Can Bell's Prescription for Physical Reality Be Considered Complete?

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An experiment is proposed to test Bell's theorem in a purely macroscopic domain. If realized, it would determine whether Bell inequalities are satisfied for a manifestly local, classical system. It is stressed why the inequalities should not be presumed to hold for such a macroscopic system without actual experimental evidence. In particular, by providing a purely classical, topological explanation for the EPR-Bohm type spin correlations, it is demonstrated why Bell inequalities must be violated in the manifestly local, macroscopic domain, just as strongly as they are in the microscopic domain.

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Despite the existence of an explicit counterexample [1], Bell's theorem is still widely believed to have proved that no physical theory can be reconciled with the notion of local reality espoused by Einstein, Podolsky, and Rosen (EPR) [2][3]. It therefore seems worthwhile to investigate the very foundations of Bell's theorem experimentally, in a purely macroscopic domain. If realized, the experiment described below would test whether or not a manifestly local, macroscopic system can violate Bell inequalities, as implied by the arguments of Ref. [1]. A physical scenario well suited for this purpose is that of the local model first considered by Bell himself [4]. The details of this model can be found also in some standard textbooks [5]. To our knowledge, no real experiment has ever been performed to check whether Bell inequalities do indeed hold in the manifestly local and realistic domain of Bell's model.

The central contention of Ref. [1] is that the sinusoidal EPR-Bohm correlations observed in the laboratory have nothing to do with entanglement or nonlocality *per se*, but stem entirely from the topological properties of the physical space. This viewpoint can be explained clearly by a closer examination of the model for spin considered by Bell. In this model the space of complete states of spin consists of unit vectors $\boldsymbol{\lambda}$ in three-dimensional Euclidean space $\mathbb{E}_3$. The local beables $A_{\mathbf{a}}(\boldsymbol{\lambda})$ and $B_{\mathbf{b}}(\boldsymbol{\lambda})$, existing at freely chosen unit directions $\mathbf{a}$ and $\mathbf{b}$, are defined by

$$A_{\mathbf{n}}(\boldsymbol{\lambda}) \,=\, -\,B_{\mathbf{n}}(\boldsymbol{\lambda}) \,=\, sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})\,, \tag{1}$$

provided $\boldsymbol{\lambda}\cdot\mathbf{n}\,\neq\,0$ for $\mathbf{n}=\mathbf{a}$ or $\mathbf{b}$, and otherwise equal to the sign of the first nonzero term from $\{n_x,\,n_y,\,n_z\}$. This simply means that $A_{\mathbf{n}}(\boldsymbol{\lambda})=+\,1$ if the two unit vectors $\mathbf{n}$ and $\boldsymbol{\lambda}$ happen to point through the same hemisphere centered at the origin of $\mathbf{n}$, and $A_{\mathbf{n}}(\boldsymbol{\lambda})=-\,1$ otherwise. As a visual aid to Bell's model [5] one can think of a bomb at rest exploding into two freely moving fragments with angular momenta $\boldsymbol{\lambda}=\mathbf{J}_1=-\mathbf{J}_2$, with $\mathbf{J}_1+\mathbf{J}_2=0$. The two functions $A_{\mathbf{a}}(\mathbf{J}_1)$ and $B_{\mathbf{b}}(\mathbf{J}_2)$ can then be taken as $sign\,(\boldsymbol{\lambda}\cdot\mathbf{a})$ and $sign\,(-\,\boldsymbol{\lambda}\cdot\mathbf{b})$, respectively. If the initial directions of the two angular momenta are uncontrollable but describable by an isotropic probability distribution $\rho(\boldsymbol{\lambda})$ (normalized on the space $\mathbb{E}_3$), then, employing the local realistic prescription provided by Bell, namely

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\int_{\mathbb{E}_3} A_{\mathbf{a}}(\boldsymbol{\lambda})\,B_{\mathbf{b}}(\boldsymbol{\lambda})\;d\rho(\boldsymbol{\lambda}), \tag{2}$$

the expectation values of the individual variables $A_{\mathbf{a}}(\boldsymbol{\lambda})$ or $B_{\mathbf{b}}(\boldsymbol{\lambda})$ can be easily shown to vanish identically [5]. Their joint correlation function on the other hand would not vanish in general, and is usually worked out to be [5]

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, -1+\frac{2}{\pi}\,\cos^{-1}(\mathbf{a}\cdot\mathbf{b})\,. \tag{3}$$

If we now substitute this linear correlation function into the CHSH string of expectation values for four arbitrarily chosen detector directions $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$, giving

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}'), \tag{4}$$

then it is easy to check that the absolute value of this string never exceeds the bound of 2, thus saturating but not violating the celebrated Bell-CHSH inequalities [6].

This often quoted result is usually considered to be well established, but in fact it is simply incorrect. The trouble is that the local realistic prescription for spin correlations provided by Bell—namely, Eq.(2) above—is incapable of accounting for the elements of physical reality envisaged by EPR *in the topologically correct order*. The situation is analogous to having taken a photograph apart pixel by pixel, keeping count of each pixel correctly, and then trying to put it back together. If the geometrical order of the pixels has been neglected in the process, then there would be little chance of recovering the photograph back. Similarly, what is missing from the prescription (2) is not so much the operational accounting of the elements of physical reality, but how these elements are coalesced together topologically. As we shall see, the correct result for the spin correlations, derived using both operationally and topologically complete prescription, works out to be

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \tag{5}$$

which extends the bound on the Bell-CHSH inequality from 2 to $2\sqrt{2}$. To fully understand this classical result let us take a closer look at the derivation of Eq.(3).

Since the initial distribution of the angular momenta is supposed to have been isotropic, in Bell's model the space of all possible directions of $\mathbf{J}_1$—that is, both the configuration space as well as the phase space of $\mathbf{J}_1$—is traditionally [5] taken to be a unit 2-sphere, defined by

$$n_x^2 + n_y^2 + n_z^2 = 1. \tag{6}$$

Next, since each point $\boldsymbol{\lambda}$ on this surface represents an EPR element of physical reality, the integration in Eq.(2) is carried out over this surface, yielding the result (3). Now the group of linear transformations that leave such a quadratic form invariant is the orthogonal group O(3), which includes both rotational and reflective symmetries of the 2-sphere. On the other hand, we know that angular momentum is not an ordinary polar vector, but a *pseudo* vector that changes sign upon reflection. One only needs to compare a spinning object with its image in a mirror to confirm this fact. This familiar fact is sufficient, however, to divulge the first sign of trouble with Bell's chosen set of observables—namely, $sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})$. Clearly, since $\boldsymbol{\lambda}$ is supposed to be the spin angular momentum whereas $\mathbf{n}$ is simply an ordinary polar vector, the dot product in the observable $sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})$ cannot be a *true* scalar, but a *pseudo* scalar—one that changes sign in the mirror.

There is of course an easy way out of this problem. All one has to do is to restrict the symmetry group of the 2-sphere to the subgroup SO(3)—i.e., to the group of non-reflective symmetries. This seems straightforward enough, but one must bear in mind that, although their Lie algebras are identical, globally the groups O(3) and SO(3) are profoundly different from each other. Globally the group SO(3) is a highly non-trivial subgroup of O(3). Indeed, topologically the space SO(3) is homeomorphic to the real projective space $\mathbb{RP}^3$, which is a connected, but not simply-connected manifold [7][8]. That is to say, there are loops in SO(3) that cannot be contracted to a point. In physical applications this fact is well known to give rise to unavoidable singularities, discontinuities, and wildly spinning trajectories [9]. In addition to this fatal defect, the group SO(3) also harbors a related conceptual defect, which is of profound significance for our concerns. The trouble is that SO(3) does not always respect the true rotational symmetries of the physical space.

To appreciate this well known fact, consider a rock in an otherwise empty universe. If such a rock is allowed to rotate by 2π radians about some axis, then it will return back to its original state. This, however, will not happen if there is at least one other object present in the universe. The rock will then have to rotate by another 2π radians (i.e., a total of 4π radians) to return back to its original state, relative to that other object. This well known fact is often demonstrated by a "belt trick" (cf. [8], p 205), which shows that what is an identity transformation for an isolated object is *not* an identity transformation for an object that is rotating in the presence of other objects. Thus, what appears to be an identity transformation in the latter case on purely operational basis, is simply an illusion. This peculiar property of the ordinary objects is not respected by the structure of SO(3). That is to say, SO(3) is capable of providing only tensor representations of the rotation group, and not its spinor representations. This is fine as long as one is concerned with rotations of only isolated objects, but it is anything but fine in our case, since we are concerned with correlations between two macroscopic bomb fragments *rotating in tandem*.

Fortunately [7][9], all of the above difficulties can be resolved by representing rotations in physical space by elements of the universal covering group of SO(3), namely the group SU(2) of unit quaternions (or spinors [8], or rotors [10]). This group can be constructed by taking *two* copies of SO(3), and gluing their boundaries together point by point, so that each $-\pi$ rotation-point on the boundary of one copy is identified with the respective $+\pi$ rotation-point on the boundary of the second copy. The resulting space is a topological 3-sphere defined by

$$n_o^2 + n_x^2 + n_y^2 + n_z^2 = 1, \tag{7}$$

where the quadruple $(\,n_o,\, n_x,\, n_y,\, n_z)$ defines a non-pure unit quaternion [7]. The 3-sphere is well known of course to have exceptionally special properties [11]. It is the only three-dimensional manifold without boundary that is not only compact and connected, but also simply-connected. And it is the only simply-connected, parallelizable sphere that is homeomorphic to a Lie group, namely SU(2) (it is also worth noting the obvious that the usefulness of this group is not exclusive to quantum mechanics [9]).

Despite its being contained in $\mathbb{R}^4$, it is in fact possible to "see" inside this sphere by means of a Hopf fibration [12]. This provides us an opportunity to appreciate the true topological structure of the elements of reality for our bomb fragments. As illustrated in Fig. 1, the 2-sphere we started out with, namely the one defined by Eq.(6), turns out to be only the base manifold of this profound structure. The points of this base manifold, namely S^2, now correspond to elements of the Lie algebra su(2), and are in fact *pure* quaternionic numbers [7][8]. The product of two such numbers on S^2 are then *general* quaternionic numbers, defined by (7), and belong to the group SU(2) itself. That is to say, they are points on the bundle space S^3, which is *completely* made up of the preimages of the points on the base S^2 [12]. These preimages are 1-spheres, S^1, called Hopf circles, or Clifford parallels ([8], p 335). Since these 1-spheres are the fibers of the bundle, they do not share a single point in common. And yet each circle threads through every other circle in the bundle, making them all linked together in a highly intricate fashion. In particular, although locally the bundle S^3 is a product space $S^2 \times S^1$, *globally it has no cross-section at all.*

It should be fairly clear by now that topologically the EPR elements of reality have far deeper structure than has been hitherto appreciated. Clearly, no prescription that ignores this structure can be expected to provide the

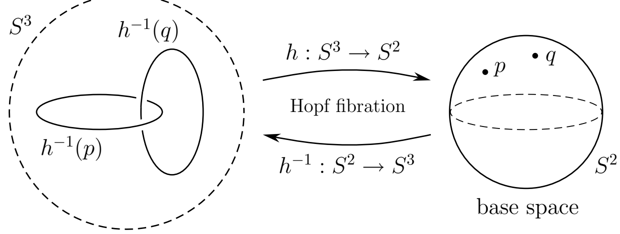

FIG. 1: The tangled web of linked Hopf circles depicting the topological intricacies of the EPR elements of physical reality.

correct correlation function for our bomb fragments. In particular, no Bell-type scalar functions of the form

$$A_{\mathbf{n}}(\lambda) : V \times \Lambda \longrightarrow S^0 \equiv \{-1, +1\}, \tag{8}$$

$$\text{with } \; A_{\mathbf{a}}(\lambda) B_{\mathbf{b}}(\lambda) \in S^0 \times S^0 = S^0 \equiv \{-1, +1\} \tag{9}$$

(where V is a vector space and Λ is a space of "complete" states), can account for the topological intricacies of the elements of physical reality, *even for our purely classical rotors*. Surely, no elements of a 0-sphere can imitate the topological profundities of the elements of a 3-sphere.

On the other hand, from the above picture, and from the fact that the groups O(3), SO(3), and SU(2) all share the same Lie algebra structure, it is clear that simply promoting the variables $A_{\mathbf{n}}(\boldsymbol{\lambda})$ to be the elements of Lie algebra su(2) cannot be sufficient to capture the global, topological features of the group SU(2). Capturing these features is mandatory, however, if we are to represent the EPR elements of physical reality faithfully within the choice of our dynamical variables. Hence, recalling the definition of a group, what we must ensure is not only that the local variables—which we write as $A_{\mathbf{n}}(\boldsymbol{\mu})$—are functions of the elements of the Lie algebra su(2), but that their group products, $A_{\mathbf{a}}(\boldsymbol{\mu})\, B_{\mathbf{b}}(\boldsymbol{\mu})$, appearing in the integrand of Eq.(2), are themselves genuine elements of the group SU(2); for what is captured by the Lie algebra su(2) is only the tangent structure of the group SU(2). At the same time, we must also ensure of course that the revised variables $A_{\mathbf{n}}(\boldsymbol{\mu})$ remain *operationally identical* to the original variables $A_{\mathbf{n}}(\boldsymbol{\lambda})$. We may then have a chance of evaluating their correlations correctly, by employing an appropriately generalized expectation functional [13]. In sum, the necessary and sufficient conditions on the local beables for obtaining the correct correlation function are:

$$\mathrm{su}(2) \ni A_{\mathbf{n}}(\boldsymbol{\mu}) = \pm 1 \text{ about } \mathbf{n}, \tag{10}$$

$$\text{and } \; \mathrm{SU}(2) \ni A_{\mathbf{a}}(\boldsymbol{\mu})\, B_{\mathbf{b}}(\boldsymbol{\mu}) = \text{a unit quaternion.} \tag{11}$$

A local realistic model for the EPR-Bohm correlations satisfying precisely these conditions has been proposed in Ref.[1]. The complete state specifying all of the elements of reality in this model is taken to be the unit trivector

$$\boldsymbol{\mu} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \pm I \equiv \pm\, \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z\,, \tag{12}$$

where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors of arbitrary length, and I is the fundamental volume form on the physical space. The specification of the complete state $\boldsymbol{\mu}$ predetermines the entire geometry of the three-dimensional Euclidean space $\mathbb{E}_3$ (encapsulated in the Clifford algebra $Cl_{3,0}$). It determines all scalars by their duality relations with $\boldsymbol{\mu}$, all vectors $\mathbf{x}$ by definition $\boldsymbol{\mu} \wedge \mathbf{x} = 0$, all bivectors by the duality relation $\boldsymbol{\mu} \cdot \mathbf{x} = \boldsymbol{\mu}\, \mathbf{x}$, and all quaternions by the Clifford product $\mathbf{x}\,\mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \boldsymbol{\mu} \cdot (\mathbf{x} \times \mathbf{y})$. The locally specified beables of the model are then taken to be the unit bivectors $\boldsymbol{\mu} \cdot \mathbf{n}$, which are elements of the Lie algebra su(2) of the group SU(2), with the following properties:

$$\boldsymbol{\mu} \cdot \mathbf{n} = \pm 1 \text{ about the dual vector } \mathbf{n}, \tag{13}$$

$$\text{and } (\boldsymbol{\mu} \cdot \mathbf{a})(\boldsymbol{\mu} \cdot \mathbf{b}) = -\,\mathbf{a} \cdot \mathbf{b} - \boldsymbol{\mu} \cdot (\mathbf{a} \times \mathbf{b}). \tag{14}$$

Note that the Clifford product $(\boldsymbol{\mu} \cdot \mathbf{a})(\boldsymbol{\mu} \cdot \mathbf{b})$ within $Cl_{3,0}$ is also a group product within SU(2), yielding a non-pure unit quaternion [7][12]. This can be verified by comparing the decomposition of the above product with Eq.(7). A generalized expectation functional analogous to (2) then gives the correct correlation function for our rotors:

$$\mathcal{E}(\mathbf{a},\, \mathbf{b}) = \int_{\mathcal{V}_3} (\boldsymbol{\mu} \cdot \mathbf{a})(\boldsymbol{\mu} \cdot \mathbf{b})\; d\boldsymbol{\rho}(\boldsymbol{\mu}) = -\,\mathbf{a} \cdot \mathbf{b}, \tag{15}$$

where the integral is a *directed* integral, $\mathcal{V}_3$ is a manifold whose "points" are *vectors* in $\mathbb{E}_3$, and the distribution $\boldsymbol{\rho}(\boldsymbol{\mu})$ is assumed to be normalized on this *vector* manifold. Clearly, unlike Eq.(2), the above prescription is not only *operationally complete*, but also *topologically complete*. As shown in Ref. [2], if we now substitute this correlation function into the CHSH string of expectation values, then the bound on its absolute value is extended to $2\sqrt{2}$.

It is worth noting here that the model described above is not only *manifestly realistic*, but also *intrinsically local*. There are several independent ways to verify the latter fact [2]. To begin with, it is evident from their bivectorial constitution that the two remote beables $\boldsymbol{\mu} \cdot \mathbf{a}$ and $\boldsymbol{\mu} \cdot \mathbf{b}$ have nothing to do with each other. In fact, since they are two genuine elements of the Lie algebra su(2), they are necessarily two independent points on the corresponding 2-sphere. Moreover, as rigorously proved in Ref. [2], the above model satisfies not only the condition of parameter independence, but also that of outcome independence.

The central message of Refs. [1] and [2] and the above discussion is that EPR-Bohm correlations have nothing to do with entanglement or non-locality *per se*, but are a vestige of geometry and topology of the physical space. This recognition almost immediately leads to prediction (15), which differs from the prediction (3) derived on the basis of Bell's prescription (2). These two predictions are clearly distinguishable. The experiment described below to distinguish them is essentially a realization of Bell's own local model discussed above [5]. It can be performed either in the outer space or in a terrestrial laboratory. In the latter case, however, the effects of gravity and air resistance would complicate matters. For simplicity we shall assume that experimental parameters can be chosen sufficiently carefully to compensate such effects.

With this assumption, consider a "bomb" made out of a hollow toy ball of diameter, say, three centimeters. The thin hemispherical shells of uniform density that make up the ball are snapped together at their rims in such a manner that a slight increase in temperature would pop the ball open into its two constituents with considerable force [5]. A small lump of density much grater than the density of the ball is attached on the inner surface of each shell at a random location, so that, when the ball pops open, not only would the two shells propagate with equal and opposite linear momenta orthogonal to their common plane, but would also rotate with equal and opposite spin momenta about a random axis in space. The volume of the attached lumps can be as small as a cubic millimeter, whereas their mass can be comparable to the mass of the ball. This will facilitate some 10^6 possible spin directions for the two shells, whose outer surfaces can be decorated with colors to make their rotations easily detectable.

Now consider a large ensemble of such balls, identical in every respect except for the relative locations of the two lumps (affixed randomly on the inner surface of each shell). The balls are then placed over a heater—one at a time—at the center of an EPR-Bohm type setup [6], with the common plane of their shells held perpendicular to the horizontal direction of the setup. Although initially at rest, a slight increase in temperature of each ball will eventually eject its two shells towards the observation stations, situated at a chosen distance in the mutually opposite directions. Instead of selecting the directions $\mathbf{a}$ and $\mathbf{b}$ for observing spin components, however, one or more contact-less rotational motion sensors—capable of determining the precise direction of rotation—are placed near each of the two stations, interfaced with a computer. These sensors will determine the exact direction of the angular momentum $\boldsymbol{\lambda}_j$ (or $-\boldsymbol{\lambda}_j$) for each shell, without disturbing them otherwise, at a designated distance from the center. The interfaced computers can then record this data, in the form of a 3D map of all such directions.

Once the actual directions of the angular momenta for a large ensemble of shells on both sides are fully recorded, the two computers are instructed to randomly choose the reference directions, $\mathbf{a}$ for one station and $\mathbf{b}$ for the other station—from within their already existing 3D maps of data—and then calculate the corresponding dynamical variables $sign\,(\boldsymbol{\lambda}_j\cdot\mathbf{a})$ and $sign\,(-\boldsymbol{\lambda}_j\cdot\mathbf{b})$. This "delayed choice" of $\mathbf{a}$ and $\mathbf{b}$ will guarantee that the conditions of parameter independence and outcome independence are strictly respected within the experiment [2]. It will ensure, for example, that the local outcome $sign\,(\boldsymbol{\lambda}_j\cdot\mathbf{a})$ remains independent not only of the remote parameter $\mathbf{b}$, but also of the remote outcome $sign\,(-\boldsymbol{\lambda}_j\cdot\mathbf{b})$. If in any doubt, the two computers can be located at a sufficiently large distance from each other to ensure local causality while selecting $\mathbf{a}$ and $\mathbf{b}$. The correlation function for the bomb fragments can then be calculated using the formula

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,=\,\frac{1}{N}\sum_{j\,=\,1}^{N}\left\{sign\,(\boldsymbol{\lambda}_j\cdot\mathbf{a})\right\}\left\{sign\,(-\boldsymbol{\lambda}_j\cdot\mathbf{b})\right\},\quad(16)$$

where N is the number of trials. This result, which would give purely local correlations, should then be compared (in $N\to\infty$ limit) with the predictions (3) and (15).

It is worth recalling here that the variables $sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})$ and $\boldsymbol{\mu}\cdot\mathbf{n}$ used in the respective derivations of equations (3) and (15) are *operationally identical* to each other:

$$sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})\,\cong\,\pm 1\ \text{about}\ \mathbf{n}\,\cong\,\boldsymbol{\mu}\cdot\mathbf{n}.\qquad(17)$$

This can be easily verified by noting that the variables $sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})$ are simply the normalized components of the angular momenta $\mathbf{J}$ along the directions $\mathbf{n}$, and so are the variables $\boldsymbol{\mu}\cdot\mathbf{n}$ (albeit in the bivector basis [10]). In other words, although mathematically $sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})$ and $\boldsymbol{\mu}\cdot\mathbf{n}$ are elements of two different grades in the algebra $Cl_{3,0}$ (one is a scalar and the other a bivector), physically they represent one and the same rotor quantity [2].

Undoubtedly, there would be many different sources of systematic errors in an experiment such as this. If it is performed carefully enough, however, then—in the light of the discussion above—we believe the experiment will vindicate prediction (15) and refute prediction (3).

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