The "Structure" of Physics: A Case Study*

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We are used to talking about the "structure" posited by a given theory of physics. We say that relativity is a theory *about* spacetime structure. Special relativity posits one spacetime structure; different models of general relativity posit different spacetime structures. We also talk of the "existence" of these structures. Special relativity says that the world's spacetime structure is Minkowskian: it posits that this spacetime structure *exists*.

Understanding structure in this sense seems important for understanding what physics is telling us about the world. But it is not immediately obvious just what this structure is, or what we mean by the existence of one structure, rather than another.

The idea of *mathematical* structure is relatively straightforward. There is geometric structure, topological structure, algebraic structure, and so forth. Mathematical structure tells us how abstract mathematical objects fit together to form different types of mathematical spaces. Insofar as we understand mathematical objects, we can understand mathematical structure. Of course, what to say about the nature of mathematical objects is not easy. But there seems to be no further problem for understanding mathematical structure.

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Modern theories of physics are formulated in terms of these mathematical structures. In order to understand "structure" as used in physics, then, it seems we must simply look at the structure of the mathematics that is used to state the physics.

But it is not that simple. Physics is supposed to be telling us about the nature of the *world*. If our physical theories are formulated in mathematical language, using mathematical objects, then this mathematics is somehow telling us about the physical make-up of the world. What is the relation between these abstract mathematical objects and the physical objects of our experience? What is the relation between the structure of a mathematical space in which we formulate a theory, and the structure of physical space according to that theory? When we infer the existence of a particular kind of structure—the spacetime structure of relativity, some other structure posited by a different theory—are we saying something about the world that theory describes, over and above the mathematics needed to formulate the theory?

These questions raise lots of big and interesting philosophical issues—about the existence of mathematical objects, the use of models in science, underdetermination theses, and more. Though relevant, these are not the questions I wish to focus on here.

I want to home in on a different issue. One of the puzzling things about the "structure" of physics is that there can be different mathematical formulations of a given physical theory—and not just gerrymandered ones with dangling bits of mathematics tacked on. Examples of this are ready at hand: Heisenberg's and Schrödinger's formulations of quantum mechanics, Lagrangian and Hamiltonian versions of classical mechanics.

Different mathematical formulations mean different mathematical structures. If there is more than one such formulation, then what can we infer from the theory about the structure of the world? Should we say that different mathematical formulations posit different structures, or are they simply different descriptions of the same underlying structure? When is one theory a mere notational variant of another, and when does it count as a distinct theory, with its own account of what the world's structure is like?

The question of what to infer about the structure of the world from the structure of the mathematics used to state the physics is already difficult.

Cases of equivalent mathematical formulations make it all the more so. I want to consider these questions with a specific example in mind. There are many to choose from. I wish to minimize technicality and limit the discussion to classical mechanics, and in particular, the Lagrangian and Hamiltonian formulations. I will suggest some general conclusions, but ultimately this really will be a case study. It remains to work through other cases to see if similar considerations apply.

1. What is *structure*?

When, and why, do we infer the existence of one structure rather than another? When, and why, do we make the jump from the structure of a mathematical space in which we formulate a theory, to the structure of the physical world according to that theory?

These questions require that we get to the bottom of what this "structure" talk is all about. What is this thing we call 'structure', such that two theories can posit different ones?

Start with the structure of a mathematical space; we will work up from there to the structure of physical space. A mathematical space can be endowed with a certain structure, say, a geometric structure. (We can think of a mathematical space as, roughly, a set of points with certain mathematical objects defined on it; these objects give the structure of the space.) We describe such a space in mathematical terms, ascribing to it various mathematical properties. But there is an important difference in the kinds of mathematical features we ascribe to such a space. There is a difference between the intrinsic features of the space, and the features it has merely because of the way we choose to describe it.[1]

Take a Euclidean plane. In order to describe the different locations in the plane, we lay down a coordinate system on it. This allows us to associate numerical values with the points in the plane. There are many ways we could do this. We might use a rectangular coordinate system.

[1]For further discussion on the difference between the coordinate-dependent features of a space, on the one hand, and the intrinsic structure of the space, on the other, see Tim Maudlin, *Quantum Non-Locality and Relativity* (Massachussetts: Blackwell, 2002, 2nd ed.), chapter 2, and "Relativity," in *Encyclopedia of Philosophy*, Donald M. Borchert, ed. (Detroit: Thompson Gale, 2006, 2nd ed.), pp. 345–357.

We might use a polar coordinate system. We could use some other, non-rectilinear coordinate system. So long as the coordinates meet certain general constraints (for example, they must vary continuously[2]), we can use any set of coordinates we like to describe the locations in the plane.

The points will get different numerical descriptions depending on which coordinate system we choose. Take a particular point in the plane. Choose a Cartesian coordinate system, and the point will get the coordinates (x, y). Rotate coordinate axes through an angle θ about the origin, and the point will have different coordinates (x', y'). (See Figure 1.) But the *point itself* does not change. Only its *description* changes.

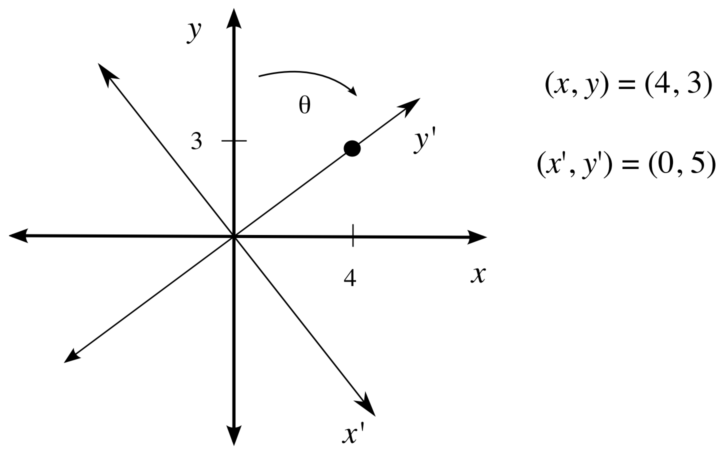

Figure 1: points

Take a slightly more complicated object, a vector. Think of a vector as an arrow, a quantity defined by both a magnitude and a direction. Choose a coordinate system, and we can describe the vector in terms of its components. A component is the part of the vector along one of the coordinate axes; the vector is equal to the sum of its components. In the x-y coordinate system we started out with above, a vector $\mathbf{v}$ has

[2]More precisely, the coordinate system within each chart covering the manifold must satisfy the continuity requirement. We could use weirder, non-numerical labels, though that will not help us much when it comes to the physics, where we need differentiable functions on these spaces. I will assume that we are interested in the above sorts of coordinates.

components v_x and v_y. Rotate coordinate axes, and the vector now has components v'_x and v'_y. (See Figure 2.) The components are different in the new coordinate system. But the vector itself, the arrow with this length and direction, does not change. Only its description changes.

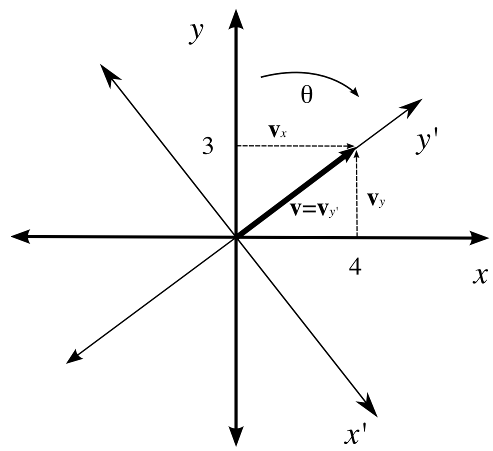

Figure 2: vectors

Objects like points and vectors are coordinate-independent, geometric objects: they exist independently of any coordinate system. Change from one coordinate system to another, and their coordinate-dependent descriptions will change; the objects themselves do not. We call these *invariant* objects, since they are invariant under, or unaffected by, changes in coordinates.

This difference between coordinate-independent objects, on the one hand, and their coordinate-dependent descriptions, on the other, is starting to bring out the idea of *structure*. Intuitively, there is a structure to these objects, some intrinsic nature that is unaffected by coordinate changes.

Apply this to the Euclidean plane. We said that there are different coordinates we can use for the plane. The reason is that, like points and vectors, the plane itself has a structure that is independent of coordinates. How do we know this? Because no matter which coordinate system we choose, some of the plane's features always remain the same. No matter what the coordinate system, the distance between any two points in the plane is unchanged. This distance measure is a coordinate-independent feature of the space. It is part of its underlying, geometric structure.

The geometric structure of a mathematical space is given by the geometric objects defined on it. Since geometric objects are invariant under coordinate changes, so too is geometric structure. Geometric structure is given by quantities that remain intact while we alter what are merely arbitrary choices of description. This is what we have in the backs of our minds when we say that we are free to choose different coordinate systems for the plane. We mean that choosing different coordinate systems does not alter the underlying structure. It only alters our *description* of that structure.

Let me briefly say something about the Euclidean distance measure, since you might be wondering about its status as a coordinate-independent object. The familiar Pythagorean theorem, $d = \sqrt{\Delta x^2 + \Delta y^2}$, is not wholly independent of coordinates. Try using non-rectangular coordinates, and this formula will not calculate the distances correctly. That is right, but there is an invariant distance on the plane. This is not given by an algebraic expression like the Pythagorean theorem. It is given by a geometric object called a *tensor*. A tensor is an abstract geometric object that is invariant under coordinate changes, just as points and vectors are. As with points and vectors, tensors get different component descriptions in different coordinate systems. In a rectangular coordinate system, the Pythagorean theorem gives the components of the metric tensor. In a different kind of coordinate system, it does not. But the Euclidean distance between any two points, this number (in a given unit), or scalar, is coordinate-independent. The metric tensor defined on the plane is an invariant object that lets us calculate distances using any allowable coordinate system. Think of it as a generalization of the Pythagorean theorem, legitimate for any set of curvilinear coordinates we can use on the plane.

In general, for any mathematical space, there will be some quantities that all allowable coordinate systems agree on. (If there are no such quantities, then we have a structureless set of points. Such a space would not have enough structure to do physics: it does not even have the topological structure needed to define continuity and differentiability.) These are features that the space will have no matter which coordinate system we use to describe it. These features are part of the intrinsic, coordinate-independent *structure* of the space.

Since quantities that stay put under coordinate changes correspond to intrinsic, structural features of a space, these invariant quantities give us a way of figuring out the structure of a space: choose an allowable coordinate system, change to another one, and see what remains the same when we do this. The fact that coordinate changes preserve Euclidean distances reveals that this metric is part of the plane's underlying structure. Invariant quantities can also indicate a symmetry, or a lack of structure. The fact that coordinate translations and rotations do not disturb the geometry of the plane indicates that this structure is uniform with respect to the different locations and directions in the plane. It indicates that there is no additional structure picking out a preferred point or direction.[3]

Of course, which are the allowable coordinate systems in the first place is determined by that structure. We know which coordinate systems we can use for the plane because we already know what structure it has: the allowable coordinate systems are precisely the ones that preserve the Euclidean distances. What we are seeing is that we also have a way of figuring out the structure of less familiar spaces. Given the allowable coordinate transformations, we can infer the structure from the invariant quantities under those transformations.

Here is where we are so far. The geometric structure of a mathematical space is given by quantities that remain intact under changes in coordinates. There is a difference, then, between the features ascribed to a space by the coordinate system being used, and the intrinsic features of the space itself. There is a difference between (genuine) *structure* and (mere) *description* of that structure.

That difference is important in physics, too. In Newtonian mechanics, any reference frame or coordinate system moving with a constant (non-accelerating) velocity is one in which the laws hold. As far as this

[3]I have been talking in terms of passive transformations, or coordinate changes. Active transformations leave the coordinates alone, acting directly on the coordinate-independent objects to yield changes in their coordinate-dependent descriptions. Both kinds of transformation capture the idea that there *is* some underlying geometric structure of which we can have different, equally good descriptions. Thinking in terms of passive transformations is more intuitive for our purposes, but either will do.

theory is concerned, any one of these frames is as good as any other for describing a given system. Choosing one frame rather than another is just an arbitrary choice in description. That is why a frame-dependent quantity like velocity is not a fundamental, objective feature of a Newtonian world. There is no "absolute velocity" on this theory. An object's velocity depends on the choice of frame, any one of which is equally legitimate.

In special relativity, the laws hold in any Lorentz frame. Since the time elapse between events depends on the choice of frame—simultaneity is "relative" to a frame—temporal distances are not in the fundamental structure of the world, according to this theory. They are "merely" frame-dependent quantities.

In physics, the frame-dependent quantities, the quantities whose values depend on the particular choice of frame, are taken to be non-fundamental; they are like the coordinate-dependent features of the plane. Frame-*in*dependent quantities, on the other hand, the quantities which are the same in all allowable reference frames, *do* correspond to fundamental, objective features of the world. In special relativity, all Lorentz frames agree on the spacetime interval between two events, just as in the Euclidean plane, all coordinate systems agree on the spatial distance between two points. The spacetime interval *is* a fundamental, objective feature of the world, according to the theory of special relativity. It gives the geometry of its spacetime, in the same way that the Euclidean metric gives the geometry of the plane.

That is why modern theories of physics are typically formulated in terms of abstract, invariant, geometric objects, rather than numerical, coordinate-dependent, algebraic ones. Objective features of the world are best represented by quantities that do not depend on our arbitrary choices of description—on choices the *theory itself* takes to be arbitrary. Since the physics tells us that many different reference frames or coordinate systems are equally legitimate, only things that hold in all of them can be genuine features of the world, apart from our changeable descriptions of it. And abstract geometric objects are the sorts of mathematical objects that are invariant under such changes in description.

Above, we saw that invariant quantities can indicate the structure of a mathematical space. In physics, too, invariant quantities can indicate underlying structure. They can tell us about the structure of the world, according to a physical theory.

An example: the invariance of the laws under time translations suggests that *time itself* is similarly symmetric—that there is no preferred point in time—just as the space-translation invariance of coordinates indicates that there is no preferred point in Euclidean space. Choosing one temporal origin rather than another is just a conventional choice in description, not an underlying distinction in reality. Another: the invariance of the laws under Lorentz transformations suggests that choosing one Lorentz frame rather than another is just a conventional choice in description, not an underlying distinction in reality. There is no temporal-distance structure in the world, according to the theory of special relativity.

By contrast, in Newtonian physics, there is a temporal-distance structure to the underlying spacetime. There is a certain preferred-point structure that is lacking from special relativity. Out of all the pairs of spatially separated spacetime points, some are picked out as being *simultaneous*, in any frame. According to Newtonian physics, there *is* a simultaneity structure in the world. The theory is not invariant under transformations failing to preserve this structure.[4]

The inference from invariances in the laws to corresponding symmetries in the world is not conclusive. There could still be a preferred Lorentz frame, say. Yet we tend to infer that there is no more structure to the world than what the fundamental laws indicate there is. Physics adheres to the methodological principle that the symmetries in the laws match the symmetries in the structure of the world. This is a principle informed by Ockham's razor; though it is not just that, other things being equal, it is best to go with the ontologically minimal theory. It is not that, other things being equal, we should go with the fewest entities, but that we should go with the *least structure*. We should not posit structure beyond that which is indicated by the fundamental dynamical laws. As a

[4]The laws would not make sense without it. Take Newton's first law: every body continues in a state of rest or uniform motion unless acted on by a net force. This law requires that there be a real distinction between the trajectories of particles at rest or in uniform motion, and the trajectories of accelerating particles. If there were no such distinction in the world (independent of our descriptions of it), then there would be no fact of the matter as to whether a given body is accelerating or in uniform motion. See Maudlin, *Quantum Non-Locality and Relativity*, p. 38.

methodological rule, this is what physics has generally done, and successfully so. This, in turn, suggests that the symmetries in the laws give us genuine *insight* into the world's underlying structure.[5]

This gives us a way of figuring out the structure of the world, according to a theory of physics. The procedure is analogous to the one we can use for a mathematical space whose structure is not already familiar. Now we look for invariances in the physical laws:

1. Take the laws of the theory. Take a system governed by these laws.

2. Transform one allowable coordinate- or frame-dependent description of the system into another.

3. See what quantities must remain the same under this transformation, in order for the laws to continue to hold. Repeat for any system governed by the theory and any type of allowed coordinate transformation.

4. These quantities give the mathematical structure needed to formulate the theory in an invariant, coordinate-independent way.

5. These quantities then indicate the structure of the world, according to the theory.[6]

In special relativity, the laws say that the speed of light is the same in all inertial reference frames. In order for this to be so, the transformations between different such frames cannot be the classical Newtonian (Galilean) ones. They must instead be the Lorentz transformations. The invariant quantity under the Lorentz transformations—the mappings which take a system's description in one allowable frame to its description in another—is the spacetime interval. This spacetime interval then

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[5]Note the two kinds of symmetry in play here: symmetries in the laws, given by mappings of solutions to the theory onto solutions, and symmetries in the spacetime (or other) structure of the world, given by mappings of the spacetime (or other structure) onto itself. The idea is that the former give us reason to infer the latter. See John Earman, *World Enough and Space-Time* (Cambridge: MIT, 1989), p. 46, who considers this a condition of adequacy on dynamical theories.

[6]I am taking for granted that there is some way in which this mathematical structure represents the world. How this works is of course a large issue, which I leave aside here.

picks out a Minkowski geometry for the underlying spacetime structure. This is the mathematical structure needed to formulate the theory in an invariant, frame-independent way. Hence this is a part of the spacetime structure of the world, according to the theory.

Note that we can also compare different degrees, or *amounts*, of structure. Compare a Euclidean plane with a similar plane that has a preferred spatial direction. The Euclidean plane without a preferred direction has less structure than the one with a preferred spatial direction. Picking out a preferred direction requires additional structure (an orientation).

In building up a mathematical space, some objects will presuppose others, in that some of the mathematical objects cannot be defined without assuming others. Starting from a structureless set of points, we can add on different "levels" of structure. A bare set of points has less structure than a topological space, a set of points together with a topology (specifying the open subsets). A topological space has less structure than a metric space: in order to define a metric, the space must already have a topology.[7] (Intuitively, a metric gives distances along curves by adding up the lengths of segments between nearby points; and without a topology, there is no sense of the "nearness," or neighborhoods, of points.) And so on.

We can compare the structures of different physical spaces in the same way. The spacetime structure of Newtonian mechanics contains a spatial metric, a temporal metric, and structure identifying spatial locations across times; whereas the spacetime structure of special relativity is fully specifiable by means of one spacetime metric. The latter spacetime has *less* structure. It lacks the structure to identify spatial locations at different times, for instance. That would require *additional* structure.

We can now say some general—admittedly imprecise—things about "structure" in physics. Structure has to do with the invariant features of a space, whether an abstract mathematical space or the physical space(time) of the world. Structure comprises the objective, fundamental, intrinsic features, the ones that remain the same regardless of arbitrary or con-

[7]For any metric (or pseudo-metric) space, we can define a topology by using the open balls as a sub-basis. The open balls $B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$ form a basis for the neighborhoods of x, where d is the metric. We say that this is the topology *induced by* the metric. See Chris J. Isham, *Modern Differential Geometry for Physicists* (Singapore: World Scientific Lecture Notes in Physics, 2003, 2nd ed., volume LXI), p. 33.

ventional choices in description. And we can learn about structure by looking at the invariant quantities under allowable transformations. I suggest that it is part of the structure of the world, according to a theory of physics, that it contain at least the amount of stuff needed to state the theory in an objective, coordinate-independent way—in a way best suited to representing the nature of the world, apart from our descriptions of it.

Structural features of the world are (somehow[8]) represented by the mathematical features of the space in which its physics is formulated. But not just any old mathematical features. In relativity, it is the invariant, geometric features, not the frame-dependent, numerical ones, which get at genuine spacetime structure. Since not just any mathematical feature indicates underlying structure, we might already be witnessing a glimmer of good things to come. If only certain kinds of mathematical object are appropriate for representing the structure of the world, then this may help us with different mathematical formulations of the physics.

For now, rest content with the following observation. It seems pretty clear why modern physics is so interested in structure. Structure consists in just the kind of things we take to be candidates for objective features of the world, for features of reality. Reality is observer-independent. It does not depend on our arbitrary descriptions or conventions. It is independent of choice of reference frame. Reality has to do with *structure*.

2. The Case: Lagrange v. Hamilton

We are not out of the woods yet. If structure is supposed to tell us about the objective features of the world according to a theory of physics, then what do we say when there are different structures we can use for the physics?

Let us turn to our concrete case. There are two formulations of classical mechanics that are generally assumed to be equivalent: the Lagrangian and Hamiltonian formulations. Each formulation has its own sets of coordinates and its own equations of motion, given in terms of those coordinates. But the Lagrangian equations of motion can be recovered from the Hamiltonian ones, and vice versa. The two formulations

[8]See note 6.

give rise to the same set of physically possible histories for any classical system. The consensus view is that these are notational variants, different mathematical presentations of one and the same theory.

Bear with me for a moment while we run through some details that will be important later. First, some features of the Newtonian formulation of classical mechanics that carry over analogously to these other formulations. In Newtonian mechanics, the total state of a system at a time is given by the position and momentum of each of its particles (in addition to the particles' intrinsic features, like mass and charge).[9] For a system moving around in three-dimensional space, the position and momentum coordinates will each have three components, one along each of the three spatial dimensions.

The *statespace* of a system is a mathematical space in which we represent all its possible fundamental states. Think of this as the fundamental possibility space of a theory. In Newtonian mechanics, a system with n particles will have a statespace of $6n$ dimensions: one dimension for each of the three position and momentum coordinates for each particle in the system.[10] Each point in this high-dimensional space represents a total possible state of the entire system. Each point represents the state of many, many objects—all the particles that make up the system. Keep in mind that a theory's statespace does not in general directly correspond to the physical space of the world. Statespaces tend to look quite different from that.

[9] I leave aside the question of whether velocity is properly included in the instantaneous state of a system. I tend to agree with arguments that it is not. (See David Albert *Time and Chance* (Cambridge, MA: Harvard, 2000), chapter 1; Frank Arntzenius, "Are There Really Instantaneous Velocities?," *The Monist*, LXXXIII, 2 (2000): 187-208.) If that view is right, then just replace the idea of a system's state at a time with what Albert calls a "dynamical condition," the state within any arbitrarily small time interval. What we need is the information required to plug into the dynamical laws to make predictions.

[10] If the system were one particle, we could represent its state by one point in a 6-dimensional mathematical space; this 6-dimensional space represents all the possible fundamental states, all possible position-momentum pairs, for this system. Two particles, and we need 12 numbers to completely specify the state—six numbers for each particle—and so a 12-dimensional statespace. Three particles, 18 numbers, and an 18-dimensional statespace. In general, the statespace of a classical system has dimension $2nr$, where n is the number of particles and r is the number of degrees of freedom, here assumed to be the three spatial dimensions of ordinary space.

A trajectory through the statespace, a curve through the statespace parameterized by time, represents a possible *history* of the system, a possible sequence of its fundamental states over time. The dynamical laws say which of these curves are possible histories. In Newtonian mechanics, $\mathbf{F} = m\mathbf{a}$, or $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, governs this evolution. There is one such equation for each particle, in each component direction. (It is a vector equation.[11]) We can group these equations into one big version of Newton's law, which says how the point representing the state of the entire system moves through the $6n$-dimensional statespace over time. Given the initial positions and momenta of the particles and the total forces acting on the system (given by a vector function on the statespace), we can (twice) integrate Newton's equation to get a unique solution.[12] A solution is the history of the system, for this initial state and subject to these forces.

Lagrangian mechanics is a bit different. It also specifies a system's state by using two sets of coordinates. It is important, though, that the Lagrangian formulation uses what are called *generalized coordinates*, or generalized positions, labeled q_i, and their first time derivatives, the *generalized velocities*, $\dot{q}_i$ (i ranging from 1 to n). Generalized coordinates are any set of independent parameters that together completely specify the state of a system. Whereas in Newtonian mechanics we use ordinary position and momentum coordinates[13], here we can use any set of generalized coordinates and their corresponding time derivatives; and there are (infinitely) many we could use. In particular, the generalized coordinates need not be ordinary position coordinates. For a pendulum, we can use

[11] Newton's law is $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) = m\frac{d^2\mathbf{x}}{dt^2}$, a solution to which is a vector function $\mathbf{x}(t)$, for initial values $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ at initial time t_0, which allows us to find also $\dot{\mathbf{x}}(t)$. There is a transformation rule for vectors, which gives their components in one frame from those in another. In order for a vector equation to be frame-independent (in Newtonian mechanics, it is Galilean-invariant), the components of all vectors must transform according to the same equation. In other words, the law must hold component-wise.

[12] I set aside the intruiging cases of indeterminism. See Earman, *A Primer on Determinism* (Boston: Reidel, 1986); John Norton, "The Dome: An Unexpectedly Simple Failure of Determinism", *Philosophy of Science (Proceedings)*, forthcoming; David Malament, "Norton's Slippery Slope", *Philosophy of Science (Proceedings)* forthcoming.

[13] The laws still hold in other coordinate systems, but their form will change if we use, say, polar rather than rectangular coordinates. Not so the Lagrangian equations.

an angle as the generalized "position"; the generalized "velocity" is the first time derivative of that angle. (See Figure 3.) It is an amazing feature

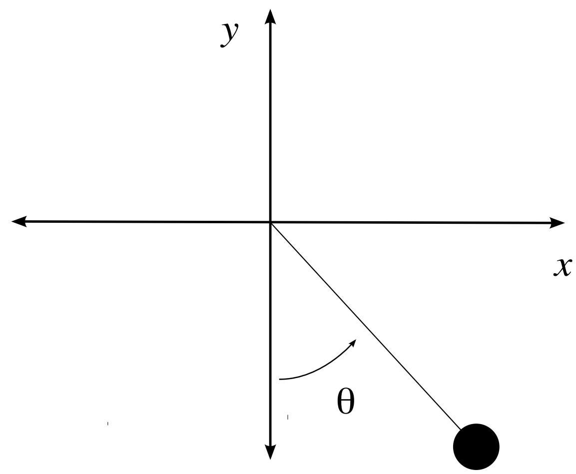

Figure 3: Pendulum with generalized coordinates $q = \theta$ and $\dot{q} = \dot{\theta}$

of the theory that we can treat this angle as if it were a rectangular coordinate of the kind we use to solve Newton's equations, and everything goes through just as if we were using ordinary rectangular coordinates.

For a Lagrangian system of n particles moving in three-dimensional space, we again need $6n$ coordinates to completely specify the state of a system; in this case, three generalized coordinates and three generalized velocities for each particle. Each possible state is represented by one point in a $6n$-dimensional statespace called *configuration space*.

This label is a bit confusing. 'Configuration space' is often used for the space of ordinary positions or configurations, that is, a $3n$-dimensional manifold, where each point corresponds to a possible set of particle positions in ordinary three-dimensional space. Each point in the Lagrangian configuration space, on the other hand, picks out both a generalized position and the corresponding generalized velocity for each particle in the system. This space comprises a set of possible configurations (which, remember, need not be given by ordinary position coordinates), plus a space in which we represent their first time derivatives. This is really the *tangent bundle* of configuration space: the $3n$-dimensional (generalized-coordinate) configuration space together with the $3n$-dimensional tangent space at each point. We need the tangent spaces in order to define the generalized velocities, which are tangent vectors. Unlike ordinary configuration space, the Lagrangian statespace is a $6n$-dimensional manifold, each point of which picks out a generalized coordinate-generalized velocity pair for all the particles in the system.

The Lagrangian equations of motion determine how the point representing the state of the system moves through the statespace over time. Whereas Newtonian mechanics requires the forces acting on a system in order to determine its behavior, here we need a scalar function, called the *Lagrangian*, L. At each point in the statespace, this function assigns a number, (typically[14]) equal to the kinetic minus potential energy of the system. Given the initial state and Lagrangian of a system, the equations determine a unique history.

Hamiltonian mechanics also uses generalized coordinates, but they

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[14]I ignore time-dependent Lagrangians and Hamiltonians and non-conservative forces, though the theories can accommodate these too. There are many sources for the physics. Some I found particularly useful: Ralph Abraham and Jerrold E. Marsden, *Foundations of Mechanics* (Reading, MA: Benjamin/Cummings, 1980, 2nd ed.); V. I. Arnol'd, *Mathematical Methods of Classical Mechanics* (New York: Springer, 1989, 2nd ed.); Luca Bombelli, chapter II.0 of Abhay Ashtekar, ed. *New Perspectives in Canonical Gravity* (Italy: Bibliopolis, 1988); Dusa McDuff, "Symplectic Structures–A New Approach to Geometry," *Notices of the American Mathematical Society*, XXXXV (1998): 952–960; Dusa McDuff and Dietmar Salamon, *Introduction to Symplectic Topology* (New York: Oxford, 1998, 2nd ed.); Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics* (Reading, MA: Pearson Education, 2004, 3rd ed.); Victor Guillemin and Shlomo Sternberg, *Symplectic Techniques in Physics* (New York: Cambridge, 1984); Isham, *Modern Differential Geometry for Physicists*; Jorge V. José and Eugene J. Saletan, *Classical Dynamics: A Contemporary Approach* (New York: Cambridge, 1998); Cornelius Lanczos, *The Variational Principles of Mechanics* (New York: Dover, 1970, 4th ed.); Jerrold E. Marsden and Tudor S. Ratiu, *Introduction to Mechanics and Symmetry* (New York: Springer, 1999, 2nd ed.); Roger Penrose, *The Road to Reality* (New York: Knopf, 2005), chapter 20; Bernard F. Schutz, *Geometrical Methods of Mathematical Physics* (New York: Cambridge, 1980); Ana Cannas da Silva, *Lectures on Symplectic Geometry* (New York: Springer, 2001); Stephanie Frank Singer, *Symmetry in Mechanics: A Gentle, Modern Introduction* (Boston: Burkhauser, 2001). See also Gordon Belot, "The Representation of Time and Change in Mechanics," in Jeremy Butterfield and John Earman, eds. *Philosophy of Physics*, Part A (Amsterdam: North-Holland, 2007), pp. 133-227; and Jeremy Butterfield, "On Symplectic Reduction in Classical Mechanics, in Butterfield and Earman, eds. *Philosophy of Physics*, Part A, pp. 1-132.

are different from the Lagrangian ones. The Hamiltonian coordinates are the *generalized positions*, labeled q_i, and the *generalized momenta*, p_i. As generalized coordinates, these need not be ordinary position and momentum coordinates. The generalized momentum need not equal the ordinary momentum of mass times velocity, for instance (and it will not, if the generalized position is not a regular Cartesian coordinate). The Hamiltonian coordinates and equations are called *canonical*, and the statespace is called *phase space*. This is (typically[15]) the *cotangent bundle* of configuration space: the (generalized) configuration space together with the cotangent space at each point. We need the cotangent spaces in order to define the generalized momenta, which are covectors, or one-forms. Once again, this is a $6n$-dimensional manifold; here, each point picks out a generalized position-generalized momentum pair for all the particles in the system. Once again, we need a scalar function on the space in order to predict a system's behavior; here, we need the *Hamiltonian*, H, (typically: note 14) equal to the total energy of the system. For a given initial state, these equations yield a unique history for a system with that Hamiltonian.

The Lagrangian and Hamiltonian formulations are both coordinate-independent versions of classical mechanics. Both are given entirely in terms of generalized coordinates; their equations of motion retain their form regardless of which set of such coordinates we use. (More, they can be formulated without mentioning coordinates at all: Appendix A.) The reason these theories can do this is that the Lagrangian and Hamiltonian functions, which determine the motion of a system, are scalar functions. By contrast, in Newtonian mechanics, the forces determine the motion, and forces are vector quantities. Vectors, we have seen, are coordinate-independent objects. But their components change with the coordinate system, and to solve a problem using Newton's law, we need to know the component forces in the chosen coordinates.[16] The Lagrangian and Hamiltonian functions manage to store this dynamical information in

[15]Hamiltonian phase spaces need not have a global vector bundle structure, though locally they are guaranteed to look like symplectic vector bundles. That their global structure is not similarly constrained is one reason for the conclusion of this paper. See Appendix B for some of the gory details.

[16]See note 11.

one scalar energy term.

This suggests that the Lagrangian and Hamiltonian formulations are both more objective, description-independent versions of classical mechanics than the Newtonian formulation. In describing a classical system's behavior, these theories allow us to use a wide range of coordinates, even ones that are not ordinary positions and momenta. These theories demonstrate that there is a wider range of variables we can use to describe classical systems than the Newtonian formulation might suggest.

There are differences between the two. Lagrangian and Hamiltonian mechanics use different sets of generalized coordinates to describe systems' states. And not just different coordinates: the generalized coordinates and velocities of the Lagrangian formulation are not related by a canonical transformation—the transformation that takes one allowable set of Hamiltonian coordinates to another—to the canonical position and momentum coordinates.[17] The difference in coordinates means a difference in statespace structure: the structure of the tangent bundle versus the structure of the cotangent bundle. (Though they are both usually fiber bundles, and at that level share an abstract structure.[18])

The equations of motion have notable differences as well. For a classical system with n degrees of freedom (such as the three spatial dimensions we are assuming here), the Lagrangian formulation gives a set of n second-order differential equations, one for each particle in each coordinate direction. The Hamiltonian gives $2n$ first-order equations, two for each particle in each coordinate direction.[19] In each case, though, the equations uniquely determine the motion, given an initial state specified by $2n$ values: for the Lagrangian, the initial n q_i's and n $\dot{q}_i$'s, and for the Hamiltonian, the initial n q_i's and n p_i's.

Despite these differences, it is universally agreed that either formu-

[17]This requires a Legendre transformation, which transforms functions on a vector space (like the tangent bundle) to functions on the dual space (the cotangent bundle).

[18]See Appendix B for precisification.

[19]The Lagrangian equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$, the Hamiltonian $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $-\dot{p}_i = \frac{\partial H}{\partial q_i}$ (i from 0 to n). See Appendix A for coordinate-free versions. Stating the difference between the two in terms of the order of the equations is a bit misleading, since the Lagrangian equations can be seen as first order equations defined on all of TQ: see José and Saletan *Classical Dynamics: A Contemporary Approach*, pp. 93-97.

lation suffices for doing classical mechanics. They are each empirically adequate. Each is completely kosher as far as classical mechanics is concerned. Physics books tell us to use whichever set of equations will make a given problem easiest to solve.

It seems we have here a distinction without a difference: a difference in mathematical formulation, not any distinction in reality according to the two theories. The two formulations simply use different variables to describe systems' fundamental states. That is what leads to their differing statespaces and differing equations of motion. We have but one theory, and two different ways of formulating it mathematically, in terms of different, but equally legitimate, coordinates. Just like using polar as opposed to Cartesian coordinates to describe the Euclidean plane.

Mere notational variants indeed.

3. Irreconcilable differences

But not so fast.

Lagrangian and Hamiltonian mechanics may be equivalent for the purpose of doing classical mechanics. Nonetheless, there are important differences between them. There are differences in *structure*.[20]

Look more closely at the structure of their statespaces. The statespace of Lagrangian mechanics—configuration space, the tangent bundle—has metric structure. Lagrangian mechanics assigns a definite, natural geometric structure to configuration space. In particular, the statespace gets naturally associated with a metric structure.[21]

How do we know it has this structure? The usual way: look at what

[20]There are also differences that crop up in the extension to other domains of physics. The Lagrangian formulation is generally taken to be more natural for generalizing to field theories and relativistic spacetimes, the Hamiltonian for statistical mechanics and quantum mechanics (though not quantum field theory). Though standard, none of this is uncontroversial. For one source of disagreement, see Penrose, *The Road to Reality*, chapter 20. I deliberately limit the discussion here to classical mechanics; things might change when we carry these considerations elsewhere.

[21]Often: not so in the completely general case, though even there, I maintain the basic conclusion about the theories' comparative structures. For ease of exposition, I continue to assume the above statespace structure, relegating technical details and precisifications to the appendices.

is invariant under allowable coordinate transformations. The invariant quantities of the Lagrangian equations determine the geometric structure of the statespace. They tell us what structure the statespace must have in order to formulate the theory in terms of it.

The Lagrangian equations of motion are invariant under a certain set of (point[22]) transformations. These transformations give the allowable sets of coordinates, the ones that preserve the equations of motion. They indicate which coordinate changes the theory takes to be mere arbitrary changes in description.

There is a quantity that is invariant under these transformations. The invariant quantity is the square of the Riemannian line element. Transformations from one set of Lagrangian coordinates to another—from one set of coordinates giving rise to the Lagrangian equations to another that gives rise to these equations—preserve this local structure, this local curvature, in the form of a quadratic differential form of the $\dot{q}$'s. This allows us to say that the structure of the Lagrangian statespace is that of a Riemannian manifold, with a Riemannian metric defined on it.[23]

The statespace of Hamiltonian mechanics—phase space, the cotangent bundle[24]—is different. Phase space has symplectic structure; it does not have metric structure. Symplectic structure comes with, or determines, a volume element, but not a distance measure.

How do we know it has this structure? In the same way: look at what is invariant under allowable transformations. There is a quantity, a fundamental differential form, that is invariant under the canonical transformations. But it is not the invariant quantity of the Lagrangian

[22]In a point transformation, each original point is in one-one correspondence with each transformed point, in such a way that the local structure, in the neighborhoods of the corresponding points, is preserved.

[23]A differential manifold with a fixed positive definite quadratic form on every tangent space is a Riemannian manifold, with quadratic form ds^2 the Riemannian metric. See Peter Szekeres, *A Course in Modern Mathematical Physics: Groups, Hilbert Space and Differential Geometry* (New York: Cambridge, 2004), p. 469, who defines a Lagrangian system as an n-dimensional Riemannian manifold (M, g), together with a function $L : TM \to \mathbb{R}$ (the Lagrangian). Note that there is a more general, coordinate-free version of the theory that does not make explicit use of this metric: see Appendix A. I thank David Malament for this point.

[24]But see Appendix B and note 15 for precisification.

transformations. It is the symplectic form, a different sort of geometric object.[25] Transformations from one set of Hamiltonian coordinates to another—from one set of coordinates giving rise to the Hamiltonian equations to another that gives rise to these equations—preserve symplectic structure. So long as a transformation leaves the symplectic structure intact, it can alter any metric structure. Hamiltonian statespaces that are bent or stretched or "floppy" with respect to one another do not count as different statespaces—not unless the symplectic form differs.[26]

Now here is the kicker. The thing is, as far as we can tell, we need only symplectic structure to do classical mechanics. This structure suffices for the theory; it does not "leave anything out."[27] And there is a clear sense in which a space with a metric structure has *more* structure than one with just a volume element. Metric structure comes with, or determines, or

[25]A symplectic form is a closed, nondegenerate, antisymmetric 2-form. Note that, although the symplectic form determines a volume form, a symplectic form is different from—and stronger than—a generic volume form.

[26]The structure of a symplectic manifold is "floppy" in that there is no local notion of curvature that would distinguish one symplectic manifold from another locally. Two real symplectic manifolds (of the same dimension and signature) are locally identical: they can be mapped onto each other so that their symplectic structures correspond. That symplectic manifolds have no local invariants such as curvature is a consequence of Darboux's theorem, which tells us that every pair of symplectic manifolds is locally isomorphic: within the neighborhood of every point, there are local (canonical) coordinates such that the symplectic form takes the canonical form. See Arnol'd *Mathematical Methods of Classical Mechanics*, p. 230; Rolf Berndt *An Introduction to Symplectic Geometry* (Providence: American Mathematical Society, 2001), 2.2; da Silva, *Lectures on Symplectic Geometry*, 8.1. A corresponding result for Riemannian manifolds does not hold unless the manifolds have zero curvature (unless they are flat). Symplectic forms are more flexible than Riemannian metrics, which can be made constant in a local chart iff they are flat. See Abraham and Marsden, *Foundations of Mechanics*, p. 177; Marsden and Ratiu, *Introduction to Mechanics and Symmetry*, p. 148. More on this in Appendix B.

[27]There is an additional topological condition for a symplectic manifold to admit a global Hamiltonian vector field (see Abraham and Marsden, *Foundations of Mechanics*, p. 189); a classical mechanical phase space cannot be compact (momentum can get arbitrarily large): see Singer, *Symmetry in Mechanics*, 3.3. Thus the 2-torus, for example, cannot be a symplectic phase space, even if it happens to be an even-dimensional symplectic manifold. There will also be the Hamiltonian function defined on the phase space. For ease of exposition, I continue to refer to this structure as 'symplectic structure', where this should be understood as including the relevant energy function and any topological conditions.

presupposes, a volume structure, but not the other way around. (In the same way that a metric space comes with, or determines, or presupposes, a topology, and not the other way around: note 7.) Intuitively, knowing the distances between the points in a space will give you the volumes of the regions, but the volumes will not determine the distances.[28] Metric structure adds a further level of structure.[29]

This suggests that the statespace of Lagrangian mechanics has *more* structure than the theory really needs. We do not need this structure for the dynamics of classical particles; therefore, we should infer that it does not exist, that it does not correspond to fundamental structure. The fact that we can do classical mechanics without the natural metric of the Lagrangian formulation suggests that this extra bit of structure is merely an artifact of the mathematics we need to formulate the theory this way. It suggests that Lagrangian mechanics contains excess, *superfluous structure*.

Note that the Lagrangian formulation cannot get by without this metric structure. The square of the Riemannian line element is what gives the allowable sets of coordinate transformations, the ones that preserve the Lagrangian equations of motion. A volume element would not suffice for this, simply because of how the theory is formulated. The relation between the generalized coordinates and their first time derivatives, the two sets of coordinates that Lagrangian mechanics uses to describe systems' fundamental states, requires this metric.[30] The canonical coordinates of the Hamiltonian formulation do not.

(An example helps to picture the canonical coordinates' relationship. The phase space for a single particle constrained to move along a line, like a bead on a straight wire, is a two-dimensional plane. The symplectic form is the area form, which we can think of as giving the oriented area of the parallelogram spanned by any two vectors. Coordinatize the plane by r and p, interpreted as the particle position and momentum, respectively. For any physically possible motion, along any allowable trajectory, these

[28] See Schutz, *Geometrical Methods of Mathematical Physics*, chapter 4. The symplectic form is not even suitable to be a metric, since for any element, the symplectic inner product taken with itself is 0 (from antisymmetry).

[29] I suggest this relationship between the two spaces as a reasonable conjecture; I do not have a proof. See Appendix B for more details.

[30] But see note 21 and Appendix A for precisification.

two coordinates will have a simple kinematic relationship: p will be a constant multiple of the time derivative of r. This relationship is coded up in the area form, the symplectic form for this system's phase space.[31])

Of course, this extra bit of structure can be *useful*. It can help solve problems, or help extend the dynamics to other realms (like field theories). Nonetheless, if our world's fundamental physical theory were the theory of classical particles, we should conclude that the structure of the mathematical space in which to best represent the theory, and of the world according to that theory, has only symplectic structure.

Consider an analogy. Newton thought that he needed a spacetime structure identifying spatial locations across times. But this is more structure than the theory really needs. We know that Newton's theory can be formulated in a completely Galilean-invariant way. So we infer that, had Newton's theory been the fundamental theory of the world, its structure would be more accurately represented by Galilean (or neo-Newtonian) spacetime, even though Newton believed otherwise.

Similarly here. All the necessary structure for the classical kinematics is contained in the structure of a symplectic manifold—and for the dynamics, with the Hamiltonian function defined on it—even though we can formulate an empirically adequate theory by utilizing more structure.[32] Hence we should infer that symplectic structure is the "real" statespace structure of the theory. We should infer that this structure corresponds to the "real" physical structure of a classical mechanical world. We should infer that Hamiltonian mechanics is *more fundamental* than Lagrangian

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[31] The canonical symplectic form is then $\omega = dp \wedge dr$; that is, p and r are canonical coordinates. Another example: the phase space of a spherical point pendulum, or a particle constrained to move on a unit sphere. The set of possible positions are represented by points on the surface of a sphere. Since the momentum vector must be tangent to the sphere, the set of all possible position-momentum pairs gives the structure of this phase space as $T^*Q^2 = \{(\mathbf{r}, \mathbf{p}) \in \mathbb{R}^3 \times (\mathbb{R}^3)^* : |\mathbf{r}| = 1 \text{ and } \mathbf{pr} = 0\}$, where Q is the configuration space and T^*Q is the cotangent bundle. See Singer, *Symmetry in Mechanics*, chapter 2.

[32] Note that the symplectic form is crucial to determining the motion, since the same Hamiltonian can induce different flows for different symplectic forms. That is, we cannot also drop this amount of structure in an effort to find a more minimal one. At what point we can stop removing structure, while still retaining an adequate theory, is a large remaining question.

mechanics.

Another way to motivate the idea. The symplectic form picks out the canonical coordinates by coding up their kinematic relationship; for any allowable motion, the two sets of canonical coordinates must be related in this way. The kinematics is then coded up in the symplectic structure of phase space. The dynamics is given by specifying the Hamiltonian, a function that depends on the particulars of the system in question (the kinds of forces among its particles, any external forces, whether the system is in a potential, etc.). Given the Hamiltonian and initial state of a system, the equations determine a unique history; different Hamiltonians yield different histories. The actual relationship that obtains between the canonical position and momentum coordinates—*which* curve through phase space describes the actual behavior—depends on the initial state and total energy of the system. Finding that relationship amounts to finding a solution, the unique curve describing the behavior of the system. Think of it this way. All the points in phase space represent possible fundamental states, for any system with that number of particles and degrees of freedom; feed in an initial state and Hamiltonian, and the differential equations spit out a unique subset of those points. These points represent the actual development of fundamental states, of position-momentum pairs, for the given system. Or this way: all curves through phase space are functions of q and p; the symplectic form picks out the physically possible such curves; the rest picks out the actual trajectory from among all the physically possible ones.

The canonical coordinates thus vary in a way that is dictated by the physics. Indeed: the relationships between the generalized positions and the generalized momenta coded up in the symplectic form just *are* the equations of motion themselves. The Lagrangian coordinates are not like this. The generalized coordinates are the only truly independent variables; the generalized velocities are simply their first time derivatives. We treat them as independent for the purpose of solving the Lagrange equations, but they are mathematically constrained, by definition of the time derivative, independent of the physical laws.

Again, this suggests that the Hamiltonian formulation is more fundamental. It suggests that *the* fundamental possibility space for any classical system is a symplectic manifold. Why? Because the relationship between the two sets of canonical coordinates, for any physically possible motion, is contained in the very structure of that space. More boldly: it suggests that a symplectic manifold just *is* the coordinate-independent, geometric representation of classical mechanics. (That is, of the classical kinematics, with the classical dynamics given once we have the Hamiltonian.) It suggests that classical dynamics is symplectic geometry.

Notice that, on this picture, momentum is a truly fundamental quantity, on a par with position. In particular, momentum is not defined as the product of mass and the time derivative of position.[33] Similarly, it is the points in canonical *position-momentum* space that are fundamental, not the points in ordinary configuration space. (Confession: this is to deny that we should understand the fundamental structure of phase space in the way that it is usually built up, by taking a configuration space and attaching the cotangent space at each point. I am arguing that the *phase space* structure is fundamental.)

We can now say a bit more about the "structure" of physics. The structure of physics is the (minimal) structure that is assumed among dynamically equivalent—in classical mechanics: canonically transformed—formulations of a theory. Structure comprises those quantities that remain intact when we transform one allowable coordinatization of the laws into another.[34]

In one sense, the orthodox view that the Lagrangian and Hamiltonian formulations are notational variants is correct: the two formulations use different sets of variables to describe the same set of dynamically possible histories for any classical system. But in another, important, sense, they are not *mere* variants: one of them contains *excess structure*. This is a

[33]This might be to endorse a modernized "impetus theory": see Arntzenius, "Are There Really Instantaneous Velocities?" Since the relationship between the two sets of canonical coordinates depends on a system's Hamiltonian, Arntzenius concludes that, on such a view, this relationship must be regarded as a physical law governing how the canonical position and momentum are causally related; that is, not as a definition of momentum as mass times the temporal derivative of position.

[34]One might wish to say that "structure" has to do with the relations among fundamental objects, rather than their intrinsic properties. I do not wish to put it this way, since I am skeptical of the distinction, for reasons argued for by Maudlin in "Suggestions from Physics for Deep Metaphysics", *The Metaphysics Within Physics* (New York: Oxford, 2001): pp. 78-103.

distinction with an all-important difference.

(The defense rests.)

4. Structural realism, or: how I learned to stop worrying, and trust the physics

When, and why, do we infer the existence of a particular structure, if there are different structures we can use to formulate a theory?

Look to the physics. Take the mathematical formulation of a given theory. Figure out what structure is required by that formulation. This will be given by the dynamical laws and their invariant quantities (and perhaps other geometric or topological constraints). Make sure that there is no other formulation getting away with less structure. Infer that this is the fundamental structure of the theory. Go on to infer that this is the fundamental structure of the world, according to the theory.

In classical mechanics, this results in the inference to symplectic structure. This is what remains intact when we change from one allowable coordinate description to another. The theory's alternate formulation tacks on more structure. Infer that the fundamental statespace of classical mechanics is a symplectic manifold. Infer that symplectic structure, the structure that is invariant up to canonical transformations, exists. In a world whose fundamental theory is classical mechanics, infer that it fundamentally contains only this amount of structure.[35]

The Lagrangian and Hamiltonian formulations thus do not have the thoroughgoing equivalence people readily assume: there is a difference in structure.

I think this gives us grounds for concluding that they are not even genuinely empirically equivalent. One of the lessons of modern geometric formulations of physics is the importance of abstract geometric objects. These invariant mathematical objects, not their coordinate-dependent descriptions, correspond to fundamental, objective features of the world.

[35] Plus the things mentioned in note 27 and, perhaps, physical space. I do not address here the relation between statespace structure and spacetime structure. For now, let us just say: posit all the fundamental structure required for our best physics, where this is not yet to say whether statespace structure is something over and above spacetime structure, or whether one of these can be gotten from the other.

And we now know that the structure given by these objects can have empirical import. This is most obvious when it comes to spacetime structure. In a special relativistic world, there may be no experiment to determine whether there is, or is not, an absolute-velocity structure.[36] Still, we think there is an empirical difference between the two: a difference in the geometric structure of the spacetimes.

You may not wish to call such a difference "empirical." Ordinarily, we say that there is an empirical difference between two things when there is a possible (idealized, perhaps) experiment to point out that difference. This is not the case here.[37] Some of this is terminological: if you do not like "empirical", substitute "physical", or some other. Yet not entirely terminological. Part of what is to come suggests that there may not be any clear-cut distinction between what counts as genuinely empirical, and not; and in particular, no clear-cut distinction on the basis of something like "in-principle experimental evidence." No difference in possible experimental evidence here, yet even so, we should say there is a difference between the above two spacetime structures. I say that we might as well call this an empirical difference: a difference in the way the world is, according to the physics. Call it some other type of difference if you prefer.

Modern geometric formulations of physics suggest that there is more to a theory's empirical content than its set of dynamically possible histories. There is also the statespace in which those histories are traced out. And there is the structure of that space. The equivalence of theories is not just a matter of physically possible histories, but of physically possible histories *through a particular statespace structure*. Hamiltonian and Lagrangian mechanics are not equivalent in terms of that structure. This means that they are not equivalent, period.

Of course, the empirical import of *phase space* structure is not immediately apparent (to say the least). Much easier to see how points in a mathematical space are supposed to represent the world in relativity: the world has a spacetime that is directly isomorphic to the mathematical

[36]But see Albert, "Special Relativity as an Open Question", in Heinz-Peter Breuer and Francesco Petruccione, eds. *Relativistic Quantum Measurement and Decoherence* (Berlin: Springer, 2000): 3-13 for suggestion that there may be such experiment.

[37]I thank Adam Elga for this point.

space representing it. Phase space is much harder to picture. The points themselves represent objects that are not easily conjured up—the dynamically possible states of a system (even the entire world). The structure of a fiber bundle is not directly isomorphic to anything familiar from our experience, such as physical space.

Still, I think we should infer that this structure somehow corresponds to the fundamental structure of a world governed by classical mechanics, and for the same reasons we infer a particular spacetime structure from the abstract geometric structure of relativity. We should infer that the world contains at least the amount of structure needed to formulate its fundamental dynamics in an invariant way. And physics has generally been successful with the further inference that the world contains no more fundamental structure than that. Just as we take seriously the structure of physical space from the mathematical structure of relativity, we should do the same for the structure of classical mechanics, even at this more abstract, not-so-easily-envisioned level of structure.[38] Just as we take seriously spacetime and its geometry, we should do the same for statespace and its geometry.

Note, for all that I have said so far, I am not necessarily a *substantivalist* about phase space, though I confess that I am happy with that view.[39] The

[38]Other areas of modern physics also suggest that our world's fundamental structure might not be isomorphic to everyday physical space. Consider the fiber bundle structure of modern formulations of classical field theories (see Maudlin, "Suggestions from Physics for Deep Metapysics"), or the configuration space of quantum mechanics (see Albert, "Elementary Quantum Mechanics", in J. T. Cushing, A. Fine, and S. Goldstein, eds. *Bohmian Mechanics and Quantum Theory: An Appraisal* (Boston: Kluwer, 1996): 277-284).

[39]Why? Two steps. (See David Malament, Review of Hartry Field's *Science Without Numbers: A Defense of Nominalism*, *Journal of Philosophy* 79 (1982): 523-534 for argument along both lines.) The first toward an anti-nominalism. Although we may be able to do science without numbers, as Hartry Field (*Science Without Numbers: A Defense of Nominalism*, New York: Blackwell, 1980) argues, it is hard to see how we can do without mathematical objects altogether. For we need the abstract—"abstract" in the sense of "non-coordinate-dependent"—invariant, geometric ones. Our best physics uses these sorts of mathematical objects to represent the fundamental structure of the world, which makes it difficult to see how we could do without them; say, by using the representation theorems Field employs to show that we can dispense with numbers. Classical mechanics is a case in point: the points of phase space themselves represent abstract objects,

claim is just this: that phase space is as much a part of the representational content of classical mechanics as the theory's spacetime is.

Why take the statespace so seriously? Rather than saying that phase space somehow corresponds to a part of reality, why not regard it as representing the world more indirectly? Perhaps it merely tells us something about the relations among fundamental properties in our world. After all, we do not infer "structures" on the basis of any old relations among physical properties, even if we can model them in a mathematical space with a certain kind of structure. Down that road lies temperature structure, color structure, altitude structure, and on and on and on.[40]

A few reasons, all of them contentious. One: I am skeptical that there are any such clear-cut distinctions between fundamental properties and fundamental relations; worse, between substances and properties.[41] Two: unlike temperature, color, and so on, in this case we are dealing in structures required by our best formulations of the fundamental physics. Three: there is no principled distinction between arguments pressing us toward a seriousness about spacetime structure, and arguments pressing us toward a seriousness about statespace structure. Three-and-a-half: if we happen to take spacetime seriously by being substantivalists, then we should not object to the mere idea of positing structures among fundamental possibilities, whether unoccupied spacetime points or unoccupied statespace points (see note 39). Four: there is no principled distinction between the kinds of considerations singling out a particular spacetime structure on the basis of the mathematical structure of relativity, and those singling out a particular statespace structure on the basis of the

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dynamically possible states. (Similarly for quantum mechanical configuration space.) First conclusion: these mathematical objects are indispensable to our best physics, and this indispensability is the best evidence we can have for these objects' existence. Second, an "anti-abstract/concrete-distinction-ism." Once we start talking about things like "spacetime points" at all, the distinction between the abstract and the concrete starts to feel pretty hazy. Second conclusion: once we have phase space structure, there is nothing to stop our going whole-hog "substantivalist" about the points as well, whether or not these are rightfully considered "abstract" in a traditional sense. The first step gets us the existence of at least some mathematical objects: those comprising the phase space structure. The second gets us the existence of the phase space points in particular.

[40] I thank Adam Elga for this question.

[41] As argued by Maudlin, "Suggestions from Physics for Deep Metaphysics."

mathematical structure of classical mechanics. Five and up: see below.

And just as we should infer the existence of the basic entities postulated by our best theories of physics, so too should we infer the existence of the basic *structures* posited by those theories. Call this 'structural realism'.[42] I think modern physics suggests that realism about scientific theories is just structural realism: realism about structure. Modern geometric formulations of physics suggest that there is such a thing as *the* fundamental structure of the world, represented by *the* structure of its fundamental physics. There is an objective fact about what structure exists, there is a privileged carving of nature at its joints: along the lines of its fundamental physical structure.

Someone like Reichenbach would disagree.[43] Reichenbach thinks that there is no non-conventional sense to be made of "the" structure of the world. There is the empirical evidence, and there are different ways of describing that evidence: in terms of one spacetime geometry, or in terms of another spacetime geometry plus a global force field. Neither description is privileged. Neither is a candidate for representing the "true" structure of the world. I do not have much to say against such a view. Insofar as we wish to be scientific realists at all, though, I think we must be structural realists. And structural realism is precisely the denial of the Reichenbachian view. Structural realism denies that different theoretical formulations, utilizing different physical structures, must be on a par—even if they yield the same results for our possible experimental evidence.

The nominalist may also bristle at my talk of the existence of structures. Structures, after all, sound awfully like mathematical objects. Still others will think I have been taking the mathematics too seriously. We must not too blithely take the physics at face value, for some of its tools may be just that—useful tools or devices, but representationally inert.[44]

Yet for all that I have said here, it is open what structure *is*, even though I have indeed argued that this structure is not mere idle wheels. At a minimum, I say that the mathematics of phase space is part of the genuine

[42] This term was independently used by James Ladyman, "What is Structural Realism?", *Studies in History and Philosophy of Science* 29 (1998): 409-424 to describe a somewhat similar view, though one very differently motivated.

[43] Hans Reichenbach, *The Philosophy of Space and Time* (New York: Dover, 1958).

[44] The phrase is from Brad Skow.

content of classical mechanics. *This* bit of the mathematics used in our physics cannot be representationally inert: it is inherent to the dynamics. One way to spell this out further would be to be a substantivalist about statespace. Again, I happen to be open to the idea. But we need not go that way. The important thing is that phase space is not just a mathematical convenience. Some part of reality is more or less directly represented by it.

There are two big inferences that I am arguing we should make here. The same considerations leading us to posit a particular spacetime structure should lead us also to posit a particular statespace structure; and we should be realists about that structure. You may want to deny either one, or both. I reply that the combination of these inferences yields the simplest, most natural interpretation of our best fundamental physics. You may demur: Surely eschewing additional possibilities and abstract structures is what really yields the better overall theory! On this, we simply disagree.

The elusive "structure" of physics is, then, something about the minimal, coordinate-independent stuff needed to formulate an empirically adequate, simple and natural, fundamental physical theory. Structure is about how the basic building blocks—which, if our current theories of physics are on the right track, are abstract, coordinate-independent, geometric objects—fit together to form the world, in such a way that all the parts and their organizational features are put to essential use; in such a way that there is no superfluous structure.

In a classical mechanical world, that structure is symplectic structure.

A. Look, Ma: No coordinates!

A.1. Hamiltonian mechanics

With Abraham and Marsden (*Foundations of Mechanics* 3.3), and others, define a Hamiltonian system as follows.[45] Take a smooth manifold M (the configuration space Q, above) together with a smooth, real-valued function H (the Hamiltonian) on the cotangent bundle of M, T^*M. There is a

[45] I thank David Malament for discussion.

natural (canonical) symplectic form, ω, on T^*M. H and ω together determine a vector field, X_H (the Hamiltonian vector field), on T^*M. (Nondegeneracy of ω, for finite-dimensional M, guarantees that X_H exists.) This vector field has an associated class of integral curves. Projecting these curves onto M gives the allowed dynamical trajectories. The Hamiltonian equations can be written as $\iota_{X_H} = -dH$, or $\omega(X_H, \cdot) = -dH$, solutions to which are integral curves of X_H. (X_H thus defines a flow on the symplectic manifold, and by Liouville's theorem, Hamiltonian flows preserve the volume form: Hamiltonian flows are canonical transformations.) We say that (M, ω, X_H) is a Hamiltonian system. See Abraham and Marsden *Foundations of Mechanics* for proofs that this is equivalent to the form of the equations given in note 19, with canonical coordinates p and q.

A.2. Lagrangian mechanics

Define a Lagrangian system analogously (see also note 23). A Lagrangian system comprises a smooth manifold M and a smooth, real-valued function L (the Lagrangian) on the tangent bundle of M, TM, where L is regular. Regularity (as defined in Abraham and Marsden, *Foundations of Mechanics* 3.5) captures the requirement that at each point in the tangent bundle to M, the second partial derivative of L with respect to the fiber (the velocity) is invertible. A curve on M is an allowed dynamical trajectory if, when we lift the curve to the tangent bundle and integrate L along the lifted curve, the result is extremal (relative to nearby curves in M). The Lagrangian equations of motion can then be written in coordinate-free terms (as $\mathbf{L}_\Delta \theta_L - dL = 0$; see José and Saletan, *Classical Dynamics: A Contemporary Approach* 3.4), though we do not need the extra details of that here. See the relevant references mentioned below.

B. Cross-structural comparisons

The invariant quantity of the Lagrangian equations is the square of the Riemannian line element. This is a quadratic differential form with components of a metric tensor. The kinetic energy term, a quadratic differential form of the generalized velocities, is what gives the definite Riemannian element of this space.

The invariant quantity of the Hamiltonian equations is the symplectic form (with canonical form $\omega = dp \wedge dq$), where the generalized q's and p's are allowed to vary independently. Since H is a function of these two sets of coordinates treated as independent variables, we do not get the invariant quadratic differential form of the Lagrangian transformations. Here, the kinetic energy is a simple linear function of the generalized momenta and positions, considered as independent coordinates.

Add to this Darboux's theorem demonstrating that any two symplectic manifolds are locally isomorphic (note 26), and it seems this should mean that not every Hamiltonian phase space is isomorphic to a Lagrangian statespace, and further, that the kind of structure had by the former is the more general, "looser" structure. Since all symplectic manifolds of the same dimension and signature are locally the same, symplectic forms are more flexible than Riemannian metrics, which can be made constant in a local chart iff they are flat.

The immediate worry is that there is a natural isomorphism between vectors and one-forms, and more generally between tangent bundles and cotangent bundles. If the Lagrangian and Hamiltonian statespaces are both vector fiber bundles—the tangent bundle TM and the cotangent bundle T^*M, respectively, for base manifold M—then we can find a natural dual basis, in which case (since they have the same dimension) they will be isomorphic as vector spaces.[46] Indeed[47] (here including abstract indices and using the usual summation convention), for a finite-dimensional[48] symplectic manifold $(M, \omega_{\alpha\beta})$, nondegeneracy guarantees that $\omega_{\alpha\beta}$ has a unique inverse $\omega^{\alpha\beta}$ (with $\omega^{\alpha\beta}\omega_{\alpha\gamma} = \delta^{\beta}_{\gamma}$), or that the mapping $\omega : TM \to T^*M$ from tangent vectors to cotangent vectors (with $\omega_{\alpha\beta}v^{\beta} = v_{\alpha}$) is an isomorphism. For a finite-dimensional symplectic vector space (M, ω), nondegeneracy of ω will imply that $\omega : TM \to T^*M$ is an isomorphism. But then how can the two spaces be said to have different amounts of structure?

[46] Isham, *Modern Differential Geometry for Physicists*, p. 122.

[47] See II.0 of Ashtekar, ed. *New Perspectives on Canonical Gravity*.

[48] That is, for ω strongly nondegenerate. For weakly nondegenerate ω, the induced map between vector fields and one-forms is one-to-one, but in general is not surjective. For the finite-dimensional statespaces here, these will be equivalent, since a linear map between finite-dimensional spaces of the same dimension is one-to-one iff it is onto.

Another worry is the more general version of Lagrangian mechanics (see Appendix A.2), which does not explicitly mention a Riemannian structure on the base manifold. Often, the Lagrangian function has a form requiring a metric—when $L(x,v) = \frac{1}{2} < v, v > - V(x)$, where V is a smooth real-valued function on M (the potential)—but it need not have this form. Given this and the above-mentioned isomorphism, how could the two statespaces have differing structures?

I do not have a proof, though I do think it plausible that, even in the most general case, the two formulations will differ in the relevant statespace structure. Two main reasons. First, notice that the isomorphism between TM and T^*M only holds (similarly, the Legendre transform can only be defined) when they are both vector spaces.[49] Therefore, if the Hamiltonian phase space is not at least a vector bundle—where the generalized momenta live in vector space fibers above the generalized-position points of the base manifold—we will not find an equivalent Lagrangian formulation via the usual procedure.[50]

[49]Marsden and Ratiu, *Introduction to Mechanics and Symmetry*, 2.2; da Silva, *Lectures on Symplectic Geometry*, chapter 20.

[50]As in Abraham and Marsden, *Foundations of Mechanics* 3.5-3.6; see also Belot, "The Representation of Time and Change in Classical Mechanics." Here is the gist of that procedure. Take a function L on TQ and solutions to a second-order equation, here the Lagrangian equation. From L we can derive an energy function E on TQ which, when translated to T^*Q by means of the fiber derivative $\mathbf{F}L : TQ \to T^*Q$, yields a suitable Hamiltonian H on T^*Q. (The fiber derivative is the derivative of L in each fiber of TQ, mapping the fiber T_qQ at $q \in Q$ to the fiber T_q^*Q. This is what is often called the Legendre transformation.) The solution curves in T^*Q and TQ will then coincide when projected to Q. An analogous procedure gets from the Hamiltonian to the Lagrangian formulation. In general, if we start with vector bundles over a common base space, then the fiber derivative of a function from L to H, say, will be smooth and fiber-preserving. Note that $\mathbf{F}L$ is not necessarily a vector bundle mapping, but it will be a fiber-preserving smooth mapping: a fiber-preserving isomorphism. This transformation requires that the Lagrangian and Hamiltonian functions be hyperregular. (Abraham and Marsden, *Foundations of Mechanics*, pp. 218-223; Marsden and Ratiu, *Introduction to Mechanics and Symmetry*, 7.4. A Lagrangian is hyperregular iff $\mathbf{F}L : TQ \to T^*Q$ is a diffeomorphism; a Hamiltonian is hyperregular iff $\mathbf{F}L : T^*Q \to TQ$ is a diffeomorphism. In other words, the fiber derivative will be locally invertible iff L (or H) is regular. Hyperregularity is thus needed for the global result.) Hyperregularity is needed for the relevant map from Lagrangians to Hamiltonians to be a bijection. Hyperregularity amounts to the requirement that the underlying statespaces be isomorphic in this way; and without

And, in general, a Hamiltonian phase space need not have the structure of a vector bundle. Though phase space is often the cotangent bundle of a configuration space,[51] it need not be possible, in a given Hamiltonian phase space, to separate the position and momentum coordinates in this way: they can be freely mixed by symplectic transformations.[52] Darboux's theorem, after all, says that every symplectic manifold is *locally* like a symplectic vector space, in suitable local coordinates. That is, locally, any symplectic phase space will look like a vector bundle. Globally, though, it can have a different structure, for the phase space might not be naturally separable in this way.

This is unlike the Lagrangian statespace, which must be a vector bundle. We build this space up by starting with the configuration space Q as the base manifold, and attaching the tangent spaces, on which we define the $\dot{q}$'s, at each point. This, after all, is how the generalized velocities are defined. The resultant structure does yield a natural symplectic form, which may lead you to ask: why not prefer the Lagrangian formulation,

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it, we cannot use this procedure to generate a Hamiltonian from a Lagrangian or vice versa.

[51] And it can be shown that cotangent bundles are equipped with a canonical symplectic structure: Abraham and Marsden, *Fundations of Mechanics* 3.2.

[52] Any $2n$-dimensional vector space Γ can be regarded as a cotangent bundle by assigning two complementary n-dimensional subspaces, Q and P, with $Q \bigcap P = \{0\}$, $Q \bigoplus P = \Gamma$. See chapter II.0 in Ashtekar, ed. *New Perspectives on Canonical Gravity*. When Γ is also a *symplectic* vector space, the symplectic form ω picks out a preferred class of pairs of complementary subspaces, namely the ones with $\omega_{\alpha\beta} v_1^\alpha v_2^\beta = \omega_{\alpha\beta} w_1^\alpha w_2^\beta = 0$, for all $v_1^\alpha, v_2^\beta \in Q$ and $w_1^\alpha, w_2^\beta \in P$. Then we can choose convenient bases $\{v_\mu^\alpha\}$ and $\{w_\mu^\alpha\}$ in Q and P, respectively, such that $\omega_{\alpha\beta} v_\mu^\alpha w_\nu^\beta = \delta_{\mu\nu}$. With any such choice of basis, the symplectic form will have the standard (canonical) form, and we have effectively identified configuration and momentum variables in Γ: we can write $v_\mu^\alpha = (\partial/\partial q^\mu)^\alpha$ and $w_\mu^\alpha = (\partial/\partial p_\mu)^\alpha$. This, however, is a special case. Here, the state of a system can be specified by means of its configuration variables q^i and momentum variables p_i, where there is this natural distinction between the two: the q^i's are coordinates on a configuration space Q, and for each q^i, the p_i's belong to the linear space of cotangent vectors at q^i (in a coordinate system). Here the phase space has a cotangent bundle structure, T^*Q. In a general Hamiltonian phase space, however, this natural splitting of Γ is not available: the phase space need not always be a cotangent bundle. An example of the more general case (from Ashtekar ed. *New Perspectives*, p. 22) is given by the choice of $\Gamma = S^2$, $\omega_{\alpha\beta} = \epsilon_{\alpha\beta}$, a volume 2-form.

where we seem to get the symplectic structure for free? Answer: the most general structure for a classical system's phase space is a kind of structure that need not be easily or naturally splitable into a configuration space plus associated tangent spaces in this way.[53] The Lagrangian statespace structure is therefore a more restrictive, less general structure.

So there is a natural isomorphism between the two statespace structures. But of course there is, if and when they are both vector fiber bundles. In that case, there will be an isomorphism that preserves the vector bundle structure: a vector bundle isomorphism. But this is not sufficient to show that the two spaces possess the same amount of structure. As a general rule, not just any isomorphism will be the relevant structure-preserving map. (A "cardinality-isomorphism" would often leave out much to be desired.) And this is one of the cases following that rule.

Once again, symplectic structure comes out as more fundamental, permitting just those sets of coordinates that allow for the equations of motion of classical particles to hold, regardless of whether those coordinates are perspicuously related to ordinary position and momentum coordinates. Symplectic structure is a simpler, more natural, more general underlying structure—but a structure that suffices for the theory all the same.[54]

Second, look at the underlying groups.[55] Start with the linear case of a $2n$-dimensional vector space, call it V. If V is equipped with a

[53]The induced symplectic form of the Lagrangian formulation requires a vector bundle structure. Peter Forrest suggested to me this characterization of what is going on. Despite the natural isomorphism between the two statespaces, there remains the following order of explanation: the symplectic form on the phase space is more natural, since the induced symplectic form on the Lagrangian configuration space is only definable by way of the Hamiltonian.

[54]Another consideration mentioned to me by Frank Arntzenius, which warrants further investigation. The Lagrangian formulation seems to presuppose a metric on time. (Recall its formulation via least action principles, where we integrate the action term with respect to the time along a path.) The Hamiltonian, it seems, does not. (Consider the Poisson bracket formulation, independent of the time coordinate and equivalent to the canonical equations: Abraham and Marsden, *Foundations of Mechanics* p. 198.)

[55]See Abraham and Marsden, chapter 9. For discussion, I thank Gordon Belot (who disagrees with my conclusions).

positive-definite symmetric bilinear form (the linear analog of a Riemannian metric), then the group of linear transformations preserving this structure is $O(V)$, a group with dimension $2n^2 - n$. If V is equipped with a symplectic form, on the other hand, then the group of linear transformations preserving the structure is $Sp(V)$, a group with dimension $2n^2 + n$. The former is a stronger structure, in that it admits a smaller group of symmetries. The difference magnifies for manifolds. The largest symmetry group of a Riemmanian metric on a $2n$-dimensional manifold is $2n^2 - n$, whereas every symplectic manifold admits an infinite-dimensional group of symmetries.

To put it another way, the Lagrangian equations of motion are invariant under a set of point transformations; the Hamiltonian, under the canonical transformations. Whereas all point transformations are canonical transformations—point transformations form a subgroup of the set of all canonical transformations[56]—point transformations are orthogonal transformations, and thus only a special type of canonical transformation. Canonical transformations are more general, precisely because the generalized momenta are treated as truly independent of the generalized positions. As Herbert Goldstein, author of the classic textbook on mechanics, puts it:[57]

> The advantages of the Hamiltonian formulation lie not in its use as a calculational tool, but rather in the deeper insight it affords into the formal structure of mechanics. The equal status accorded to coordinates and momenta as independent variables encourages a greater freedom in selecting the physical quantities to be designated as "coordinates" and "momenta." As a result we are led to newer, more abstract ways of presenting the physical content of mechanics.

[56] Abraham and Marsden, p. 181.

[57] Goldstein et. al., *Classical Mechanics*, p. 369.