On the Existence of 'Time Machines' in General Relativity*

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Abstract

Within the context of general relativity, we consider one definition of a 'time machine' proposed by Earman, Smeenk, and Wüthrich (2009). They conjecture that, under under their definition, the class of time machine spacetimes is not empty. Here, we prove this conjecture.

1 Introduction

One peculiar feature of general relativity concerns the existence of closed timelike curves in some cosmological models permitted by the theory. In such models, a massive point particle may both commence and conclude a journey through spacetime at one and the same point. In this respect, these models allow for "time travel".

Naturally, the existence of closed timelike curves in some relativistic models prompts fascinating questions.[1] One issue, recently addressed by Earman, Smeenk, and Wüthrich (2009), concerns what it might mean to say that a model allows for the operation of a "time machine" in some sense.[2] They propose a precise definition and then conjecture that, under their formulation, there exist cosmological models which count as time machines. In this paper, we provide a proof of this conjecture.

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[1]For a thorough investigation of many of these questions see Earman (1995).

[2]See also Earman and Wüthrich (2004).

2 Background Structure

We begin with a few preliminaries concerning the relevant background formalism of general relativity.[3] An n-dimensional, relativistic *spacetime* (for $n \geq 2$) is a pair of mathematical objects (M, g_{ab}). M is a connected n-dimensional manifold (without boundary) that is smooth (infinitely differentiable). Here, g_{ab} is a smooth, non-degenerate, pseudo-Riemannian metric of Lorentz signature $(+, -, ..., -)$ defined on M. Each point in the manifold M represents an "event" in spacetime.

For each point $p \in M$, the metric assigns a cone structure to the tangent space M_p. Any tangent vector ξ^a in M_p will be *timelike* (if $g_{ab}\xi^a\xi^b > 0$), *null* (if $g_{ab}\xi^a\xi^b = 0$), or *spacelike* (if $g_{ab}\xi^a\xi^b < 0$). Null vectors create the cone structure; timelike vectors are inside the cone while spacelike vectors are outside. A *time orientable* spacetime is one that has a continuous timelike vector field on M. A time orientable spacetime allows us to distinguish between the future and past lobes of the light cone. In what follows, it is assumed that spacetimes are time orientable.

For some interval $I \subseteq \mathbb{R}$, a smooth curve $\gamma : I \to M$ is *timelike* if the tangent vector ξ^a at each point in $\gamma[I]$ is timelike. Similarly, a curve is *null* (respectively, *spacelike*) if its tangent vector at each point is null (respectively, spacelike). A curve is *causal* if its tangent vector at each point is either null or timelike. A causal curve is *future-directed* if its tangent vector at each point falls in or on the future lobe of the light cone. Given a point $p \in M$, the *causal future of* p (written $J^+(p)$) is the set of points $q \in M$ such that there exists a future-directed causal curve from p to q. Naturally, for any set $S \subseteq M$, define $J^+[S]$ to be the set $\cup\{J^+(x) : x \in S\}$. A *chronology violating region* $V \subseteq M$ is the set of points $p \in M$ such that there is a closed timelike curve through p.

A point $p \in M$ is a *future endpoint* of a future-directed causal curve $\gamma : I \to M$ if, for every neighborhood O of p, there exists a point $t_0 \in I$ such that $\lambda(t) \in O$ for all $t > t_0$. A *past endpoint* is defined similarly. For any set $S \subseteq M$, we define the *past domain of dependence of* S (written $D^-(S)$) to be the set of points $p \in M$ such that every causal curve with past endpoint p and no future endpoint intersects S. The *future domain of dependence of* S (written $D^+(S)$) is defined analogously. The entire *domain of dependence of* S (written $D(S)$) is just the set $D^-(S) \cup D^+(S)$.

A set $S \subset M$ is *achronal* if no two points in S can be connected by a

[3] The reader is encouraged to consult Hawking and Ellis (1973) and Wald (1984) for details. An outstanding (and less technical) survey of the global structure of spacetime is given by Geroch and Horowitz (1979).

timelike curve. A set $S \subset M$ is a *slice* if it is closed, achronal, and without edge. A set $S \subset M$ is a *spacelike surface* if S is an (n - 1)-dimensional submanifold (possibly with boundary) such that every curve in S is spacelike.[4]

Two spacetimes (M, g_{ab}) and (M', g'_{ab}) are *isometric* if there is a diffeomorphism $\phi : M \to M'$ such that $\phi_*(g_{ab}) = g'_{ab}$. We say a spacetime (M', g'_{ab}) is a (proper) *extension* of (M, g_{ab}) if there is a proper subset N of M' such that (M, g_{ab}) and $(N, g'_{ab|N})$ are isometric. We say a spacetime is *inextendible* if it has no proper extension.

3 A Time Machine

In their recent paper, Earman, Smeenk, and Wüthrich attempt to clarify what it might mean to say that a time machine operates within a relativistic spacetime.

First, in order to count as a time machine, a spacetime (M, g_{ab}) must contain a spacelike slice S representing a "time" before the time machine is switched on. Next, they note that a time machine should operate within a finite region of spacetime. Accordingly, they require that the time machine region $T \subset M$ have compact closure. In addition, so as to guarantee that instructions for the operation of the time machine (set on S) are followed, they require that $T \subset D^+(S)$. Of course, the spacetime (M, g_{ab}) must also have a chronology violating region V to the causal future of the time machine region T.

Finally, in oder to capture the idea that a time machine must "produce" closed timelike curves, Earman, Smeenk, and Wüthrich demand that every suitable extension of $D(S)$ contain a chronology violating region V'. For them, a suitable extension must be inextendible and satisfy a condition known as "hole-freeness". This condition, introduced by Geroch (1977), essentially requires that the domain of dependence $D(\Sigma)$ of each spacelike surface Σ be "as large as it can be". Here, hole-freeness serves to rule out extensions of $D(S)$ which fail to have closed timelike curves only because of the formation of seemingly artificial "holes" in spacetime.[5] Formally, we say a spacetime (M, g_{ab}) is *hole-free* if, for any spacelike surface Σ in M there

[4] Allowing S to have a boundary is non-standard but the formulation introduces no difficulties. In particular, one may consider initial data on S and determine its domain of dependence $D(S)$. See Hawking and Ellis (1973, 201).

[5] A result due to Kasnikov (2002) seems to indicate that, without the assumption of hole-freeness, one can always find extensions of $D(S)$ bereft of closed timelike curves. For a discussion of whether hole-freeness is a physically reasonable condition to place on spacetime, see Manchak (2009).

is no isometric embedding $\theta : D(\Sigma) \to M'$ into another spacetime (M', g'_{ab}) such that $\theta(D(\Sigma)) \neq D(\theta(\Sigma))$. We can now state the definition of a time machine.

Definition. A spacetime (M, g_{ab}) is an *ESW time machine* if (i) there is a spacelike slice $S \subset M$, a set $T \subset M$ with compact closure, and a chronology violating region $V \subset M$ such that $T \subset D^+(S)$ and $V \subset J^+[T]$ and (ii) every hole-free, inextendible extension of $D(S)$ contains some chronology violating region V'.

4 An Existence Theorem

With a definition in place, Earman, Smeenk, and Wüthrich then conjecture that, under their formulation, there exist spacetimes which count as time machines. Here we prove this conjecture by showing that the well-known example of Misner spacetime satisfies the conditions of the definition.[6] We have the following theorem.

Theorem. There exists an ESW time machine.

Proof. Let (M, g_{ab}) be Misner spacetime. So, $M = \mathbb{R} \times \mathbb{S}$ and $g_{ab} = 2\nabla_{(a} t \nabla_{b)} \varphi + t \nabla_a \varphi \nabla_b \varphi$ where the points (t, φ) are identified with the points $(t, \varphi + 2\pi n)$ for all integers n.

Let S be the spacelike slice $\{(t, \varphi) \in M : t = -1\}$. It can be easily verified that, $D^+(S) = \{(t, \varphi) \in M : -1 \leq t < 0\}$. Let T be the compact set $\{(t, \varphi) \in M : t = -1/2\}$. So, $T \subset D^+(S)$. Note that the set $\{(t, \varphi) \in M : t > 0\}$ is a chronology violating region. Call it V. Clearly, $V \subset J^+[T]$. Thus, we have satisfied condition (i) of the definition of an ESW spacetime. For future reference, let N be the set $\{(t, \varphi) \in M : t \leq 0\}$.

Now, let (M', g'_{ab}) be an inextendible extension of $D(S) = \{(t, \varphi) \in M : t < 0\}$ which does not contain closed timelike curves. We show that it must fail to be hole-free. Now, for every $k \in [0, 2\pi]$, let γ_k be the null geodesic curve whose image is the set $\{(t, \varphi) \in M : \varphi = k \ \& \ -1 < t < 0\}$. Now, for each k, γ_k either has a future endpoint p_k or not. Clearly, for (M', g'_{ab}) to be inextendible, there is some k, such that p_k exists. Let K be the set of

[6] For details concerning Misner spacetime, including a diagram, see Hawking and Ellis (1973, p. 171-174). Note, however, that because of the sign conventions used in that reference, the diagram there is an upside down representation of the version of Misner spacetime considered here.

all the endpoints p_k. We can extend the coordinate system used in Misner spacetime to a neighborhood $K' \subset M'$ of K. Under this coordinate system, we have $K = \{(t, \varphi) \in K' : t = 0\}$. For future reference, let the set N' be defined as $\{(t, \varphi) \in M' : t < 0 \text{ or } (t, \varphi) \in K\}$.

Next, we show that for any distinct points $u, v \in K$, if $u \in J^-(v)$ then $v \notin J^-(u)$. It suffices to show that for some $k \in [0, 2\pi]$, γ_k has no future endpoint (in that case, K cannot be a closed null curve). Assume that for all $k \in [0, 2\pi]$, there is a future endpoint p_k of γ_k in K. We show a contradiction. Consider any point $p_k \in K$ and a neighborhood $U_k \subset K'$ of p_k. Let $f_k : U_k \to \mathbb{R}$ be the function defined by $f_k(t, \varphi) = g'_{ab}(t, \varphi)(\frac{\partial}{\partial \varphi})^a(\frac{\partial}{\partial \varphi})^b$. Of course, when the domain of f_k is restricted to the set of points $(t, \varphi) \in U_k$ where $t \leq 0$, then $f_k(t, \varphi) = t$. The smoothness of g'_{ab} ensures the boundary conditions $f_k(0, \varphi) = 0$ and $\frac{\partial}{\partial t} f_k(0, \varphi) = 1$ are satisfied. Clearly then, there must be an ϵ_k such that $f_k(t, \varphi) > 0$ for all $t \in (0, \epsilon_k]$. Now, let $\epsilon : K \to \mathbb{R}$ be the function defined by $\epsilon(p_k) = \epsilon_k$. Note that the smoothness of g'_{ab} allows us to choose our ϵ_k so that ϵ is a continuous function. Because K is compact, ϵ takes on a minimum value (call it ϵ_{min}).[7] Next, let V' be the set $\{(t, \varphi) : 0 < t \leq \epsilon_{min}\}$. Clearly, on V', we have $g'_{ab}(\frac{\partial}{\partial \varphi})^a(\frac{\partial}{\partial \varphi})^b > 0$. Now, let w be any point in V' and consider the curve $\gamma : I \to V'$ through w with tangent vector $\xi^a = (\frac{\partial}{\partial \varphi})^a$ at every point. Because γ is contained entirely within V', we know that $g'_{ab}\xi^a\xi^b > 0$. Thus, γ is a closed timelike curve and we have a contradiction. So, we now know that for any distinct points $u, v \in K$, if $u \in J^-(v)$ then $v \notin J^-(u)$.

Now, let q be a point in K. Without any loss of generality, we may assume that $q \in K$ is the origin point $(0, 0)$. Consider the spacelike surface Σ in (M', g'_{ab}) which is defined as the set $\{(t, \varphi) \in N' : -2\pi \leq t \leq 0 \ \& \ \varphi = -t\}$. Note that $q \in \Sigma$.

Now, we show that $D(\Sigma) \subseteq N'$. Let r be any point in $D(\Sigma)$. We show that r must also be in N'. It is easy to see that if $r \in D^-(\Sigma)$ then $r \in N'$. We turn to the other case: $r \in D^+(\Sigma)$. Assume $r \notin N'$. We show a contradiction. If $r \in D^+(\Sigma)$, then every past inextendible timelike curve through r must intersect Σ.[8] Since $r \notin N'$, every past inextendible timelike curve through r must intersect some $s \in K$. So, we know that $s \in I^-(r)$ and $s \in D^+(\Sigma)$. Now, let $\lambda : I \to K$ be the past inextendible null geodesic from s with tangent $(\frac{\partial}{\partial \varphi})^a$.[9] Note that the image of γ is contained entirely within K. (It can't enter the $t > 0$ region of K' for then γ must become spacelike.

[7] See Wald (1984, 425).

[8] A past inextendible timelike or null curve has no past endpoint.

[9] For details concerning geodesics, see Wald (1984, 41-47).

Similarly, γ cannot enter the $t < 0$ region of K' for then it must become timelike. So, it must remain in the t=0 region, which by definition, is just K.) On pain of contradiction, λ must intersect Σ. Since, λ is contained within K, this means that $q \in J^-(s)$. Because $s \in I^-(r)$, this means that $q \in I^-(r)$.[10] Since $I^-(r)$ is open, we can find a point $q' \in K \cap J^-(q)$ in the neighborhood of q (distinct from q) such that $q' \in I^-(r)$. Clearly, we then can find a past-directed timelike curve from r to q' which fails to intersect q and hence Σ (no past-directed timelike curve may enter and then leave the $t < 0$ region of M'). So, this means that $q' \in D^+(\Sigma)$. Now, let $\lambda' : I' \to K$ be the past inextendible null geodesic from q' with tangent $(\frac{\partial}{\partial\varphi})^a$. On pain of contradiction, λ' must intersect Σ. Since λ' is contained entirely within K, this means that $q \in J^-(q')$. But, we have shown above that for any distinct points $u, v \in K$, if $u \in J^-(v)$ then $v \notin J^-(u)$. So, because q, q' are distinct points in K and $q' \in J^-(q)$, we know that $q \notin J^-(q')$. However, this contradicts the fact that $q \in J^-(q')$. So, $D(\Sigma) \subseteq N'$.

Because $D(\Sigma) \subseteq N'$ and N' may be isometrically embedded, via the identity map, into Misner spacetime (M, g_{ab}) we know there exists an isometric embedding $\theta : D(\Sigma) \to M$. It is easily verified that $D(\theta(\Sigma)) = N$. We have already shown that K cannot be a closed null curve. So clearly, N' contains no closed null curves. Since $D(\Sigma) \subseteq N'$, there can be no closed null curves in $D(\Sigma)$, and hence none in $\theta(D(\Sigma))$. But there is a closed null curve in N. So $\theta(D(\Sigma)) \neq N$. So $D(\theta(\Sigma)) \neq \theta(D(\Sigma))$. This implies that (M', g'_{ab}) is not hole-free and we are done. □

5 Conclusion

So, we have shown one sense in which there exist "time machines" within general relativity. We conclude with a few remarks about other ways one might interpret the result presented here.

Following Earman, Smeenk, and Wüthrich, we have assumed that spacetime is hole-free and then shown that certain initial conditions "force" the production of closed timelike curves. But instead, we may have taken for granted that spacetime is free of closed timelike curves. In fact, this is routinely done (e.g. the singularity theorems of Hawking and Penrose (1970) proceed under this assumption). But, then the logical structure of our result can be reworked to show that certain initial conditions "force" the production of "holes" in spacetime. So, in this way, the theorem demonstrates the existence of "hole machines" rather than "time machines".

[10] See Hawking and Ellis (1973, 183).

We prefer to think of the theorem as a type of no-go result. It seems that some initial conditions force us to give up either (i) our intuition that spacetime is inextendible, (ii) our intuition that spacetime is hole-free, or (iii) our intuition that spacetime is free of closed timelike curves.

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