Concepts of Symmetry
in the Work of Wolfgang Pauli

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Abstract

"Symmetry" was one of the most important methodological themes in 20th-century physics and is probably going to play no lesser role in physics of the 21st century. As used today, there are a variety of interpretations of this term, which differ in meaning as well as their mathematical consequences. Symmetries of crystals, for example, generally express a different kind of invariance than gauge symmetries, though in specific situations the distinctions may become quite subtle. I will review some of the various notions of "symmetry" and highlight some of their uses in specific examples taken from Pauli's scientific œvre.

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Contents

1 General Introduction

In the Introduction to Pauli's Collected Scientific Papers, the editors, Ralph Kronig and Victor Weisskopf, make the following statement:

> *It is always hard to look for a leading principle in the work of a great man, in particular if his work covers all fundamental problems of physics. Pauli's work has one common denominator: his striving for symmetry and invariance. [...] The tendency towards invariant formulations of physical laws, initiated by Einstein, has become the style of theoretical physics in our days, upheld and developed by Pauli during all his life by example, stimulation, and criticism. For Pauli, the invariants in physics where the symbols of ultimate truth which must be attained by penetrating through the accidental details of things. The search for symmetry and general validity transcend the limits of physics in Pauli's work; it penetrated his thinking and striving throughout all phases of his life, in all fields of philosophy and psychology.*"([38], Vol. 1, p. viii)

Indeed, if I were asked to list those of Pauli's scientific contributions which make essential use of symmetry concepts and applied group theory, I would certainly include the following, which form a substantial part of Pauli's scientific œvre:[1]

> Relativity theory and Weyl's extension thereof (1918-1921), the Hydrogen atom in matrix mechanics (1925), exclusion principle (1925), anomalous Zeeman effect and electron spin (1925), non-relativistic wave-equation for spinning electron (1927), covariant QED (1928, Jordan), neutrino hypothesis (1930), Kaluza-Klein theory and its projective formulation (1933), theory of γ-matrices (1935), Poincaré-invariant wave equations (1939, Fierz), general particle statistics and Lorentz invariance (1940, Belinfante), spin-statistics (1940), once more General Relativity and Kaluza-Klein theory (1943, Einstein), meson-nucleon interaction and differential geometry (1953), CPT theorem (1955), β-decay and conservation of lepton charge ('Pauli group', 1957), unifying non-linear spinor equation (collaboration with Heisenberg, 1957-58), group structure of elementary particles (1958, Touschek).

Amongst the theoretical physicists of his generation, Pauli was certainly outstanding in his clear grasp of mathematical notions and methods. He had

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[1] Two of the listed themes, "meson-nucleon interaction and differential geometry" and "unifying non-linear spinor equation", were never published in scientific journals (in the second case Heisenberg published for himself without Pauli's consent) but can be followed from his letters and manuscripts as presented in [45].

a particularly sober judgement of their powers as well as their limitations in applications to physics and other sciences. Let us once more cite Kronig and Weisskopf:

> *Pauli's works are distinguished by their mathematical rigour and by a thorough and honest appraisal of the validity of assumptions and conclusions. He was a true disciple of Sommerfeld in his clear mathematical craftsmanship. By example and sharp criticism he constantly tried to maintain a similarly high standard in the work of other theoretical physicists. He was often called the living conscience of theoretical physicists.* ([38], Vol. 1, p. viii)

It seems plausible that this critical impregnation dates back to his school-days, when young Pauli read, for example, Ernst Mach's critical analysis of the historical development of the science of mechanics, a copy of which Pauli received as a present from his Godfather (Mach) at around the age of fourteen. Mach's "Mechanik", as this book is commonly called, starts out with a discussion of Archimedes' law of the lever, thereby criticising the following symmetry consideration ([43], p. 11-12): Imagine two equal masses, M, and a perfectly stiff and homogeneous rod of length L, both being immersed into a static homogeneous vertical gravitational field, where the rod is suspended at its midpoint, m, from a point p above; see Fig. 1. What happens if we attach the two equal masses to the ends of the rod and release them simultaneously without initial velocity? An immediate symmetry argument suggests that it stays horizontal; it might be given as follows: Everything just depends on the initial geometry and distribution of masses, which is preserved by a reflection at the plane perpendicular to the rod through p and m. Suppose that after release the rod dropped at one side of the suspension point m, then the mirror image of that process would have the same initial condition with the rod dropping to the other side. This is a contradiction if the laws governing the process are assumed to be reflection symmetric and deterministic (unique outcome for given initial condition). This argument seems rigorous and correct. Now, how does one get from here to the law of the lever? The argument criticised by Mach is as follows: Assume that the condition for equilibrium depends only on the amount of mass and its suspension point on the rod, but not on its shape. Then we may replace the mass to the left of m by two masses of half the amount each on a small rod in equilibrium, as shown in the second (upper right) picture. Then replace the suspension of the small rod by two strings attached to the left arm of the original rod, as shown in the third (lower left) picture, and observe that the right one is just under the suspension point m, so that it does not disturb the equilibrium if it were cut away as in the last (lower right) picture of Fig. 1. The weak point in the argument is clearly the transition from the second to the third picture: There is no *global* symmetry connecting them, even though locally, i.e. regarding the small rod only, it connects two
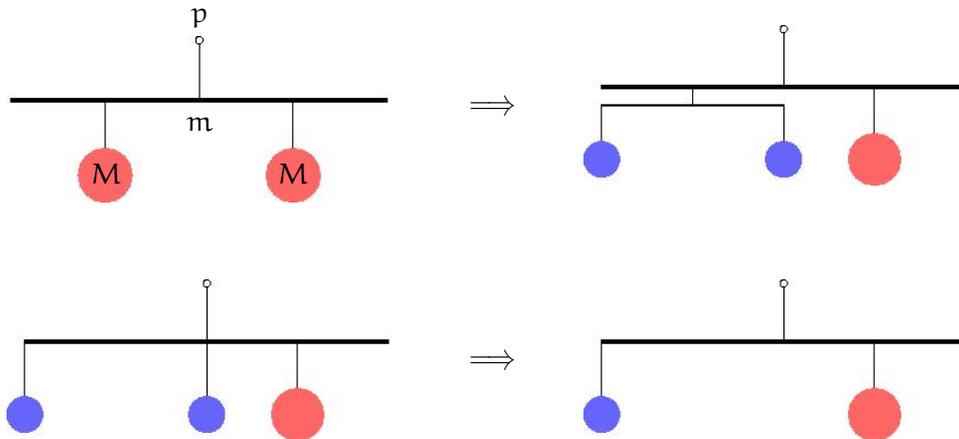
Figure 1: The law of the lever 'derived' from alleged symmetry considerations. The step from the upper right (second) to the lower left (third) picture does not follow. The small (blue) balls represent half the mass of the big (red) balls.

equilibrium positions. It is easy to see that, in fact, the assumption that a global equilibrium is maintained in this change is *equivalent* to Archimedes' law of the lever. This example shows (in admittedly a fairly trivial fashion) that alleged symmetry properties can work as a *petitio principii* for the law to be derived. This is essentially the criticism of Mach.

The reason why we consider this 'derivation' of the law of the lever to be a *petitio principii* is that we have other, physically much more direct ways to actually derive it from *dynamical* first principles. From that point of view the alleged symmetry is to be regarded as an artifact of the particular law and certainly not vice versa. The observed symmetry requires an explanation in terms of the dynamical laws, which themselves are to be established in an independent fashion. This is how we look upon, say, the symmetry of crystals or the symmetric shape of planetary orbits.

On the other hand, all fundamental dynamical theories of 20th century physics are motivated by symmetry requirements. They are commonly looked at as particularly simple realisations of the symmetries in question, given certain *a priori* assumptions. It is clear that, compared to the previous example, there are different concepts of symmetry invoked here. However, there also seems to be a shift in attitude towards a more abstract understanding of 'physical laws' in general.

What makes Pauli an interesting figure in this context is that this shift in attitude can be traced in his own writings. Consider Special Relativity as an example, thereby neglecting gravity. One may ask: What is the general relation between the particular symmetry (encoded by the Poincaré group) of spacetime and that very same symmetry of the fundamental interactions

(weak, strong, and electromagnetic, but not gravity)? Is one to be considered as logically prior to the other? For example, if we take Einstein's original operationalist attitude, we would say that the geometry of spacetime is defined through the behaviour of 'rods' and 'clocks', which eventually should be thought of as physical systems obeying the fundamental dynamical laws. In fact, Einstein often complained about the fact that rods and clocks are introduced as if they were logically independent of the dynamical laws, e.g., in a discussion remark at the 86th meeting of the Gesellschaft Deutscher Naturforscher und Ärzte in Bad Nauheim in 1920:[2]

> *It is a logical shortcoming of the Theory of Relativity in its present form to be forced to introduce measuring rods and clocks separately instead of being able to construct them as solutions to differential equations.* ([66], Vol. 7, Doc. 46, p. 353)

From that viewpoint, symmetry properties of spacetime are nothing but an effective codification of the symmetries of the fundamental laws. Consequences like 'length contraction' and 'time dilation' in Special Relativity are then only *effectively* described as due to the geometry of spacetime, whereas a fundamental explanation clearly has to refer to the dynamical laws that govern clocks and rods. This was clearly the attitude taken by H. A. Lorentz and H. Poincaré, though in their case still somehow afflicted with the idea of a material æther that, in principle, defines a preferred rest frame, so that the apparent validity of the principle of relativity must be interpreted as due to a 'dynamical conspiracy'.[3] In his famous article on Relativity for the Encyclopedia of Mathematical Sciences, the young Pauli proposes to maintain this view, albeit without the idea on a material æther. He writes:[4]

> *Should one, then, in view of the above remarks, completely abandon any attempt to explain the Lorentz contraction atomistically? We think that the answer to this question should be No. The contraction of a measuring rod is not an elementary but a very complicated process. It would not take place except for the covariance*

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[2] German original: "Es ist eine logische Schwäche der Relativitätstheorie in ihrem heutigen Zustande, daß sie Maßstäbe und Uhren gesondert einführen muß, statt sie also Lösungen von Differentialgleichungen konstruieren zu können."

[3] H. A. Lorentz still expressed this viewpoint well after the formulation of Special Relativity, for example in [41], p. 23.

[4] German original: "Ist aber das Bestreben, die Lorentz-Kontraktion atomistisch zu verstehen, vollkommen zu verwerfen? Wir glauben diese Frage verneinen zu müssen. Die Kontraktion des Maßstabes ist kein elementarer, sondern ein sehr verwickelter Prozeß. Sie würde nicht eintreten, wenn nicht schon die Grundgleichungen der Elektronentheorie sowie die uns noch unbekannten Gesetze, welche den Zusammenhalt des Elektrons selbst bestimmen, gegenüber der Lorentz-Gruppe kovariant wären. Wir müssen eben postulieren, daß dies der Fall ist, wissen aber auch, daß dann, wenn dies zutrifft, die Theorie imstande sein wird, das Verhalten von bewegten Maßstäben und Uhren atomistisch zu erklären." ([58], p. 30.)

> *with respect to the Lorentz group of the basic equations of elec-*
> *tron theory, as well as those laws, as yet unknown to us, which*
> *determine the cohesion of the electron itself. We can only postu-*
> *late that this is so, knowing that then the theory will be capable of*
> *explaining atomistically the behaviour of moving measuring rods*
> *and clocks."* ([54], p. 15.)

Very recently, this traditional view has once more been defended under the name of 'Physical Relativity' [7] against todays more popular view, according to which Special Relativity is about the symmetry properties of spacetime itself. Clearly, the latter view only makes sense if spacetime is endowed with its own ontological status, independently of the presence of rods and clocks.

This shift in emphasis towards a more abstract point of view is also reflected in Pauli's writings, for example in the Preface to the English edition of his 'Theory of Relativity' of 1956, where the abstract group-theoretic properties of dynamical laws are given an autonomous status in the explanation of phenomena:

> *The concept of the state of motion of the 'luminiferous æther',*
> *as the hypothetical medium was called earlier, had to be given up,*
> *not only because it turned out to be unobservable, but because it*
> *became superfluous as an element of a mathematical formalism,*
> *the group-theoretical properties of which would only be disturbed*
> *by it. By the widening of the transformation group in general*
> *relativity the idea of a distinguished inertial coordinate system*
> *could also be eliminated by Einstein, being inconsistent with the*
> *group-theoretical properties of the theory.*

Pushed to an extreme, this attitude results in the belief that the most fundamental laws of physics are nothing but realisations of basic symmetries. Usually this is further qualified by adding that these realisations are the most 'simple' ones, at least with respect to some intuitive measure of simplicity. Such statements are well known from Einstein's later scientific period and also from Heisenberg in connection with his 'unified theory' of elementary particles, for which he proposed a single non-linear differential equation, whose structure was almost entirely motivated by its symmetry properties. Heisenberg made this point quite explicitly in his talk entitled *Planck's discovery and the foundational issues of atomism*[5], delivered during the celebrations of Max Planck's 100th anniversary—at which occasion Wolfgang Pauli received the Max-Planck medal in absentia—, where he also

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[5] German original: "Die Plancksche Entdeckung und die philosophischen Grundfragen der Atomlehre".

talked about his own 'unified theory':[6]

> *The mentioned equation contains, next to the three natural units [$c, \hbar, l$], merely mathematical symmetry requirements. These requirements seem to determine everything else. In fact, one should just regard this equation as a particularly simple representation of the symmetry requirements, which form the actual core of the theory.*

Pauli, who briefly collaborated with Heisenberg on this project, did not at all share Heisenberg's optimism that a consistent quantum-field theory could be based on Heisenberg's non-linear field equation. His objections concerned several serious technical aspects, overlayed with an increasing overall dislike of Heisenberg's readiness to make premature claims, particularly when made publicly.

However, I think it is fair to say that the overall attitude regarding the heuristic rôle and power of symmetry principles in fundamental physics, expressed by Heisenberg in the above quote, was also to a large extent shared by Pauli, not only in his later scientific life. This is particularly true for symmetry induced conservation laws, towards which Pauli had very strong feelings indeed. Examples from his later years will be discussed in later sections (e.g. Sect. 3.7). An example from his early scientific life is his strong resistance against giving up energy-momentum conservation for individual elementary processes, while keeping it on the statistical average. Such ideas were advocated in the "new radiation theory" of Bohr, Kramers, and Slater of early 1924 [5] and again by Bohr in connection with β-decay, which Pauli called *spiritual somersaults* in a letter to Max Delbrück. A week after his famous letter suggesting the existence of the neutrino, Pauli wrote to Oskar Klein in a letter dated Dec. 12th 1930:[7]

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[6] German original: "Die erwähnte Gleichung enthält neben den drei natürlichen Maßeinheiten nur noch mathematische Symmetrieforderungen. Durch diese Forderungen scheint alles weitere bestimmt zu sein. Man muß eigentlich die Gleichung nur als eine besonders einfache Darstellung der Symmetrieforderungen, aber diese Forderung als den eigentlichen Kern der Theorie betrachten." ([45], Vol. IV, Part IV B, p. 1168)

[7] German original: "Erstens scheint es mir, daß der Erhaltungssatz für Energie-Impuls dem für die Ladung doch sehr weitgehend analog ist und ich kann keinen theoretischen Grund dafür sehen, warum letzterer noch gelten sollte (wie wir es ja empirisch für den β-Zerfall wissen), wenn ersterer versagt. Zweitens müßte bei einer Verletzung des Energiesatzes auch mit dem *Gewicht* etwas sehr merkwürdiges passieren. [...] Dies widerstrebt meinem physikalischen Gefühl auf das äußerste! Denn es muß dann sogar auch für das Gravitationsfeld, das von dem ganzen Kasten (samt seinem radioaktiven Inhalt) selber *erzeugt* wird (...), angenommen werden, daß es sich ändern kann, während wegen der Erhaltung der Ladung das nach außen erzeugte elektrostatische Feld (beide Felder scheinen mir doch analog zu sein; das wirst Du ja übrigens auch aus deiner fünfdimensionalen Vergangenheit noch wissen) unverändert bleiben soll." ([45], Vol. II, Doc. [261], p. 45-46)

> *First it seems to me, that the conservation law for energy-momentum is largely analogous to that for electric charge, and I cannot see a theoretical reason why the latter should still be valid (as we know empirically from β-decay) if the former fails. Secondly, something strange should happen to the* weight *if energy conservation fails. [...] This contradicts my physical intuition to an extreme! For then one has to even assume that the gravitational field* produced *[...] by the box (including the radioactive content) can change, whereas the electrostatic field must remain unchanged due to charge conservation (both fields seem to me analogous; as you will remember from your five-dimensional past).*

This is a truly remarkable statement. Not many physicists would nowadays dare suggesting such an intimate connection between the conservation laws of charge and energy-momentum. What Pauli hints at with his last remarks in brackets is the Kaluza-Klein picture, in which electric charge is interpreted as momentum in an additional space dimension in a five-dimensional spacetime.

It is not difficult to find explicit commitments from Pauli's later scientific life expressing his belief in the heuristic power of symmetry considerations. Let me just select two of them. The first is from his introduction to the International Congress of Philosophers, held in Zürich in 1954, where Pauli states:[8]

> *"It seems likely to me, that the reach of the mathematical group concept in physics is not yet fully exploited."*

The second is from his closing remarks as the president of the conference "50 Years of Relativity" held in Berne in 1955, where with respect to the still unsolved problem of whether and how the gravitational field should be described in the framework of Quantum-Field-Theory he remarks:[9]

> *It seems to me, that the heart of the matter [the problem of quantising the gravitational field] is not so much the linearity or non-linearity, but rather the fact that there is present a more general group than the Lorentz group.*

This, in fact, implicitly relates to much of the present-day research that is concerned with that difficult problem.

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[8] German original: "Es ist mir wahrscheinlich, dass die Tragweite des mathematischen Gruppenbegriffes in der Physik heute noch nicht ausgeschöpft ist." ([38], Vol. 2, p. 1345)

[9] German original: "Es scheint mir also, daß nicht so sehr die Linearität oder Nichtlinearität der Kern der Sache ist, sondern eher der Umstand, daß hier eine allgemeinere Gruppe als die Lorentzgruppe vorhanden ist." ([38], Vol. 2, p. 1306)

Before we can discuss specific aspects of 'symmetry' in Pauli's work in Section 3, we wish to recall various aspects of symmetry principles as used in physics.

2 Remarks on the notion of symmetry

2.1 Spacetime

The term 'symmetry' is used in such a variety of meanings, even in physics, that it seems appropriate to recall some of its its main aspects. One aspect is that which mathematicians call an 'automorphism' and which basically means a 'structure preserving self-map'. Take as an example (conceptually not an easy one) the modern notion of spacetime. First of all it is a set, M, the members of which are events, or better, 'potential events', since we do not want to assume that every spacetime point to be an actual physical event in the sense that a material happening is taking place, or at least not one which is dynamically relevant to the problem at hand.[10] That set is endowed with certain structures which are usually motivated through operational relations of actual physical events.

One such structure could be that of a preferred set of paths, which represent inertial (i.e. force free) motions of 'test bodies', that is, localised objects which do not react back onto spacetime structure. This defines a so-called 'path-structure' (compare [12][10]), which in the simplest case reduces to an *affine structure* in which the preferred paths behave, intuitively speaking, like 'straight lines'. This can clearly be said in a much more precise form (see, e.g., [60]). Under very mild technical assumptions (not even involving continuity) one may then show that the only automorphisms of that 'inertial structure' can already be narrowed down to the inhomogeneous Galilei or Lorentz groups, possibly supplemented by constant scale transformations (cf. [24][28]).[11]

Another structure to start with could have been that of a causal relation on M. That is, a partial order relation which determines the pairs of points on spacetime which, in principle, could influence each other in form of a propagation process based on ordinary matter or light signals. The automorphism group of that structure is then the subgroup of bijections on M that, together with their inverse, preserve this order relation. For example, in case of Minkowski space, where the causal relation is determined by the

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[10] Minkowski was well aware that empty domains of spacetime may cause conceptual problems. Therefore, in his famous 1908 Cologne address *Space and Time* (German original: "Raum und Zeit"), he said: *In order to not leave a yawning void, we wish to imagine that at every place and at every time something perceivable exists.* German original: "Um nirgends eine gähnende Leere zu lassen, wollen wir uns vorstellen, daß allerorten und zu jeder Zeit etwas Wahrnehmbares vorhanden ist". ([47], p. 2)

[11] We shall from now on use 'Poincaré group' for 'inhomogeneous Lorentz group' and 'Lorentz group' for 'homogeneous Lorentz group'.

light-cone structure, it may be shown that the most general automorphism is given by a Poincaré transformation plus a constant rescaling[1][73]. Since, according to Klein's *Erlanger Programm* [36], any geometry may be characterised by its automorphism group, the geometry of Minkowski space is, up to constant rescalings, entirely encoded in the causal relations.

The same result can be arrived at through topological considerations. Observers (idealised to be extensionless) move in spacetime on timelike curves. Take the set $\mathcal{C}$ of all (not necessarily smooth) timelike curves which are continuous in the standard (Euclidean) topology $\mathcal{T}_\mathsf{E}$ of Minkowski spacetime M. Now endow M with a new topology, $\mathcal{T}_\mathsf{P}$, called the *path topology*, which is the finest topology on M which induces the same topology on each path in $\mathcal{C}$ as the standard (Euclidean) topology $\mathcal{T}_\mathsf{E}$. The new topology $\mathcal{T}_\mathsf{P}$ is strictly finer than $\mathcal{T}_\mathsf{E}$ and has the following remarkable property: The automorphism group of $(\mathsf{M}, \mathcal{T}_\mathsf{P})$[12], i.e. the group of bijections of M which, together with their inverses, preserve $\mathcal{T}_\mathsf{P}$, is just the Poincaré group extended by the constant rescalings [32]. This is possibly the closest operational meaning one could attribute to the topology of spacetime, since in $\mathcal{T}_\mathsf{P}$ a set in spacetime is open if and only if every observer "times" it to be open.

All this is meant to illustrate that there are apparently different ways to endow spacetime with structures that are, physically speaking, more or less well motivated and which lead to the same automorphism group. That group may then be called the group of *spacetime symmetries*. So far, this group seems to bear no direct relation to any dynamical law. However, the physical meaning of such statements of symmetry is tight to an ontological status of spacetime points. We assumed from the onset that spacetime is a *set* M. Now, recall that Georg Cantor, in his first article on transfinite set-theory [8], started out with the following definition of a set:[13]

> By a 'set' we understand any gathering-together M of determined well-distinguished objects $\mathfrak{m}$ of our intuition or of our thinking (which are called the 'elements' of M) into a whole.

Hence we may ask: Is a point in spacetime, a 'potential event' as we called it earlier, a "determined well-distinguished object of our intuition or of our thinking"? This question is justified even though modern axiomatic set theory is more restrictive in what may be called a set (for otherwise it runs into the infamous antinomies) and also stands back from any characterisation of elements in order to not confuse the axioms themselves with their

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[12] In the standard topological way of speaking this is just the 'homeomorphism group' of $(\mathsf{M}, \mathcal{T}_\mathsf{P})$.

[13] German original: "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten $\mathfrak{m}$ unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen." ([8], p. 481)

possible *interpretations*.[14] However, applications to physics require interpreted axioms, where it remains true that elements of sets are thought of as definite as in Cantors original definition. But it is just this definiteness that seems to be physically unwarranted in application to spacetime. The modern general-relativistic viewpoint takes that into account by a quotient construction, admitting only those statements as physically meaningful that are invariant under the group of (differentiable) permutations of spacetime points. This is possible only because all other structures on spacetime, in particular the metric and with it the causal structure, are not fixed once and for all but are subsumed into the dynamical fields. Hence no non-dynamical background structures remain, except those that are inherent in the definition of a differentiable manifold. The group of automorphisms is therefore the whole diffeomorphism group of spacetime, which, in some sense, comes sufficiently close to the group of all permutations.[15]

2.2 Dynamical symmetries versus covariance

What is the relation between spacetime automorphisms and symmetries of dynamical laws? Before we can answer this, we have to recall what a symmetry of a dynamical law is.

For definiteness, let us restrict attention to dynamical laws in classical (i.e. non-quantum) physics. The equations of motion generally take the form of systems of differential equations, which we here abbreviate with **EM** (**E**quation of **M**otion). These equations involve two types of quantities: 1) background structures, collectively abbreviated here by Σ, and 2) dynamical entities, collectively abbreviated here by Φ. The former will typically be represented by geometric objects on M (tensor fields, connections, etc), which are taken from a somehow specified set $\mathcal{B}$ of 'admissible backgrounds'. Typical background structures are external sources, like currents, and the geometry of spacetime in non-general-relativistic field theories. Dynamical entities typically involve 'particles' and 'fields', which in the simplest cases are represented by maps to and from spacetime,

$$\gamma : \mathbb{R} \to M \qquad (\text{'particle'}), \qquad (1a)$$

$$\psi : M \to V \qquad (\text{'field'}), \qquad (1b)$$

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[14] This urge for a clean distinction between the axioms and their possible interpretations is contained in the famous and amusing dictum, attributed to David Hilbert by his student Otto Blumenthal: "One must always be able to say 'tables', 'chairs', and 'beer mugs' instead of 'points, 'lines', and 'planes". (German original: "Man muß jederzeit an Stelle von 'Punkten', 'Geraden' und 'Ebenen' 'Tische', 'Stühle' und 'Bierseidel' sagen können.")

[15] There are clearly much more general bijections of spacetime than continuous or even differentiable ones. However, the diffeomorphism group is still n-point transitive, that is, given any two n-tuples of mutually distinct spacetime points, $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$, there is a diffeomorphism ϕ such that for all $1 \leq i \leq n$ we have $\Phi(p_i) = q_i$; this is true for all positive integers n.

and were V is usually some vector space.

In order to state the equations of motion, one has to first specify a set of so-called[16]**kinematically possible trajectories** out of which the dynamical entities Φ are taken and solutions to the equations of motion are sought. Usually this involves particle trajectories which are sufficiently smooth (typically piecewise twice continuously differentiable) and fields which are sufficiently smooth and in addition have a sufficiently rapid fall-off at large spatial distances, so as to give rise to finite quantities of energy, angular-momentum, etc. This space of kinematically possible trajectories will be denoted by $\mathcal{K}$. According to the discussion above, the equation of motion takes two arguments, one from $\mathcal{B}$ the other from $\mathcal{K}$, and is hence written in the form

$$\mathbf{EM}\{\,\Sigma \mid \Phi\,\} = 0\,, \tag{2}$$

where the zero on the right-hand side may be a many-component object. Equation (2) should be read as a selection criterion on the set $\mathcal{K}$, depending on the externally specified values of Σ. We shall sometimes write $\mathbf{EM}_\Sigma$ for $\mathbf{EM}\{\,\Sigma \mid \cdot\,\}$ to denote the particular equation of motion for Φ corresponding to the choice Σ for the background structures. In general, the sets of solutions to (2) for variable Σ are Σ-dependent subset $\mathcal{D}_\Sigma \subset \mathcal{K}$, whose elements are called the **dynamically possible trajectories**[16]. We can now say more precisely what is usually meant by a symmetry:

Definition 1 *An abstract group G is called a **symmetry group** of the equations of motion iff[17] the following conditions are satisfied:*

1. *There is an effective (see below) action $\mathsf{G} \times \mathcal{K} \to \mathcal{K}$ of G on the set of kinematically possible trajectories, denoted by $(\mathsf{g}, \Phi) \mapsto \mathsf{g} \cdot \Phi$.*

2. *This action leaves the subset $\mathcal{D}_\Sigma \subset \mathcal{K}$ invariant; that is, for all g in G we have:*

$$\mathbf{EM}\{\,\Sigma \mid \Phi\,\} = 0 \quad \Longleftrightarrow \quad \mathbf{EM}\{\,\Sigma \mid \mathsf{g} \cdot \Phi\,\} = 0\,. \tag{3}$$

Recall that an action is called effective if no group element other than the group identity fixes all points of the set it acts on. Effectiveness is required in order to prevent mathematically trivial and physically meaningless extensions of G. What really matters are the orbits of G in $\mathcal{K}$, that is, the subsets $\mathsf{O}_\Phi = \{\mathsf{g} \cdot \Phi \mid \mathsf{g} \in \mathsf{G}\}$ for each $\Phi \in \mathcal{K}$. If the action were not effective, we could simply reduce G to a smaller group with an effective action and the same orbits in $\mathcal{K}$, namely the quotient group G/G', where G' is the normal subgroup of elements that fix all points of $\mathcal{K}$.

It should be noted that this definition is still very general due to the fact that no further condition is imposed on the action of G, apart from the

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[16] This terminology is due to James Anderson [2].
[17] Throughout we use "iff" as abbreviation for "if and only if".

obvious one of effectivity. For example, for fields one usually requires the action to be 'local', in the sense that for any point p of spacetime, the value $(\mathsf{g} \cdot \psi)(\mathsf{p})$ of the g-transformed field should be determined by the value of the original field at some point p' of spacetime, and possibly *finitely many* derivatives of ψ at p'. If there are no dependencies on the derivatives, the action is sometimes called 'ultralocal'. Note that the point p' need not be identical to p, but it is assumed to be uniquely determined by g and p. A striking example of what can happen if locality is not imposed is given by the vacuum Maxwell equations (no external currents), which clearly admit the Poincaré group as ultralocally acting symmetry group. What is less well known is the fact that they also admit the inhomogeneous Galilei group as symmetry group[18], albeit the action is non-local; see [20] or Chap. 5.9 of [21]. (There are also other non-local symmetries of the vacuum Maxwell equations [19].)

To be strictly distinguished from the notion of symmetry is the notion of covariance, which we define as follows:

Definition 2 *An abstract group* G *is called a **covariance group** of the equations of motion iff the following conditions are satisfied:*

1. *There is an effective action* $\mathsf{G} \times \mathcal{K} \to \mathcal{K}$ *of* G *on the set of kinematically possible trajectories, denoted by* $(\mathsf{g}, \Phi) \mapsto \mathsf{g} \cdot \Phi$.

2. *There is also an action (this time not necessarily effective)* $\mathsf{G} \times \mathcal{B} \to \mathcal{B}$ *of* G *on the set of background structures, likewise denoted by* $(\mathsf{g}, \Sigma) \mapsto \mathsf{g} \cdot \Sigma$.

3. *The solution-function* $\Sigma \mapsto \mathcal{D}_\Sigma \subset \mathcal{K}$ *from* $\mathcal{B}$ *into the subsets of* $\mathcal{K}$ *is* G*-equivariant. This means the following: If* $\mathsf{g} \cdot \mathcal{D}_\Sigma$ *denotes the set* $\{\mathsf{g} \cdot \Phi \mid \Phi \in \mathcal{D}_\Sigma\}$*, then, for all* g *in* G*, we have*

$$\mathsf{g} \cdot \mathcal{D}_\Sigma = \mathcal{D}_{\mathsf{g} \cdot \Sigma}. \tag{4}$$

An alternative way to say this is that the relation that **EM** *establishes on* $\mathcal{B} \times \mathcal{K}$ *via (2) is* G *invariant, that is, for all* g *in* G*, we have*

$$\mathbf{EM}\{\Sigma \mid \Phi\} = 0 \quad \Longleftrightarrow \quad \mathbf{EM}\{\mathsf{g} \cdot \Sigma \mid \mathsf{g} \cdot \Phi\} = 0. \tag{5}$$

The obvious difference between (3) and (5) is that in the former case the background structure is not allowed to change. The transformed dynamical entity is required to satisfy the *very same* equation as the untransformed one,

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[18] This is different from, and certainly more surprising than, the better known (ultra local) Galilei symmetry of Maxwell's equations in the presence of appropriate constitutive relations between the electric field $\vec{E}$ and the electric displacement-field $\vec{D}$ on one side, and between the magnetic induction-field $\vec{B}$ and the magnetic field $\vec{H}$ on the other; see e.g. [39] and [27].

whereas for a covariance it is only required to satisfy a suitably changed set of equations. Here 'changed' refers to the fact that $g \cdot \Sigma$ is generally different from Σ. Hence it is clear that a symmetry group is automatically also a covariance group, by just letting it act trivially on the set $\mathcal{B}$ of background structures. The precise partial converse is as follows: Given a covariance group G with action on $\mathcal{B}$, then for each $\Sigma \in \mathcal{B}$ define the 'stabiliser subgroup' of Σ in G as the set of elements in G that fix Σ,

$$\mathrm{Stab}_G(\Sigma) := \{ g \in G \mid g \cdot \Sigma = \Sigma \} . \tag{6}$$

Then the subgroup $\mathrm{Stab}_G(\Sigma)$ of the covariance group is also a symmetry group of the equation of motion $\mathbf{EM}_\Sigma$.

The requirement of covariance is a rather trivial one, since it can always be met by suitably taking into account all the background structures and a sufficiently general action of G on $\mathcal{B}$. To see how this works in a specific example, consider the ordinary 'heat equation' for the temperature field T (κ is a dimensionful constant):

$$\partial_t T - \kappa \Delta T = 0 . \tag{7}$$

Let $G = E_3 \times \mathbb{R}$ be the 7-parameter group of Euclidean motions (rotations and translations in $\mathbb{R}^3$) and time translations, whose defining representation on spacetime ($\mathbb{R}^3 \times \mathbb{R}$) is denoted by $g \rightarrow \rho_g$, then G acts effectively on the set of temperature fields via $g \cdot T := T \circ \rho_{g^{-1}}$ (the inverse being just introduced to make this a *left* action). It is immediate from the structure of (7) that this implements G as symmetry group of this equation. The background structures implicit in (7) are: a) a preferred split of spacetime into space and time, 2) a preferred measure and orientation of time, and c) a preferred distance measure on space. There are many ways to parametrise this structure, *depending on the level of generality one starts from.* If, for example, we start from Special Relativity, we only list those structural elements that we need *on top of* the Minkowski metric $\{\eta_{\mu\nu}\} = \mathrm{diag}(1, -1, -1, -1)$ in order to write down (7). They are given by a single constant and normalised timelike vector field n, by means of which we can write (7) in the form

$$\mathbf{EM}\{ n \mid T \} := n^\mu \partial_\mu T - \kappa(n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu \partial_\nu T = 0 . \tag{8}$$

In the special class of inertial reference frames in which $n^\mu = (1, 0, 0, 0)$ equation (8) reduces to (7). From the structure of (8) it is obvious that this equation admits the whole Poincaré group of Special Relativity as covariance group. However, the symmetry group it contains is the stabiliser subgroup of the given background structure. The latter is given by the vector field n, whose stabiliser subgroup within the Poincaré group is just $E_3 \times \mathbb{R}$, the same as for (7).

Had we started from a higher level of generality, in which no preferred coordinate systems are given to us as in Special Relativity, we would write

the heat equation in the form

$$\mathbf{EM}\{\mathfrak{n}, \mathfrak{g} \mid \mathsf{T}\} := \mathfrak{n}^\mu \nabla_\mu \mathsf{T} - \kappa(\mathfrak{n}^\mu \mathfrak{n}^\nu - \mathfrak{g}^{\mu\nu})\nabla_\mu \nabla_\nu \mathsf{T} = 0\,, \qquad (9)$$

where now $\mathfrak{n}$ as well as $\mathfrak{g}$ feature as background structures. $\mathfrak{n}$ is again specified as unit timelike covariant-constant vector field, $\mathfrak{g}$ as a flat metric, and ∇ as the unique covariant derivative operator associated to $\mathfrak{g}$ (i.e. torsion free and preserving $\mathfrak{g}$). Since ∇ is here taken as a unique function of $\mathfrak{g}$, it does not count as independent background structure. Once again it is clear from the structure of (9) that the covariance group is now the whole diffeomorphism group of spacetime. However, the symmetry group remains the same as before since the stabiliser subgroup of the pair $(\mathfrak{g}, \mathfrak{n})$ is $\mathsf{E}_3 \times \mathbb{R}$.

This example should make clear how easy it is to almost arbitrarily inflate covariance groups by starting from higher and higher levels of generality and adding the corresponding extra structures into ones list of background structures. This possibility is neither surprising nor particularly disturbing. Slightly more disturbing is the fact that a similar game can be played with symmetries, at least on a very formal level. The basic idea is to simply declare background structures to be dynamical ones by letting their values be determined by equations. We may do this since we have so far not qualified 'equations of motion' as any special sort of equations. For example, in the special relativistic context we may just take (8) and let $\mathfrak{n}$ be determined by

$$\mathfrak{n}^\mu \mathfrak{n}^\nu \eta_{\mu\nu} = 1\,, \qquad \partial_\mu \mathfrak{n}^\nu = 0\,. \qquad (10)$$

Then (8) and (10) together define a background free (from the special relativistic point of view) system of equations for $\mathsf{T}, \mathfrak{n}$ which has the full Poincaré group as symmetry group. Its symbolic form is

$$\mathbf{EM}\{\emptyset \mid \mathsf{T}, \mathfrak{n}\} = 0\,, \qquad (11)$$

where the 0 on the right-hand side has now 18 components: one for (8), one for the first equation in (10), and 16 ($= 4 \times 4$) for the second equation in (10). But note that its T-sector of solution space is *not* the same as that of (7), as it now also contains solutions for different $\mathfrak{n}$. However, as the equations (10) for $\mathfrak{n}$ do not involve T, the total solution space for $(\mathfrak{n}, \mathsf{T})$ can be thought of as fibred over the space of allowed $\mathfrak{n}$, with each fibre over $\mathfrak{n}$ being given by the solutions T of (8) for that given $\mathfrak{n}$. Each such fibre is a faithful image of the original solution space of (7), suitably transformed by a Lorentz boost that relates the original $\mathfrak{n}$ in (7) (i.e. $\{\mathfrak{n}^\mu\} = (1, 0, 0, 0)$) to the chosen one.

Even more radically, we could take (9) and declare $\mathfrak{n}$ and $\mathfrak{g}$ to be dynamical entities obeying the extra equations

$$\mathfrak{n}^\mu \mathfrak{n}^\nu \mathfrak{g}_{\mu\nu} = 1\,, \qquad \nabla_\mu \mathfrak{n}^\nu = 0\,, \qquad \mathrm{Riem}[\mathfrak{g}] = 0\,, \qquad (12)$$

16

where Riem is the Riemann curvature tensor of $\mathfrak{g}$, so that the last equation in (12) just expresses flatness of $\mathfrak{g}$. The system consisting of (9) and (12) has no background structures and admits the full diffeomorphism group as symmetry group. It is of the symbolic form

$$\mathbf{EM}\{\,\emptyset \mid \mathsf{T}, \mathfrak{n}, \mathfrak{g}\,\} = 0\,, \tag{13}$$

which now comprises 36 components: the 16 as above and an additional set of 20 for the independent components of Riem. Again, note that the T-sector of solution space of (9) is now much bigger that of (7) of or (8). With any solution T it also contains its diffeomorphism-transformed one, $\mathsf{T}' = \mathsf{T} \circ \varphi^{-1}$, where $\varphi \in \mathrm{Diff}(\mathsf{M})$. Again, since the equations for $\mathfrak{n}$ and $\mathfrak{g}$ do not involve T, the total solution space is fibred over the allowed $\mathfrak{n}$ and $\mathfrak{g}$ fields, with each fibre corresponding to a faithful image of the original solution space for (7).

Finally we remark that, in principle, constants appearing in equations of motion could also be addressed as background structures whose values might eventually be determined by more general dynamical theories. For example, one might speculate (as was done some time ago in the so-called Brans-Dicke theories) that the gravitational constant is actually the value of some field that only in the present epoch of our Universe has settled to a spatially constant and quasi-static value, but whose value at much earlier times was significantly different. Another example from Quantum Field Theory concerns the idea that masses of elementary particles are dynamically generated by the so-called Higgs field (whose existence is strongly believed but not yet experimentally confirmed).

In any case, the important message from the considerations of this subsection is the following: symmetries emerge or disappear if, respectively, background structures become dynamical $(\Sigma \to \Phi)$ or dynamical structures 'freeze' $(\Phi \to \Sigma)$.

2.3 Observable versus gauge symmetries

Within the concept of symmetry as explained so far, an important distinction must be made between *observable* symmetries on one hand, and *gauge* symmetries on the other. An observable symmetry transforms a state or a history of states (trajectory) into a *different*, that is, *physically distinguishable* state or history of states. On the other hand, a gauge symmetry transforms a state or a history of states into a *physically indistinguishable* state or a history of states. In this case there is a redundancy in the mathematical description, so that the map from mathematical labels to physical states is not faithful. This is usually associated with a group, called the *group of gauge transformations*, denoted by $\mathsf{G}_{\mathrm{gau}}$, which acts on the set of state labels such that two such labels correspond to the same physical state iff they lie in the same orbit of $\mathsf{G}_{\mathrm{gau}}$.

It is clear that the notion of 'distinguishability' introduced here refers to the set of observables, i.e. functions on state space that are physically realisable in the widest sense. Assuming for the moment that this was well defined, we could attempt a definition as follows:

Definition 3 *Let* G *be a symmetry group in the sense of Definition 1. Then* $\mathfrak{g} \in \mathsf{G}$ *is called an **observable or physical symmetry** iff there exists a* $\Phi \in \mathcal{D}_\Sigma$ *and a physical observable that separates* $\mathfrak{g} \cdot \Phi$ *from* Φ. *If no such observable exists,* $\mathfrak{g}$ *is called a **gauge symmetry**.*

It is clear that for a theoretician the stipulation of what functions on state space correspond to physically realisable observables is itself of hypothetical nature. However, what is important for us at this point is merely that *relative* to such a stipulation the distinction between observables and gauge symmetries makes sense. In the mathematical practice gauge symmetries are often signalled by an underdeterminedness of the equations of motion, which sometimes simply fail to restrict the motion in certain degrees of freedom which are then called 'gauge degrees of freedom'. In that case, given any solution $\Phi \in \mathcal{D}_\Sigma$, we can obtain another solution, Φ', by just changing Φ in those non-determined degrees of freedom in an arbitrary way. For example, if the equations of motion are obtained via an action principle, such spurious degrees of freedom will typically reveal their nature through the property that motions in them are not associated with any action. As a result, the equations of motion, which are just the condition for the stationarity of the action, will not constrain the motion in these directions. Conversely, if according to the action principle the motion in some degree of freedom costs action, it can hardly be called a redundant one. In this sense an action principle is not merely a device for generating equations of motion, but also contains some information about observables.

The combination of observable and gauge symmetries into the total symmetry group G need not at all be just that of a semi-direct or even direct product. Often, in field theory, the gauge group $\mathsf{G}_{\mathrm{gau}}$ is indeed a subgroup of G, in fact an invariant (normal) one, but the observable symmetries, $\mathsf{G}_{\mathrm{obs}}$, are merely a quotient and not a subgroup of G. In standard group theoretic terms one says that G is a $\mathsf{G}_{\mathrm{gau}}$−extension of $\mathsf{G}_{\mathrm{obs}}$. This typically happens in electromagnetism or more generally in Yang-Mills type gauge theories or General Relativity with globally charged configurations. In this case only the 'gauge transformations' with sufficiently rapid fall-off at large spatial distances are proper gauge transformations in our sense, whereas the long ranging ones cost action if performed in real time[19] and therefore have to be interpreted as elements of $\mathsf{G}_{\mathrm{obs}}$; see e.g. [23] and Chap. 6 of [33].

This ends our small excursion into the realm of meanings of 'symmetry'. We now turn to the discussion of specific aspects in Pauli's work.

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[19] By the very definition of global charge, which is just the derivative of the action with respect to a long-ranging 'gauge transformation'.

3 Specific comments on symmetries in Pauli's work

The usage of symmetry concepts in Pauli's work is so rich and so diverse that it seems absolutely hopeless, and also inappropriate, to try to present them in a homogeneous fashion with any claim of completeness. Rather, I will comment on various subjectively selected aspects without in any way saying that other aspects are of any lesser significance. In fact, I will not include some of his most outstanding contributions, like, for example, the formulation of the exclusion principle, the neutrino hypothesis, or his anticipation of Yang-Mills Gauge Theory for the strong interaction. There exist excellent reviews and discussions of these topics in the literature. Specifically I wish to refer to Bartel van der Waerden's contribution *Exclusion Principle and Spin* to the Pauli Memorial Volume ([18], pp. 199-244), Norbert Straumann's recent lecture on the history of the exclusion principle [68], Pauli's own account of the history of the neutrino (in English: [57], pp. 193-217; in German: [38], Vol. 2, pp. 1313-1337 and [55], p. 156-180), Chien-Shiung Wu's account *The Neutrino* in the Pauli Memorial Volume ([18], pp. 249-303), and the historical account of gauge theories by Lochlainn O'Raifeartaigh and Norbert Straumann [50]. A non-technical overview concerning *Pauli's Belief in Exact Symmetries* is given by Karl von Meyenn [46]. Last, but clearly not least, I wish to mention Charles Enz's fairly recent comprehensive scientific biography [16] of Wolfgang Pauli, which gives a detailed discussion of his scientific œvre.

In this contribution I rather wish to concentrate on some particular aspects of the notion of symmetry that are directly related to the foregoing discussion in Sections 2.2 and 2.3, as I feel that they are somewhat neglected in the standard discussions of symmetry.

3.1 The hydrogen atom in matrix mechanics

In January 1926 Pauli managed to deduce the energy spectrum for the Hydrogen atom from the rules of matrix mechanics. For this he implicitly used the fact that the mechanical problem of a point charge moving in a spherically symmetric force-field with a fall-off proportional to the square of the inverse distance has a symmetry group twice as large (i.e. of twice the dimension) as the group of spatial rotations alone, which it contains. Hence the total symmetry group is made half of a 'kinematical' part, referring to space, and half of a 'dynamical' part, referring to the specific force law ($1/r^2$ fall-off). Their combination is a proper physical symmetry group that transforms physically distinguishable states into each other. In the given quantum-mechanical context one also speaks of 'spectrum generating' symmetries.

Let us recall the classical problem in order to convey some idea where the

symmetries and their associated conserved quantities show up, and how they may be employed to solve the dynamical problem. Consider a mass-point of mass m and position coordinate $\vec{r}$ in the force field $\vec{F}(\vec{r}) = -(K/r^2)\vec{n}$, where r is the length of $\vec{r}$, $\vec{n} := \vec{r}/r$, and K is some dimensionful constant. Then, according to Newton's 3rd law (an overdot stands for the time derivative),

$$\ddot{\vec{r}} = -\frac{k}{r^2}\vec{n} \qquad (k = K/m). \tag{14}$$

Next to energy, there are three obvious conserved quantities corresponding to the three components of the angular-momentum vector (here written per unit mass)

$$\vec{\ell} = \vec{r} \times \dot{\vec{r}}. \tag{15}$$

But there are three more conserved quantities, corresponding to the components of the following vector (today called the Lenz-Runge vector),

$$\vec{e} = k^{-1}\dot{\vec{r}} \times \vec{\ell} - \vec{n}. \tag{16}$$

Conservation can be easily verified by differentiation of (16) using (14) and $\dot{\vec{n}} = \vec{\ell} \times \vec{n}/r^2$. Hence on has ($\ell$ = length of $\vec{\ell}$)

$$\vec{\ell} \cdot \vec{r} = 0, \quad \vec{\ell} \cdot \vec{e} = 0, \quad r + \vec{r} \cdot \vec{e} - k^{-1}\ell^2 = 0, \tag{17}$$

from which the classical orbit immediately follows: Setting $\vec{r} \cdot \vec{e} = re\cos\varphi$, the last equation (17) reads

$$r = \frac{\ell^2/k}{1 + e\cos\varphi}, \tag{18}$$

which is the well known equation for a conic section in the plane perpendicular to $\vec{\ell}$, focus at the origin, eccentricity e (= length of $\vec{e}$), and *latus rectum* $2\ell^2/k$. The vector $\vec{e}$ points from the origin to the point of closest approach (periapsis). The few steps leading to this conclusion illustrate the power behind the method of working with conservation laws which, in turn, rests on an effective exploitment of symmetries.

The total energy per unit mass is given by $E = \frac{1}{2}\dot{\vec{r}}^2 - k/r$. A simple calculation shows that

$$e^2 - 1 = 2E\ell^2/k^2, \tag{19}$$

which allows to express E as function of the invariants e^2 and ℓ^2. This is the relation which Pauli shows to have an appropriate matrix analogue, where it allows to express the energy in terms of the eigenvalues of the matrices for ℓ^2 and e^2 which Pauli determines, leading straight to the Balmer formula.

From a modern point of view one would say that, for fixed energy $E < 0$,[20] the state space of this problem carries a Hamiltonian action of the Lie algebra

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[20] For $E > 0$ one obtains a Hamiltonian action of $\mathfrak{so}(1,3)$.

$\mathfrak{so}(4)$, generated by the 3+3 quantities $\vec{\ell}$ and $\vec{e}$. Quantisation then consists in the problem to represent this Lie algebra as a commutator algebra of self-adjoint operators and the determination of spectra of certain elements in the enveloping algebra. This is what Pauli did, from a modern point of view, but clearly did not realise at the time. In particular, even though he calculated the commutation relations for the six quantities $\vec{\ell}$ and $\vec{e}$, he did not realise that they formed the Lie algebra for $\mathfrak{so}(4)$, as he frankly stated much later (1955) in his address on the occasion of Hermann Weyl's 70th birthday:[21]

> *Similarly I did not know that the matrices which I had derived from the new quantum mechanics in order to calculate the energy values of the hydrogen atom were a representation of the 4-dimensional orthogonal group.*

This may be seen as evidence for Pauli's superior instinct for detecting relevant mathematical structures in physics. Much later, in a CERN-report of 1956, Pauli returned to the representation-theoretic side of this problem [53].

3.2   Particles as representations of spacetime automorphisms

The first big impact of group theory proper on physics took place in quantum theory, notably through the work or Eugene Wigner [72] and Hermann Weyl [70]. While in atomic spectroscopy the usage of group theory could be looked upon merely as powerful mathematical tool, it definitely acquired a more fundamental flavour in (quantum) field theory. According to a dictum usually attributed to Wigner, every elementary system (particle) in special-relativistic quantum theory corresponds to a unitary irreducible representation of the Poincaré group.[22] In fact, all the Poincaré invariant linear wave equations on which special-relativistic quantum theory is based, known by the names of Klein & Gordon, Weyl, Dirac, Maxwell, Proca, Rarita & Schwinger, Bargmann & Wigner, Pauli & Fierz, can be understood as projection conditions that isolate an irreducible sub-representation of the Poincaré group[23] within a reducible one that is easy to write down. More concretely, the latter is usually obtained as follows: Take a field ψ on spacetime M with

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[21] German original: "Ebensowenig wußte ich, daß die Matrices, die ich ausgerechnet hatte, um die Energiewerte des Wasserstoffatoms aus der neuen Quantenmechanik abzuleiten, eine Darstellung der 4-dimensionalen othogonalen Gruppe gewesen sind". ([45], Vol. IV, Part III, Doc. [2183], p. 402) Note that, in modern terminology, Pauli actually refers to a representation of the *Lie algebra* of the orthogonal group.

[22] The converse is not true, since there exist unitary irreducible representations which cannot correspond to (real) particles, for example the so-called 'tachyonic' ones, corresponding to spacelike four-momenta.

[23] More precisely, its universal cover $\mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C})$, or sometimes an extension thereof by the discrete transformations of space and time reversal.

values in a finite-dimensional complex vector space V. Let D be a finite-dimensional irreducible representation of the (double cover of the) Lorentz group $SL(2,\mathbb{C})$ on V.[24] It is uniquely labelled by a pair (p,q) of two positive integer- or half-integer-valued numbers. In the standard terminology, $2p$ corresponds to the number of unprimed, $2q$ to the number of primed spinor indices of ψ. The set of such fields furnishes a linear representation of the (double cover of the) Poincaré group, $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, where the action of the group element $g = (a, A)$ is given by

$$g \cdot \psi := D(A)(\psi \circ g^{-1}), \qquad (20)$$

or for the Fourier transform $\tilde{\psi}$,

$$g \cdot \tilde{\psi} := \exp(ip_\mu a^\mu)D(A)(\tilde{\psi} \circ A^{-1}). \qquad (21)$$

One immediately infers from (21) that irreducibility implies that $\tilde{\psi}$ must have support on a single $SL(2,\mathbb{C})$ orbit in momentum space. Here one usually restricts to those orbits consisting of non-spacelike p (those with spacelike p give rise to the tachyonic representations which are deemed unphysical), which are labelled by $p_\mu p^\mu = m^2$ with non-negative m. For ψ this means that it obeys the Klein-Gordon equation $(\Box + m^2)\psi = 0$. This is already half the way to an irreducible representation, insofar as it now contains only modes of fixed mass. But these modes still contains several spins up to the maximal value $p + q$. A second and last step then consists of projecting out one (usually the highest) spin, which gives rise to the equations named above. In this fashion the physical meanings of *mass* and *spin* merge with the abstract mathematical meaning of mere labels of irreducible representations. Mass and spin are the most elementary attributes of physical objects, so that objects with no other attributes are therefore considered *elementary*. As just described, these elementary attributes derive from the representation theory of a group whose significance is usually taken to be that it is the automorphism group of spacetime. However, as already discussed in Sections 1 and 2.1, this point of view presupposes a hierarchy of physical thinking in which spacetime (here Minkowski space) is considered an entity prior to (i.e. more fundamental than) matter, which may well be challenged. A more consistent but also more abstract point of view would be to think of the abstract[25] Poincaré group as prior to the matter content *as well as* the spacetime structure and to derive both simultaneously. Here 'deriving'

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[24] The representation D is never unitary, simply because the Lorentz group has no non-trivial finite-dimensional unitary irreducible representations. But it will give rise to an infinite-dimensional representation on the linear space of fields ψ which will indeed be unitary.

[25] 'Abstract' here means to consider the isomorphicity class of the group as mathematical structure, without any interpretation in terms of transformations of an underlying set of objects.

a spacetime structure (geometry) from a group would be meant in the sense of Klein's *Erlanger Programm* [36].

We have already discussed in Section 1 Pauli's shift in emphasis towards a more abstract point of view as regards spacetime structure. But also as regards to matter he was, next to Wigner, one of the proponents to put symmetry considerations first and to derive the wave equations of fundamental fields as outlined above. Based on previous work by Fierz on the theory of free wave equations for higher spin [17], Fierz and Pauli published their very influential paper *On Relativistic Equations for Particles of Arbitrary Spin in an Electromagnetic Field* ([38], Vol. 2, pp. 873-894) which is still much cited today.

In fact, much earlier, in his 1927 paper *Quantum Mechanics of the Magnetic Electron*[26], Pauli succeeded to implement the electron's spin into non-special-relativistic quantum mechanics in an entirely representation-theoretic fashion as regards the (Lie algebra of) spatial rotations. In contrast to the other (translational) degrees or freedom, spin does not appear as the quantisation of an already existent classical degree of freedom. This must have appeared particularly appealing to Pauli, who never wanted the electron's 'spin' to be understood as an intrinsic angular momentum due to a spatial rotation of a material structure. When Pauli introduced the new spin quantum-number for the electron in his 1924 paper *On the Influence of the Velocity Dependence of the Electron Mass on the Zeeman Effect*[27] he deliberately stayed away from any model interpretation and cautiously referred to it as *a peculiar, classically indescribable disposition of two-valuedness of the quantum-theoretic properties of the light-electron*[28]. At that time an understandable general scepticism against possible erroneous prejudices imposed by the usage of classical models had already firmly established itself in Pauli's (and others) thinking.

As much justified as this is in view of Quantum Mechanics, this had also led to overstatements to the effect that spin has no classical counterpart and that any classical model is even classically contradictory in the sense of violating Special Relativity. As regards the second point, which was also pushed by Pauli, we refer to the detailed discussion in [22]. To the first point we first wish to mention that composite models with half-integer angular momentum states exist in ordinary Quantum Mechanics (without spin), as, e.g., pointed out by Bopp & Haag in 1950 [6]. This is possible if their classical configuration space contains the whole group $SO(3)$ of spatial rotations.

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[26] German original: "Zur Quantenmechanik des magnetischen Elektrons". ([38], Vol. 2, pp. 306-330)

[27] German original: "Über den Einfluß der Geschwindigkeitsabhängigkeit der Elektronenmasse auf den Zeemaneffekt". ([38], Vol. 2, pp. 201-213)

[28] German original: "eine eigentümliche, klassisch nicht beschreibbare Art von Zweideutigkeit der quantentheoretischen Eigenschaften des Leuchtelektrons" ([38], Vol. 2, p. 213).

Pauli himself showed in his 1939 paper *On a Criterion for Single- or Double-Valuedness of the Eigenfunctions in Wave Mechanics*[29] the possibility of double-valued wavefunctions, which are the ones that give rise to half-integer angular momentum states. Moreover, in classical mechanics there is also a precise analog of Wigner's notion of an elementary system. Recall that the space of states of a mechanical system is a symplectic manifold (phase space). The analog of an irreducible and unitary representation of the group of spacetime automorphisms is now a transitive and Hamiltonian action of this group on the symplectic manifold. It is interesting to note that this classical notion of an elementary system was only formulated much later than, and in the closest possible analogy with, the quantum mechanical one. An early reference where this is spelled out is [4]. The classification of elementary systems is now equivalent to the classification of symplectic manifolds admitting such an action. An early reference where this has been done is [3]. Here, as expected, an intrinsic angular momentum shows up as naturally as it does in Quantum Mechanics. What makes it slightly unusual (but by no means awkward or even inconsistent) is the fact that it corresponds to a phase space[30] that is not the cotangent bundle (space of momenta) over some configuration space of positions.

Pauli's later writings also show this strong inclination to set the fundamentals of (quantum) field theory in group-theoretic terms. In his survey *Relativistic Field Theories of Elementary Particles* ([38], Vol. 2, pp. 923-952), written for the 1939 Solvay Congress, Pauli immediately starts a discussion of "transformation properties of the field equations and conservation laws". His posthumously published notes on *Continuous Groups in Quantum Mechanics* [53] focus exclusively on Lie-algebra methods in representation theory.

Today we are used to *define* physical quantities like energy, momentum, and angular momentum as the conserved quantities associated to spacetime automorphisms via Noether's theorem. Here, too, Pauli was definitely an early advocate of this way of thinking. Reviews on the subject written shortly after Pauli's death show clear traces of Pauli's approach; see e.g. [35].

3.3 Spin and statistics

Pauli's proof of the spin-statistics correlation [51] (also [38], Vol. 2, pp. 911-922), first shown by Markus Fierz in his habilitation thesis [17], is a truly impressive example for the force of abstract symmetry principles. Here we wish to recall the basic lemmas on which it rests, which merely have to do with classical fields and representation theory.

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[29] German original: "Über ein Kriterium für Ein- oder Zweiwertigkeit der Eigenfunktionen in der Wellenmechanik". ([38], Vol. 2, pp. 847-868)

[30] The phase space for classical spin is a 2-sphere, which is compact and therefore leads to a finite-dimensional Hilbert space upon quantisation.

We begin by replacing the proper orthochronous Lorentz group by its double (= universal) cover SL(2,$\mathbb{C}$) in order to include half-integer spin fields. We stress that everything that follows merely requires the invariance under this group. No requirements concerning invariance under space- or time reversal are needed!

We recall from the previous section that any finite-dimensional complex representation of SL(2,$\mathbb{C}$) is labelled by an ordered pair (p, q), where p and q may assume independently all non-negative integer or half-integer values. $2p$ and $2q$ correspond to the numbers of 'unprimed' and 'primed' spinor indices, respectively. The tensor product of two such representations decomposes as follows:

$$D^{(p,q)} \otimes D^{(p',q')} = \bigoplus_{r=|p-p'|}^{p+p'} \bigoplus_{s=|q-q'|}^{q+q'} D^{(r,s)}, \tag{22}$$

where—and this is the important point in what follows—the sums proceed in *integer* steps in r and s. With each $D^{(p,q)}$ let us associate a 'Pauli Index', given by

$$\pi : D^{(p,q)} \to ((-1)^{2p}, (-1)^{2q}) \in \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{23}$$

This association may be extended to sums of such $D^{(p,q)}$ proceeding in integer steps, simply by assigning to the sum the Pauli Index of its terms (which are all the same). Then we have[31]

$$\pi(D^{(p,q)} \otimes D^{(p',q')}) = \pi(D^{(p,q)}) \cdot \pi(D^{(p',q')}). \tag{24}$$

According to their representations, we can associate a Pauli Index with spinors and tensors. For example, a tensor of odd/even degree has Pauli Index $(-,-)/(+,+)$. The partial derivative, ∂, counts as a tensor of degree one. Now consider the most general linear (non interacting) field equations for integer spin (here and in what follows $\sum(\cdots)$ simply stands for "sum of terms of the general form $(\cdots)$"):

$$\begin{aligned} \sum \partial_{(-,-)} \Psi_{(+,+)} &= \sum \Psi_{(-,-)}, \\ \sum \partial_{(-,-)} \Psi_{(-,-)} &= \sum \Psi_{(+,+)}. \end{aligned} \tag{25}$$

These are invariant under

$$\Theta : \begin{cases} \Psi_{(+,+)}(x) &\mapsto + \Psi_{(+,+)}(-x), \\ \Psi_{(-,-)}(x) &\mapsto - \Psi_{(-,-)}(-x). \end{cases} \tag{26}$$

Next consider any current that is a polynomial in the fields and their derivatives:

$$\begin{aligned} J_{(-,-)} = \sum \Psi_{(-,-)} &+ \Psi_{(+,+)} \Psi_{(-,-)} + \partial_{(-,-)} \Psi_{(+,+)} \\ &+ \Psi_{(+,+)} \partial_{(-,-)} \Psi_{(+,+)} + \Psi_{(-,-)} \partial_{(-,-)} \Psi_{(-,-)} + \quad \cdots \end{aligned} \tag{27}$$

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[31] This may be expressed by saying that the map π is a homomorphism of semigroups. One semigroup consists of direct sums of irreducible representations proceeding in integer steps with operation $\otimes$, the other is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is actually a group.

Then one has

$$(\Theta J)(x) = -J(-x)\,. \tag{28}$$

This shows that for any solution of the field equations with charge Q for the conserved current J (Q being the space integral over J^0) there is another solution (the Θ transformed) with charge $-Q$. It follows that charges of conserved currents cannot be sign-definite in any SL$(2,\mathbb{C})$-invariant theory of non-interacting integer spin fields. In the same fashion one shows that conserved quantities, stemming from divergenceless symmetric tensors of rank two, bilinear in fields, cannot be sign-definite in any SL$(2,\mathbb{C})$ invariant theory of non-interacting half-integer spin fields. In particular, the conserved quantity in question could be energy!

An immediate but far reaching first conclusion (not explicitly drawn by Pauli) is that there cannot exist a relativistic generalisation of Schrödinger's one- particle wave equation. For example, for integer-spin particles, one simply cannot construct a non-negative spatial probability distribution derived from conserved four-currents. This provides a general argument for the need of second quantisation, which in textbooks is usually restricted to the spin-zero case.

Upon second quantisation the celebrated spin-statistics connection for free fields can now be derived in a few lines. It says that integer spin fields cannot be quantised using anti-commutators and half-integer spin field cannot be quantised using commutators. Here the so-called Jordan-Pauli distribution plays a crucial role[32] in the (anti)commutation relations, which ensures causality (observables localised in spacelike separated regions commute). Also, the crucial hypothesis of the existence of an SL$(2,\mathbb{C})$ invariant stable vacuum state is adopted. Pauli ends his paper by saying:

> *In conclusion we wish to state, that according to our opinion the connection between spin and statistics is one of the most important applications of the special relativity theory.* ([51], p. 722)

It took almost 20 years before first attempts were made to generalise this result to the physically relevant case of interacting fields by Lüders & Zumino [42].

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[32] The Jordan-Pauli distribution was introduced by Jordan and Pauli in their 1927 paper *Quantum Electrodynamics of Uncharged Fields* ("Zur Quantenelektrodynamik ladungsfreier Felder"; [38], Vol. 2, pp. 331-353) in an attempt to formulate Quantum Electrodynamics in a manifest Poincaré invariant fashion. It is uniquely characterised (up to a constant factor) by the following requirements: (1) it must be Poincaré invariant under simultaneous transformations of both arguments; (2) it vanishes for spacelike separated arguments; (3) it satisfies the Klein-Gordon equation. The (anti)commutators of the free fields must be proportional to the Jordan-Pauli distribution, or to finitely many derivatives of it.

3.4 The meaning of 'general covariance'

General covariance is usually presented as *the* characteristic feature of General Relativity. The attempted meaning is that a generally covariant law takes the 'same form' in all spacetime coordinate systems. However, in order to define the 'form' of a law one needs to make precisely the distinction between background entities, which are constitutive elements of the law, and the dynamical quantities which are to be obey the laws so defined (cf. Section 2.1). In the language we introduced above, 'general covariance' cannot just mean simple covariance under all smooth and invertible transformations of spacetime points, i.e. that the spacetime diffeomorphism group is a covariance group is the sense of Definition 2, for that would be easily achievable without putting any restriction on the intended law proper, as was already pointed out by Erich Kretschmann in 1917 [37]. Einstein agreed with that criticism of Kretschmann's, which he called "acute" (German original: "scharfsinnig")([66], Vol. 7, Doc. 4, pp. 38-41), and withdrew to the view that the principle of general covariance has at least some heuristic power in the following sense:[33]

> *Between two theoretical systems which are compatible with experience, that one is to be preferred which is the simpler and more transparent one from the standpoint of the absolute differential calculus. Try to bring Newton's gravitational mechanics in the form of generally covariant equations (four dimensional) and one will surely be convinced that principle a)[34] is, if not theoretically, but practically excluded.*

But the *principle of general covariance* is intended as a non-trivial selection criterion. Hence modern writers often characterise it as the requirement of diffeomorphism invariance, i.e. that the diffeomorphism group of spacetime is a symmetry group in the sense of Definition 1. But then, as we have seen above, the principle is open to trivialisations if one allows background structures to become formally dynamical. This possibility can only be inhibited if one limits the amount of structure that may be added to the

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[33] German original: "Von zwei mit der Erfahrung vereinbaren theoretischen Systemen wird dasjenige zu bevorzugen sein, welches vom Standpunkte des absoluten Differentialkalküls das einfachere und durchsichtigere ist. Man bringe einmal die Newtonsche Gravitationsmechanik in die Form von kovarianten Gleichungen (vierdimensional) und man wird sicherlich überzeugt sein, daß das Prinzip a) diese Theorie zwar nicht theoretisch, aber praktisch ausschließt." ([66], Vol. 7, Doc. 4, p. 39)

[34] Einstein formulates principle a) thus: "Principle of relativity: The laws of nature exclusively contain statements about spacetime coincidences; therefore they find their natural expression in generally covariant equations." ([66], Vol. 7, Doc. 4, p. 38)

dynamical fields.[35]

The reason why I mention all this here is that Pauli's Relativity article is, to my knowledge, the only one that seems to address that point, albeit not as explicitly as one might wish. After mentioning Kretschmann's objection, he remarks (the emphases are Pauli's):[36]

> *The generally covariant formulation of the physical laws acquires a physical content only through the principle of equivalence, in consequence of which gravitation is described* solely *by the* g_{ik} *and the latter are not given independently from matter, but are themselves determined by the field equations. Only for this reason can the* g_{ik} *be described as* physical quantities. ([54], p. 150)

Note how perceptive Pauli addresses the two central issues: 1) that one has to limit the the amount of dynamical variables and 2) that dynamical structures have to legitimate themselves as physical quantities through their back reaction onto other (matter) structures. It is by far the best few-line account of the issue that I know of, though perhaps a little hard to understand without the more detailed discussion given above in Section 2.2. Most modern textbooks do not even address the problem. See [25] for more discussion.

3.5 General covariance and antimatter

In this section I wish to give a brief but illustrative example from Pauli's work for the non-trivial distinction between observable physical symmetries on one hand, and gauge symmetries on the other (cf. Section 2.3). The example I have chosen concerns an argument within the (now outdated) attempts to understand elementary particles as regular solutions of classical field equations. Pauli reviewed such attempts in a rather detailed fashion in his Relativity article, with particular emphasis on Weyl's theory, to which he had actively contributed in two of his first three published papers in 1919.

The argument proper says that in any 'generally covariant'[37] theory, which allows for regular static solutions representing charged particles, there

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[35] Physically speaking, one may be tempted to just disallow such formal 'equations of motions' whose solution space is (up to gauge equivalence) zero dimensional. But this would mean that one would have to first understand the solution space of a given theory before one can decide on its 'general covariance' properties, which would presumably render it a practically fairly useless criterion.

[36] German original: Einen physikalischen Inhalt bekommt die allgemein kovariante Formulierung der Naturgesetze erst durch das Äquivalenzprinzip, welches zur Folge hat, daß die Gravitation durch die g_{ik} *allein* beschrieben wird, und daß diese nicht unabhängig von der Materie gegeben, sondern selbst durch Feldgleichungen bestimmt sind. Erst deshalb können die g_{ik} als *physikalische Zustandsgrößen* bezeichnet werden. ([58], p. 181)

[37] Here 'general covariance' is taken to mean that the diffeomorphism group of spacetime acts as symmetry group.

exists for any solution with mass m and charge e another such solution with the same mass but opposite charge $-e$. Pauli's proof looks like an almost trivial application of general covariance and runs as follows: Let $g_{\mu\nu}(x^\lambda)$ and $A_\mu(x^\lambda)$ represent the gravitational and electromagnetic field respectively. The hypothesis of staticity implies that coordinates (and gauges for A_μ) can be chosen such that all fields are independent of the time coordinate, x^0, and that $g_{0i} \equiv 0$ as well as $A_i \equiv 0$ for $i = 1, 2, 3$.[38] Now consider the orientation-reversing diffeomorphism $\phi : (x^0, \vec{x}) \mapsto (-x^0, \vec{x})$. It maps the gravitational field to itself while reversing the sign of A_0 and hence of the electric field. General covariance assures these new fields to be again solutions with the same total mass but opposite total electric charge.

Pauli presents this argument in his second paper addressing Weyl's theory, entitled *To the Theory of Gravitation and Electricity by Hermann Weyl*[39] ([38], Vol. 2, pp. 13-23, here p. 18) and also towards the end of Section 67 of his Relativity article. The idea of this proof is due to Weyl who communicated it (without formulae) in his first two letters to Pauli ([45], Vol. 1, Doc. [1] and [2]), as Pauli also acknowledges in his paper ([38], Vol. 2, p. 18, footnote 2).

It is interesting to note that Einstein rediscovered the very same argument in 1925 and found it worthy of a separate communication [13]. At the time it was common to all, Weyl, Pauli, and Einstein, to regard the argument a nuisance and of essentially destructive nature. This was because at this time antiparticles had not yet been discovered so that the apparent asymmetry as regards the sign of the electric charges of fundamental particles was believed to be a fundamental property of Nature. Already in his first paper on Weyl's theory ([38], Vol. 2, pp. 1-9), entitled *Perihelion Motion of Mercury and Deflection of Rays in Weyl's Theory of Gravitation*[40], Pauli emphasised:[41]

> The main difficulty [with Weyl's theory] is – apart from Einstein's objection, which appears to me not yet sufficiently disproved – that the theory cannot account for the asymmetry between the two sorts of electricity.

Now, there is an interesting conceptual point hidden in this argument that relates to our discussions in Sections 2.2 and 2.3. First of all, the two

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[38] The latter conditions distinguish staticity from mere stationarity. The condition on A_i may, in fact, be relaxed.

[39] German original: "Zur Theorie der Gravitation und der Elektrizität von Hermann Weyl".

[40] German original: "Merkurperihelbewegung und Strahlenablenkung in Weyls Gravitationstheorie".

[41] German original: "Die Hauptschwierigkeit ist – neben Einstein's Einwand, der mir durchaus noch nicht hinreichend widerlegt scheint –, daß die Theorie von der Asymmetrie der beiden Elektrizitätsarten nicht befriedigend Rechenschaft zu geben vermag." ([38], Vol. 2, p. 8)

solutions are clearly considered physically distinct, otherwise the argument could not be understood as contradicting the charge asymmetry in Nature. Hence the diffeomorphism involved cannot be considered a gauge transformation but rather corresponds to a proper physical symmetry. On the other hand, we know that diffeomorphisms within bounded regions must be considered as gauge transformations, for otherwise one would run into the dilemma set by the so-called "hole argument"[42]. Hence one faces the problem of how one should characterise those diffeomorphisms which are not to be considered as gauge transformations (cf. Section 2.3). It is conceivable that this question is not decidable without contextual information. (See e.g. [23] and Chapter 6 of [33] for more discussion of this point.) The historical sources have almost nothing to say about this, though there are suggestions by all three mentioned authors how to circumvent the argument by adding more non-dynamical structures, as a result of which general covariance is lost. Einstein, being most explicit here, suggested the existence of a time-like vector field which fixes a time orientation. At least the so-defined time orientation would then have to be considered as non-dynamical structure of type Σ (cf. Section 2.2) in order to break the symmetry group down to the stabiliser group of Σ. The time-orientation-reversing transformation used above would then not be a symmetry anymore. Similar suggestions were made by Weyl, who also hinted at a structure to distinguish past and future:[43]

> *Their essential difference [of past and future] I take, contrary to most physicists, to be a fact of much more fundamental meaning than the essential difference between positive and negative charge.*

In the last (5th) edition of *Raum Zeit Materie*, Hermann Weyl writes re-

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[42] Let Ω be a bounded region in spacetime which is disjoint from a spacelike hypersurface Σ. Consider two solutions to the field equations which merely differ by the action of a diffeomorphism ϕ with support in Ω. If they are considered distinct, then the theory cannot have a well posed initial-value problem, since then for any Σ distinct solutions exist with identical data on Σ. This is a rephrasing of Einstein's original argument ([66], Vol. 4, Doc. 25, p. 574, Doc. 26, p 580, Vol. 6, Doc. 2, p. 10), which did not construct a contradiction to the existence of a well posed *initial-value problem*, but rather to the requirement that the gravitational field be determined by the matter content (more precisely: its energy momentum tensor). But this requirement is clearly never fulfilled in any generally covariant theory in which the gravitational field has its own degrees of freedom, independent of whether one regards diffeomorphisms as gauge. Slightly later he rephrased it so as to construct a contradiction to the existence of a well posed *boundary-value problem* ([66], Vol. 6, Doc. 9, p. 110), which is also not the right thing to require from equations that describe the propagation of fields with own degrees of freedom.

[43] German original: "Ihren Wesensunterschied [von Vergangenheit und Zukunft] halte ich, im Gegensatz zu den meisten Physikern, für eine Tatsache von noch viel fundamentalerer Bedeutung als der Wesensunterschied zwischen positiver und negativer Elektrizität." ([45], Vol. 1, Doc. [2], p. 6)

garding his unified theory (the emphases are Weyl's): [44]

> *The theory gives no clue as regards the* disparity of positive
> and negative electricity. *But that cannot be taken as a reproach*
> *against the theory. For that disparity is based without doubt on*
> *the fact that of both fundamental constituents of matter, the elec-*
> *tron and the hydrogen nucleus, the positively charged one is tight*
> *to another mass then the negatively charged one; it originates*
> *from the nature of matter and not of the field.*

Given that Weyl is talking about his unified field-theory of gravity and
electricity, whose original claim was to explain all of matter by means of
field theory, this statement seems rather surprising. It may be taken as a
sign of Weyl's beginning retreat from his once so ambitious programme.

3.6 Missed opportunities

3.6.1 Supersymmetry

One issue that attracted much attention during the 1960s was, whether the
observed particle multiplets could be understood on the basis of an all em-
bracing symmetry principle that would combine the Poincaré group with
the internal symmetry groups displayed by the multiplet structures. This
combination of groups should be non-trivial, i.e., not be a direct product, for
otherwise the internal symmetries would commute with the spacetime sym-
metries and lead to multiplets degenerate in mass and spin (see, e.g., [49]).
Subsequently, a number of no-go theorems appeared, which culminated in
the now most famous theorem of Coleman & Mandula [11]. This theorem
states that those generators of symmetries of the S-matrix belonging to the
Poincaré group necessarily commute with those belonging to internal sym-
metries. The theorem is based on a series of assumptions [45] involving the
crucial technical condition that the S-matrix depends analytically on stan-
dard scattering parameters. What is less visible here is that the structure of
the Poincaré group enters in a decisive way. This result would not follow for

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[44] German original: "Die Theorie gibt keinen Aufschluß über die *Ungleichartigkeit von*
positiver und negativer Elektrizität. Das kann ihr aber nicht zum Vorwurf gemacht
werden. Denn jene Ungleichartigkeit beruht ohne Zweifel darauf, daß von den beiden
Urbestandteilender der Materie, Elektron und Wasserstoffkern, der positiv geladene mit
einer anderen Masse verbunden ist als der negaiv geladene; sie entspringt aus der Natur
der Materie und nicht des Feldes." ([71], p. 308)

[45] The assumptions are: (1) there exists a non-trivial (i.e., $\neq \mathbf{1}$) S-matrix which depends
analytically on s (the squared centre-of-mass energy) and t (the squared momentum
transfer); (2) the mass spectrum of one-particle states consists of (possibly infinite)
isolated points with only finite degeneracies; (3) the generators (of the Lie algebra) of
symmetries of the S-matrix contains (as a Lie-sub algebra) the Poincaré generators;
(4) some technical assumptions concerning the possibility of writing the symmetry gen-
erators as integral operators in momentum space.

the Galilean group, as was explicitly pointed out by Coleman & Mandula ([11], p. 159).

One way to avoid the theorem of Coleman & Mandula is to generalise the notion of symmetries. An early attempt was made by Golfand & Likhtman [29], who constructed what is now known as a Super-Lie algebra, which generalises the concept of Lie algebra (i.e. symmetry generators obeying certain commutation relations) to one also involving anti-commutators. In this way it became possible for the first time to link particles of integer and half-integer spin by a symmetry principle. It is true that Supersymmetry still maintains the degeneracy in masses and hence cannot account for the mass differences in multiplets. But its most convincing property, the symmetry between bosons and fermions, suggested a most elegant resolution of the notorious ultraviolet divergences that beset Quantum Field Theory.

It is remarkable that the idea of a cancellation of bosonic and fermionic contributions to the vacuum energy density occurred to Pauli. In his lectures *Selected Topics in Field Quantization*, delivered in 1950-51 (in print again since 2000, [59]), he posed the question

> *..whether these zero-point energies [from Bosons and Fermions] can compensate each other.* ([59], p. 33)

He tried to answer this question by writing down the formal expression for the zero point energy density of a quantum field of spin j and mass $m_j > 0$ (Pauli restricted attention to spin 0 and spin $1/2$, but the generalisation is immediate):

$$4\pi^2 \frac{E_j}{V} = (-1)^{2j}(2j+1)\int dk\, k^2 \sqrt{k^2 + m^2}\,. \tag{29}$$

Cancellation should take place for high values of k. The expansion

$$4\int_0^K dk\, k^2 \sqrt{k^2 + m^2} = K^4 + m_j^2 K^2 - m_j^4 \log(2K/m_j) + O(K^{-1}) \tag{30}$$

shows that the quartic, quadratic, and logarithmic terms must cancel in the sum over j for the limit $K \to \infty$ to exist. This implies that for $n = 0, 2, 4$ one must have

$$\sum_j (-1)^{2j}(2j+1)m_j^n = 0 \quad \text{and} \quad \sum_j (-1)^{2j}(2j+1)\log(m_j) = 0\,. \tag{31}$$

Pauli comments that

> *these requirements are so extensive that it is rather improbable that they are satisfied in reality.* ([59], p. 33)

Unless enforced by an underlying symmetry, one is tempted to add! This would have been the first call for a supersymmetry in the year 1951.

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However, the real world does not seem to be as simple as that. Supersymmetry, if at all existent, is strongly broken in the phase we live in. So far no supersymmetric partner of any existing particle has been detected, even though some of them (e.g., the neutralino) are currently suggested to be viable candidates for the missing-mass problem in cosmology. Future findings (or non-findings) at the Large Hadron Collider (LHC) will probably have a decisive impact on the future of the idea of supersymmetry, which—whether or not it is realised in Nature—is certainly very attractive; and Pauli came close to it.

3.6.2 Kaluza-Klein Monopoles

Ever since its first formulation in 1921, Pauli as well as Einstein were much attracted by the geometric idea of Theodor Kaluza and its refinement by Oskar Klein, according to which the classical theories of the gravitational and the electromagnetic field could be unified into a single theory, in which the unified field has the same meaning as Einsteins gravitational field in General Relativity, namely as metric tensor of spacetime, but now in five instead of four dimensions. The momentum of a particle in the additional fifth direction (which is spacelike) is now to be interpreted as its charge. Charge is conserved because the geometry of spacetime is *a priori* restricted to be independent of that fifth direction. The combined field equations are exactly the five-dimensional analog of Einstein's equations for General Relativity.

A natural question to address in this unified classical theory was whether it admits solutions that could represent particle-like objects. More precisely, the solution should be stationary, everywhere regular, and possess long-ranging gravitational and electromagnetic fields (usually associated with aspects of mass and charge). Pauli, who was very well familiar with this theory since its first appearance[46], kept an active interest in it even after the formulations of Quantum Mechanics and early Quantum Electrodynamics, which made it unquestionable for him that the problem of matter could not be adequately addressed in the framework of a classical field theory, unlike Einstein, who maintained such a hope in various forms until the end of his life in 1955.

It is therefore remarkable that in 1943 Einstein and Pauli wrote a paper in which they proved the non-existence of such solutions. The introduction contains the following statement:

> *When one tries to find a unified theory of the gravitational and electromagnetic fields, he cannot help feeling that there is some truth in* Kaluza's *five-dimensional theory. ([14], p. 131)*

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[46] It came out too late to be considered in the first edition of Pauli's Relativity article. But he devoted to it a comparatively large space in his Supplementary Notes written in early 1956 for the first English edition ([54], Suppl. Note 23, pp. 227-232; [58], pp. 276-282)

In fact, Einstein and Pauli offered a proof for the more general situation with an arbitrary number of additional space dimensions, fulfilling the generalised Kaluza-Klein "cylinder-condition" that the gravitational field should not depend on any of these extra directions. Note that this extra condition introduces non-dynamical background structures, so that of the 5-dimensional diffeomorphism group only those diffeomorphisms preserving this condition can act as symmetries, a point Pauli often emphasised as a deficiency regarding the Kaluza-Klein approach.

Restricting attention to five dimensions, the explicitly stated hypotheses underlying the proof were these ([14], p. 131; annotations in square brackets within quotations are mine):

H1 "The field is stationary (i.e the g_{ik} [the five-dimensional metric] are independent of x^4 [the time coordinate])." Clearly, g_{ik} is also assumed to be independent of the fifth coordinate x^5.

H2 "It [the field g_{ik}] is free from singularities."

H3 "It is imbedded in a Euclidean space (of the Minkowski type), and for large values of r (r being the distance from the origin of the spatial coordinate system) g_{44} has the asymptotic form $g_{44} = -1 + \mu/r$, where $\mu \neq 0$." The last condition is meant to assure the non-triviality of the solution, i.e. that there really is an attracting object at the spatial origin. This becomes clear if one recalls that in the lowest weak-field and slow-motion approximation $1 + g_{44}$ just corresponds to the Newtonian gravitational potential. Unfortunately, the other statement: "It is imbedded in a Euclidean space (of the Minkowski type)" seems ambiguous, since the solution is clearly not meant to be just (a portion of) 5-dimensional flat Minkowski space. Hence the next closest reading is presumably that the underlying five-dimensional spacetime manifold is (diffeomorphic to) $\mathbb{R}^5$, with some non-flat metric of Minkowskian signature $(-, +, +, +, +)$.[47]

The elegant method of proof makes essential use of the fact that the suitably restricted group of spacetime diffeomorphisms (to those preserving the cylinder condition) is a symmetry group for the full set of equations in the sense of (3) of Definition 1. More precisely, two types of diffeomorphisms from that class are considered separately by Einstein and Pauli:

D1 Arbitrary ones in the three coordinates (x^1, x^2, x^3) which leave invariant the (x^4, x^5) coordinates.

D2 Linear ones in the (x^4, x^5) coordinates, leaving invariant the (x^1, x^2, x^3).

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[47] In fact, it turns out that formally the proof does not depend on whether the fifth dimension is space- or time-like, as noted by Einstein and Pauli ([14], p. 134).

Now, as a matter of fact, this innocent looking split introduces a further and, as it turns out, crucial restriction, over and above the hypotheses H1-H3. The point is that the split and, in particular, the set D2 of diffeomorphisms simply do not exist unless the spacetime manifold, which in H3 was assumed to be $\mathbb{R}^5$, globally splits into $\mathbb{R}^2 \times \mathbb{R}^3$ such that the first factor, $\mathbb{R}^2$, corresponds to the x^4x^5-planes of constant spatial coordinates (x^1, x^2, x^3) and the second factor, $\mathbb{R}^3$, corresponds to the $x^1x^2x^3$-spaces of constant coordinates (x^4, x^5). But this need not be the case if H1-H3 are assumed. The identity derived by Einstein and Pauli from the requirement that transformations of the field induced by diffeomorphisms of the type D2 are symmetries are absolutely crucial in proving the non-existence of regular solutions.[48]

We now know that this additional restriction is essential to the non-existence result: There do exist solutions of the type envisaged that satisfy H1-H3, but violate the extra (and superfluous) splitting condition.[49] They are called *Kaluza-Klein Monopoles* [65][31] and carry a gravitational mass as well as a magnetic charge. It is hard to believe that Pauli as well as Einstein would not have been much impressed by those solutions, though possibly with different conclusions, had they ever learned about them. It is also conceivable that these solutions could have been found at the time, had real attempts been made, rather than—possibly—discouraged by Pauli's and Einstein's result. In fact, Kurt Gödel, who was already in Princeton when Pauli visited Einstein, found his famous cosmological solution [26] in 1949 by a very similar geometric insight that also first led to the Kaluza-Klein monopole [65].[50]

3.7 Irritations and psychological prejudices

One of Pauli's major interests were discrete symmetries, in particular the transformation of space inversion, $\vec{x} \mapsto -\vec{x}$, also called *parity* transformation. Given a linear wave equation which is symmetric under the proper orthochronous (i.e. including no space and time inversions) Poincaré group,

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[48] Specifically we mean their identity (13), which together with spatial regularity implies the integral form (13a), which in turn leads directly to vanishing mass in (22-23a). (All references are to their formulae in [14].)

[49] The somewhat intricate topology of the Kaluza-Klein spacetime is this: The x^5 coordinate parametrises circles which combine with the 2-spheres (polar coordinates (θ, φ)) of constant spatial radius, r, into 3-spheres (Hopf fibration) which are parametrised by (θ, φ, x^5), now thought of as Euler angles. The radii of these 3-spheres appropriately shrink to zero as r tends to zero, so that $(r, \theta, \varphi, x^5)$ define, in fact, polar coordinates of $\mathbb{R}^4$. Together with time, x^4, we get $\mathbb{R}^5$ as global topology. Now the submanifolds of constant (x^1, x^2, x^3) are those of constant (r, θ, φ) and have a topology $\mathbb{R} \times S^1$ rather than $\mathbb{R}^2$, so that the linear transformations D2 in the x^4x^5 coordinates do not define diffeomorphisms of the Kaluza-Klein spacetime manifold.

[50] Both use invariant metrics on 3-dimensional group manifolds, $SU(2)$ in the KK case, $SU(1,1)$ in Gödels case. This simplifies the calculations considerably.

one may ask whether it is also symmetric under space and time inversions. For this to be a well defined question one has to formulate conditions on how these inversions interact with Poincaré transformations. Let us focus on the operation of space inversion. If this operation is implementable by an operator $\mathcal{P}$, it must conjugate each rotation and each time translation to their respective self, and each boost and each space translation to their respective inverse. This follows simply from the geometric meaning of space inversion. Hence, generally speaking, we need to distinguish the following three possible scenarios (recall the notation from Section 2.2):

(a) $\mathcal{P}$ acts on $\mathcal{K}$ and is a symmetry, i.e. leaves $\mathcal{D}_\Sigma \subset \mathcal{K}$ invariant;

(b) $\mathcal{P}$ acts on $\mathcal{K}$ and is no symmetry, i.e. leaves $\mathcal{D}_\Sigma \subset \mathcal{K}$ not invariant;

(c) $\mathcal{P}$ is not implementable on $\mathcal{K}$.

It is clear that when one states that a certain equation is not symmetric under $\mathcal{P}$ one usually addresses situation (b), though situation (c) also occurs, as we shall see.

Consider now the field of a massless spin-$\frac{1}{2}$ particle, that transforms irreducibly under the proper orthochronous Poincaré group. The field is then either a two-component spinor, ϕ^A, which in the absence of interactions obeys the so-called Weyl equation[51]

$$\partial_{AA'}\phi^A = 0 \,. \tag{32}$$

Alternatively, one may also start from a four-component Dirac spinor,

$$\psi = \begin{pmatrix} \phi^A \\ \overline{\chi}_{A'} \end{pmatrix} \tag{33}$$

which carries a reducible representation of the proper orthochronous Poincaré group: If ϕ^A transforms with $A \in \mathsf{SL}(2,\mathbb{C})$ then $\overline{\chi}_{A'}$ transforms with $(A^\dagger)^{-1}$ (being an element of the complex-conjugate dual space), so that the space of the upper two components ϕ^A of ψ and the space of the lower two components $\overline{\chi}_{A'}$ of ψ are separately invariant. One may then eliminate two of the four components by the so-called Majorana condition, which requires the state ψ to be identical to its charge-conjugate, ψ^c, where

$$\mathcal{C}: \psi \mapsto \psi^c := i\gamma^2\psi^* = \begin{pmatrix} \chi^A \\ \overline{\phi}_{A'} \end{pmatrix} \,. \tag{34}$$

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[51] Here I use the standard Spinor notation where upper-case capital Latin indices refer to (components of) elements in spinor space (2-dimensional complex vector space), lower case indices to the dual space, and primed indices to the respective complex-conjugate spaces. Indices are raised and lowered by using a (unique up to scale) $\mathsf{SL}(2,\mathbb{C})$ invariant 2-form. An overbar denotes the map into the complex-conjugate vector space. Unless stated otherwise, my conventions are those of [64].

Hence for a Majorana spinor one has $\phi = \chi$ and the interaction-free Dirac equation reads

$$\gamma^\mu \partial_\mu \psi := \sqrt{2} \begin{pmatrix} 0 & \partial^{AA'} \\ \partial_{A'A} & 0 \end{pmatrix} \begin{pmatrix} \phi^A \\ \overline{\phi}_{A'} \end{pmatrix} = 0 \,. \tag{35}$$

One can now either regard (32) or (35) as the interaction-free equation for a neutrino.

Here I wish to briefly recall a curious discussion between Pauli and Fierz on whether or not these two equations describe physically different state of affairs. Superficially this discussion is about a formal and, mathematically speaking, rather trivial point. But, as we will see, it relates to deep-lying preconceptions in Pauli's thinking about issues of symmetry. This makes it worth looking at this episode in some detail.

First note that there is an obvious bijection, β, between two-component spinors and Majorana spinors, given by

$$\beta : \phi^A \mapsto \begin{pmatrix} \phi^A \\ \overline{\phi}_{A'} \end{pmatrix} \,. \tag{36}$$

Note also that the set of Majorana spinors is *a priori* a *real*[52] vector space, though it has a complex structure, j, given by

$$j : \begin{pmatrix} \phi^A \\ \overline{\phi}_{A'} \end{pmatrix} \mapsto \begin{pmatrix} i\phi^A \\ -i\overline{\phi}_{A'} \end{pmatrix} \,, \tag{37}$$

with respect to which the bijection (36) satisfies $\beta \circ i = j \circ \beta$, where here i stands for the standard complex structure (multiplication with imaginary unit i) in the space $\mathbb{C}$ of two-component spinors. However, regarded as a map between complex vector spaces, the bijection β is *not* linear.

Now, Pauli observed already in 1933 (see quotation below) that the Weyl equation (32) is not symmetric under parity. Hence he concluded it could not be used to describe Nature. In fact, what is actually the case is that parity cannot even be implemented as a *linear* map on the space of two-component spinors (case (c) above). This is easy to see and in fact true for any irreducible representation of the Lorentz group that stays irreducible if restricted to the rotation group (i.e. for purely primed or purely unprimed spinors).[53]

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[52] The reality structure on the complex vector space of Dirac spinors is provided by the charge conjugation map.

[53] As stated above, the geometric meaning of space inversion requires that the parity operator (if existent) commutes with spatial rotations and conjugates boosts to their inverse. The first requirement implies (via Schur's Lemma) that it must be a multiple of the identity in any irreducible representation that stays irreducible when restricted to the rotation subgroup, which contradicts the second requirement. Hence it cannot exist in such representations, which are precisely those with only unprimed or only primed indices.

On the other hand, the Dirac equation *is* symmetric under space inversions. Indeed, the spinor-map corresponding to the inversion in the spatial plane perpendicular to the timelike normal n is given by

$$\mathcal{P} : \psi \mapsto \psi^p := \eta\, n_\mu \gamma^\mu (\psi \circ \rho_n)\,, \qquad (38)$$

where $\rho_n : x^\mu \mapsto -x^\mu + 2n^\mu(n_\nu x^\nu)$ and where η is a complex number of unit modulus, called the *intrinsic parity* of the particular field ψ. It is easy to see that $\mathcal{P}$ is a symmetry of (35) for any η. Note that $\mathcal{P}^2 = \eta^2 \mathbf{1}$ so that $\eta \in \{1, -1, i, -i\}$, since for spinors one only requires $\mathcal{P}^2 = \pm\mathbf{1}$ (rather than $\mathcal{P}^2 = \mathbf{1}$). It is also easy to verify that $\mathcal{P}$ commutes with $\mathcal{C}$ iff $\eta = \pm i$. So if we assign imaginary parity to the Majorana field[54], the operator $\mathcal{P}$ also acts on the subspace of Majorana spinors. We conclude that the free Majorana equation *is* parity invariant.

Hence it seems at first that the Weyl formulation and the Majorana formulation differ since they have different symmetry properties. But this is not true. Using the bijection (36), we can pull-back the parity map (38) to the space of two-component spinors, where it becomes (now either $\eta = i$ or $\eta = -i$)

$$\phi^A \mapsto \eta\sqrt{2}\, n^{AA'}(\overline{\phi}_{A'} \circ \rho_n)\,, \qquad (39)$$

which is now an anti-linear map on the space of two-component spinors.

All this was essentially pointed out to Pauli by Markus Fierz in a letter dated February 6th 1957 ([45] Vol. IV, Part IV A, Doc. [2494], p. 171) in connection with Lee's and Yang's two-component theory of the neutrino. Fierz correctly concluded from this essential equivalence[55] that the 2-component theory as such (i.e. without interactions) did not warrant the conclusion of parity violation; only interactions could be held responsible for that.

This was a relevant point in the theoretical discussion at the time, as can be seen from the fact that there were two independent papers published in *The Physical Review* shortly after Fierz's private letter to Pauli, containing the very same observation. The first paper was submitted on February 13th by McLennan [44], the second on March 25th by Case [9]. In fact, Serpe made this observation already in 1952 [62] and emphasised it once more in 1957 [63].

One might be worried about the anti-linearity of the transformation in (39). In that respect, also following Fierz, an illuminating analogy may be mentioned regarding the vacuum Maxwell equations, which can be written in the form

$$i\partial_t \vec{\Phi} - \vec{\nabla} \times \vec{\Phi} = 0\,, \qquad \vec{\nabla} \cdot \vec{\Phi} = 0\,, \qquad (40)$$

where

$$\vec{\Phi} := \vec{E} + i\vec{B} \qquad (41)$$

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[54] Which is also the standard choice in QFT; see e.g. [69], pp. 126,226.

[55] Meaning the existence of a bijection that maps all quantities of interest (states, currents, symmetries) of one theory to the other.

is a complex combination of the electric and magnetic field. Both equations (40) are clearly equivalent of the full set of Maxwell's equations. It can be shown that spatial inversions cannot be implemented as complex-linear transformations on the complex-valued field $\vec{\Phi}$.[56] But, clearly, we know that Maxwell's equations are parity invariant, namely if we transform the electric field as $\vec{E} \mapsto -\vec{E} \circ \rho$ ('polar' vector-field) and the magnetic field as $\vec{B} \mapsto \vec{B} \circ \rho$ ('axial' vector-field), where $\rho : (t, \vec{x}) \mapsto (t, -\vec{x})$. This corresponds to an *anti*linear symmetry of (40), given by $\vec{\Phi} \mapsto -\overline{\vec{\Phi}} \circ \rho$.

Coming back to Fierz's (and other's) original observation for the spinor field, they were accepted without much ado by others. For example, in her survey on the neutrino in the Pauli Memorial Volume, Madame Wu states that "It is the interaction and the interaction only that violates parity" ([18], footnote on p. 270.). In note 25c of that paper she explicitly thanks Fierz for "enlightening discussions" on the two-component theory of the neutrino. Clearly Fierz expected his observation to be of interest to Pauli, who had already in the 1933 first edition of his handbook article on wave mechanics propagated the view that Weyl's two-component equations are[57]

> ...*not invariant under reflections (interchange of left and right) and, as a consequence, not applicable to the physical reality.*

But instead, Pauli reacts with a surprising plethora of ridiculing remarks:[58]

> *Dear Mr. Fierz! Your letter from the 6th is the biggest blunder you ever commited in your life! (Probably this afternoon you will send a correction). Have only read the first paragraph of your letter which originated in the asylum and was shaking with laughter. [...] When this letter arrives (yours I will frame!) you probably will already know everything.*

Personal irritations emerged which lasted about one week through several exchanges of letters and a phone-call. Finally Pauli essentially conceded

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[56] Equations (40) are equivalent to $\partial^{AA'} f_{AB} = 0$, where f_{AB} is the unprimed spinor equivalent of the tensor $F_{\mu\nu}$ for the electromagnetic field strength. Parity cannot be linearly implemented on this purely unprimed spinor, for reasons already explained in footnote 53.

[57] German original, full sentence: "Indessen sind diese Wellengleichungen, wie ja aus ihrer Herleitung hervorgeht, nicht invariant gegenüber Spiegelungen (Vertauschung von links und rechts) und infolge dessen sind sie auf die physikalische Wirklichkeit nicht anwendbar" ([56], p. 234, note 54). The conclusion concerning non applicability to the physical reality is cancelled in the 1958 edition; cf. [56], p. 150.

[58] German original: "Lieber Herr Fierz! Ihr Brief vom 6. ist der größte Bock den Sie im Laufe Ihres Lebens geschossen haben! (Wahrscheinlich kommt heute Nachmittag schon eine Berichtigung von Ihnen.) Habe nur den ersten Absatz Ihres der Anstalt entsprungenen Briefes gelesen und mich geschüttelt vor lachen. [...] Wenn dieser Brief ankommt (Ihren rahme ich ein!), wissen Sie wohl schon alles!" ([45], Vol. IV, Part IV A, Doc. [2497], p. 179).

Fierz's point in a long letter of February 12th 1957 that also contains first hints at Pauli's psychological resistances (the emphasis is Pauli's):[59]

> *Your presentation creates in me a feeling of "formal boredom",
> to which the fusillade of laughter was of a* compensatory *nature.*

This is a curious episode and not easy to understand. Pauli's point seems to have been that he wanted to maintain the particle-antiparticle distinction *independently* of parity, whereas Fierz pointed out that the two-component theory provided no corresponding structural element: In Weyl's form the operations $\mathcal{C}$ and $\mathcal{P}$ simply do not exist separately, in the Majorana form $\mathcal{P}$ exists and $\mathcal{C}$ is the identity (hence not distinguishing). Psychologically speaking, Pauli's point becomes perhaps more understandable if one takes into account the fact that since the fall of 1956 he was thinking about the question of lepton-charge conservation. Intuitively he had therefore taken as self-evident that opposite helicities also corresponded to the particle-antiparticle duality (cf. [45], Vol. IV, Part IV A, Doc. [2497]), even though this mental association did not correspond to anything in the equations. In a letter dated February 15th 1957 he offered the following in-depth psychological explanation to Fierz (the emphases are Pauli's):[60]

> *Well, the fusillade of laughter occurred with the expression "Majorana Theory" of your first letter. After this catchword I could not go on reading. The immediate association with Majorana clearly has been this: "aha, particles and antiparticles should no longer exist, these one intends to take away from me (as one takes away a symbol from somebody)!" This causes me* anxiety.
> *I also know that since last fall the conservation of lepton charge*

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[59] German Original: "Ihre Darstellung erzeugt bei mir das Gefühl der 'formalistischen Langeweile'', zu der die Lachsalve *kompensatorisch* war" ([45], Vol. IV, Part IV A, Doc. [2510], p. 197).

[60] German original: "Also die 'Lachsalve' erfolgte beim Wort 'Majorana Theorie' Ihres ersten Briefes, ich konnte nach diesem Stichwort nicht mehr weiterlesen. Die unmittelbare Assoziation zu Majorana war natürlich 'aha, Teilchen und Antiteilchen soll es nicht mehr geben, die will man wir wegnehmen (wie man jemandem ein Symbol wegnimmt)!' Davor habe ich *Angst*. Ich weiss auch, daß mir schon seit Herbst die Erhaltung der Leptonladung in der Physik ungeheuer wichtig ist – rational betrachtet, vielleicht *zu wichtig*. Ich habe *Angst*, sie könnte sich als unrichtig herausstellen und, psychologisch gesehen, ist 'Unzufriedenheit' ein Euphemismus für Angst. Die CP - ($\equiv$ Majorana P+ Vertauschung von Elektron und Positron) Invarianz ist mir auch wichtig, aber weniger wichtig als die Erhaltung der Leptonladung. Es ist *sicher wahr*, daß 'mein platonischer Spiegelkomplex angestochen' war. Teilchen und Antiteilchen sind das Symbol für jene allgemeine Spiegelung (wie weit sie speziell *platonisch* ist, dessen bin ich nicht sicher). [...] Offenbar hat der 'Spiegelungskomplex' bei mir etwas mit Tod und Unsterblichkeit zu tun. *Daher* die Angst! Wäre die Beziehung zwischen dem schlafenden Spiegelbild und dem Wachenden gestört, oder wären sie gar identisch (Majorana), so gäbe es, psychologisch gesprochen, weder Leben (Geburt) noch Tod." ([45], Vol. IV, Part IV A, Doc. [2517], p. 225)

in physics was tremendously important to me—looked upon ra-
tionally probably too important*. I am* anxious *it could turn out*
to be incorrect and, psychologically speaking, "discontentedness"
is a euphemism for anxiety. The CP *(≡ Majorana* P+ *exchange*
between electron and positron) invariance is also important to
me, but less so than the conservation of lepton charge. It is
certainly true *that it "hit upon my Platonic mirror complex".*
Particles and antiparticles are the symbol for that more general
mirroring (I am not sure to what extent it is particularly pla-
tonic*).[...] Mirroring is also a* gnostic symbol *for life and death.*
There light is extinguished *at birth and* lightened up *at death.*
[..] Obviously, for me the "mirroring complex" has something to
do with death and immortality. Hence *the anxiety! If the rela-*
tion between the sleeping mirror image and the one awake would
be disturbed, or if they would even be identical (Majorana), then,
psychologically speaking, there would neither be life (birth) nor
death.

Fierz later commented on that episode in a personal letter to Norbert Strau-
mann, parts of which are quoted in [67].

3.8 β-Decay and related issues

3.8.1 CPT

In 1955 a collection of essays by distinguished physicists appeared to cele-
brate Niels Bohr's 70th birthdays [52]. Pauli's contribution ([52], p. 30-51) is
entitled "Exclusion Principle, Lorentz Group and Reflection of Space-Time
and Charge", whose introduction contains the following remarks:

After a brief period of spiritual and human confusion, caused by
provisional restriction to "Anschaulichkeit", a general agreement
was reached following the substitution of abstract mathematical
symbols, as for instance psi, for concrete pictures. Especially the
concrete picture of rotation has been replaced by mathematical
characteristics of the representations of the group of rotations
in three dimensional space. This group was soon amplified to
the Lorentz group in the work of Dirac. [...] The mathemat-
ical group was further amplified by including the reflections of
space and time. [...] I believe that this paper also illustrates the
fact that a rigorous mathematical formalism and epistemological
analysis are both indispensable in physics in a complementary
way in the sense of Niels Bohr. While I try to use the former to
connect all mentioned features of the theory with help of a richer
"fullness" of plus and minus signs in an increasing "clarity", the

latter makes me aware that the final "truth" on the subject is still
"dwelling in the abyss".[61] ([52], p. 30-31)

In some sense this paper of Pauli's can be seen as a follow-up to his spin-statistics paper already discussed above, the main difference being that Pauli now considers *interacting* fields. Pauli now *assumes* (1) the validity of the spin-statistics correlation for interacting fields (for which there was no proof at the time), (2) invariance under (the universal cover of) the proper orthochronous Lorentz group SL(2,$\mathbb{C}$) (as in the spin-statistics paper), and (3) locality of the interactions (i.e. involving only finitely many derivatives). Then Pauli shows that this suffices to derive the so-called CPT theorem that states that the combination of charge conjugation (C) and spacetime reflection (PT) is a symmetry.[62]

At the time (1955) Pauli wrote his paper it was not known whether any of the operations of C, P, or T would separately *not* be a symmetry. This changed when in January 1957 through the experiments of Madame Wu *et al.*, in which explicit violations of P and C were seen in processes of beta-decay, following a suggestion that this should be checked by Lee and Yang in mid 1956 [40]. Pauli had still offered a bet that this would not happen on January 17th 1957 (the emphases are Pauli's):[63]

I do not believe that God is a weak left-hander and would be prepared to bet a high amount that the experiment will show a symmetric angular distribution of the electrons (mirror symmetry). For I cannot see a logical connection between the strength of an interaction and its mirror symmetry.

In view of this firm belief in symmetry the following is remarkable: In his CPT paper Pauli takes great care to writes down the most general ultralocal (i.e. no derivatives) four-fermion interaction (for the neutron, proton, electron and neutrino), which is *not* P invariant. In contains 10 essentially different terms with ten coupling constants $C_1, \cdots C_{10}$, only the first five of which are parity invariant (scalars), whereas the other five are pseudoscalars, i.e change sign under spatial inversions. Apparently this he did just for the

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[61] Here Pauli sets the following footnote: "I refer here to Bohr's favourite verses of Schiller: 'Nur die Fülle führt zur Klarheit / Und im Abgrund wohnt die Wahrheit'".

[62] Pauli used a now outdated terminology: instead of CPT he uses SR (strong reflection), instead of PT he uses WR (weak reflection), and instead of C he uses AC (antiparticle conjugation). Preliminary versions of the CPT theorem appeared in papers by Julian Schwinger (1951)and Gerhard Lüders (1954) to which Pauli refers. Two years after Pauli's 1955 paper Res Jost gave a very elegant proof in the framework of axiomatic quantum field theory [34].

[63] German original: "Ich glaube aber *nicht, daß der Herrgott ein schwacher Linkshänder* ist und wäre bereit hoch zu wetten, daß das Experiment symmetrische Winkelverteilung der Elektronen (Spiegelinvarianz) ergeben wird. Denn ich sehe keine logische Verbindung von *Stärke* einer Wechselwirkung und ihrer Spiegelinvarianz." ([45], Vol. IV, Part IV A, Doc. [2455], p. 82)

sake of mathematical generality without any physical motivation, as he explicitly stated in a letter to Madame Wu dated January 19th 1957 (the emphases are Pauli's):

> *When I considered such formal possibilities in my paper in the Bohr-Festival Volume (1955), I did not think that this could have something to do with Nature. I considered it merely as a mathematical play, and, as a matter of fact, I did not believe in it when I read the paper of Yang and Lee. [...] What prevented me* until now *from accepting this formal possibility is the question why this restriction of mirroring appears only in the 'weak' interactions, not in the strong ones.* Theoretically, *I do not see any interpretation of this fact, which is empirically so well established.* ([45], Vol. IV, Part IV A, Doc. [2460], p. 89)

Lee and Yang took this possibility more serious: In an appendix to their paper they also write down all ten terms for the full, parity non-invariant interaction ([40], p. 258), without any citation of Pauli.

Pauli first learnt that the experiments by Madame Wu *et al.* had led to an asymmetric angular distribution from a letter by John Blatt from Princeton, dated January 15th 1957. There Blatt writes:

> *I don't know whether anyone has written you as yet about the sudden death of parity. Miss Wu has done an experiment with beta-decay of oriented Co nuclei which shows that parity is* not *conserved in* β *decay. [...] We are all rather shaken by by the death of our well-beloved friend, parity.* ([45], Vol. IV, Part IV A, Doc. [2451], p. 74)

Pauli, too, was shocked as he stated in his famous letter to Weisskopf dated January 27/28 1957 ([45], Vol. IV, Part IV A, Doc. [2476]). In that very same letter Pauli already started speculating how symmetry could be restored by letting the constants C_i become dynamical field, scalar fields for $i = 1, \cdots, 5$ and pseudo-scalar ones for $i = 6, \cdots, 10$:[64]

> *Let us imagine, for example, the terms with $C_1, \cdots, C_5$ being multiplied with a scalar field $\phi(x)$, the terms $C_6, \cdots, C_{10}$ multiplied with a pseudo-scalar field $\hat{\phi}(x)$. For God Himself, Who can change the sign of $\hat{\phi}(x)$, such a theory would be left-right-invariant—not for us mortal men, however, who do not know*

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[64] German original: "Denken wir uns z.B. die Terme mit $C_1, \cdots, C_5$ mit einem Skalarfeld $\phi(x)$, die Terme mit $C_6, \cdots, C_{10}$ mit einem Pseudo-Skalarfeld $\hat{\phi}(x)$ multipliziert. Für den Herrgott, der das Vorzeichen von $\hat{\phi}(x)$ umdrehen kann, wäre eine solche Theorie natürlich rechts-links-invariant – nicht aber für uns sterbliche Menschen, die wir gar nichts wissen über jenes hypothetische neue Feld, außer daß es praktisch auf der Erde raum-zeitlich konstant (statisch-homogen) ist, und die wir noch kein Mittel haben, es zu ändern." ([45], Vol. IV, Part IV A, Doc. [2476], pp. 122-123)

> *anything about that new hypothetical field, except that it is prac-*
> *tically constant in space and time on earth (static-homogeneous),*
> *and that we do not yet[65] have any means to change it.*

The mechanism envisaged here to restore symmetry is just that discussed in Section 2.2, where non-dynamical backgrounds structures, Σ, are (formally) turned into dynamical quantities, Φ.

3.8.2   The Pauli group

As already mentioned, since fall of 1956 Pauli's thinking about beta-decay was dominated by the lepton-charge conservation. In a paper submitted on March 14th 1957, entitled *On the Conservation of Lepton Charge* ([38], Vol. 2, pp. 1338-1349), Pauli once more showed his mastery of symmetry considerations while keeping everything at the largest possible degree of generality.

He starts by considering the most general ultralocal four-fermion interactions (not necessarily preserving parity or lepton charge) in which the neutrino field is represented by a Dirac 4-spinor, ψ. For what follows it is convenient to think of the four components of ψ as comprising the following four particle states (per momentum): a left-handed neutrino, ψ_L, a right-handed neutrino, ψ_R, and their antiparticles ψ_L^c and ψ_R^c respectively. Note that this means $\psi_{L,R}^c := (\psi_{L,R})^c$ and that accordingly ψ_L^c is right- and ψ_R^c is left-handed. Here we follow the convention of [35].

Next Pauli considers a four-parameter group of canonical transformations (i.e. they leave the anticommutation relations between the fermion fields invariant) of the neutrino field, henceforth called the *Pauli group*, whose interpretation will be given below. These transformations define a symmetry of the interaction-free equations of motions (assuming a massless neutrino throughout), but will generally *not* define a symmetry once the interaction is taken into account. Rather, the following is true (cf. [48]): Suppose that the general interaction depends on a finite number of coupling constants c_i for $i = 1, \cdots, n$ and that the equations of motion follow from an action principle with Lagrange density $\mathbf{L}\{\Sigma \mid \Phi\}$, where Σ represents the array of coupling constants (we notationally ignore other non-dynamical structures here) and Φ the dynamical fields. Then the Pauli group acts as covariance in a slightly stronger sense than (5), namely so that

$$\mathbf{L}\{g \cdot \Sigma \mid g \cdot \Phi\} = \mathbf{L}\{\Sigma \mid \Phi\}. \tag{42}$$

This means that on the level of the Lagrange density (or the Hamiltonian), and hence in particular at the level of the equations of motion, the transformation of the dynamical fields can be compensated for by a transformation

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[65] The "yet" is incorrectly omitted in the official translation ([45], Vol. IV, Part IV A, p. 126).

of the coupling constants. A large part of Pauli's paper is actually devoted to the determination of that compensating action of the Pauli group on the array of coupling constants.

Next suppose the initial state is chosen to be invariant under the Pauli group, i.e. $\mathfrak{g} \cdot \Phi = \Phi$ for all $\mathfrak{g}$. Then (42) implies that its evolution with interaction parametrised by Σ (the array of c_i's) is identical to the evolution parametrised by $\mathfrak{g} \cdot \Sigma$ for any $\mathfrak{g}$. Hence the outcome of the evolution can only depend on the c_i's through their Pauli-invariant combinations.[66] In particular, since the neutrinoless double beta-decay simply has no initial neutrino, this reasoning can be applied to it. If this lepton-charge-conservation violating process is deemed impossible, the corresponding Pauli-invariant combination of coupling constants to which the scattering probability is proportional[67] must vanish. This, in turn, gives the sought-after constraint on the possible four-fermion interaction. For (massless) neutrinos in Majorana representation Pauli finally arrived at the result that either only the left- or the right-handed component enters the interaction. It should be added that this clever sort of reasoning was shortly before used by Pursey in a less general setting [61] in which the interaction was specialised *a priori* to conserve lepton charge.[68] More on the history of the search for the right form of the four-fermion interaction may be found in [67]. It should also be mentioned that the possibility of neutrinoless double beta-decays is currently still under active experimental investigation at the National Gran Sasso Laboratory, where the 2003-2005 CUORICINO experiment set upper bounds for the Majorana mass of the electron neutrino well below one eV. The upcoming next-generation experiment, CUORE, is designed to lower this bound to $5 \cdot 10^{-2}\,meV$; compare [30].

What is the interpretation of the Pauli group? Mathematically it is isomorphic to $U(2)$, the group of 2×2 unitary matrices acting on a two-dimensional complex vector space. Here there are two such spaces (per 4-momentum) in which it acts: the 'left-handed subspace' that is spanned by the two left-handed components ψ_L and ψ_R^c, and the 'right-handed subspace' that is spanned by the two right-handed components ψ_R and ψ_L^c. The two

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[66] For illustrative purposes we argue here as if all fields were classical and obeyed classical equations of motion, though Pauli clearly considers the quantum theory where the fields become operators. The principal argument is the same, though what makes a big difference between the classical and the quantum case is that in the latter we can more easily ascertain the existence of invariant initial states. This is because in quantum theory, assuming there are no superselection rules at work, the superposition principle always allows us to construct invariant initial states by group-averaging any given state over the group (which is here compact, so that the averaging is unambiguously defined). Such states would, for example, appropriately represent physical situations where those observables that distinguish between the states in the group orbit are not measured, may it be for reasons of practice or of principle.

[67] It will be a quadratic combination in leading order of perturbation theory. Explicit calculations had been done by Pauli's assistant Charles Enz [15].

[68] In terms of the Pauli group, Pursey did not consider the $U(1)$ part.

actions of $U(2)$ in these spaces are complex conjugate to each other (see equation (43)). Usually one thinks of the Pauli group as $U(1) \times SU(2)$, which is a double cover of $U(2)$, so that the four real parameters are written as a phase $\exp(i\alpha)$, parametrising $U(1)$, and two complex parameters a, b satisfying $|a|^2 + |b|^2 = 1$, which give three real parameters when split into real and imaginary part and which parametrise a 3-sphere that underlies $SU(2)$ as group manifold. In this parametrisation the action of the Pauli group reads (an asterisk stands for complex conjugation):[69]

$$\begin{pmatrix} \psi_L \\ \psi_R^c \end{pmatrix} \mapsto \exp(+i\alpha) \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R^c \end{pmatrix}, \qquad (43a)$$

$$\begin{pmatrix} \psi_L^c \\ \psi_R \end{pmatrix} \mapsto \exp(-i\alpha) \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} \psi_L^c \\ \psi_R \end{pmatrix}. \qquad (43b)$$

Invariance under the Pauli group is now seen to correspond to an ambiguity in the particle-antiparticle distinction. This ambiguity would only be lifted by interactions that allowed to distinguish the two left and the two right states respectively. In the absence of such interactions the various definitions of 'particle' and 'antiparticle' are physically indistinguishable, so that the Pauli group acts by *gauge* symmetries in the sense of Section 2.3.

Also, the different presentations of the two-component theory, already discussed in Section 3.7 can be seen here. The Majorana condition reads $\psi = \psi^c$, which in terms of the four components introduced above leads to $\psi_L = \psi_R^c$ and $\psi_R = \psi_L^c$. This can be read in two different ways, depending on whether one addresses ψ_L, ψ_L^c or ψ_L, ψ_R as independent basic states. In the first case one would say that there is a left-handed neutrino and its right-handed antiparticle, whereas in the second case one regards the tuple (ψ_L, ψ_R) as respectively the left- and right-handed components of a single particle which is identical to its antiparticle.

Beyond weak interaction and beta-decay, the Pauli group played a very important rôle in Pauli's brief participation in Heisenberg's programme for a unified field theory. It was Pauli who first showed that the (so far classical) non-linear spinor equation proposed by Heisenberg was invariant under the Pauli group (cf. Heisenberg's account in his letter to Zimmermann from Jan. 7th 1958 in [45], Vol. IV, Part IV B, p. 779). In this new context the $U(1)$ part of the Pauli group was connected to conservation of baryon charge and the $SU(2)$ part acquired the meaning of isospin symmetry.[70] The central

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[69] Usually the Pauli group is written in terms of the 4-component neutrino field ψ as $\psi \mapsto \exp(i\alpha\gamma_5)(a\psi + b\gamma_5\psi^c)$, where $\psi^c := i\gamma_2\psi^*$ is the charge conjugate field. But this is easily seen to be equivalent to (43) if one sets $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi$ and $\psi_{R,L}^c = \frac{1}{2}(1 \pm \gamma_5)\psi^c$. The more explicit form (43) is better suited for the interpretational discussion; cf. [35]. The two-to-one homomorphism from $U(1) \times SU(2)$ to $U(2)$ is given by $(\exp(i\alpha), A) \mapsto \exp(i\alpha)A$ whose kernel is $\{(1, \mathbf{1}), (-1, -\mathbf{1})\}$.

[70] The non-linear spinor equation was at that stage not designed to include weak interaction.

importance of isospin for this programme may already be inferred from the title of the proposed common publication by Heisenberg and Pauli, which reads: *On the Isospin Group in the Theory of Elementary particles*. However, due to Pauli's later retreat from this programme, the manuscript (cf. [45], Vol. IV, Part IV B, pp. 849-861) for this publication never grew beyond the stage of a preprint.

3.8.3 Cosmological speculations

In his last paper on the subject of discrete symmetries, entitled *The Violation of Mirror-Symmetries in the Laws of Atomic Physics*[71] ([38], Vol. 2, pp. 1368-1372), Pauli comes back to the question which bothered him most: How is the strength of an interaction related to its symmetry properties? He says that having established a violation of C and P symmetry for weak interactions, we may ask why they are maintained for strong and electromagnetic interactions, and whether the reason for this is to be found in particular properties of these interactions. He ends with some speculations on possible connections between violations of C and P symmetry in the laws of microphysics on one hand, and properties of theories of gravitation and its cosmological solutions on the other:[72]

> *Second, one can try to find and justify a connection between symmetry violation in the small with properties of the Universe in the large. But this exceeds the capabilities of the presently known theory of gravity. [...] New ideas are missing to go beyond vague speculations. But this shall not be taken as definite expression for the impossibility of such a connection.*

It may be of interest to contrast this expression of a certain open-mindedness for speculations concerning the physics of elementary particles on one side and large-scale cosmology on the other, with a more critical attitude from Pauli's very early writings. In Section 65 of his Relativity article, where Pauli discussed Weyl's attempt for a unifying theory of gravity and electromagnetism (to which Pauli himself actively contributed), he observes that in Weyl's theory (as well as in Einstein's own attempts from that time) it is natural to suspect a relation between the size of the electron and the size

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[71] German original: "Die Verletzung von Spiegelungs-Symmetrien in den Gesetzen der Atomphysik".

[72] German original: "Zweitens kann man versuchen, einen Zusammenhang der Symmetrieverletzungen in Kleinen mit Eigenschaften des Universums im Grossen aufzufinden und zu begründen. Dies überschreitet aber die Möglichkeiten der jetzt bekannten Theorien der Gravitation. [...] Um bei der Frage des Zusammenhangs zwischen dem Kleinen und dem Grossen über vage Spekulationen hinauszugelangen, fehlen daher noch wesentlich neue Ideen. Hiermit soll jedoch nicht die Unmöglichkeit eines solchen Zusammenhanges bestimmt behauptet werden." ([38], Vol. 2, p. 1371)

(mean curvature radius) of the universe. But then he comments somewhat dismissively that this *might seem somewhat fantastic*[73]([54], p. 202).

4   Conclusion

I have tried to display some of the aspects of the notion of symmetry in the work of Wolfgang Pauli which to me seem sufficiently interesting in their own right. In doing this I have drawn freely from Pauli's scientific œvre, irrespectively of whether the particular part is commonly regarded as established part of present-day scientific knowledge or not. Pauli's faith in the explanatory power of symmetry principles clearly shows up in all corner of his œvre, but it also appears clearly rooted beyond the limits of his science.

In the editorial epilogue to the monumental collection of Pauli's scientific correspondence, Karl von Meyenn reports that many physicists he talked to at the outset of his project spoke against the publication of those letters that contained ideas which did not stand the test of time ([45], Vol. IV, Part IV B, p. 1375). Leaving aside that this must clearly sound outrageous to the historian, it is, in my opinion, also totally misguided as far as the scientific endeavour is concerned. Science is not only driven by the urge to *know* but also, and perhaps most importantly, by the urge to *understand*. No one who as ever actively participated in science can deny that. One central aspect of scientific understanding, next to offering as many as possible alternative and complementary explanations for the actual occurrences in Nature, is to comprehend why things could not be different from what they appear to be. The insight into a theoretical or an explanatory failure can be as fruitful as an experimental failure. What makes Pauli a great scientist, amongst the other most obvious reasons, is not that he did not err—such mortals clearly do not exist—, but that we can still learn much from where he erred and how he erred. In that sense, let me end by the following words from Johann Wolfgang von Goethe's *Maximen und Reflexionen* (#1292):

> *Wenn weise Männer nicht irrten, müßten die Narren verzweifeln.*
>
> *(If wise men did not err, fools should despair.)*

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[73] German original: "...was immerhin etwas phantastisch erscheinen mag." ([58], p. 249)

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