Algebraic aspects of quantum indiscernibility

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Abstract

We show that using quasi-set theory, or the theory of collections of indistinguishable objects, we can define an algebra that has most of the standard properties of an orthocomplete orthomodular lattice, which is the lattice of all closed subspaces of a Hilbert space. We call the mathematical structure so obtained $\mathfrak{I}$-lattice. After discussing (in a preliminary form) some aspects of such a structure, we indicate the next problem of axiomatizing the corresponding logic, that is, a logic which has $\mathfrak{I}$-lattices as its Lindembaum algebra, which we postpone to a future work. Thus we conclude that the initial intuitions by Birkhoff and von Neumann that the "logic of quantum mechanics" would be not classical logic (a Boolean algebra), is consonant with the idea of considering indistinguishability right from the start, that is, as a primitive concept. In the first sections, we present the main motivations and a "classical" situation which mirrors that one we focus on the last part of the paper. This paper is our first analysis of the algebraic structure of indiscernibility.

1 Introduction

Indiscernibility is a typical concept of quantum physics, and some facts implied by indiscernibility, as the properties of a Bose-Einstein condensate, have no parallel in classical physics. Without considering that quanta are indiscernible, no explanation of colors would be done, no vindication to the periodic table of elements would result, and (among other things), Planck would not arrive to his formula for the black body radiation. Some authors have sustained that quantum indiscernibility results from the raise of quantum "statistics" (really, ways of counting), while others think that they can *explain* quantum statistics without presupposing indiscernibility, but at the expenses of rejecting equiprobability.[1] The discussion is still alive, and we have much to do in the philosophical, epistemological, logical, and on ontological aspects of quantum indiscernibility, mainly if we agree (with Arthur Fine) that philosophy of science should be engaged with on-going science (*apud* [13]). This poses us directly to the quantum field theories, and perhaps more, to string theories and to quantum gravitation. Acknowledging this naturalistic claim, we shall be here quite modest in discussing some algebraic aspects of a mathematical theory which was conceived to deal with indistinguishable objects, termed *quasi-set theory*. Without revising all the details of such a theory (to which we refer to chapter 7 of [14]), we shall keep the paper self-contained so that the reader can understand the basic ideas, although sometimes in the intuitive sense, and only the really necessary concepts and postulates are mentioned.

It should be recalled that indiscernibility enters in the standard quantum formalism by means of symmetry postulates. The relevant functions for systems of many quanta ought to be either symmetrical or anti-symmetrical, and this assumption makes the expectation values to assume the same values before and after a permutation of indiscernible elements. Thus, physicists (and philosophers accept that) say that "the individuality was lost", as if there would be something to lose. In this work, we enlarge our research program of providing a mathematical basis for quantum theory that takes indiscernibility "right from the start", as claimed by Heinz Post [24], see [14], with the algebraic discussion of indiscernibility. All the considerations are performed within quasi-set theory, which we revise in its main ideas below.

2 Quasi-sets

Quasi-set theory (denoted $\mathfrak{Q}$) was conceived to handle collections of indistinguishable objects, and was motivated by some considerations taken from quantum physics, mainly in what respects Schrödinger's idea that the concept of identity cannot be applied to elementary particles [28, pp. 17-18]. Of course the theory can be developed independently of quantum mechanics, but here we shall have this motivation always in mind. Our way of dealing with indistinguishability is to assume that expressions like $x = y$ are not well formed in all situations involving x and y. We express that by saying that the concept of identity does not apply to the entities denoted by x and y in these situations. Here, quantum objects do not mean necessarily *particles*, but ought to be thought as representing the basic objects of quantum theories, which although differ form one theory to another ([12, cap.6]), have some common characteristics, as those

[1] This is in particular van Fraassen's view; for instance, he supposes two particles 1 and 2 in two possible states A and B, and the possible cases are (i) 1 and 2 in A, (ii) 1 and 2 in B, both cases with probability 1/3 each, (iii) 1 in A and 2 in B, and (iv) 1 in B and 2 in A, both with probability 1/6. According to this author, this way we can arrive at Bose-Einstein statistics [30]. But the problem is that situations (iii) and (iv) need to be distinguished from one another, and if the involved quanta are indiscernible, this can be done only either by the assumption of some kind of hidden variable or by some form of *substratum*, and we know that both possibilities conduce to well known problems.

related to indiscernibility (with the exception of some hidden variable theories, like Bohm's, which will be not discussed here).[2] Due to the lack of sense in applying the concept of identity to certain elements, informally, a quasi-set (qset), that is, a collection of such objects, may be such that its elements cannot be identified by names, counted, ordered, although there is a sense in saying that these collections have a cardinal (not defined by means of ordinals, as usual – but see below). But we aim at to keep standard mathematics intact,[3] so the theory is developed in a way that ZFU (and hence ZF, perhaps with the axiom of choice, ZFC) is a subtheory of $\mathfrak{Q}$ (in other words, there is a "copy" of ZFU in $\mathfrak{Q}$). In other words, the theory is constructed so that it extends standard Zermelo-Fraenkel with *Urelemente* (ZFU); thus standard sets (of ZFU) can be viewed as particular qsets (that is, there are qsets that have all the properties of the sets of ZFU, and we call then $\mathfrak{Q}$-sets; the objects in $\mathfrak{Q}$ corresponding to the *Urelemente* of ZFU are termed M-atoms). But quasi-set theory encompasses another kind of *Urelemente*, the m-atoms, to which the standard theory of identity does not apply (that is, expressions like $x = y$ are not well formed if m-atoms are involved).

When $\mathfrak{Q}$ is used in connection with quantum physics, these m-atoms are thought of as representing quantum objects (henceforth, q-objects), and not necessarily they are 'particles', as mentioned above; waves or perhaps even strings (and whatever 'objects' sharing the property of indistinguishability of pointlike elementary particles) can be also be values of the variables of $\mathfrak{Q}$. The lack of the concept of identity for the m-atoms make then *non-individuals* in a sense, and it is mainly (but not only) to deal with collections of m-atoms that the theory was conceived. So, $\mathfrak{Q}$ is a theory of generalized collections of objects, involving non-individuals. For details about $\mathfrak{Q}$ and about its historical motivations, see [5, p. 119], [9], [14, Chap. 7], [16], [19].

In $\mathfrak{Q}$, the so called 'pure' qsets have only q-objects as elements (although these elements may be not always indistinguishable from one another), and to them it is assumed that the usual notion of identity cannot be applied (that is, $x = y$, so as its negation, $x \neq y$, are not a well formed formulas if x and y stand for q-objects). Notwithstanding, there is a primitive relation $\equiv$ of indistinguishability having the properties of an equivalence relation, and a concept of *extensional identity*, not holding among m-atoms, is defined and has the properties of standard identity of classical set theories. Since the elements of a qset may have properties (and satisfy certain formulas), they can be regarded as *indistinguishable* without turning to be *identical* (that is, being *the same* object), that is, $x \equiv y$ does not entail $x = y$. Since the relation of equality (and the concept of identity) does not apply to m-atoms, they can also be thought of as

[2]Since such theories present difficulties due to results like Kochen-Specker theorem and Bell's inequalities, so as due to the fact that apparently they cannot be extended to quantum field theories, we shall leave them outside our discussion.

[3]So respecting the quite strange rule what Birkhoff and von Neumann call "Henkel's principle of the 'perseverance of formal laws' ", explained by Rédei as "a methodological principle that is supposed to regulate mathematical generalizations by insisting on preserving certain laws in the generalization" [25]; of course we are 'preserving' all standard mathematics built in ZFC.

entities devoid of individuality. We remark further that if the 'property' $x = x$ (to be identical to itself, or *self-identity*, which can be defined for an object a as $I_a(x) =_{\mathsf{def}} x = a$) is included as one of the properties of the considered objects, then the so called Principle of the Identity of Indiscernibles (PII) in the form $\forall F(F(x) \leftrightarrow F(y)) \rightarrow x = y$ is a theorem of classical second order logic, and hence there cannot be indiscernible but not identical entities (in particular, non-individuals). Thus, if self-identity is linked to the concept of non-individual, and if quantum objects are to be considered as such, these entities fail to be self-identical, and a logical framework to accommodate them is in order (see [14] for further argumentation).

We have already discussed at length in the references given above (so as in other works) the motivations to build a quasi-set theory, and we shall not return to these points here,[4] but before to continue we would like to make some few remarks on a common misunderstanding about PII and quantum physics. People generally think that spatio-temporal location is a sufficient condition for individuality. Thus, an electron in the South Pole and another one in the North Pole *are* discernible, hence *distinct individuals*, so that we can call "Peter" one of them and "Paul" the another one. Leibniz himself prevented us about this claim (yet not directly about quantum objects of course), by saying that "it is not possible for two things to differ from one another in respect to place and time alone, but that is always necessary that there shall be some other internal difference" [20]. Leaving aside a possible interpretation for the word 'internal', we recall that even in quantum physics, where fermions obey the Pauli Exclusion Principle, which says that two fermions (yes, they 'count' as more than one) cannot have all their quantum numbers (or 'properties') in common, two electrons (which are fermions), one in the South Pole and another one in the North Pole, *are not individuals in the standard sense.*[5] In fact, we can say that the electron in the South Pole is described by the wave function ψ_S, while the another one is described by ψ_N (words like 'another' in the preceding phrase are just ways of speech). But the joint system is (in a simplified form) given by $\psi_{SN} = \psi_S - \psi_N$ (the function must be anti-symmetric in the case of fermions, that is, $\psi_{SN} = -\psi_{NS}$), a superposition of the two first wave functions, and this last function cannot be factorized. Furthermore, in the quantum formalism, the important thing is the square of the wave function, which gives the joint probability density; in the present case, we have $||\psi_{SN}||^2 = ||\psi_S||^2 + ||\psi_N||^2 - 2\mathrm{Re}\psi_S\psi_N$. This last term, called 'the interference term' cannot be dispensed with, and says that nothing, not even *in mente Dei*, can tell us which is the particular electron in the South Pole (and the same happens for the North Pole), that is, we never will know who is Peter and who is Paul, and in the limits of quantum mechanics, this is not a matter of epistemological ignorance, but it is rather an ontological question. As far as quantum physics is concerned in its main interpretations, they seem to be really and truly objects

[4] But see [6], [7], [8], [14], [17], [19].

[5] Without aiming at to extend the discussion on this topic here (but see [14], by an individual we understand an object that obeys the classical theory of identity of classical logic (extensional set theory included).

without identity.

In the next sections, we shall discuss from an algebraic point of view some issues of non-individuality. It should be interesting to recall that the 'qset'-operations of intersection ($\cap$), union ($\cup$), difference ($-$) work similarly in $\mathfrak{Q}$ as the standard ones in usual set theories.

3 Algebraic aspects: the lattice of indiscernibility

Quantum logic was born with Birkhoff and von Neumann's paper from 1936 [1]. Today it consists in a wide field of knowledge, having widespread to domains never thought by the two celebrated forerunners. For a look on the state of the art, see [10]. The main idea is that the typical algebraic structures arising from the mathematical formalism of quantum mechanics is not a Boolean algebra, but an orthocomplete (σ-orthocomplete in the general case [10, p. 39]) orthomodular lattice. We shall see below that in quasi-set theory, by considering indiscernibility right from the start, a similar structure 'naturally' arises. Let us provide the details before ending with some comments and conclusions.

We shall be working in the theory $\mathfrak{Q}$, and use the equality symbol $=$ to stand for the extensional equality of $\mathfrak{Q}$. Intuitively speaking, $x = y$ holds when x and y are both qsets and have the same elements (in the sense that an object belongs to x iff it belongs to y) or they are both M-objects and belong to the same qsets. It can be proven that $=$ has all properties of standard identity of first-order ZFC. Qsets which may have m-atoms as elements are written (in the metalanguage) with square brackets "[" and "]", and $\mathfrak{Q}$-sets (qsets whose transitive closure have no m-atoms) with the usual curly braces "{" and "}".

Definition 3.1 (Closure) *Let U be a non empty qset and A be a subqset of U. The* cloud *of A, here called the* closure *of A, is de qset*

$$\overline{A} =_{\mathsf{def}} [y \in U : \exists x(x \in A \wedge y \equiv x)].$$

Intuitively speaking, $\overline{A}$ is the qset of the elements of U (the universe) which are indistinguishable from the elements of A. If A is a $\mathfrak{Q}$-set, that is, a copy of a set of ZFU, then of course the only indistinguishable of a certain x is x itself, thus $\overline{A} = A$. According to this definition, we have that $\overline{\emptyset} = \emptyset$. From now on, we shall suppose that U is closed, that is, it contains all the indistinguishable objects of its elements. Some interpretations linked to physical situations are possible. For instance, $\overline{A}$ can be thought as the region where the wave function A of a certain physical system is different from zero. Another possible interpretation is to suppose that the clouds describe the systems plus the cloud of virtual particles that accompany those of the considered system. But in this paper we shall be not considering these motivations, but just to explore its algebraic aspects.

It is immediate to prove the following theorem:

Theorem 3.1 *The application*[6] *that associates to every subqset of U its closure is a Tarski's operator, that is, for all A and B in $\mathcal{P}(U)$, we have: (i) $A \subseteq \overline{A}$; (ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; (iii) $\overline{\overline{A}} \subseteq \overline{A}$.*

Proof: (i) Let $t \in A$. Then, by the reflexivity of $\equiv$, we have $t \equiv t$, hence $t \in \overline{A}$. (ii) Let $A \subseteq B$, and let $t \in \overline{A}$. Then there exists $x \in A$ such that $t \equiv x$. Since $x \in B$, then $t \in \overline{B}$. (iii) Let $t \in \overline{\overline{A}}$. Then there exists $x \in \overline{A}$ such that $t \equiv x$. But then there exists $y \in A$ such that $x \equiv y$. By the transitivity of $\equiv$, we have $t \equiv y$, hence $t \in \overline{A}$. ▮

Corolary 3.1 *It follows that $\overline{A} = \overline{\overline{A}}$, that is, the closure of a closed qset is closed.*

Proof: Consequence of (i) and (iii) of the preceding theorem. ▮

Definition 3.2 (Closed qset) *A qset A is closed if $\overline{A} = A$.*

Theorem 3.2 *From the above results and definitions, we have that:*

(i) $\overline{A} \cup \overline{B} = \overline{A \cup B}$;

(ii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ (if A and B are closed, then the equality holds);

(iii) $\overline{\overline{A} \cup \overline{B}} = \overline{A \cup B}$;

(iv) $\overline{A} \cap \overline{B} = \overline{\overline{A} \cap \overline{B}}$.

Proof:

(i) $A \subseteq A \cup B$, so $\overline{A} \subseteq \overline{A \cup B}$. In the same way, $B \subseteq A \cup B$ and $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Conversely, suppose $t \in \overline{A \cup B}$, then there is $x \in A \cup B$ such that $t \equiv x$. So there is $t \equiv x$ such that $x \in A$ or $x \in B$. In this way $t \in \overline{A}$ or $t \in \overline{B}$ and therefore $t \in \overline{A} \cup \overline{B}$.

(ii) $A \cap B \subseteq A$, hence $\overline{A \cap B} \subseteq \overline{A}$. Also $A \cap B \subseteq B$, then $\overline{A \cap B} \subseteq B$. Thus $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. That we don't have equality may be seem with the following argument. Let A and B such that $A \not\subseteq B$ but $A \cap B \neq \emptyset$. Furthermore, let $a \in A$ and $b \in B$. If $c \in \overline{A} \cap \overline{B}$ is such that $c \equiv a$ and $c \equiv b$, it does not follow that $c \in A \cap B$. This will happen iff $a, b \in A \cap B$.

[6] In $\mathfrak{Q}$, the concept of function must be generalized, for if there are m-atoms involved, a mapping in general does not distinguish between arguments and values. Thus we use the notion of q-function, which leads indistinguishable objects into indistinguishable objects, and which reduces to standard functions when there are no m-atoms involved. Thus, from the formal point of view, the defined mapping may associate to A whatever qset from a collection of indistinguishable qsets. But this does not matter. As in quantum physics, it is not the extension of the collections which are important; informally saying, *any* elementary particle of a certain kind serves for all purposes involving it. This is the principle of the invariance of permutations.

(iii) Item (i) entails that $\overline{\overline{A} \cup \overline{B}} \subseteq \overline{\overline{A \cup B}} \subseteq \overline{A \cup B}$. Furthermore, since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, then $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$.

(iv) Firstly, from $A \subseteq \overline{A}$, we have $\overline{A} \cap \overline{B} \subseteq \overline{\overline{A} \cap \overline{B}}$. Secondly, $\overline{A} \cap \overline{B} \subseteq \overline{A}$ entails $\overline{\overline{A} \cap \overline{B}} \subseteq \overline{\overline{A}} = \overline{A}$, while $\overline{A} \cap \overline{B} \subseteq \overline{B}$ entails $\overline{\overline{A} \cap \overline{B}} \subseteq \overline{\overline{B}} = \overline{B}$. Hence $\overline{\overline{A} \cap \overline{B}} \subseteq \overline{A} \cap \overline{B}$. Both results imply the enunciated. ▮

The next definition introduces the lattice operations on subqsets of a qset U, the universe.

Definition 3.3 ($\mathfrak{I}$-lattice operations) *Let A and B be subqsets of U. Then:*

($\sqcap$) $A \sqcap B =_{\mathsf{def}} \overline{A \cap B}$;

($\sqcup$) $A \sqcup B =_{\mathsf{def}} \overline{A} \cup \overline{B}$;

(0) $\mathbf{0} =_{\mathsf{def}} \emptyset$;

(1) $\mathbf{1} =_{\mathsf{def}} U$.

We note that even if $A \cap B = \emptyset$, may be that $\overline{A} \cap \overline{B} \neq \emptyset$.

Theorem 3.3 *For any A and B in $\mathcal{P}(U)$:*

(i) $A \sqcap B \subseteq \overline{(\overline{A} \cap \overline{B})}$;

(ii) $A \sqcap B \subseteq A \sqcup B$;

(iii) If A and B are closed, $A \cup B$ and $A \cap B$ are closed, and $A \sqcap B = \overline{A} \cap \overline{B}$.

Proof:

(i) Immediate, since $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ (Theorem 3.2 (ii));

(ii) $A \sqcap B = \overline{A \cap B} \subseteq$ (Theorem 3.2 (ii)) $\subseteq \overline{A} \cap \overline{B} \subseteq \overline{A} \cup \overline{B} = A \sqcup B$;

(iii) If $\overline{A} = A$ and $\overline{B} = B$, then $A \cup B = \overline{A} \cup \overline{B} = \overline{A \cup B}$ (Theorem 3.2 (iii)). Furthermore, the same hypothesis entails that $\overline{A \cap B} = \overline{\overline{A} \cap \overline{B}} =$ (Theorem 3.2 (iv)) $= \overline{A} \cap \overline{B} = A \cap B$. Finally, since $\overline{A} \cap \overline{B} = \overline{(\overline{A} \cap \overline{B})} = \overline{A \cap B} = A \sqcap B$ (Theorem 3.2 (iv) and the hypothesis). ▮

Theorem 3.4 *Let $\mathcal{C}$ be the qset of all closed subqsets of U (that is, such that $\overline{A} = A$). Then the structure $\mathfrak{C} = \langle \mathcal{C}, \sqcap, \sqcup, \mathbf{0}, \mathbf{1} \rangle$ is a lattice with 0 and 1. But, if we consider also the sub-qsets of U that are not closed, then some of the properties of such a structure do not hold, as we emphasize in the proof below.*

Proof: Firstly, it is immediate to see that if $U \neq \emptyset$, then $\mathcal{P}(U) \neq \emptyset$. Furthermore, we can prove that $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$ for closed qsets.

(a) Idempotency (restricted to closed qsets): $A \sqcap A = \overline{A \cap A} = \overline{A}$ (= A if A is closed). Also, $A \sqcup A = \overline{A} \cup \overline{A} = \overline{A}$ (= A if closed). If A is not closed, then $A \sqcap A = \overline{A}$ and $A \sqcup A = \overline{A}$;

(b) Commutativity (unrestricted): $A \sqcap B = \overline{A \cap B} = \overline{B \cap A} = B \sqcap A$. In the same way, $A \sqcup B = \overline{A} \cup \overline{B} = \overline{B} \cup \overline{A} = B \sqcup A$;

(c) Associativity (unrestricted): (we shall be using items (iii) and (iv) of Theorem 3.2 without mentioning):

(i) $A \sqcap (B \sqcap C) = A \sqcap (\overline{B} \cap \overline{C}) = \overline{A} \cap \overline{(\overline{B} \cap \overline{C})} = \overline{A} \cap (\overline{B} \cap \overline{C}) = (\overline{A} \cap \overline{B}) \cap \overline{C} = \overline{(\overline{A} \cap \overline{B}) \cap \overline{C}} = \overline{(\overline{A} \cap \overline{B})} \sqcap C = (A \sqcap B) \sqcap C$;

(ii) $A \sqcup (B \sqcup C) = A \sqcup (\overline{B} \cup \overline{C}) = \overline{A} \cup \overline{(\overline{B} \cup \overline{C})} = \overline{\overline{A}} \cup \overline{(\overline{B} \cup \overline{C})}$ = (Theorem 3.2 (i)) $\overline{\overline{A} \cup (\overline{B} \cup \overline{C})} = \overline{(\overline{A} \cup \overline{B}) \cup \overline{\overline{C}}}$ = (Theorem 3.2 (i)) $\overline{(\overline{A} \cup \overline{B})} \cup \overline{\overline{C}} = \overline{(\overline{A} \cup \overline{B}) \cup \overline{C}} = (\overline{A} \cup \overline{B}) \sqcup C = (A \sqcup B) \sqcup C$;

(d) Absorption (restricted):

(i) $A \sqcap (A \sqcup B) = A \sqcap (\overline{A} \cup \overline{B}) = \overline{A \cap (\overline{A} \cup \overline{B})}$. But $A \subseteq \overline{A}$, so $A \subseteq \overline{A} \cup \overline{B}$, then $\overline{A \cap (\overline{A} \cup \overline{B})} = \overline{A}$ (= A for closed qsets);

(ii) $A \sqcup (A \sqcap B) = A \sqcup \overline{(A \cap B)} = \overline{A} \cup \overline{\overline{(A \cap B)}} = \overline{A} \cup \overline{(A \cap B)} = \overline{A}$, for $A \cap B \subseteq A \subseteq \overline{A}$ (= A for closed qsets);

(e) The properties of **0** and **1**:

(i) $\mathbf{0} \sqcap A = \overline{\emptyset \cap A} = \overline{\emptyset} = \emptyset = \mathbf{0}$;

(ii) $\mathbf{0} \sqcup A = \overline{\emptyset} \cup \overline{A} = \emptyset \cup \overline{A} = \overline{A}$ (= A for closed qsets);

(iii) $A \sqcap \mathbf{1} = \overline{A \cap U} = \overline{A}$ (= A for closed qsets);

(iv) $A \sqcup \mathbf{1} = \overline{A} \cup \overline{U} = \overline{U} = U = \mathbf{1}$ (recall our initial hypothesis that U is closed). ▮

Theorem 3.5 *The lattice $\mathfrak{C}$ of the closed qsets of U is distributive.*

Proof: We shall emphasize those passages which make use of the hypothesis that the qsets are closed.

(i) $A \sqcup (B \sqcap C) = A \sqcup \overline{(B \cap C)} = \overline{A} \cup \overline{\overline{(B \cap C)}}$ = (th. 3.2(i)) $\overline{A \cup \overline{(B \cap C)}}$ = (A is closed) = $\overline{\overline{A} \cup \overline{(B \cap C)}}$ = (th. 3.2(iii)) $\overline{A \cup (B \cap C)} = \overline{(A \cup B) \cap (A \cup C)}$ = (th. 3.2(ii) and being $A \cup B$ and $A \cup C$ both closed, for otherwise the equality does not hold) = $\overline{(A \cup B)} \cap \overline{(A \cup C)}$ = (th. 3.2(i)) $(\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})$ = (closed) $\overline{(\overline{A} \cup \overline{B})} \cap \overline{(\overline{A} \cup \overline{C})}$ = (th. 3.2(iv)) $\overline{(\overline{A} \cup \overline{B}) \cap (\overline{A} \cup \overline{C})} = (\overline{A} \cup \overline{B}) \sqcap (\overline{A} \cup \overline{C}) = (A \sqcup B) \sqcap (A \sqcup C)$;

(ii) $(A \sqcap B) \sqcup (A \sqcap C) = \overline{(A \cap B)} \sqcup \overline{(A \cap C)} = \overline{\overline{(A \cap B)}} \cup \overline{\overline{(A \cap C)}}$ = (th. 3.2(i)) $\overline{\overline{(A \cap B)} \cup \overline{(A \cap C)}}$ = (for $A \cap B$ and $A \cap C$ are closed) = $\overline{(A \cap B) \cup (A \cap C)}$ = $\overline{A \cap (B \cup C)}$ = (for closed qsets) $\overline{\overline{A} \cap \overline{(B \cup C)}}$ = (th. 3.2(i)) $\overline{\overline{A} \cap (\overline{B} \cup \overline{C})}$ = (since A is closed) = $\overline{A \cap (B \sqcup C)} = A \sqcap (B \sqcup C)$. ▮

This result is not surprising, for we are dealing with set theoretical operations which, defined on the closed qsets of U, act as the usual set theoretical properties on standard sets. But if we consider *all* qsets in U and not only the closed ones, the distributive laws do not hold, as we can see from the above proof, which makes essential use of the fact that the involved qsets are closed (without such an hypothesis, the proof does not follow). Since the corresponding structure $\mathfrak{I} = \langle \mathcal{P}(U), \sqcap, \sqcup, \mathbf{0}, \mathbf{1} \rangle$ has similarities with a lattice with 0 and 1, we propose to call it *the lattice of indiscernibility*, or just $\mathfrak{I}$-lattice for short. Other distinctive characteristics of this "quasi-lattice" are obtained when we introduce other operations similar to those of order and involution, or generalized complement [10, p. 11]. At Section 4, we sum up the main properties of an $\mathfrak{I}$-lattice.

Definition 3.4 ($\mathfrak{I}$-order) $A \leq B =_{\mathsf{def}} A \sqcup B = \overline{B}$.

Theorem 3.6 *The order relation obeys the following properties:*

(i) $A \leq A$ and $A \leq \overline{A}$;

(ii) $A \leq B$ and $B \leq A \Rightarrow \overline{A} = \overline{B}$ (and $A = B$ if they are both closed);

(iii) $A \leq B$ and $B \leq C \Rightarrow A \leq C$

(iv) $A \sqcap B \leq A$, and $A \sqcap B \leq B$;

(v) $C \leq A$ and $C \leq B \Rightarrow C \leq A \sqcap B$

(vi) $A \leq A \sqcup B$, $B \leq A \sqcup B$;

(vii) $A \leq C$ and $B \leq C \Rightarrow A \sqcup B \leq C$;

(viii) $\mathbf{0} \leq A$, and $A \leq \mathbf{1}$ (recall that $\mathbf{1} = U$ is closed);

(ix) $A \leq B \Rightarrow A \sqcap B = \overline{A}$.

Proof:

(i) $A \sqcup A = \overline{A} \cup \overline{A} = \overline{A}$, so $A \leq A$; and $A \sqcup \overline{A} = \overline{A} \cup \overline{\overline{A}} = \overline{A} \cup \overline{A} = \overline{A}$, so $A \leq \overline{A}$;

(ii) $A \leq B \Rightarrow \overline{A} \cup \overline{B} = \overline{B}$, while $B \leq A \Rightarrow \overline{B} \cup \overline{A} = \overline{A}$, since $\overline{A} \cup \overline{B} = \overline{B} \cup \overline{A}$, then $\overline{A} = \overline{B}$ ($A = B$ for closed qsets);

(iii) If $\overline{A} \cup \overline{B} = \overline{B}$ and $\overline{B} \cup \overline{C} = \overline{C}$, then $\overline{A} \cup \overline{C} = \overline{A} \cup (\overline{B} \cup \overline{C}) = (\overline{A} \cup \overline{B}) \cup \overline{C} = \overline{B} \cup \overline{C} = \overline{C}$;

(iv) $A \sqcap B \leq A$ iff $\overline{(A \cap B)} \cup \overline{A} = \overline{A}$. But, by Theorem 3.2 (i), $\overline{(A \cap B)} \cup \overline{A} = \overline{(A \cap B) \cup A} = \overline{A}$. Equivalently, $A \sqcap B \leq B$ iff $\overline{(\overline{A} \cap \overline{B})} \cup \overline{\overline{B}} = \overline{B}$. But, by Theorem 3.2 (i), $\overline{(\overline{A} \cap \overline{B})} \cup \overline{\overline{B}} = \overline{(\overline{A} \cap \overline{B}) \cup \overline{B}} = \overline{\overline{B}} = \overline{B}$;

(v) $C \leq A \sqcap B$ iff $\overline{C} \cup \overline{(A \sqcap B)} = \overline{(A \sqcap B)} =$ (Theorem 3.2 (iv)) $\overline{A} \cap \overline{B}$. But the hypothesis tells us that $\overline{C} \cup \overline{A} = \overline{A}$ and $\overline{C} \cup \overline{B} = \overline{B}$, hence $\overline{C} \subseteq \overline{A}$ and $\overline{C} \subseteq \overline{B}$, that is, $\overline{C} \subseteq (\overline{A} \cap \overline{B})$. So, $\overline{C} \cup (\overline{A} \cap \overline{B}) = \overline{A} \cap \overline{B}$, that is, $C \leq A \sqcap B$;

(vi) $A \leq A \sqcup B$ iff $A \sqcup (A \sqcup B) = \overline{A \sqcup B} = \overline{(\overline{A} \cup \overline{B})} = \overline{A \cup B}$ by Theorem 3.2 (i). But $A \sqcup (A \sqcup B) = A \sqcup (\overline{A} \cup \overline{B}) = \overline{A} \cup \overline{(\overline{A} \cup \overline{B})} = \overline{A} \cup \overline{(A \cup B)} = \overline{A \cup (A \cup B)} = \overline{A \cup B})$, using the same theorem.;

(vii) The hypothesis says that $A \sqcup C = \overline{C}$ and $B \sqcup C = \overline{C}$, that is, $\overline{A} \cup \overline{C} = \overline{C}$, hence $\overline{A} \subseteq \overline{C}$. In the same vein, $\overline{B} \subseteq \overline{C}$. But these results entail that $\overline{A} \cup \overline{B} \subseteq \overline{C}$, hence $\overline{(\overline{A} \cup \overline{B})} \subseteq \overline{\overline{C}}$, then $\overline{(\overline{A} \cup \overline{B})} \cup \overline{C} = \overline{C}$;

(viii) $\mathbf{0} \sqcup A = \overline{\mathbf{0}} \cup \overline{A} = \overline{A}$, and $A \sqcup \mathbf{1} = \overline{A} \cup \overline{U} = U = \mathbf{1}$ (recall that U is closed);

(ix) If $A \leq B$, then $A \sqcup B = \overline{B}$. But this entails that $\overline{A} \subseteq \overline{B}$. Thus $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B} = \overline{A}$ (Theorem 3.2 (ii)), that is, $A \sqcap B = \overline{A}$. ▮

Alternatively, we could define $A \leq_1 B$ iff $A \sqcap B = \overline{A}$. The theorem above follows, with the exception of item (ix), which should be substituted by $A \sqcup B = \overline{B}$. Really, assuming this definition, we have $A \sqcup B = \overline{A} \cup \overline{B} = \overline{A} \cup \overline{B} = \overline{B}$, for the hypothesis entails that $A \sqcap B = \overline{A}$, that is, $\overline{A} = \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ by Theorem 3.2 (ii). So $\overline{A} \subseteq \overline{B}$, then $\overline{A} \cup \overline{B} = \overline{B}$, that is, $A \sqcup B = \overline{B}$. Item (ix) of the theorem and this result show that $A \leq B$ iff $A \leq_1 B$.

We have proved that $\leq$ is both reflexive and transitive ((i) and (iii) above), but only "partially" anti-symmetric, that is, $A \leq B$ and $B \leq A$ entail $\overline{A} = \overline{B}$. Thus, $\langle \mathcal{P}(U), \leq \rangle$ is a kind of "weak" poset. Since it contains $\mathbf{0}$ and $\mathbf{1}$ and since any two elements of U have a supremum (namely, $A \sqcup B$) and an infimum (namely, $A \sqcap B$). $\mathfrak{I}$ is a "weak lattice", but of course it is a lattice stricto sensu if we consider only closed qsets.

The complement of a qset A relative to the universe U is the sub-qset of U, termed $A^{\perp}$, which has no element indistinguishable from any element of A.

Definition 3.5 ($\mathfrak{I}$-involution, or Generalized $\mathfrak{I}$-complement)

$$A^{\perp} =_{\mathsf{def}} U - \overline{A}.$$

Theorem 3.7 *Let $A, B \in \mathcal{P}(U)$. Then:*

(i) $\emptyset^{\perp} = U$;

(ii) $U^{\perp} = \emptyset$;

(iii) $U - A^{\perp} = \overline{A}$;

(iv) $\overline{A^{\perp}} = A^{\perp} = \overline{A}^{\perp}$;

(v) $A^{\perp^{\perp}} = \overline{A}$ $\quad$ ($= A$ for closed qsets);

(vi) $A \leq B \Rightarrow B^{\perp} \leq A^{\perp}$.

Proof:

(i) $\emptyset^{\perp} = U - \overline{\emptyset} = U - \emptyset = U$;

(ii) $U^{\perp} = U - \overline{U} = U - U = \emptyset$ (U is closed);

(iii) $U - A^{\perp} = U - (U - \overline{A}) = \overline{A}$ for they are all closed;

(iv) $\overline{A^{\perp}} = \overline{U - \overline{A}} = U - \overline{A} = A^{\perp} = \overline{A}^{\perp}$. Informally speaking, in $U - \overline{A}$ there are no elements indiscernible from the elements of $\overline{A}$ (according to Definition 3). Thus, it is closed, and coincides with $\overline{A}^{\perp} = U - \overline{\overline{A}}$.

(v) $A^{\perp^{\perp}} = U - \overline{A^{\perp}} = U - A^{\perp} = \overline{A}$ $\;$ ($= A$ for closed qsets);

(vi) $A \leq B \Rightarrow A \sqcup B = \overline{B}$, hence $\overline{A} \cup \overline{B} = \overline{B}$ and $\overline{A} \subseteq \overline{B}$. But this implies that $U - \overline{B} \subseteq U - \overline{A}$, that is, $B^{\perp} \subseteq A^{\perp}$. So $B^{\perp} \cup A^{\perp} = A^{\perp}$, then, by Theorem 3.2 (i), $\overline{B^{\perp}} \cup \overline{A^{\perp}} = \overline{A^{\perp}}$, hence $B^{\perp} \sqcup A^{\perp} = \overline{A^{\perp}}$, that is, $B^{\perp} \leq A^{\perp}$. ∎

Properties (v) and (vi) of the preceding theorem show that $^{\perp}$ is an involution for closed qsets. For qsets in general, we shall call it $\mathfrak{I}$-involution, in the spirit of the above discussion.

Theorem 3.8 *If $A, B \in \mathcal{P}(U)$, then:*

(i) $A \sqcup A^{\perp} = \mathbf{1}$;

(ii) $A \sqcap A^{\perp} = \mathbf{0}$;

(iii) $A \sqcup (B \sqcap B^{\perp}) = \overline{A}$ ($= A$ for closed qsets);

(iv) $A \sqcap (B \sqcup B^{\perp}) = \overline{A}$ $\;$ ($= A$ for closed qsets);

(v) (De Morgan) $(A \sqcup B)^{\perp} = A^{\perp} \sqcap B^{\perp}$;

(vi) ("Partial" De Morgan) $(A \sqcap B)^{\perp} \subseteq A^{\perp} \sqcup B^{\perp}$ (equality holds for closed qsets).

Proof:

(i) $A \sqcup A^{\perp} = \overline{A} \cup \overline{A^{\perp}} = \overline{A} \cup A^{\perp} = \overline{A} - (U - \overline{A}) = U = \mathbf{1}$;

(ii) $A \sqcap A^{\perp} = A \sqcap (U - \overline{A}) = \overline{A} \cap \overline{(U - \overline{A})} = \overline{A} \cap (U - \overline{A}) = \emptyset = \mathbf{0}$ $\;$ (for $\overline{(U - \overline{A})}$ is closed);

(iii) $A \sqcup (B \sqcap B^{\perp}) = \overline{A} \cup \mathbf{0} = \overline{A} \cup \emptyset = \overline{A}$ $\;$ ($= A$ for closed qsets);

(iv) $A \sqcap (B \sqcup B^{\perp}) = A \sqcap (\overline{B} \cup \overline{B^{\perp}}) = A \sqcap (\overline{B} \cup \overline{B}^{\perp}) = \overline{A \cap (\overline{B} \cup \overline{B}^{\perp}) = A \cap \mathbf{1}} = \overline{A}$ $\;$ ($= A$ for closed qsets);

(v) $(A \sqcup B)^\perp = (\overline{A} \cup \overline{B})^\perp = U - \overline{(\overline{A} \cup \overline{B})} =$ (th. 3.2(iii)) $= U - \overline{(A \cup B)} =$ (th. 3.2(i)) $= U - (\overline{A} \cup \overline{B}) = (U - \overline{A}) \cap (U - \overline{B}) =$ (th. 3.2(i) and the fact that the involved qsets are closed) $= \overline{(U - \overline{A}) \cap (U - \overline{B})} = A^\perp \sqcap B^\perp$;

(vi) $(A \sqcap B)^\perp = U - \overline{(A \sqcap B)} = U - \overline{\overline{(A \cap B)}} = U - \overline{(A \cap B)} \subseteq$ (th. 3.2(ii); equality holds for closed qsets) $\subseteq U - (\overline{A} \cap \overline{B}) = (U - \overline{A}) \cup (U - \overline{B}) = A^\perp \cup B^\perp =$ (th. 3.7(iv)) $= \overline{A^\perp} \cup \overline{B^\perp} = A^\perp \sqcup B^\perp$. ▮

The lattice $\mathfrak{I}$ is $\mathfrak{I}$-orthomodular, that is, if $A \leq B$, we have that $\overline{B} = A \sqcup (A \sqcup B^\perp)^\perp$.

Theorem 3.9 ($\mathfrak{I}$-orthomodularity) *For all $A, B \in \mathcal{P}(U)$: $A \leq B \Rightarrow A \sqcup (A \sqcup B^\perp)^\perp = \overline{B}$.*

Proof: $A \sqcup (A \sqcup B^\perp)^\perp = A \sqcup (B \sqcap A^\perp) = A \sqcup \overline{(B \cap A^\perp)} = \overline{A} \cup \overline{(B \cap A^\perp)} = \overline{\overline{A} \cup \overline{(B \cap A^\perp)}} = \overline{A} \cup \overline{(B \cap A^\perp)} =$ (th. 3.2 (iii)) $\overline{A \cup (B \cap A^\perp)} = \overline{(A \cup B) \cap (A \cup A^\perp)} = \overline{(A \cup B) \cap \mathbf{1}} = \overline{A \cup B} =$ (th. 3.2 (i)) $\overline{A} \cup \overline{B} = A \sqcup B =$ (by the hypothesis) $= \overline{B}$. ▮

Definition 3.6 (Orthogonality) *Let $A, B \subseteq U$. We say that A is orthogonal to B, and write $A \perp B$, as follows: $A \perp B =_{\mathsf{def}} A \leq B^\perp$. Furthermore, a collection S of elements of $\mathcal{P}(U)$ is called pairwise orthogonal iff for any $A, B \in S$ such that $A \neq B$, it results that $A \perp B$.*

Theorem 3.10 *$A \perp B$ iff $A \cap \overline{B} = \emptyset$.*

Proof: If $A \perp B$, then $A \leq B^\perp$, that is, $A \sqcap B^\perp = \overline{B^\perp}$. Thus, $\overline{A} \cup \overline{B^\perp} = \overline{B^\perp}$, so $\overline{A} \subseteq \overline{B^\perp}$, hence $A \subseteq \overline{B^\perp} = B^\perp$ (by Theorem 2.7 (iv)), so $A \cap \overline{B} = \emptyset$. Conversely, if $A \cap \overline{B} = \emptyset$, then $A \subseteq B^\perp$, hence $\overline{A} \cup \overline{B^\perp} = \overline{B^\perp} = B^\perp$ by Theorem 3.7 (iv), that is, $A \perp B$. ▮

Intuitively speaking, $A \cap \overline{B} = \emptyset$ (by the way, this could be an alternative definition) says that A has no element indistinguishable from elements of B.

In quantum logic, the operations $\leq$ and $^\perp$ are usually understood as an *implication relation* and a *negation relation* respectively. Thus, we may introduce the concept of *logical incompatibility* just using the idea of orthogonality ([10, p. 12]): A is incompatible with B iff A implies the negation of B, that is iff they are orthogonal. The negation of the relation $\perp$ is called *accessibility* (ibid.), written $A \not\perp B$.

All of this show that our structure $\mathfrak{I}$ resembles a non-distributive orthocomplete orthonormal lattice, and it is a Boolean lattice if we consider only the closed qsets. Since every modular ortholattice is orthomodular [10, p. 15], it is an open question whether our lattice has some similarity with modular lattices, that is, $A \leq B$ entails $A \sqcup (C \sqcap B) = (A \sqcup C) \sqcap B$ (we still need to check this and other results).

4 Summing up

We resume here the properties of the quasi-lattice $\mathfrak{I} = \langle \mathcal{P}(U), \sqcap, \sqcup, ^{\perp}, \mathbf{0}, \mathbf{1} \rangle$:

($\mathfrak{I}$-idempotency) $A \sqcap A = \overline{A}$, $A \sqcup A = \overline{A}$

(Commutativity) $A \sqcap A = B \sqcap A$, $A \sqcup B = B \sqcup A$

(Associativity) $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$, $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$

($\mathfrak{I}$-absorption) $A \sqcap (A \sqcup B) = \overline{A}$, $A \sqcup (A \sqcap B) = \overline{A}$

($\mathfrak{I}$-minimum) $\mathbf{0} \sqcap A = \mathbf{0}$, $\mathbf{0} \sqcup A = \overline{A}$

($\mathfrak{I}$-maximum) $A \sqcap \mathbf{1} = \overline{A}$, $A \sqcup \mathbf{1} = \mathbf{1}$

($\mathfrak{I}$-involution - 1) $A^{\perp^{\perp}} = \overline{A}$

($\mathfrak{I}$-involution - 2) $A \leq B \Rightarrow B^{\perp} \leq A^{\perp}$

(Complementation) $A \sqcap A^{\perp} = \mathbf{0}$, $A \sqcup A^{\perp} = \mathbf{1}$

($\mathfrak{I}$-absorption -1) $A \sqcup (B \sqcap B^{\perp}) = \overline{A}$

($\mathfrak{I}$-absorption-2) $A \sqcap (B \sqcup B^{\perp}) = \overline{A}$

($\mathfrak{I}$-De Morgan) $(A \sqcup B)^{\perp} = A^{\perp} \sqcap B^{\perp}$, $(A \sqcap B)^{\perp} \subseteq A^{\perp} \sqcup B^{\perp}$

($\mathfrak{I}$-orthomodularity) $A \sqcup (A \sqcup B^{\perp})^{\perp} = \overline{B}$.

As we see, it is a rather unusual mathematical structure which resembles the non-distributive ortholattice of quantum mechanics. What the specific $\mathfrak{I}$-properties show is that sometimes we need to consider the closure of a certain qset for getting the desired result. If we interpret the qsets of elements of U as extensions of certain predicates, which might stand for physical properties, the necessity of considering the closure of the qsets show that some fuzzy characteristic of these properties are been shown. In fact, take for instance a qset A as the extension of a certain property P, that is, A should stand for the collection of objects having the property P.[7] Then, for instance, if we transform A twice by the operation $^{\perp}$ ($\mathfrak{I}$-involution - 1), we do not obtain A anymore, but the qset of the indiscernible of its elements. It seems that something is changed when we operate with the collections of objects of the physical systems: we really *transform* them, as we really do with quantum systems. But we remark that the physical interpretation of such a structure and its consequences is still being investigated. For the moment, let us keep with its mathematical counterpart only.

[7]By the way, this is something that is lacking in the usual discussion on quantum theories, that is, a right "semantics", which would enable us to talk of the extension of the relevant predicates.

5 The corresponding logic

In this section, we shall be dealing with the first ideas for an alternative axiomatization of a logic that has as its algebraic counterpart the lattice $\mathfrak{I}$, based on the above assumptions and definitions. We remark once more that this is only a preliminary sketch, and maybe some modifications would need to be done, but let us continue even so. As before, we shall be working within the theory $\mathfrak{Q}$. The concepts introduced below, which mirror the standard ones, can be developed in the "standard part" of $\mathfrak{Q}$, so that we can use the usual mathematical terminology. Here, as before, the equality symbol "=" stands for the extensional equality of $\mathfrak{Q}$.

Let us take our algebra $\mathfrak{I} = \langle \mathcal{P}(U), \sqcap, \sqcup, ^{\perp}, \mathbf{0}, \mathbf{1} \rangle$. Now we shall introduce a generalized (or abstract) logic $\mathcal{L} = \langle F, \mathcal{T}, \sim \curlywedge, \curlyvee, \twoheadrightarrow \rangle$ in the sense of [4], and we shall continue to use use $\to$ $\wedge$, $\vee$, $\neg$, $\forall$ and $\exists$ as metalinguistic symbols for implication, conjunction, disjunction, negation, the universal quantifier, and the existential quantifier respectively. The elements of the $\mathfrak{Q}$-set F will be called *formulas*, and denoted by small Greek letters, while the elements of $\mathcal{T}$ ($\mathcal{T} \subseteq \mathcal{P}(F)$) are the *theories* of $\mathcal{L}$, and denoted by uppercase Greek letters (indices can be used in both cases).

To begin with, let us see how we link such a logic with the quasi-lattice $\mathfrak{I}$. Suppose that there is a valuation $v : F \mapsto \mathcal{P}(U)$ such that:

(i) For any $\alpha \in F$, $v(\alpha) \in \mathcal{P}(U)$;

(ii) $\curlywedge$ and $\curlyvee$ are binary operations on F, and we denote the corresponding images of the pair $\langle \alpha, \beta \rangle$ respectively by $\alpha \curlywedge \beta$ and $\alpha \curlyvee \beta$. These operations obey the following rules:

(a) $v(\alpha \curlywedge \beta) = v(\alpha) \sqcap v(\beta)$

(b) $v(\alpha \curlyvee \beta) = v(\alpha) \sqcup v(\beta)$;

(iii) $\sim$ is a mapping from F into F, and we define $v(\sim \alpha) = (v(\alpha))^{\perp}$, for any $\alpha \in F$. This means that if $v(\alpha) = A$, then $v(\sim \alpha) = U - \overline{A}$ according to the above definitions;

(iv) $F \in \mathcal{T}$;

(v) If $\{\Gamma_i\}_{i \in I}$ is a collection of elements of $\mathcal{T}$, then $\bigcap \Gamma_i \in \mathcal{T}$.

It is clear that this definition is an algebraic characterization of our logic $\mathcal{L}$ by means of the lattice $\mathfrak{I}$. Some immediate consequences of this definition are: $v(\alpha \curlywedge \sim \alpha) = \mathbf{0}$, $v(\alpha \curlyvee \sim \alpha) = \mathbf{1}$, $v(\sim\sim \alpha) = v(\alpha)$, etc.

It is well known that in standard quantum logics there is an "implication-problem", to use Dalla Chiara et al.'s words [10, p.164]. That is, all conditional connectives "that can be reasonably introduced" in quantum logics are "anomalous" (ibid.), and this was taken by some authors as a motive to criticize quantum logics as not being "real logics". As Dalla Chiara et al. say, there are some conditions that a conditional would satisfy to be classified as an implication.[8] These conditions are:

Conditions for an Implication

(i) identity, that is, $\alpha \xrightarrow{*} \alpha$, being $\xrightarrow{*}$ the considered conditional;

(ii) modus ponens, that is, is α is true and $\alpha \xrightarrow{*} \beta$ is true, then β is true (op. cit., p. 164);

(iii) In an algebraic semantics, a sufficient condition is: for any structure $\mathcal{A} = \langle A, v\rangle$, $\mathcal{A} \models \alpha \xrightarrow{*} \beta$ iff $v(\alpha) \leq v(\beta)$.

We say that a formula α is true in the structure $\mathfrak{I}$, and write $\mathfrak{I} \models \alpha$ iff $v(\alpha) = \mathbf{1}$, for any valuation v. In this case, $\mathfrak{I}$ is a model of α. We write $\Gamma \models \alpha$ to mean that every model of (the formulas of) Γ is model of α. Finally, α is valid iff it is true in every structure which is an $\mathfrak{I}$-lattice. In this case, we write $\models \alpha$. It is quite obvious that our aim is to prove a completeness theorem for our logic relative to the given semantic, but to do so we need to introduce the concept of deduction from a set of premises. To begin the issue we shall finish only in a forthcoming paper, let us define implication.

Definition 5.1 ($\mathfrak{I}$-conditional) $\alpha \twoheadrightarrow \beta =_{\mathsf{def}} \beta \curlyvee (\sim \alpha \curlywedge \sim \beta)$

This conditional is quite similar to that one called "Dishlkant implication" in [22]. Using the above definitions and Theorem 3.2(i), it is immediate to see that $v(\alpha \twoheadrightarrow \beta) = \overline{v(\beta) \cup (v(\alpha)^{\perp} \cap v(\beta)^{\perp})}$. Thus,

$$v(\alpha \twoheadrightarrow \alpha) = \overline{v(\alpha) \cup (v(\alpha)^{\perp} \cap v(\alpha)^{\perp})} = \overline{v(\alpha) \cup v(\alpha)^{\perp}} = \overline{\mathbf{1}} = \mathbf{1},$$

for $\mathbf{1} = U$ is closed. So, $\models \alpha \twoheadrightarrow \alpha$. Furthermore, if $v(\alpha) = \mathbf{1}$ and $v(\alpha \twoheadrightarrow \beta) = \mathbf{1}$, then $\overline{v(\beta) \cup (v(\alpha)^{\perp} \cap v(\beta)^{\perp})} = \mathbf{1}$ and, since $v(\alpha) = \mathbf{1}$, we get that $v(\beta) = \mathbf{1}$. Thus, our conditional obeys conditions (i) and (ii) of the Conditions for an Implication. In addition, we can see that condition (iii) is also fulfilled. In fact, by the hypothesis, we have $\models \alpha \twoheadrightarrow \beta$, so $v(\beta \curlyvee (\sim \alpha \curlywedge \sim \beta)) = \mathbf{1}$. Call $v(\alpha) = A$ and $v(\beta) = B$. Then $B \sqcup \overline{(A^{\perp} \sqcap B^{\perp})} = U$, that is, $\overline{B} \cup \overline{(A^{\perp} \cap B^{\perp})} = U$. For this equality to hold, we need that $\overline{(A^{\perp} \cap B^{\perp})} = \overline{B}^{\perp} = \overline{B^{\perp}}$. Then $A^{\perp} \cap B^{\perp} = B^{\perp}$, so $B^{\perp} \subseteq A^{\perp}$, that is (for $\mathfrak{Q}$-sets), $A \subseteq B$. By Theorem 3.7 (vii), $A \leq B$, that is, $v(\alpha) \leq v(\beta)$.

Let us make a further remark on this definition. We say that $A \in \mathcal{P}(U)$ is *definable* by a formula $\alpha \in F$ if $v(\alpha) = A$. Let β be such that $v(\beta) = \overline{A}$. Is there such a β? The answer is in the affirmative. Since $A \subseteq \overline{A}$, then $v(\alpha) \leq v(\beta)$,

[8] Really, several "quantum implications" can be defined, as shown in [21], [22], [26], but we shall not continue with this discussion here. One of the first works (to our knowledge) that proposed an axiomatization of the lattice of quantum mechanics is [15], in which other conditionals are defined. We had no access to this paper, but know it from indirect sources, namely, [11] and [27].

hence by condition (iii) above, $\models \alpha \twoheadrightarrow \beta$. So, β is any formula implied by α. This affirmative makes sense, for $v(\alpha \twoheadrightarrow \beta) = \mathbf{1}$ and $v(\beta) = \overline{A}$ say that $\overline{A} \sqcup (A^{\perp} \sqcap \overline{A^{\perp}}) = U$, that is, $\overline{\overline{A}} \cup \overline{(A^{\perp} \cap \overline{A^{\perp}})} = A \cup (\overline{A^{\perp}} \cap \overline{A^{\perp}}) = \overline{A} \cup \overline{A}^{\perp} = U$. This fact will be important for the definition of the connectives of the logic $\mathcal{L}$. Finally, let us say that $\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta =_{\mathsf{def}} (\alpha \twoheadrightarrow \beta) \curlywedge (\beta \twoheadrightarrow \alpha)$.

Next we introduce the notion of syntactical consequence from a set of premises, written $\Gamma \vdash \alpha$, as follows, where $v(\Gamma) = \bigcup[v(\alpha) : \alpha \in \Gamma]$ (the terminology is from $\mathfrak{Q}$ – see again Section 2, if necessary).

Definition 5.2 (Syntactical Consequence) $\Gamma \vdash \alpha$ *iff any theory containing* Γ *(really, the formulas of* Γ*) contains* α.

Let $\vdash \alpha$ abbreviates $\emptyset \vdash \alpha$, while $\alpha \vdash \beta$ abbreviates $\{\alpha\} \vdash \beta$ (recall that they are $\mathfrak{Q}$-sets, so the standard notation can be used), and $\Gamma \nvdash \alpha$ says that it is not the case that $\Gamma \vdash \alpha$. It is immediate to prove the following theorem:

Theorem 5.1 *In* $\mathcal{L}$*, we have*

(i) $\alpha \in \Gamma$ entails $\Gamma \vdash \alpha$. In particular, $\alpha \vdash \alpha$;

(ii) $\Gamma \vdash \alpha$ entails $\Gamma \cup \Delta \vdash \alpha$;

(iii) If $\Gamma \vdash \alpha$ and for every $\beta \in \Gamma$, we have that $\Delta \vdash \beta$, then $\Delta \vdash \alpha$;

(iv) If $\{\Gamma_i\}_{i \in I}$ is a family of subqsets of F such that $\forall \alpha (\alpha \in \Gamma \leftrightarrow \Gamma_i \vdash \alpha)$, then $\forall \alpha (\alpha \in \bigcap_{i \in I} \Gamma_i \leftrightarrow \bigcap_{i \in I} \Gamma_i \vdash \alpha)$.

Proof: Immediate, for the definition of consequence is standard (see [4], [23]). ∎

We shall not continue to develop the syntactical aspects of this logic (in algebraic terms, but see [4]), but just try to link it with the semantic aspects sketched above. The least theory containig α is denoted T_α, and it coincides with the intersection of all theories containing α (op. cit.). Thus, $\Gamma \vdash \alpha$ iff $v(\alpha) \subseteq v(T_\alpha)$. In particular, if Γ is a theory, that is, $\mathrm{Cn}(\Gamma) =_{\mathsf{def}} [\alpha : \Gamma \vdash \alpha] = \Gamma$, then $v(\alpha) \subseteq v(\Gamma)$, and in particular $v(\alpha) \subseteq \overline{v(\Gamma)}$. Finally, let us recall that since no deduction theorem holds in quantum logics [21], the same seems to happen here due to the nature of our implication (but this is still a open problem).

The last point of this paper, which conduces us to another work, is the question: how to characterize the logic $\mathcal{L}$ axiomatically? We shall follow the approach of generalized logics in the sense of [4], but not here.

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