only that their intuition at the heuristic level, which usually takes the accepted understanding of the foundations for granted, is enough for daily researches. However, in a situation in which complicated conceptual problems cannot be understood properly at the heuristic level, or in a period of crisis when basic conceptions have to be radically revised for the future development of the discipline, a clarification of the foundations is badly needed and cannot be avoided without hindering the discipline from going further. Nonetheless, the difficulties in grasping the questions in a mathematical precise way and in the conceptual analysis of the unfamiliar logical and ontological foundations would deter most practising physicists from doing so." [2]

The case involving non-individuals is of course one of the typical cases where conceptual problems appear. We could go further and say that there are also logical and mathematical problems involved in the sense anticipated above. Another disputable topic involved with the admissibility of non-individuals is the logical sense of quantifiers. In what sense is it legitimate to assert that "There exists an electron with some property"? Intuitively, quantifiers seem to involve us with equality. In effect, to assert that there is an object with property P seems equivalent to affirm that there exists a distinguishable object with P. Similarly, the universal quantifier is linked to a classical set of distinct entities. Since quasi-set theory involves quantification, we need to explain the meaning of quantification in this theory, as we shall do below.

Thus, the informal talk of the physicists, say in speaking of "this" quanta in distinction to "another" one, cannot make sense if these entities are non-individuals. This kind of talk must be understood metaphorically. In other words, we are faced with a situation that resembles Berkeley's dictum recalled at the beginning of this paper: we speak with a 'vulgar' language (based on classical logic and mathematics), saying things like "this", "that", etc. but, rigorously, reason with the 'learned' language, that is, with a language that does not involve such terms.

3 The resources of quasi-set theory

A metaphysics of non-individuals is possible, and perhaps even necessary to enlighten the philosophical foundations of quantum physics. The formal aspects of such a metaphysics can be captured by quasi-set theory. Within quasi-set theory, we suppose that the essential difference between individuals and non-individuals concerns the fact that equality makes sense for the former but not for the latter. Thus, non-individuals are basically objects refracting to identity. Hence, an "as if" semantics is provided by the observation that our non-individuals, in our present case, quantum objects, can be classified in equivalence classes (electrons, protons, etc.), so to say composing a partition of the collection of non-individuals. This is the way we understand, in particular, quantification within quasi-set theory. Formally, this device of thinking of the objects as 'classical', that is, as obeying classical logic, and then making then indiscernible simply for allocating them into some equivalence classes, reminds us to Weyl's approach [24, App. B] (see [10]); but such a procedure, for us, constitute only a *façon de parler*, since Weyl starts with (standard) sets, that is, with collections of *distinguishable* individuals. In synthesis, from an intuitive point of view, classical logic without equality can be applied as if the non-individuals where individuals.

Furthermore, we remark that the theory of quasi-sets is a formal system whose semantics cannot be constructed in the usual set theories like Zermelo-Fraenkel, as it has become clear, since collections of q-objects would not obey the ZF postulates. Anyhow, taking into account an intuitive "interpretation" via equivalence classes (in some classical set theory), one is allowed to discuss topics related to non-individuals, such as elementary particles, in everyday language, with the resort to equality. This is, in some circumstances, the way of the physicist, who employs a figurative language (or 'vulgar', according to Berkeley's dictum at the beginnings); however, the only rigorous way to deal with them seems to appeal to our formal system, if one does want to treat non-individuals seriously.

4 Quasi-sets: general ideas

As we have said, intuitively speaking a quasi-set is a collection of indistinguishable (but not identical) objects. This of course is not a strict 'definition' of a quasi-set, acting more or less as Cantor's 'definition', according to which a set is "any collection into a whole M of definite and separate objects m of our intuition or our thought" [1, p. 85], but see also the discussion in [10, §6.4]), giving no more than an intuitive account of the concept. But we should realise that it seems reasonable, due to the above argumentation (which of course does not cover all the situations presented by modern physics),[6] to search for a mathematical theory which considers, without dodges, collections of *truly* indistinguishable objects. In characterizing such collections (quasi-sets), we have followed specially Schrödinger's opinion that the concept of identity cannot be applied to elementary particles, and developed the theory by posing that the expression $x = y$ is not generally a well-formed formula (and, as a consequence, its negation $x \neq y$ is also not a formula). This enable us to consider logico-mathematical systems in which identity and indistinguishability are separated concepts; that is, these concepts do not reduce to one another as in standard set theories.

In particular, forms of logic –called Schrödinger logics– have been introduced for which '$a = a$' cannot be inferred for certain objects a (see [4], [5], [10, Chap. 8]). For all other entities classical logic is maintained.[7] Correspondingly, quasi-set theory is the 'set-theoretical' version of this idea.

4.1 The basic ideas of quasi-set theory

Quasi-set theory $\mathfrak{Q}$ allows two kinds of Urelemente:[8] the m-atoms, whose intended interpretation are the quanta, and the M-atoms, which stand for macroscopic objects,

[6] The interested reader should have a look in the papers by Dalla Chiara listed in our references. She and Toraldo di Francia developed an alternative theory to deal with quantum objects; a comparison between their theory an quasi-set theory is given in [6] (and also mentioned in [10]).

[7] This is a characteristic of these systems, but in principle it is possible to define Schrödinger logics associated with other non-classical logics. In [5], a system encompassing modal operators is considered.

[8] This section is based in [13].

to which classical logic is supposed to apply.[9] Quasi-sets are the collections obtained by applying ZFU-like (Zermelo-Fraenkel plus *Urelemente*) axioms to a basic domain composed of m-atoms, M-atoms and aggregates of them. The theory still admits a primitive concept of quasi-cardinal which intuitively stands for the 'quantity' of objects in a collection. This is made so that certain quasi-sets x (in particular, those whose elements are q-objects) may have a quasi-cardinal, written $qc(x)$, but not an ordinal. It is also possible to define a translation from the language of ZFU into the language of $\mathfrak{Q}$ in such a way so that there is a 'copy' of ZFU in $\mathfrak{Q}$ (the 'classical' part of $\mathfrak{Q}$). In this copy, all the usual mathematical concepts can be defined, and the 'sets' (in reality, the '$\mathfrak{Q}$-sets') turn out to be those quasi-sets whose transitive closure (this concept is like the usual one) does not contain m-atoms.[10]

To understand the basic involved ideas, let us consider the three protons and the four neutrons in the nucleus of a ^{7}Li atom. As far as quantum mechanics goes, nothing distinguishes these *three* protons. If we regard these protons as forming a quasi-set, its quasi-cardinal should be 3, and there is no apparent contradiction in saying that there are also 3 subquasi-sets with 2 elements each, despite we can't distinguish their elements, and so on. So, it is reasonable to postulate that the quasi-cardinal of the power quasi-set of x is $2^{qc(x)}$. Whether we can distinguish among these subquasi-sets is a matter which does not concern logic.

In other words, we may consistently (with the axiomatics of $\mathfrak{Q}$) reason as if there are three entities in our quasi-set x, but x must be regarded as a collection for which it is not possible to discern its elements as individuals. The theory does not enable us to form the corresponding singletons. The grounds for such kind of reasoning has been delineated by Dalla Chiara and Toraldo di Francia as partly theoretical and partly experimental. Speaking of electrons instead of protons, they note that in the case of the helium atom we can say that there are two electrons because, *theoretically*, the appropriate wave function depends on six coordinates and thus "we can therefore say that the wave function has the same degrees of freedom as a system of two classical particles".[11] Dalla Chiara and Toraldo di Francia have also noted that, "[e]xperimentally, we can ionize the atom (by bombardment or other means) and extract two separate electrons ..." (ibid.).

Of course, the electrons can be counted as two only at the moment of measurement; as soon as they interact with other electrons (in the measurement apparatus, for example) they enter into entangled states once more. It is on this basis that one can assert that there are two electrons in the helium atom or six in the 2p level of the sodium atom or (by similar considerations) three protons in the nucleus of a ^{7}Li atom (and it may be contended that the 'theoretical' ground for reasoning in this way also depends on these experimental considerations, together with the legacy of classical metaphysics). On this basis it is stated the axiom of 'weak extensionality' of $\mathfrak{Q}$, which says that those quasi-sets that have the same quantity of elements of the same sort (in the sense that they belong to the same equivalence class of indistinguishable objects) are indistinguishable.

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[9] Of course we could develop theories based also on other logics.

[10] So, we can make sense to the primitive concept of quasi-cardinal of a quasi-set x as being a cardinal defined in the 'classical' part of the theory. The reason to take the concept of quasi-cardinal as a primitive concept will appear below, when we make reference to the distinction between cardinals and ordinals.

[11] Op. cit., p. 268. This might be associated to the legacy of Schrödinger, who says that this kind of formulation "gets off on the wrong foot" by initially assigning particle labels and then permuting them before extracting combinations of appropriate symmetry [23].

This axiom has interesting consequences. As we have said, there is no space here for the details, but let us mention just one of them which is related to the above discussion on the non observability of permutations in quantum physics, which is one of the most basic facts regarding indistinguishable quanta. In standard set theories, if $w \in x$, then of course $(x - \{w\}) \cup \{z\} = x$ iff $z = w$. That is, we can 'exchange' (without modifying the original arrangement) two elements iff they are *the same* elements, by force of the axiom of extensionality. But in $\mathfrak{Q}$ we can prove the following theorem, where z' (and similarly w') stand for a quasi-set with quasi-cardinal 1 whose only element is indistinguishable from z (respectively, from w –the reader shouldn't think that this element *is identical to either* z or w, for the relation of equality doesn't apply here; the set theoretical operations can be understood according to their usual definitions):

[Unobservability of Permutations] Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that

$$(x - z') \cup w' \equiv x$$

Supposing that x has n elements, then if we 'exchange' their elements z by correspondent indistinguishable elements w (set theoretically, this means performing the operation $(x - z') \cup w'$), then the resulting quasi-set remains *indistinguishable* from the original one. In a certain sense, it is not important whether we are dealing with x or with $(x - z') \cup w'$. This of course gives a 'set-theoretical' sense to the following claim made by Roger Penrose:

> "[a]ccording to quantum mechanics, any two electrons must necessarily be completely identical [in the physicist's jargon, that is, indistinguishable], and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person's brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would be happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves" [18, p. 32].

Within $\mathfrak{Q}$ we can express that 'permutations are not observable', without necessarily introducing symmetry postulates, and in particular to derive 'in a natural way' the quantum statistics (see [16], [10, Chap. 7]).

4.2 An alternative quasi-set theory

Suppose we are to assume a more radical metaphysical position by saying that identity cannot be applied significantly even to macroscopic objects. Thus, the identification of *that* pencil as mine is only a product of subsidiary assumptions (which are not physical laws properly speaking), for after a brief absence, I have no more grounds to assert with absolute certainty that that pen is in fact *that one* I saw before.[12] Can we treat even situations like these? The answer is yes, for there is an alternative version of quasi-set theory in which we do not have identity for any object whatever, which can be motivated in the fact that since macroscopic objects are such that the replacement of elementary q-objects that constitute them by q-objects of the same species do not "alter" these objects (as Penrose's phrase mentioned above shows), one could argue that identity does not even apply to macroscopic items, at least on the basis of sole physics.

However, in certain cases, owing to particular reasons, we say that a certain man is identical to himself or that I am identical to myself, but these statements are not founded only on physical laws (these assertions belong to the 'vulgar' language, not to the 'learned' one). This way, identity, properly speaking, do not apply to any object whatsoever, but only in some cases it can be *ad hoc* characterized.

Therefore, an alternative theory to treat the foundations of quantum mechanics, in some of its formulations, would be the theory we term $ZFU^{\equiv}$, i.e., ZFU in which the equality relation is substituted by an equivalence relation $\equiv$ whose field of application is the the domain of all objects; let us recall that, in $\mathfrak{Q}$, $\equiv$ is an equivalence relation. $ZFU^{\equiv}$ may be obtained from $\mathfrak{Q}$, if we drop the distinction between m-atoms and M-atoms and consider all objects as q-objects. To explore this possibility within a more general metaphysics of non-individuality even of macroscopic objects seems to be interesting.

5 Logical issues on individuation

Regarding present day studies in metaphysics, Jonathan Lowe is an important philosopher. He characterized individuality in terms of countability. According to him, "the items to be counted should possess determinate identity conditions, since each should be counted just once and this presupposes that it be determined distinct from any other item that is to be included in the count" (see [10, p. 371]). This is one of the common ways of doing that. In considering it, a question arises: how are we to understand a 'countable plurality' of items about which we cannot deny that it is indeterminate whether they are identical to themselves or not? In particular, in what sense can such a plurality be said to be countable? Countability is precisely what is problematic about quantum entities. Teller, for example, has emphasized that quantum objects cannot be counted but only *aggregated*, and invokes the analogy of money in a bank account to exemplify his claims (as exposed in [10]). In this sense, I can say that I have 100 pounds in my current account but I can't go in and point to a particular pound and say, 'that's mine'. This offers a further perspective on the way in which quantum particles might be regarded as non-individuals.

How might we represent this distinction between a 'countable plurality' and an aggregate? One way is to note that it corresponds to which holds between ordinality and cardinality respectively. Hence an 'aggregate' of quanta would have the latter but not the former, since we can say on experimental and theoretical grounds that there are, say, seven electrons in an atom, but we cannot count them, in the sense of putting them in a series and establishing an ordering.

[12] David Hume has put this identification in the habit, as it is well known.

Thus, in our opinion quasi-set theory, in all its forms, can function as a foundation for non-relativistic quantum mechanics and, as we can show, to relativistic approaches too [7], [10, Chap. 9].[13] A book like [17] can be developed o the basis of quasi-set theoretical ideas ideas, as we shall show in future works. So, today we have at our disposal the traditional versions of quantum theories and the quasi-set version. We think that present day physics, being compatible with both views, is unable to decide which metaphysics is the better; as discussed in [10, §4.5], quantum mechanics support different metaphysical positions, and physics keeps metaphysics underdetermined.[14]

On the other hand, quasi-set theory is the tool appropriate to philosophers who are interested in an ontology or in a metaphysics of non-individuality. These topics will be postponed to be discussed in other works.

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