

**TIME AND THE FOUNDATIONS  
OF QUANTUM MECHANICS**

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# TIME AND THE FOUNDATIONS OF QUANTUM MECHANICS

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University of Pittsburgh, 2014

Quantum mechanics has provided philosophers of science with many counterintuitive insights and interpretive puzzles, but little has been written about the role that time plays in the theory. One reason for this is the celebrated argument of Wolfgang Pauli against the inclusion of time as an observable of the theory, which has been seen as a demonstration that time may only enter the theory as a classical parameter. Against this orthodoxy I argue that there are good reasons to expect certain kinds of ‘time observables’ to find a representation within quantum theory, including clock operators (which provide the means to measure the passage of time) and event time operators, which provide predictions for the time at which a particular event occurs, such as the appearance of a dot on a luminescent screen. I contend that these time operators deserve full status as observables of the theory, and on reflection provide a uniquely compelling reason to expand the set of observables allowed by the standard formalism of quantum mechanics. In addition, I provide a novel association of event time operators with conditional probabilities, and propose a temporally extended form of quantum theory to better accommodate the time of an event as an observable quantity. This leads to a proposal to interpret quantum theory within an event ontology, inspired by Bertrand Russell’s *Analysis of Matter*. On this basis I mount a defense of Russell’s relational theory of time against a recent attack.

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## PREFACE

It has been my great pleasure to have been involved in the philosophy of physics for over 10 years now, and it continues to be a source of fascinating and challenging intellectual adventures. This particular endeavor began with an invitation by John Earman to attend a workshop in the foundations of physics (as I am sure have many before). The title of this workshop was “The Nature of Time in Fundamental Science,”<sup>1</sup> and it became clear in listening to the presentations and conversations of the participants that many of their concerns were overtly philosophical. However, the approaches they advocated involved sophisticated mathematical arguments and theoretical apparatus which was often unfamiliar to even the intended audience of theoretical and mathematical physicists.

Investigating further, I found a significant literature in the foundations of physics regarding the status of time within quantum mechanics, which had been mostly overlooked by philosophers of science. The arguments began by the progenitors of quantum mechanics such as Heisenberg, Dirac, Pauli and Schrödinger have continued up to the present day, and have generated a variety of ‘no go’ theorems, and ways around them. While these may appear at first sight to be only of formal interest, it quickly became clear to me that genuinely philosophical questions are being raised (and, perhaps, answered) that have rarely been considered in the philosophy of physics literature.

Another significant event in my intellectual development began soon after my arrival in Pittsburgh: I became engaged with the history of science, and in particular developed a fascination with the early development of relativistic quantum mechanics by Dirac. This project convinced me of the essential value of the history of science for the continued vitality of the philosophy of science. In pursuing the philosophical and foundational goals of this

<sup>1</sup>Center for Fundamental Theory, Penn State, College Park. Nov. 9–10, 2009.

dissertation, I became drawn into the early history of the theory and the distinctive approaches to time in quantum mechanics that formed a rich topic for discussion at the famous Solvay conference of 1927. My interest was also piqued by the ground-breaking work of Jan Hilgevoord which, although I found much to disagree with, formed a vital and informative guide for my studies. I hope he will view my attempt to contribute to this field of study in the spirit of friendly debate in which it is intended.

I should also mention the scholars whose approach to the topic of time in quantum mechanics forms the backbone of this dissertation: Paul Busch, Romeo Brunetti, Klaus Fredenhagen, and Marc Hoge. Although I have not had the opportunity to communicate with them directly, I have learnt a great deal from them. The intellectual debt I owe to Marc Hoge in particular through my engagement with the rich ideas of his Masters thesis will be quite evident to him should he ever read this document. For others, I hereby acknowledge Hoge's work as the inspiration for my conditional probability interpretation of event times. I am not sure whether or not he would agree with the direction in which I have taken these ideas, but I hope at some point to find out.

I also owe a great deal to those whose engagement with my dissertation has been more direct. First, of course, to my dissertation co-advisors John Earman and John D. Norton. There is little in this document that has not benefited from my engagement with this advisory 'dream team' in comments, conversations, and advice. I feel especially fortunate to have been part of the great tradition of philosophy of physics at Pitt, and it heartens me greatly that John Earman is continuing in his role as dissertation advisor for another generation of students, despite his richly deserved recent retirement. I would also like to thank my committee members Bob Batterman, Gordon Belot, and Giovanni Valente, whose friendly advice has also benefited me greatly.

This also gives me an opportunity to thank the generation of graduate students that I was lucky enough to be a part of at Pitt. While everyone I have known and become friends with over the last six years deserve thanks, I would like to thank in particular those who have materially assisted with this project. Conversations and sessions at the whiteboard with philosophers of physics Balazs Gyenis, Mike Miller and Mike Tamir were particularly valuable, but Bryan W. Roberts deserves special recognition for the close collaboration he and I have enjoyed with over the past few years on these topics. I have tried to acknowledge

in the text where Byran's influence was especially direct, particularly in Chapters 2 and 4, but I owe a great deal of my (admittedly partial) understanding of the finer points of time in quantum and classical mechanics to helpful conversations and correspondence with my former officemate.

It would be remiss of me not to mention by name Eric Hatleback, William Lebing, Thomas Cunningham, Peter Distelzweig, Benny Goldberg, Elay Shech, Katie Tabb, Bihui Li, Preston Stovall, Julia Bursten, and Joe McCaffrey. They will know why. I'd also like to thank fellow travelers Kerry McKenzie, Elise Crull, Matt Gorski, Pablo Ruiz de Olano, Alex Blum, Adrian Wuthrich, Matt Farr, Sam Fletcher, Noel Swanson, and especially my great friend Adam Caulton. Comments by Adam made in conversation about the role of time in quantum theory have proved suggestive, provocative, and insightful. I owe many thanks to James Yearsley, who was a sometime (and crucial) collaborator in explorations of the idea of space-time localization.

I'd also like to thank (in no particular order) some more senior members of the community, whose assistance has been gratefully received: Katherine Brading, Laura Reutsche, Jeremy Butterfield, Chris Wuthrich, Don Howard, Guido Bacciagaluppi, Erik Curiel, Steve French, Mark Wilson, Michel Janssen, and Jos Uffink. I must single out Tony Duncan (of the Physics Department at Pitt) in particular for his generous assistance in my historical endeavors, and some hearty dinners. James Ladyman deserves special thanks for his crucial role in my philosophical education.

This brings me, finally, to address the contributions of emotional (and often financial) support of those closest to me. To my parents, Carolyn and David, I thank you for your unstinting faith in my abilities, your constant support and love, and the relaxed way that you have dealt with my continuing presence across the Atlantic. To my darling wife Jean: your day-to-day companionship, love, devotion, and—occasionally—cajoling or threats, have been more important to the successful completion of this project than you will ever know, and I offer you my heartfelt gratitude. Our dissertations have been the silent partners in our relationship so far, and I am so glad these first projects begun together are complete while our lives together have just begun.

Thanks must also go to my sister Laura, and to my extended family, including those who first inspired me to think that graduate school was a worthy endeavor, and that emigration to North America was not such a bad idea.

Although several names I have mentioned above are no doubt more worthy, I dedicate this work to my nephews: Elias, Rowan, and Seth. May you never stop asking, “Why?”

## 1.0 INTRODUCTION

This dissertation concerns the intersection of two fascinating and contentious philosophical topics: one as old as the history of Western philosophy (the metaphysics of time), and the other arising relatively recently (the interpretation and foundations of quantum mechanics). Unlike most philosophy of time informed by modern physics, our focus will not be on Einstein’s relativistic achievements, but on the quantum revolution of which he was also part. However, we will not be concerned here with quantum gravity—one of the great unsolved problems of 21st century physics—at least, not directly. Our concern here is more prosaic: I will be addressing the good old quantum mechanics of graduate physics textbooks. That is, the quantum theory of finitely many particles in non-relativistic space-time, often called Gallilei invariant quantum theory. This is not a theory of space-time—as in special or general relativity—but a theory of *matter*. What could such a theory have to teach philosophers about time?

As it happens, the role of time in quantum mechanics was the subject of much discussion in the early history of the theory, and has drawn the attention of foundationally-minded theoretical and mathematical physicists ever since. Traditionally, the most prominent topic has been the status of the time-energy uncertainty relation, posited by Heisenberg in 1927. However, there is a rich and varied tradition of addressing various ‘problems of time’ that arise in quantum theories. Among philosophers of science, ‘the’ problem of time in canonical quantum gravity is the best known, and has been subject to recent philosophical analysis.<sup>1</sup> It is my contention that the problems of time in humble non-relativistic quantum mechanics have been unjustly overlooked by philosophers of physics.

In regarding the conjunction of time and quantum mechanics as particularly fertile ground for foundational work, I am in accord with the editors of a recent two volume series of

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<sup>1</sup>See Belot & Earman (2001); Butterfield & Isham (1999) for an introduction, or the recent dissertation of Wüthrich (2006). The scare quotes indicate the common wisdom that there are several distinct problems going under that name, see Kuchař (1992).

collected papers on the topic of “Time in Quantum Mechanics”. In their preface, they introduce the topic as follows:

Time and quantum mechanics have, each of them separately, captivated scientists and layman alike, as shown by the abundance of popular publications on “time” or on the many quantum mysteries or paradoxes. We too have been seduced by these two topics, and in particular by their combination. Indeed, the treatment of time in quantum mechanics is one of the important and challenging open questions in the foundations of quantum theory. (Muga et al., 2002)

It is my hope that in pursuing these topics this dissertation will break fertile new ground for the foundations of physics, while demonstrating to philosophers the relevance of these foundational issues for metaphysical and interpretational concerns.

The first aim of the dissertation concerns a ‘no go’ result known as Pauli’s Theorem. Straightening out the content and interpretation of this result will require several chapters, but the import among physicists has commonly been taken to be this:

[T]ime is just a parameter in quantum mechanics, and *not* an operator. In particular, time is not an observable . . . It is nonsensical to talk about the time operator in the same sense as we talk about the position operator. . . . when we now look at quantum mechanics in its finished form, there is no trace of a symmetrical treatment between time and space.<sup>2</sup> (Sakurai, 1994, p. 68, original emphasis)

Recent authors Giannitrapani (1997), Hilgevoord & Atkinson (2011), and Olkhovsky (2011) have gone so far as to call this ‘the problem of time in quantum mechanics.’ I will argue here that the statement of the problem is not so simple: Pauli’s Theorem merely restricts the form of possible time operators, rather than banning them outright. In addressing the implications of Pauli’s Theorem, I am answering the recent call of Halvorson, who concludes a presentation of a related result as follows:

A common response is to claim that time *is* a quantity in quantum theory, but that it is represented by a parameter (*c*-number) rather than by an operator.

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<sup>2</sup>Other statements of this sort include: “time is a parameter in quantum mechanics and *not* an operator” (Duncan & Janssen, 2013, p. 216, original emphasis); “Since the very beginning of quantum mechanics it is not so easy to define time at a quantum level; in the ordinary theory, in fact, it is not an observable, but an external parameter, in other words, time is *classical*.” (Giannitrapani, 1997, p. 1575, original emphasis). The introduction of Aharonov & Bohm (1961) contains a similar statement, and may be the source of this concordance.

But that distinction is merely verbal, and does nothing to help us understand the special role of time in quantum theory. What is the difference between quantities that can be represented by operators, and those—such as ‘amount of time’—that cannot? And why is time the only such parametric quantity? What is special about time? (Halvorson, 2010, p. 3)

I will argue that time (in its various roles) can be represented by operators, but nonetheless retains something of a special status in quantum mechanics.

The second aim of the dissertation is to provide an analysis of a common approach to introducing time operators in terms of mathematical objects not subject to Pauli’s Theorem, known as Positive Operator Valued Measures (or POVMs). This approach makes use of tools developed by mathematical physicists such as Davies & Lewis (1970), Holevo (1973, 2011), Kraus (1983), and Ludwig (1983) to solve certain problems in quantum measurement theory. Following the work of Werner (1986), Busch et al. (1994), and Giannitrapani (1997), among others, this approach to time in quantum mechanics became well enough established to deserve the name ‘the standard approach’ in the recent collection of Muga et al. (2002). These tools and techniques have, however, received minimal attention from philosophers,<sup>3</sup> or have been spuriously dismissed as irrelevant (Hilgevoord & Atkinson, 2011).<sup>4</sup> The several chapters I devote to the use of POVMs as a means to define time operators and their significance for quantum foundations provide a much needed introduction of these topics into the philosophy of science.

The third aim of the dissertation is interpretative. I provide an interpretation of the use of these time POVMs as quantum clocks and as event time observables, and an assessment of the possible implications for familiar issues in the interpretation of quantum mechanics. This is a subject that has received little attention from physicists, and is ripe for philosophical analysis. Event times prove particularly difficult to account for on the conventional interpretation of quantum mechanics. This leads to my major positive contribution: a novel interpretation of event times as conditional probabilities, which I use to give a consistent interpretation of the proposal of Brunetti & Fredenhagen (2002b) for generalized event time POVMs. I supply an interpretation of these probabilities formally in terms of an extended Hilbert space

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<sup>3</sup>Butterfield (2013) deserves an honorable mention here for acknowledging their importance. A recent review of time in quantum mechanics, Zeh (2009), fails to mention them at all.

<sup>4</sup>These last authors state: “POVMs are interesting in their own right, having many practical applications, but we shall not discuss them here, since we believe their use as a way of nullifying Pauli’s objection to be fundamentally misdirected.” I will argue against some specific reasons for this claim in §§2.2–2.3, but this dissertation as a whole can be read as an argument in favor of their relevance for this purpose.

of functions of time and space, and philosophically as Lewisian chances (i.e. objectified credences). This suggests an account of quantum theory in terms of events rather than persisting objects, and I conclude with an examination of some ontological and metaphysical concerns from this point of view, including some implications for the metaphysics of time.

A surprisingly broad cross-section of these topics and aims were considered or shared by the authors of quantum mechanics in the years leading up to the establishment of the standard (Dirac-von Neumann) formalism in use today. This is because the role of time in quantum theory was far from settled in those early years, before the foundations of the theory were set hard. The speculations and arguments of the founders of quantum mechanics regarding time, therefore, often provide a richer source of possibilities and concerns than can be found in the foundations literature today. I believe we can learn a lot from the struggles of our forebears in this regard, and the views developed at length in this dissertation owe much to the inspiration of quantum luminaries such as Werner Heisenberg, Wolfgang Pauli, Erwin Schrödinger and Paul Dirac. I also owe those who have walked these paths more recently a sincere debt. As will be evident to the reader, this dissertation would not have been possible without the work of those intrepid explorers in the foundations of quantum theory who thought to ask, “What about time?”

## 1.1 SUMMARY OF CHAPTERS

Chapter 2 begins with a history of time in quantum theory, or rather a pre-history of time in quantum mechanics, since this history ends more or less where the modern form of quantum mechanics begins. This provides an introduction to many of the topics that later chapters are devoted to, set in the context in which they originally arose. For example, this chapter contains our first introduction to Pauli’s Theorem, which became the driving force behind the common wisdom that time must play a limited role in quantum mechanics. It also transpires that the close links with classical mechanics of the early quantum theory were both a source of inspiration and confusion during those years. This chapter concludes with an contemporary assessment of the role of time in classical mechanics, and contains details of the relationship between some simple classical and quantum systems and the way time enters into their description.



In Chapter 3, I introduce the formalism of modern quantum mechanics, paying particular attention to the role of Positive Operator Valued Measures (POVMs). I provide a characterization of what I call Ordinary Quantum Mechanics (i.e. the standard textbook account) and explore the motivations that have led many researchers in the foundations of measurement to go beyond it. This chapter contains a critical assessment of the role of the Spectral Theorem in excluding POVMs from the range of observables of the theory, i.e. the quantities thought to have empirical significance in concrete experimental situations. That is, I review some results that encourage us to go beyond the ambit of Ordinary Quantum Mechanics and embrace the expanded theoretical framework of POVMs. Doing so leaves open the possibility of finding a substantive role for observables concerning temporal quantities and processes in quantum theory.

Chapter 4 contains a detailed analysis of the statement, scope, and interpretation of Pauli's Theorem, which purports to rule out time observables in quantum mechanics. I first point out that, strictly speaking, Pauli did not present a theorem at all since his original argument admits significant classes of counterexamples. Nonetheless, Pauli's argumentative strategy was sound and there are closely related results that appear to support his conclusion. I discuss two versions: a somewhat limited result that sticks closely to Pauli's strategy but excludes periodic systems from consideration, and a recent strengthening of that result which relies on a premise further restricting the allowed energies of the system. Roughly put, this premise (known as the spectrum condition) limits the allowed values of energy of a quantum system to be positive, or at least have a lower bound. This result applies quite comprehensively, but, I argue, it doesn't support the negative conclusions often drawn regarding the prospects for time observables in quantum theory.

The idea that some time observables may avoid Pauli's Theorem is not new, and there are some well-established examples which have drawn much attention in the foundations literature. Chapter 5 provides an introduction to those examples by exploring the ideas of the previous two chapters in these specific contexts. These examples make clear the role of POVMs in avoiding Pauli's Theorem, and provide concrete applications of this framework which demonstrate its usefulness for defining time observables. These applications are often paradigmatic examples of the more general approaches considered in later chapters. This chapter also contains a critique of the common interpretation of a particular quantum system, the freely falling particle, as defining a time observable of this sort.

Drawing on this detailed analysis of some specific cases, Chapter 6 provides a critical look at attempts to define ideal quantum clocks: quantum systems which perfectly track the passage of time. I begin by presenting some ‘no go’ results for quantum clocks derived from Pauli’s Theorem, and argue against a particular proposal for getting around them by discarding the spectrum condition. In its place I endorse the use of the phase of a quantum harmonic oscillator (a system that respects the spectrum condition) to provide a limiting notion of an ideal clock. I show how the two proposals have much in common, and demonstrate that the harmonic oscillator clock has much going in its favor—not least the availability of a concrete physical realization. I conclude with a skeptical assessment of recent claims that every system can serve as a quantum clock.

Chapters 7 and 8 provide a novel account of the use and interpretation of POVMs in accounting for event times in quantum mechanics. Event times arise in common experimental situations where the possible outcomes are distributed in time. Making predictions for these temporal distributions from the instantaneous form of quantum mechanics in common use presents many practical and conceptual difficulties. In Chapter 7 I review some recent proposals for providing these temporal probability distributions using POVMs. Despite some impressive progress, the interpretation of these event time POVMs as observables of the theory has remained somewhat opaque. I argue that the difficulties arise specifically from the failure of the so-called Standard Model of Measurement in these cases.

In Chapter 8 I propose a new interpretation in terms of conditional probabilities: an event time POVM predicts the occurrence of an event during a specific time interval, given the occurrence of the event at some time. The need to give an account of the occurrence of these events independently of the notion of measurement leads to their interpretation as resulting from an observer-independent physical process. Whatever the form this process takes, I contend that the conditional probabilities supplied by the theory have the form of Lewisian chances. That is, they supply an assignment of probabilities to future events, considered as occurring in distinct possible worlds.

I point out that the probabilities supplied by Ordinary Quantum Mechanics involve an implicit conditionalization on the occurrence of some event at a particular instant of time. Furthermore, I argue that the usual interpretation of instantaneous projections as the possessed properties of a persisting quantum system combines with the results of Chapter 4 to lead to some absurd consequences. As an alternative to the interpretation of the quantum

state in terms of properties, I suggest a view where the state gives the probabilities for the occurrence of events, interpreted as physical happenings situated in time and space.

Chapter 9 concludes the dissertation. I begin by exploring some of the implications of the alternative event-based interpretation I suggest in Chapter 8. First, I consider some recent advocates of this event-based view of quantum theory and bring them into contact with some other extant interpretations of quantum mechanics that seem to make room for events. I contend that the account of quantum theory presented in Chapter 8 is interestingly distinct, and suggest some areas of empirical investigation that may reveal the benefits of this approach. I also examine the relevance of the results of Chapter 6 for the metaphysics of time, and advocate a particular relational view of time inspired by Leibniz and held by Russell. I defend Russell's view against a recent attack by making use of his proposal for an event ontology, given in *The Analysis of Matter* (1927). I conclude by (very) briefly suggesting a further application of this proposal to recent structuralist philosophies of science.

## 2.0 A PRELUDE TO TIME IN QUANTUM MECHANICS

This chapter provides some valuable historical and conceptual background to the issues to be addressed in the remainder of the dissertation. The aim of the dissertation as a whole is to address the representation of time and temporal processes within quantum theory, and to do so we will use the mathematical and conceptual resources currently associated with the theory. However, some of the most interesting claims and arguments regarding the proper role of time in the theory were made during the early history of the theory (approximately pre-WWII) when those tools were often not available or, if they were, not under widespread use. In evaluating and interpreting these arguments, therefore, we must take care to ensure that we do not project onto these early authors a modern view of quantum mechanics that they either could not or did not possess.

These, then, are some of the costs of this work: if we want to gain a genuine understanding of the insights of our forebears, we must be prepared to approach their work on their own terms. But in bridging this conceptual gap between their era and our own, we also come to understand the conceptual possibilities that were open to those for whom the theory was nascent and as yet unfixed. If these possibilities were closed off by the development of quantum mechanics into the form in which we know it today, we should not judge harshly those whose momentary opinions and expectations were hostage to fortune. Examining their thwarted expectations may even bring us to see the formalism and interpretation of quantum theory we possess today in a new light: as a conceptual artifact that not only bears signs of the struggles of its discovery and refinement, but that also may be open to further refinement or extension.

The first part of this chapter provides what I call a pre-history of time in quantum mechanics, which respects the fact that these authors were often drawing conclusions from a theoretical formalism that is not the one in use today. I begin with what is often marked as the birth of quantum mechanics (from its ancestor the ‘old quantum theory’ of Bohr

and Sommerfeld): Heisenberg’s famous *Umdeutung* (Reinterpretation) paper of 1925. The history I have to tell concerns the role of time in the theory, how the expectations of early authors regarding that role came about, and how those expectations were thwarted (or not) by what came to be known as ‘Pauli’s Theorem.’

The details of my account conflict with the recent history provided by Hilgevoord (2005), since Hilgevoord argues that there was a fundamental confusion that lay behind many of these expectations. I address this conflict in Section 2.2, arguing that Hilgevoord’s interpretation misrepresents the views of the historical actors. In addition, I contend that Hilgevoord’s complaint misrepresents the diverse roles that time can play in classical mechanics, and that attending to these distinctions in the classical context leads to a revised expectation of the role that time might play in quantum theory. In particular, in Section 2.3 I argue that Hilgevoord ignores the reasonable expectation that the role played by the time of an event in classical mechanics might come to be played by an observable of quantum theory. This leads to a brief discussion of some specific cases, and thence, in Section 2.4, to an analysis of time covariance in modern (analytical) classical mechanics.

## 2.1 A PRE-HISTORY OF TIME IN QUANTUM MECHANICS

The early years of quantum mechanics (1925-1927) were a period during which close links to classical mechanics were routinely posited and exploited, and then discarded if they conflicted with the further development of the theory. Some of these links were very suggestive of a role for time as an observable (or operator, or  $q$ -number) of the theory. The ‘modern era’ for time in quantum mechanics begins with the influential dismissal of this idea by Wolfgang Pauli in 1933 in a footnote appearing in the second edition of his *Encyclopedia of Physics* article on quantum mechanics. This argument came to be known as ‘Pauli’s Theorem.’

Chapter 4 provides an ahistorical assessment of the argument he provides, couched in our modern language of functional analysis. Here, I am concerned with assessing the historical origins of the claims against which Pauli was arguing, and the important antecedent of his argument in the work of Erwin Schrödinger on the ideal quantum clock. I also point out, in Section 2.1.5, that one of the supposed targets of this argument (Paul Dirac) had motivations and methods that are entirely untouched by Pauli’s argument. Dirac’s concern

was with the quantum analogue of a particular form of classical mechanics in which the dynamical variables were extended to explicitly include time, and Pauli’s argument is silent about the role of time in this quantum formalism.

Although when Pauli was writing Dirac had already abandoned those methods, he was to return to them much later in his work on the quantization of constrained Hamiltonian systems (Dirac, 1964). In Chapter 7 I will introduce another reason to consider the extended formalism in quantum mechanics: giving an account of the time of an event. The idea that the time of an event would be an observable quantity of the theory also arose in the early history of the theory. In particular, advocates of matrix mechanics initially suggested that the dynamics of the theory would consist of a series of discrete ‘quantum jumps’ between states of fixed energy, reminiscent of the orbital transitions of an electron in Bohr’s early atomic model. For a brief period in 1927, Heisenberg and Jordan advocated an interpretation of the time-energy uncertainty relation in these terms. This is addressed in Section 2.1.2.

### 2.1.1 The Road to Pauli’s Theorem

This section aims to give a limited account of the origins of Pauli’s Theorem against the backdrop of the development of the Dirac-von Neumann formalism of quantum mechanics. The development of quantum mechanics is neatly book-ended by Heisenberg’s famous Umdeutung of 1925 on one end, with Dirac and Jordan’s transformation theory in 1927 (in two distinct forms) at the other. Notably, 1927 also saw von Neumann’s trio of papers introducing the Hilbert space formalism and the 1927 Solvay conference (famed for the legendary Bohr-Einstein debate). This two year period 1925–1927 saw the rapid development of the formalism along several different directions. This period is often characterized as a two-pronged development of Heisenberg’s matrix mechanics and Schrödinger’s wave mechanics, which were unified in Dirac’s and Jordan’s transformation theories and subsequently given a firm mathematical foundation by von Neumann. For our purposes here, this neat picture will suffice.<sup>1</sup>

Despite the revolutionary implications of quantum mechanics, its development was made possible by maintaining a close eye on its relationship with parts of classical mechanics. One of those parts, which played an essential role in the initial development of matrix

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<sup>1</sup>To add some detail and depth to this barebones sketch, see (e.g.) the classic account of Jammer (1966). For recent noteworthy additions and amendments see Darrigol (1993) on Dirac, Joas & Lehner (2009) on Schrödinger, Duncan & Janssen (2013) on Jordan and von Neumann, and references therein.

mechanics by Heisenberg, Born and Jordan (Born & Jordan, 1925; Born et al., 1926) was the action-angle form of Hamiltonian dynamics. In classical mechanics, the action-angle coordinates are introduced by means of a canonical transformation and result in a remarkable simplification of the dynamics. In the periodic systems that matrix mechanics began by treating, the action variable  $J$  (with units of energy) was constant in time while the angle variable  $w$  (with units of time) varies linearly with time. Writing a (real) solution of the equations of motion  $x(t)$  in terms of action and angle variables meant specifying a Fourier series decomposition in terms of a (countably infinite) set of amplitudes  $A(J_n)$  and complex phases  $v_n$ .

$$x(t) = \sum_{n=-\infty}^{\infty} \sum_{\tau \pm 1} A_{\tau}(J_n) e^{2\pi i \tau v_n t}.$$

In the matrix mechanics of Born & Jordan (1925) kinematical quantities are represented by Hermitian matrices whose time dependence takes the same form,

$$\mathbf{p}(t) = p(nm) e^{2\pi i v(nm)t}; \quad \mathbf{q}(t) = q(nm) e^{2\pi i v(nm)t}.$$

The crucial difference was that these equations now gave the time development of the dynamical variables  $\mathbf{p}(t)$  (momentum)  $\mathbf{q}(t)$  (position) rather than the time development of a single solution of the classical equations. Nonetheless, the dynamics for these matrix operators could be put into the same form as Hamilton's equations:

$$\left( \frac{\partial H}{\partial q} = \right) \dot{\mathbf{p}} = \frac{i}{\hbar} [\mathbf{H}\mathbf{p} - \mathbf{p}\mathbf{H}]; \quad \left( -\frac{\partial H}{\partial p} = \right) \dot{\mathbf{q}} = \frac{i}{\hbar} [\mathbf{H}\mathbf{q} - \mathbf{q}\mathbf{H}].$$

This suggested the following general dynamical equation for an arbitrary matrix  $\mathbf{g}$ , the so-called Heisenberg equation of motion,

$$\dot{\mathbf{g}} = \frac{i}{\hbar} [\mathbf{H}\mathbf{g} - \mathbf{g}\mathbf{H}] = \frac{i}{\hbar} [\mathbf{H}, \mathbf{g}], \quad (2.1)$$

where  $[\cdot, \cdot]$  is the quantum commutator.

Working independently in Cambridge, Dirac recognized in Heisenberg's original paper a strong resemblance to Poisson's equations in classical mechanics,

$$\dot{f} = \sum_r \left\{ \frac{\partial H}{\partial q_r} \frac{\partial f}{\partial p_r} - \frac{\partial f}{\partial q_r} \frac{\partial H}{\partial p_r} \right\} = \{H, f\},$$

where  $\{\cdot, \cdot\}$  is the Poisson bracket and  $H$  is the classical Hamiltonian. In classical mechanics, the relationship between a configuration variable  $q_r$  and its conjugate momentum  $p_r$  leads

to their Poisson bracket being unity,  $\{q, p\} = 1$ . This relation of conjugacy was replicated in quantum mechanics by replacing the Poisson bracket with the quantum commutator,

$$[\mathbf{q}, \mathbf{p}] = i\hbar.$$

This is the Canonical Commutation Relation (CCR) of quantum theory. In classical mechanics, a canonical transformation of conjugate variables is one that preserves their Poisson bracket. (Another way of saying this: the Poisson bracket is a canonical invariant). Thus action-angle variables, reached by a canonical transformation, are canonically conjugate  $\{J, w\} = 1$ . Given the close resemblance of the quantum commutator to the Poisson bracket, there was a strong suggestion that quantum mechanics would likewise allow for an expression in action-angle form.

In the matrix mechanics of Born et al. (1926), a “canonical transformation” was defined as a matrix transformation  $\mathbf{S}$  that preserves the quantum commutation relation, i.e.,

$$[\mathbf{S}\mathbf{q}\mathbf{S}^{-1}, \mathbf{S}\mathbf{p}\mathbf{S}^{-1}] = i\hbar.$$

To solve the dynamical equations of matrix mechanics, the crucial mathematical problem was to find a matrix transformation that brought the Hamiltonian matrix  $\mathbf{H}(\mathbf{q}, \mathbf{p})$  (a function of the position and momentum matrices) into diagonal form,  $\mathbf{S}\mathbf{H}\mathbf{S}^{-1} = W$ . In turn, this transformation gave the time evolution of an arbitrary matrix by means a complex phase  $v(nm) = (W_n - W_m)/\hbar$ , i.e.,  $\mathbf{g}(t) = a(nm)e^{iv(nm)t}$ . Evidently this gave the right result since then

$$\frac{d\mathbf{g}}{dt} = iv(nm)a(nm)e^{iv(nm)t} = \frac{i}{\hbar}(W_n\mathbf{g} - \mathbf{g}W_m).$$

The similarity of form to the classical solutions of the equations of motion in action-angle form encouraged the idea that  $\mathbf{S}\mathbf{H}\mathbf{S}^{-1}$  corresponds to the quantum equivalent of the classical action variable.<sup>2</sup> Classically, the canonically conjugate variable, angle, can be defined by means of the Poisson bracket as the variable conjugate to action. The equivalent quantum variable  $\mathbf{w}$  would, therefore, obey the equation  $[\mathbf{S}\mathbf{w}\mathbf{S}^{-1}, \mathbf{S}\mathbf{H}\mathbf{S}^{-1}] = i\hbar$ . Since the relation of conjugacy is preserved under the transformation  $\mathbf{S}$ , we have  $[\mathbf{w}, \mathbf{H}] = i\hbar$  and thus  $\mathbf{w}$  is conjugate to  $\mathbf{H}$  and varies linearly with time,

$$\dot{\mathbf{w}} = \frac{i}{\hbar}[\mathbf{H}, \mathbf{w}] = 1.$$

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<sup>2</sup>See in particular Born et al. (1926, p. 330).



However, the analogy is less than perfect. While the classical action variable was reached by a canonical transformation of the configuration variables  $q_r$  and their conjugate momenta  $p_r$ , the “quantum action”  $W$  was reached by a “canonical” matrix transformation that takes the Hamiltonian  $\mathbf{H}$  into a special form. In particular, the transformation to a basis in which the Hamiltonian operator is diagonal (i.e. the energy representation). This means that  $\mathbf{S}$  is a unitary operator.<sup>3</sup> The same transformation applied to the quantum position and momentum operators leads not to a distinct set of canonical variables but the *same* operators written in a different basis of vectors.<sup>4</sup> So the quantum “angle” operator  $\mathbf{w}$  defined above is canonically conjugate to the Hamiltonian operator  $\mathbf{H}$ , which represents the energy of the system, not some other “action operator”.

This is a crucial distinction between the situation in classical and quantum mechanics. Whereas in classical mechanics the action variables and the Hamiltonian are distinct, in quantum mechanics the “action variable” is nothing but the Hamiltonian operator expressed in a different basis. In defining the “quantum angle variable” through the relation of conjugacy to the Hamiltonian, it is therefore conjugate to *energy* rather than action. This makes it tempting to regard energy and time as conjugates in quantum mechanics, whereas the analogous situation in classical mechanics maintains a sharp distinction between the notions of action and energy, and—crucially—angle and time. Recall that, classically, angle (the conjugate variable to action) covaries with time. However, a variable that covaries with time is not therefore an angle variable. In the quantum case this distinction is lost: a quantum operator conjugate to energy (or “angle”) covaries with time, and an operator that covaries with time is conjugate to energy.<sup>5</sup>

### 2.1.2 Heisenberg’s Time-Energy Uncertainty Principle

As we have seen, the quantum version of the action-angle variable form of mechanics led directly to (and was indistinguishable from) a relationship between energy and time. On the other hand, in classical mechanics the relationship was between a distinct quantity with units

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<sup>3</sup>An operator is unitary if  $UU^{-1} = U^{-1}U = \mathbb{I}$ . We will investigate further the link between the existence of such a unitary transformation and a corresponding self-adjoint operator in Chapter 3.

<sup>4</sup>The failure to recognize this distinction was endemic in early quantum mechanics. See Duncan & Janssen (2013) for details of how this confusion played out.

<sup>5</sup>This is a rough and ready statement of a fact about conjugate observables that will be explored in detail in later chapters. Also note that there is a directly analogous situation in classical mechanics, i.e. if we demand of a classical observable that it is conjugate to the Hamiltonian then it also covaries with time, and vice versa. This is not the same thing, however, as introducing action-angle variables. See Section 2.4.

of action, and a quantity that *covaries* with time (angle). In his famous uncertainty paper of 1927 Heisenberg introduces separate expressions  $\mathbf{E}\mathbf{t} - \mathbf{t}\mathbf{E} = -i\hbar$ , relating energy and time, and  $\mathbf{J}\mathbf{w} - \mathbf{w}\mathbf{J} = -i\hbar$ , relating action and angle. These are given distinct interpretations in his explanations of the physical content of these expressions. Since these ultimately express the *same* relation, this undoubtedly leads to confusion, and Heisenberg’s initial discussion of the time-energy uncertainty principle is justly criticized on these grounds by Hilgevoord (2005). This confusion arises prior to (and independently of) the fact that the existence of any such operator  $\mathbf{t}$  is thrown into doubt by Pauli’s Theorem.

Despite these confusions, I contend that the physical core of Heisenberg’s interpretation of the time-energy principle (circa 1927) may nevertheless be discerned without much difficulty. What is absolutely crucial to bear in mind is that, at this time, Heisenberg was still deeply wedded to the basic physical picture of Bohr’s atomic model. According to Bohr’s semi-classical model (and later developments by Sommerfeld going under the name ‘the old quantum theory’) the possible states of an atomic system—for hydrogen just a single orbiting electron—were quantized into a discrete series of orbits (or stationary states) each associated with a definite energy of the system. At any given time, the system was to be found in one of those states.

By 1927, it was realized that solutions of the time-independent Schrödinger equation corresponded to stationary states of definite energy.<sup>6</sup> However, such a system was believed to also display the characteristic quantum behavior of ‘jumping’ from one state to another unpredictably and (apparently) instantaneously. Thus, on this picture, there was no doubt that a closed system had a definite stationary state at all times. We could, however, be uncertain as to *which* state the system takes at any particular time and, correspondingly, about the time at which the system jumped between two stationary states.

It is only with this simple physical picture in place that we can make sense of Heisenberg’s interpretation of the energy-time uncertainty principle (as opposed to the action-angle principle), which he expresses as follows:<sup>7</sup>

According to the physical interpretation of quantum theory aimed at here, the time of transitions or “quantum jumps” must be as concrete and determinable

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<sup>6</sup>By Heisenberg, in fact. See the discussion of Darrigol (1993, pp. 329-331).

<sup>7</sup>It is also of interest that Jordan’s view at this time is very similar to Heisenberg’s, and one imagines that this is something that they had discussed together. In his Habilitation lecture (later published in English as a Nature article of 1927) Jordan chose to address the question of indeterminism with reference to the time of a single quantum jump. See the footnote of Duncan & Janssen (2013, pp. 187–188).

by measurement as, say, energies in stationary states. The spread within which such an instant is specifiable is given according to  $[\mathbf{E}t - t\mathbf{E} = -i\hbar]$  by  $\hbar/\Delta E$ , if  $\Delta E$  designates the change of energy in a quantum jump. (Heisenberg, 1927, pp. 76–77. author’s translation)

He then proceeds to describe a simple experiment to demonstrate this relation, which he calls “quite in the spirit of the old formulation of quantum theory founded by Planck, Einstein and Bohr when we speak of the discontinuous change of the energy.” However, the theoretical basis for this relation relies on the existence of appropriate operators  $t$  and  $\mathbf{E}$ , for which an uncertainty relation can be derived along the lines that Heisenberg sketched for position and momentum. Another (somewhat archaic) way of saying this is that, for the time-energy uncertainty relation to have a theoretical basis,  $t$  must be a  $q$ -number rather than a  $c$ -number.<sup>8</sup>

Now, in identifying the stationary states of Bohr’s theory with the energy eigenstates of Schrödinger’s wave mechanics, this amounts to the supposition that a closed system has a definite value for energy at all times, a supposition that was rejected by Schrödinger (and quite rightly).<sup>9</sup> As we know today, a closed system is not restricted to an eigenstate of energy, and its instantaneous state will in general correspond to a superposition of such states. A system in a stationary state (i.e. in an energy eigenstate) will remain in that state indefinitely. Remarkably, this fact was later recognized and embraced by Heisenberg, leading to the view that quantum theory served to *eliminate* time (Bacciagaluppi & Valentini, 2009, p. 185).

According to this view, ‘quantum jumps’ correspond to energy exchanges between interacting systems, and it is this behavior alone that is responsible for our experience of time, which is a statistical phenomenon that only emerges at a macroscopic level, like temperature. Thus for a closed system, like the universe, there could be no time. But, as Schrödinger recognized, this dramatic conclusion rests on a false presupposition: an isolated quantum system does not have a definite value of energy at all times. This realization brought about the end of the quantum jump interpretation of the time-energy uncertainty principle, although the language of quantum jumps was to re-emerge in later discussions of measurement.

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<sup>8</sup>This claim is cast into doubt by the existence of time-energy uncertainty relations that do not rely on a time operator for their derivation (Hilgevoord, 1996). However, there remains a distinction to be drawn with position and momentum, which provide two families of uncertainty relations (corresponding to shifts in momentum and in position) whereas there is only a single time-energy relation corresponding to shifts in time (Uffink & Hilgevoord, 1985).

<sup>9</sup>See the introduction of his presentation at Solvay (Bacciagaluppi & Valentini, 2009, p. 409).

### 2.1.3 Quantum Clocks

What, then, was the meaning of the energy-time uncertainty relation? Schrödinger was led to examine this question through consideration of the complications that quantum uncertainty relations would introduce when setting up an inertial frames of reference.<sup>10</sup> According to Einstein's clock synchronization protocol, to set up a common notion of time requires a collection of distant ideal clocks exchanging light signals with one another. Schrödinger (1931b) argued that the uncertainty relations of quantum mechanics would lead to in principle limitations on the accuracy to which this procedure could be carried out. Ultimately, he called for a revision of the Lorentz transformations, which should be 'quantized' by replacing time and space co-ordinates with operators. As Hilgevoord (2005) convincingly argues, this last claim derives from a conceptual error: in the transition to quantum mechanics position variables are quantized, not spatio-temporal co-ordinates, nor transformations thereof.

However, there is nothing misguided about an investigation of the epistemic limits quantum theory may place on our ability to *know* that we are in an inertial frame.<sup>11</sup> In this vein, Schrödinger (1931a) was led to explore the characteristics of an ideal quantum clock: a system that evolves through a succession of states that mimic the values of the external time parameter to which the instantaneous state of the system (i.e. the state of the clock) is referred. He objects that such a system would be unphysical on the following grounds: since the clock is ideal, its state at each time would have to be perfectly distinguishable from every other state, i.e. the successive states of the system must be mutually orthogonal. However, because of the relationship between energy and time, this entails that the energy of the system is completely uncertain: all values of energy from  $+\infty$  to  $-\infty$  are equally probable. This leads him to conclude that:

Such a state of the system is physically meaningless; the ideal clock, that quantum mechanics uses by the application of the variable  $t$ , is in contradiction with the foundations of quantum mechanics. (From Hilgevoord (2005, p. 50).)

This brings us to Pauli's (1933) discussion of time in quantum theory, which is confined to two footnotes of his Handbuch article on quantum mechanics. Although it is to this article<sup>12</sup> that citations for 'Pauli's Theorem' invariably refer, as it happens neither of these footnotes

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<sup>10</sup>The following two paragraphs rely heavily on Hilgevoord's presentation, (Hilgevoord, 2005, §3.4).

<sup>11</sup>In this connection, see Salecker & Wigner (1958).

<sup>12</sup>Or its English translation, (Pauli, 1958), later published as a book.

contains the proof of a theorem. However, one of them contains the outline of an argument for the conclusion discussed above, that time is a parameter (or  $c$ -number) not an operator ( $q$ -number). As it stands, unfortunately, that argument is not strictly valid. Nonetheless, there are closely related results (for which see Chapter 4), and we will encounter Pauli's argument as it appeared in 1933 in due course.

The first of Pauli's footnoted comments is specifically directed towards Schrödinger's analysis of the ideal clock. I will quote from the main text for context before giving the remark. Following a discussion of measurement uncertainty in the Compton effect (involving the stimulated emission of x-rays by a moving electron), Pauli concludes:

The measurement of position with great accuracy is always possible since the wave-length of the material particle [varies inversely with its mass]. At least in non-relativistic quantum mechanics, where  $v \ll c$ , the following basic assumption is, therefore, natural: *In every state of a system and indeed for free particles there exists, at each instant of time  $t$ , a probability  $W(x_i; t)dx_i$  that the particle is found in the volume interval  $(x_i, x_i + dx_i)$ .* This basic assumption is neither self-evident nor is it a direct consequence of the uncertainty relations. . . . In the formulation of [this basic assumption], a distinction is made between space and time since the position co-ordinates are considered within a margin  $dx_i$  while the time co-ordinate is exactly fixed. Actually, as we have seen, this time point cannot be fixed more accurately than  $\Delta t = \Delta x/c$ . (Pauli, 1958, p. 10)

It is the penultimate sentence to which the footnote refers.<sup>13</sup> In the footnote, Pauli addresses the problem of knowing what time it is, and so Schrödinger's worry about the ideal clock:

In this connection, it is stressed [by Schrödinger] that an ideal clock, i.e. one which gives the time exactly, will possess an infinitely large uncertainty in energy and hence also an infinite energy. According to us, this does *not* mean that the use of the usual time concept contradicts quantum mechanics, since such an ideal clock can be approximated arbitrarily. (Pauli, 1958, p. 10–11, original emphasis)

He draws an analogy here with his earlier argument regarding the uncertainty in position of a massive particle, and what we might call the 'ideal quantum rod': an infinitely massive material particle. On the face of it, this seems quite reasonable. However, possessing an

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<sup>13</sup>It is worth remarking that the probabilities introduced by Pauli here do not concern how accurately one could know one's position in space (as would be required to set up a system of inertial co-ordinates), but rather the likelihood of 'finding' something in a spatial region at a time. When uncertainty in time is introduced here, it is as a consequence of relativistic concerns, and so corresponds to a notion of finding something within a spatio-temporal region rather than a spatial region.

“infinitely large uncertainty in energy” and possessing “an infinite energy” are resolutely *not* the same thing in quantum mechanics, and in this the ideal quantum clock differs crucially from the ideal quantum rod. Possessing an infinite energy in quantum mechanics means, roughly, that a system has the expectation value  $\infty$  for its energy observable. But having an infinitely large *uncertainty* in energy corresponds not to having a large energy, but having a completely uncertain energy: essentially, being in a state in which no value of energy is more likely than any other. Thus the expectation value of energy for Schrödinger’s ideal clock, having possible values of energy that run from  $-\infty$  to  $+\infty$ , is zero (if defined). This is quite different to the situation concerning a (free) massive particle, which can have energy values from 0 to  $\infty$ .

Whether or not Pauli’s conclusion may be supported by other means, the fact remains that an ideal clock in quantum mechanics is not on a par with an infinitely massive particle. Again: Schrödinger’s problem with the ideal quantum clock is that its energy spectrum is unbounded, not that it may possess infinite energy. Most (if not all systems) in quantum mechanics possess a ground state, a state of lowest energy below which the energy of the system cannot drop. Famously, the ground state of the quantum harmonic oscillator corresponds to an energy greater than zero (the zero point energy). The ideal clock, however, is not like this. Its energy can take any value, positive or negative. There is a simple argument for why this leads to a violation (in spirit, if not in letter) of that central canon of modern physics, energy conservation. Take a system which has no lower bound for its energy, and extract from it enough energy to power the whole of Pittsburgh for a day. Do the same thing tomorrow, and the day after, and so on. No matter how much energy you extract from the system, there is always more to be extracted. Since there is no lower bound on the energy of this system it can be used as an energy source indefinitely, and there is thus no total quantity of energy that can be assigned to that system.<sup>14</sup>

This is certainly grounds to be suspicious of the ideal quantum clock. But, according to Schrödinger, something even more alarming is true of this system: not only does the value of energy that the system can possess fail to have a lower bound, but the ideal clock has a completely uncertain value for energy. This means that on interaction with another system, like the poor lab tech who tries to use it to read the time, it can exchange an arbitrarily large amount of energy—either positive or negative. This would make the ideal clock something

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<sup>14</sup>This argument is adapted from Malament (1996).

like a ticking bomb. Fortunately, there is a common understanding in quantum mechanics today that such systems are unphysical. The *spectrum condition* states that any physical Hamiltonian must have a spectrum bounded from below, i.e. a lower bound for the value of energy that the system whose evolution it describes can take.<sup>15</sup>

If this principle is accepted, then Schrödinger’s ideal clock is automatically ruled out. However, the status of Pauli’s claim that “an ideal clock can be approximated arbitrarily” may be unaffected by these arguments. If Pauli is right about this, then there should be a series of physically allowed systems whose behavior approaches that of an ideal clock while avoiding the difficulties of the unbounded energy spectrum that this concept seems to entail. This is the topic of a Chapter 6, where I argue that Pauli’s claim may be substantiated, but that Schrödinger’s ideal clock does not lie at the limit of this sequence. It is a further question how the existence of such a family of systems bears on the claim that “the use of the usual time concept contradicts quantum mechanics.”

The problem for “the usual time concept” posed by Schrödinger’s ideal clock seems to be one for an operationalist, who believes that physical quantities must be associated with concrete experimental procedures. For the operationalist, therefore, the concept of time is meaningless without the means to measure it. Pauli’s argument, it seems, is that the operationalist can be content with a procedure that provides the means to measure time approximately, with an accuracy that may be increased without limit. This seems to be a fair response, particularly in the context of quantum theory, which (as Pauli points out) introduces a similar problem for measurements of position in space. Whether or not this is ultimately a satisfactory analysis of time will depend on one’s philosophical bent, but it does seem that Pauli and Schrödinger were onto something here: a satisfactory notion of time in physics should provide some relation to the means by which it may be measured, even if time itself is ultimately to be defined independently of the means of its measurement. See Section 9.2 for further discussion of these topics.

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<sup>15</sup>The history behind the use of quantum Hamiltonians that do not obey the spectrum condition is fascinating, and begins with Dirac’s relativistic electron equation of 1928. In the ‘hole’ theory that Dirac used to predict the existence of antimatter, the Hamiltonian of the entire system has no lower bound, but the Hamiltonian of a single particle or antiparticle (a ‘hole’ in the completely filled negative energy ‘sea’) has a positive energy. In the field theoretic formalism that replaced it, the Hamiltonian is written as a sum of positive energy terms that are combinations of creation and annihilation operators, each corresponding to a particle or antiparticle with positive energy. See Pashby (2012) for more details of this episode (and its implications for the realism debate).

### 2.1.4 Introducing Pauli's Theorem

We are now ready to discuss Pauli's famous 'theorem,' the detailed discussion of which is the subject of Chapter 4. The argument appears in a footnote to a passage that presents a derivation of Heisenberg's equation of motion. I quote in full:

In the older literature on quantum mechanics we often find the operator equation

$$Ht - tH = \frac{\hbar}{i}I, \quad (2.2)$$

which arises from  $[\frac{\hbar}{i}\dot{F} = HF - FH]$  by substituting  $t$  for  $F$ . It is generally not possible, however, to construct a Hermitian operator (e.g. as a function of  $p$  and  $q$ ) which satisfies this equation. This is so because, from the C.R. [Commutation Relation] written above, it follows that  $H$  possesses continuously all eigenvalues from  $-\infty$  to  $+\infty$  (cf. Dirac, *Quantum Mechanics*, First edition (1930), 34 and 56) whereas on the other hand, discrete eigenvalues of  $H$  can be present. *We, therefore, conclude that the introduction of an operator  $t$  is basically forbidden and the time  $t$  must necessarily be considered as an ordinary number ("c-number") in Quantum Mechanics* (cf. for this Schrödinger 1931). (Pauli, 1958, p. 63, original emphasis)

The sketch that Pauli provides here can be fleshed out into a fully formed argument by following the references to Dirac's *Principles of Quantum Mechanics*. However, this argument cannot be strictly valid since there exist counterexamples to Pauli's claim. That is, (to mention just one class) there exist pairs of self-adjoint operators  $(H, t)$  that satisfy the commutation relation given by Pauli (which, given the Heisenberg equation of motion, entails that the rate of change of  $t$  is unity) even though  $H$  is a Hamiltonian with a discrete spectrum. These pairs of operators can be thought of as describing periodic clocks: a discrete Hamiltonian is conjugate to a phase or angle observable, which is periodic.<sup>16</sup>

One might suppose that Pauli's argument could be rescued by replacing the mistaken assumption he uses for his attempted reductio. In particular, we might suppose that the spectrum condition (see Section 2.1.3) provides a more robust foe for a time operator than the existence of a Hamiltonian with discrete spectrum. However, even this does not suffice to guard against counterexamples, since there exist pairs of self-adjoint operators  $(H, t)$  obeying (2.2) where  $H$  is the Hamiltonian for a quantum harmonic oscillator, and  $t$  is the phase observable. This Hamiltonian has a discrete spectrum that is bounded from below

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<sup>16</sup>This will be analyzed in great detail in Chapters 4, 5 and 6.



since a harmonic oscillator has a ground state. (See Chapters 4 and 5). In Chapter 6 I will argue that the existence of such pairs of operators provides just the sort of approximation to an ideal clock that Pauli appeals to in his footnote considered previously. But what of the physicist’s folklore that Pauli proved conclusively that ‘time is not a  $q$ -number but a  $c$ -number’?

It turns out that there is a closely related argument that may be reached by strengthening the premises of Pauli’s proof, and so it seems plausible that it is this argument (which follows from a single foundational result: the Stone-von Neumann Theorem) that physicists refer to when they speak of ‘Pauli’s Theorem.’ That is, the conclusion that a conjugate pair  $H, t$  can only exist if “ $H$  possesses continuously all eigenvalues from  $-\infty$  to  $+\infty$ ” (i.e. has the spectrum of the reals) can be reached from the stronger assumption that  $H$  and  $t$  are conjugate in the *Weyl sense*, as well as the Heisenberg sense given by (2.2). Given this strengthened premise, the proof proceeds much as Pauli had envisioned and returns precisely this conclusion. This claim will be explained in full in Chapter 4.

For now, note that this result (i.e. the one that follows from a valid argument) must be reckoned with by anyone who wishes to introduce a time operator into quantum mechanics, and I will argue there is good cause to do so. Indeed, we have already met the idea that such operators are required to give an operational definition of time through the use of quantum clocks. It is worthy of note, however, that one of the likely intended targets of Pauli’s argument, Paul Dirac, had introduced time operators into quantum theory in a way that was left untouched by this result. The approach taken by Dirac in his early papers on quantum theory was to introduce time as an additional canonical variable in classical mechanics and then promote it to an operator through his quantization procedure. By following this procedure, he obtained a time operator that was not conjugate to the Hamiltonian, and so this method is left untroubled by Pauli’s Theorem.

### 2.1.5 The Path Less Traveled: Dirac’s Extended Phase Space

To see how things could have been different, that is, how quantum theory could have come equipped with a canonical time operator, we turn to Dirac’s first papers on quantum theory. There we find a heavy reliance on the hypothesized close correspondence between quantum theory and the action-angle form of classical mechanics. This led to Heisenberg’s confusion about the roles of action, angle, time and energy in quantum mechanics. However, Dirac

(1926b) does not display the same confusion in exploring forms of classical mechanics that explicitly promote time to the position of a dynamical variable. This technique involves extending the phase space of the classical theory by two dimensions, corresponding to the time variable  $t$  and its canonical conjugate  $-W$ , i.e. minus the (numerical value of) energy. This variable  $-W$  is quite distinct from the Hamiltonian  $H$  that is defined on the unextended phase space, and  $H$  takes on a new role within the extended phase space.

In particular, fixing the energy of the system involves fixing the value of  $-W$ . The dynamical equations are obtained by setting  $H$  equal to  $W$ , that is, by introducing what is known as the *constraint* equation<sup>17</sup>

$$H - W = 0.$$

Dirac used this extended dynamical equation to treat relativistic dynamics, but also regarded it as necessary when introducing time-varying Hamiltonians that could not be addressed by perturbative methods.<sup>18</sup> As a matter of more than historical interest, it was by this route that Dirac (1926a) introduced the time-dependent Schrödinger equation into quantum mechanics.

From Schrödinger's wave mechanics Dirac had learned that values of energy (eigenvalues of the Hamiltonian matrix) corresponded to eigenfunctions, i.e. functions on configuration space. He had also learned that  $q$ -numbers with a matrix representation could be written as operators on those functions. The relationship of conjugacy that held between position and momentum was revealed to correspond to a particular relationship between the operations by which they were represented. In particular, when position acts by multiplication, momentum acts by differentiation, and vice versa.<sup>19</sup> This correspondence licensed Dirac's substitution of differential operators for classical momentum variables as follows:

$$p = -i\hbar \frac{\partial}{\partial x}; \quad -W = -i\hbar \frac{\partial}{\partial t}.$$

Writing the constraint equation (above) as a wave equation acting on functions  $\psi(x)$ , and making this substitution for  $-W$ , Dirac (1926a) obtained

$$H\psi(x, t) - i\hbar \frac{\partial}{\partial t} \psi(x, t) = 0, \tag{2.3}$$

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<sup>17</sup>For technical reasons that we will not go into here, this is not a strict equality. Instead, we require that dynamical solutions lie on the constraint surface of the extended phase space on which this equality holds. For a philosophical introduction to Dirac's constrained Hamiltonian formalism (a product of his later work on this topic) see Earman (2003).

<sup>18</sup>See Pashby (2014) for more on this topic, and a more detailed analysis of Dirac's approach to time.

<sup>19</sup>Take an arbitrary (well-behaved) function  $f(x)$ . Making the substitutions  $Q = xf(x)$  and  $P = -i\hbar df(x)/dx$  it is easily seen that  $[QP - PQ]f(x) = i\hbar f(x)$  holds, as required.

which we recognize as the time-dependent Schrödinger equation.

However, this is not (repeat, *not*) the Schrödinger equation of standard quantum mechanics, as it is presented in textbooks today. Dirac's equation involves a partial derivative that acts on functions of time and space. The dynamical equation of contemporary quantum theory does not concern functions of time and space but functions of (configuration) space acted on by a unitary group parameterized by time. Explicitly, we have

$$HU_t\psi(x) = i\hbar\frac{d}{dt}U_t\psi(x), \quad (2.4)$$

where  $U_t = e^{iHt}$  is the unitary group uniquely generated by  $H$ , the (self-adjoint) Hamiltonian operator. Note the difference: the differentiation of  $t$  concerns the unitary group  $U_t$  alone and the function  $\psi(x)$  is left alone by this differential operation. In contrast, (2.3) involves a partial derivative acting on functions of  $x$  and  $t$ .<sup>20</sup>

Physical relevance (or otherwise) of (2.3) notwithstanding, this distinction is worth noting because, by introducing  $-W$  as an operator distinct from the Hamiltonian, Dirac provides the means to resist Pauli's Theorem. Classically,  $W$  represents the (numerical value of the) energy of the system, whereas  $H$  is a function of the other phase space variables. In standard quantum mechanics, the Hamiltonian operator  $H$  does double duty as both the generator of time translations and the energy operator, whose spectrum determines the possible values of energy which the system may take. As we will see in Chapter 4, the fact that the Hamiltonian operator is (a) required to be self-adjoint, and (b) has a semi-bounded spectrum rules out the existence of a canonically conjugate time observable (with the spectrum of the reals). However, in Dirac's extended phase space the variable conjugate to  $-W$  is  $t$ , time. The generator of time translations is, therefore, not  $H$  but  $-W$ , and there is nothing to stop  $t$  and  $-W$  from being promoted to canonically conjugate quantum observables on an *extended* Hilbert space.

This procedure is well-established today as a means to reach a quantization of a constrained classical system. In fact, some of these procedures are due to Dirac's later work on this topic, motivated by the desire to quantize general relativity (Dirac, 1964). However, the extended Hilbert space (often referred to as the auxiliary or kinematical Hilbert space) is not the final destination of Dirac's later quantization procedure, and it is typical to seek

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<sup>20</sup>As an aside, I note here that the equation on Schrödinger's gravestone involves  $\dot{\psi}$  i.e.  $(d/dt)\psi$  whereas, on the other hand, the Dirac Equation appears on Dirac's gravestone (in Westminster Abbey) with a partial derivative.

a ‘physical’ Hilbert space containing only functions  $f$  that are annihilated by the quantum constraint operator  $\hat{H}f = (H - W)f = 0$ . Applying this procedure in the present case would result in a Hilbert space equipped with an orthonormal basis of momentum states, but without a time operator.<sup>21</sup> One way to avoid Pauli’s Theorem, then, is to do physics in the *extended* Hilbert space, without seeking a reduced space of physical states. I explore this possibility in Chapter 8.

## 2.2 A STORY OF CONFUSION?

In telling the story of time and energy in the pre-history of quantum mechanics I am following the lead of Hilgevoord (2005), who provides (to my knowledge) the first such history. Hilgevoord, however, takes rather a dim view of the actors involved. He even goes so far as to entitle his essay, “Time in Quantum Mechanics: A Story of Confusion.” I am not sure that the subtitle is entirely justified by its content, and I am inclined to take a rather more favorable view of the motivations and interests of those early quantum pioneers, and the researchers who followed their trail.<sup>22</sup> Hilgevoord’s motivations, however, are admirable: he wishes to provide a dissolution of the problem posed by Pauli’s Theorem, which has been called ‘the problem of time in quantum mechanics’ (Olkhovsky, 2011; Hilgevoord & Atkinson, 2011).

The idea is this: the non-existence of time observables in quantum mechanics is a problem only if there is a legitimate need for them. Hilgevoord seeks to dissolve the problem by arguing that there is no need for time observables in quantum mechanics. There are several components to this project. First, and most impressive, is the understanding of the time-energy uncertainty relations supplied by the Hilgevoord-Uffink analysis of the uncertainty principle (Hilgevoord, 1996). These time-energy uncertainty relations do not require a time observable since the only operators involved are the unitary operators that implement time shifts, which together form a unitary group generated by the Hamiltonian. With

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<sup>21</sup>The state of the art is essentially given by Ashtekar et al. (1995), where they explicitly address the example of a free particle.

<sup>22</sup>This may come down to merely a choice of emphasis. Hilgevoord writes: “When I criticize some of their work of this period, I am fully aware, of course, that these were the heroic years when modern quantum mechanics was taking shape. I have confined myself to what was known at the time and have refrained from discussing important later developments.” (Hilgevoord, 2005, p. 30).

these relations in hand, there is good reason to suppose that we can understand the time-energy uncertainty principle without recourse to a canonical (or general, or external) time observable.

Second is the analysis Hilgevoord provides of ‘internal times’ through an analysis of quantum clocks—quantum systems that possess time observables that function like the hands of a clock. Since Hilgevoord chooses systems that do not obey the spectrum condition, Pauli’s Theorem poses no problem for these clock observables, which are thus self-adjoint. These observables are also intended to supply an operational meaning to time in quantum mechanics, in line with the discussion of quantum clocks in Section 2.1.3. This operational understanding of time is thought by Hilgevoord to suffice for understanding the role of time in the theory, and these systems supply concrete instances of time-energy uncertainty relations (however, in failing to obey the spectrum condition, they represent exceptional systems). On my view, the spectrum condition is not optional, and I devote Chapter 6 to the problem of defining clock observables in quantum theory while respecting Pauli’s Theorem.

The third component of Hilgevoord’s attempted resolution of this problem of time, and the issue that concerns us here, involves his rejection of a mistaken analogy supposed to hold between time and space. Hilgevoord’s argument is essentially this: there is nothing problematic about time being represented by a parameter rather than an operator since *space* is represented by a parameter rather than an operator as well. Hilgevoord claims that the expectation of the authors of quantum mechanics that time should be an observable was due to a confusion between space and position: looking at the role of position as an observable of the theory, they were mistakenly led to the idea that time should be observable too. He traces the source of the confusion to the frequent use of the spatial co-ordinates  $(x, y, z)$  to denote the spectral values of the position observable of a single particle  $(q_x, q_y, q_z)$ .

When presented with an operator whose spectral values appear to correspond to points of space, it is natural to expect also an operator whose spectral values correspond to instants of time. And given the expectation of these authors that quantum mechanics would ultimately be a relativistic theory, it seemed reasonable to demand of a theory set in space-time that time and space should appear on the same footing. However, as Hilgevoord points out, the spectral values of position are *not* identical with spatial points—this correspondence is only valid for a system consisting of a single particle. In general the dimension of configuration space (and so the spectrum of the position observable) is  $3n$ , where  $n$  is the number of

particles. Once this confusion is made apparent, and it is realized that time  $t$  (a parameter) is to be contrasted with space  $(x, y, z)$  (also parameters), the apparent asymmetry is removed and so the justification for defining a time operator (i.e. a canonical time observable) is removed, or so Hilgevoord claims. This leads him to dismiss later developments, such as the recent use of POVMs (Positive Operator Valued Measures) to define (generalized) time observables, as conceptually confused for the same reason (Hilgevoord & Atkinson, 2011).

Now, with regard to this particular justification for regarding time as an operator, I agree that Hilgevoord offers an apt diagnosis: what is being contrasted here is not time and space, but time and position. But while I agree wholeheartedly that it would be a mistake to confuse space, time and position in this way, I am not convinced that this was a confusion to which many (or perhaps any) of the authors of quantum theory were prone. Indeed, Hilgevoord (2005) acknowledges that there were other reasons which led to the expectation that temporal properties were apt for representation by operators. It is my view that these other reasons for defining time operators were more important to those authors—I will claim that some remain compelling today—and are not so easily dismissed as resulting from a simple conceptual error.

Of the authors discussed above (Heisenberg, Schrödinger, Pauli and Dirac) only Schrödinger fell victim to this particular confusion, and he redeemed himself through his perspicacious discussion of quantum clocks. Even von Neumann, apparently regarded by Hilgevoord as an exceptionally egregious offender, does not make this particular mistake, or so I now argue. Here is the passage Hilgevoord quotes:

... we must admit that this objection [against the possibility of instantaneous measurements] points to an essential weakness which is, in fact, the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes the time  $t$  from the three space coordinates  $x, y, z$ , and presupposes an objective simultaneity concept. In fact, while all other quantities (especially those  $x, y, z$  closely connected with  $t$  by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter  $t$ , just as in classical mechanics. Or: a system consisting of 2 particles has a wave function which depends on its  $2 \times 3 = 6$  space coordinates, and only upon one time  $t$ , although, because of the Lorentz transformation, two times would be desirable. (From Hilgevoord (2005, p. 51))

There are several aspects of this passage that can be brought to von Neumann's defense. First, the problematic aspect of the non-relativistic character of quantum mechanics lies in

the requirement that measurement is instantaneous, not the absence of time observables.<sup>23</sup> While von Neumann's claim that all other quantities, including the spatial co-ordinates, are represented by operators does display the mistake that Hilgevoord alleges, the paraphrase he gives of the problem provides the means to exonerate him.

If von Neumann truly regarded a putative time operator as a partner to the 'spatial operators,' why would he assert that *two* times would be desirable, one per particle? It seems far more plausible that he is taking the point of view that was to lead to Dirac's (1932) 'many-time' theory. The idea here is that a (classical) system of many relativistic particles would require  $4n$  variables, rather than  $3n$ . The fourth co-ordinate describing each particle parameterizes the world-line followed by the particle, and can be thought of as the time read by a co-moving clock, displaying the 'age' of the particle.

This appeal to a need for  $4n$  co-ordinates, then, does display a sensitivity to the distinction between the position of a particle in space-time (given classically by its progress along its worldline) and the background spatio-temporal co-ordinates. It is not clear that there is any conceptual confusion in such an idea and, although as a theory of physics it turned out not to be especially useful, there nothing conceptually problematic here, or so I would claim.<sup>24</sup> Admittedly (as Hilgevoord reports) history was not kind to such relativistic particle theories, even in the classical context, and they became something of a niche concern in theoretical physics.

However, there is another reason to expect time observables to find a home in quantum theory. The crux of my argument is the idea that time observables in quantum theory need not 'measure time' (as would a physical clock) but may instead serve to provide probability measures for the occurrence of events at particular (sets of) times, just as the position observable provides probability measures for the occurrence of events at particular (sets of) spatial points. This provides the means to resist Hilgevoord's accusation of pursuing a false analogy since if 'the' time observable concerns the location of an event in time, then it is no false contrast to draw an analogy with the position observable, which concerns the location of an event in space (the event in question being, at first blush, something like 'the particle's

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<sup>23</sup>In Chapter 8 I consider a way in which this requirement is troublesome in non-relativistic quantum theory too.

<sup>24</sup>Dirac's many-time theory was an attempt to find an alternative to the quantum field theoretic approach to interacting matter. He later showed, with Fock and Podolsky, that these approaches are equivalent. Note, however, that Dirac's many-time relativistic quantum mechanics was the inspiration for Tomonaga's 'super-many-time' formulation of quantum field theory, so it did possess some theoretical fecundity, at least.

being here'). This possibility is excluded from consideration by Hilgevoord, even in classical mechanics.

### 2.3 TIMES, CLOCKS, EVENTS

As we have seen, there is some considerable potential for confusion in making sense of the role of time in quantum mechanics. The root of the problem is the fact that the word 'time' is multivalent in the context of a theory of mechanics. To avoid confusion, it will be useful to sharply distinguish these separate roles for time. This can be done most straightforwardly in the classical context. Here is a simple tripartate classification of the ways that time arises in classical mechanics:<sup>25</sup>

**External** External (or parameter) time corresponds to the co-ordinate  $t$  that parameterizes the temporal dimension of the co-ordinates with respect to which the motion of the system is given. Its natural partners are the spatial co-ordinates  $x, y, z$ . The co-ordinates  $(x, y, z, t)$  are usually taken to correspond to an inertial frame, in which the dynamical equations take a particularly simple form.

**Clock** A system may be thought of as a clock to the extent that it possesses a variable whose values over time are functionally related to the value of external time. A system may possess many such variables. A clock variable is ideal if it varies linearly (covaries) with the value of external time.<sup>26</sup>

**Event** An event time is the value of the time parameter that corresponds to the possession of some property of the system. To the extent that we can associate particular values of the time parameter (external time) with particular values of the variables describing the system (or with relations to other variables or external parameters) we may say that the system has *event times*. Given the specification of an event in these terms, we should be able (given the initial data) to determine the time(s) at which that event occurs.

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<sup>25</sup>Inspired by Busch (1990, 2007).

<sup>26</sup>Note that this definition may also be turned around so that external time can be operationally defined as the time read by an ideal clock.



With these distinctions on the table, we can see that external time is to be given independently of the dynamics of the system in question. Operationally, the time co-ordinate corresponds to something like the time read by a good clock, with respect to which the dynamics of the system of interest may be defined. Analogously, the spatial co-ordinates correspond to an independent system of rigid rods with respect to which the position of the system at a time may be given. This is not to say, however, that we shouldn't use our dynamical theory to describe the clocks and rods by means of which these co-ordinates are defined. The point is, rather, that in describing a particular dynamical system in classical mechanics, we must set up some external system of co-ordinates with respect to which the dynamics of the system of interest may be given.

Assuming our dynamical theory to be deterministic, the equations of motion will determine the values of the dynamical variables at all times (given sufficient initial data). The tripartate analysis above reserves two distinct roles for how times may be given in terms of those dynamical variables: clocks and event times. The distinction is important, and turns on the way that clock times and event times vary with the relevant dynamical symmetry, in this case *time translation* symmetry. The defining characteristic of an ideal clock variable is the correlation of the time read by the clock with the value of external time; an ideal clock will *covary* with external time. That is to say that the values of an ideal clock variable  $\tau(t)$  will display the simple linear relationship  $\tau(t) = \tau(0) + t$ . Thus, 'reading the clock' at times  $t'$  and  $t''$  to obtain values  $\tau(t'), \tau(t'')$  will return the corresponding (positive) interval of time,  $t'' - t' = \tau(t'') - \tau(t')$ .

An event time, on the other hand, does not vary with time since it picks out the time at which some relation between the dynamical variables and/or external co-ordinates obtains. The value that an event time takes depends on the dynamics of the system, but it is not therefore a dynamical variable. Event times are defined in terms of the initial data,  $q_i(0), p_i(0)$ , which does not vary with time; an event time  $\epsilon(q_i(0), p_i(0))$  has units of time but does not depend on the value of the time co-ordinate. There is, therefore, no sense in which an event time is a clock time since, for a given solution of the equations of motion, the event time is the same at all times. Nonetheless, there is also a sense in which  $\epsilon$  covaries with time translations, but it does so through its dependence on the initial data,  $\epsilon(q_i(t), p_i(t)) = \epsilon(q_i(0), p_i(0)) - t$ . It is easiest to illustrate this contrast in a specific case.

### 2.3.1 The Classical Free Particle

Classically, the dynamics of the free particle (in one dimension) may be given by its Hamiltonian function  $h = \frac{1}{2m}p^2$ , where  $p$  is its momentum and  $m$  its mass. Solving the equations of motion, we find that momentum is constant,  $p(t) = p(0)$  and the position varies with time as  $q(t) = q(0) + \frac{p(0)}{m}t$ . Thus the dynamical curves are straight lines with gradient  $\frac{p(0)}{m}$ , intersecting the  $x$ -axis at  $q(0)$ . It is straightforward to invert this relationship to obtain the time as a function of position. Call this function  $\tau$ , which depends on the position and the initial data as follows

$$\tau(q(t); q(0), p(0)) = \frac{m}{p(0)} (q(t) - q(0)).$$

Holding the initial data fixed, we may easily see that the value of  $\tau$  acts as a clock variable since

$$\begin{aligned} \tau(q(t''); q(0), p(0)) - \tau(q(t'); q(0), p(0)) &= \frac{m}{p(0)} (q(t'') - q(0)) - \frac{m}{p(0)} (q(t') - q(0)) \\ &= \frac{m}{p} \left( (q(0) + \frac{p}{m}t'') - (q(0) + \frac{p}{m}t') \right) = t'' - t'. \end{aligned}$$

In other words, the function  $\tau$  acts as a clock because it covaries with time through its dependence on  $q(t)$ , the position of the particle at time  $t$ .

In contrast, an event time varies with time through its dependence on the initial data  $q(0), p(0)$ . A simple example is provided by considering the time at which the particle crosses a point in space  $x = 0$ , i.e. the value of  $t$  at the intersection of  $q(t)$  with the  $t$ -axis. At this point, the value of the position variable  $q(t) = 0$ . So we have  $0 = q(t) = q(0) + \frac{p}{m}t$ . Rearranging this expression gives a function of the initial data  $\epsilon_{x=0}$  which returns the time at which  $q(t) = 0$ ,

$$\epsilon_{x=0}(q(0), p(0)) = -\frac{m}{p(0)}q(0).$$

This is known as the *time of arrival* of the particle at the point  $x = 0$ . It is easy to see that this expression, although it does not depend on the value of the time co-ordinate, will covary with time through its dependence on the initial data,

$$\epsilon_{x=0}(q(t_0), p(t_0)) = -\frac{m}{p(0)}q(t') = -\frac{m}{p(0)} \left( q(0) + \frac{p(0)}{m}t_0 \right) = \epsilon_{x=0}(q(0), p(0)) - t_0.$$

This is intuitively exactly what we would expect: by shifting the initial data forward in time by  $t'$  we have reduced the time it takes the particle to reach the point  $x = 0$  accordingly. In contrast to a clock variable, we are not considering two points along the same dynamical

trajectory, but rather comparing two dynamical trajectories related by a time translation so that the second trajectory passes through the point  $(q(t_0), p(t_0))$  at time  $t = 0$  rather than time  $t = t_0$ .

### 2.3.2 Introducing Event Time Observables

Let us return to Hilgevoord's (2005) rejoinder that time already has an appropriate representation in the theory as a parameter, and so the use of extended phase space is misguided. In order to understand (and sidestep) these complaints it will be helpful to outline Hilgevoord's classification of the ways that time may arise in a physical theory (his Section 1.1). The main distinction that Hilgevoord draws is between external time and internal times. External (or co-ordinate) time  $t$  is the partner of the spatial co-ordinates  $x, y, z$  in a co-ordinatization of the space-time in which the system is situated,  $(x, y, z, t)$ . Hilgevoord contends that this role for external time suggests that it should remain a parameter rather than a variable, and regards the use of the extended phase space in classical mechanics as 'unnatural' (Hilgevoord, 2005, p. 36).

Internal times, by contrast, are to be found among

... the internal spatial and temporal variables connected with the specific physical systems the theory aims to describe, such as the position variables of particles and time variables of clocks. These variables are dynamical: they obey equations of motion. ... if we look for time operators we must look for internal times provided by special physical systems, 'clocks.' (Hilgevoord, 2005, p. 31)

Thus, for Hilgevoord, the only valid motivation for defining time operators comes from the canonical quantization of variables that describe physical clocks. However, this ignores a large body of foundational work in quantum theory regarding what I will term *event time* observables.

For example, in a recent review of the time-energy uncertainty relation Busch (2007) describes another role for time in addition to external and internal (or intrinsic) time, which he calls *observable* time. He motivates this definition as follows:

[T]he study of dynamics often involves experimental questions about the time of an event, the time difference between events, or the duration of a process associated with the object system. This [also] raises the quest for a treatment of time as an observable. (Busch, 2007, p. 76)

Note that this motivation is quite distinct from the idea that external time should be promoted to a variable (for whatever reason). In this case, we begin with an experimental quantity—the time of an event, such as the moment that a Geiger counter clicks, or the duration of a process, such as tunneling time—and demand that the theory provide a suitable prediction for the distribution of these events in time (or durations). This motivation is, therefore, most closely related to Heisenberg’s notion of the time of a quantum jump, regarded as an event occurring at a definite moment of time (discussed in Section 2.1.2). However, there is an important difference since proponents of event time observables do not purport to be measuring the time of a quantum jump, or collapse, or what have you. The idea is that the events in question correspond to determinable outcomes of an experiment, which are considered to be distributed in time rather than, say, space.

In this case, then, the analogy to position is a good one: an event time observable concerns the position of an event in time just as the position observable concerns the position of an event in space. In this connection it may help to visualize a standard diffraction experiment, in which the experimentally determinable positions concern the location of a dot on, say, a photoluminescent screen. In this context, the event time observable concerns the distribution of times at which a dot appears (after emission), whereas the position observable concerns the distribution of spatial locations at which a dot appears. In modern quantum theory, to derive such a distribution from the system state we are required to define a suitable observable, customarily a self-adjoint operator. In effect, Pauli’s Theorem serves to show that this cannot be done in this case, and thus measurement of an event time observable cannot correspond to measurement of a self-adjoint operator. It is towards this ‘problem of time’ that this dissertation is addressed, culminating in a positive account of event times as conditional probabilities in Chapter 8.

### 2.3.3 Event Times and Clocks in Quantum Mechanics

Although the distinction between clocks and event times is easily drawn and recognized in classical mechanics, there has been an unfortunate tendency to conflate these cases when considering quantum mechanics. The source of this confusion is often a failure to properly distinguish between the Heisenberg and Schrödinger pictures when considering how an event time observable should act on the state. In the Schrödinger picture, states vary with time

and observables remain fixed; in the Heisenberg picture, observables vary with time and states remain fixed. The equivalence between the two pictures can be expressed as follows, where  $A$  is a Schrödinger picture observable and  $A(t)$  its Heisenberg picture equivalent:

$$\langle \psi_t | A \psi_t \rangle = \langle U_t \psi_0 | A U_t \psi_0 \rangle = \langle \psi_0 | U_{-t} A U_t \psi_0 \rangle = \langle \psi_0 | A(t) \psi_0 \rangle.$$

Here  $\psi_0$  is the Heisenberg state of the system or, equivalently, the Schrödinger picture state  $\psi_t$  at time  $t = 0$ . The unitary group of time translations  $U_t$  implements the time translation symmetry of quantum mechanics, and it is through this group that the equivalence is defined.

When considering clock times, either picture will suffice since the value of the clock observable can either be seen to vary as the state changes, or itself can be seen to vary with time. Let  $T_c$  be a Schrödinger picture ideal clock observable, then through the corresponding Heisenberg picture observable we may express its time covariance as follows:

$$T_c(t) = U_{-t} T_c U_t = T_c + \mathbb{I}t.$$

However, if event times are of interest then the dependence on the state arises through dependence on the initial data, not the time. Only the Heisenberg picture suffices to capture this idea. Let  $E_0$  be an event time observable. Then  $\langle \psi_0 | E_0 \psi_0 \rangle$  gives the probability distribution for the time of the corresponding event, with  $\psi_0$  being the Heisenberg state of the system. If we vary the initial state to  $\psi_{t_0} = U_{t_0} \psi_0$  then we should expect to see a corresponding shift in the distribution,  $\langle \psi_{t_0} | E_0 \psi_{t_0} \rangle = \langle U_{t_0} \psi_0 | E_0 U_{t_0} \psi_0 \rangle$ . Calling this operator  $E_{t_0} = U_{-t_0} E_0 U_{t_0}$ , we can see that  $E_{t_0} = E_0 - t_0 \mathbb{I}$  since the initial state  $\psi_{t_0}$  amounts to a reparameterization of the family of Schrödinger picture states  $\psi_t = U_t \psi_0$  such that  $\psi_{t_0}$  now gives the state at time  $t = 0$ . Thus we have,

$$E_0(t) = U_{-t} E_0 U_t = E_0 - \mathbb{I}t,$$

and we see again that an event time observable covaries in the opposite sense. While this relationship may be numerically identical to a corresponding relationship expressed in the Schrödinger picture, the important conceptual dependence of the distribution on the *initial* state is lost once the Heisenberg picture is abandoned. The value of an event time observable does not covary with the external time co-ordinate, it only covaries with the initial state of the system. Each entire evolution of the system through time corresponds to just one event time distribution; thus there is no Schrödinger picture event time observable, which would (in effect) have to be measured at an instant of time, and whose value would change with time.

### 2.3.4 The Quantum Free Particle

To illustrate this point, and the confusion that a failure to appreciate it may lead to, let us consider the quantum equivalents of the free particle clock and event time functions. Here we consider the (Heisenberg picture) position observable  $Q(t)$  and momentum observable  $P(t)$  as the equivalents to the classical position and momentum  $q(t), p(t)$ . The Hamiltonian is  $H = \frac{1}{2m}P^2$ . As before, the momentum is constant,  $P(t) = P$ . The position varies in much the same way,

$$Q(t) = U_{-t}Q(0)U_t = Q(0) + \frac{1}{m}Pt,$$

where  $U_t = e^{-iHt}$  is the (one-parameter strongly continuous) unitary group generated by the Hamiltonian  $H$ .<sup>27</sup> Let us begin with the event time observable corresponding to a quantization of the classical time of arrival,  $\epsilon = -\frac{m}{p(0)}q(0)$ . Simply transcribing the expression in quantum form won't suffice, since this would introduce an ordering ambiguity of  $Q$  and  $P$ . To reach a symmetric operator  $T_a$  (with a real spectrum) we take instead the symmetric product as follows:

$$T_a = -\frac{m}{2}\langle(Q(0)P^{-1} + P^{-1}Q(0))\rangle.$$

This is commonly described as the time of arrival operator, and has been the subject of much foundational study (and controversy) over the years.<sup>28</sup> What concerns us here is the behavior of this operator under time shifts of the Heisenberg state  $\psi_0$ . It can easily be seen that (formally, at least) we have<sup>29</sup>

$$T_a(t) = U_{-t}T_aU_t = T_a - \mathbb{I}t,$$

as expected. Let us turn to the quantum version of the free particle quantum clock  $\tau(t)$ . By transcription we have

$$T'_c = mP^{-1}(Q(t) - Q(0)) = mP^{-1}(U_{-t}Q(0)U_t - Q(0)).$$

Following the usual symmetrization procedure, we obtain

$$T_c(t) = \frac{m}{2}(U_{-t}(P^{-1}Q(0) + Q(0)P^{-1})U_t - (P^{-1}Q(0) + Q(0)P^{-1})) = T_a - T_a(t).$$

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<sup>27</sup>We will consider these notions in quite a bit more detail in subsequent chapters. For now, the idea is just to get enough on the table so we may discuss the quantization of a classical system.

<sup>28</sup>For more on the quantum time of arrival see in particular Chapter 5.

<sup>29</sup>This follows from the rule  $[Q, f(P)] = f'(P)[Q, P]$ .

But since  $T_a$  covaries with time translations, this merely says that  $T_c(t) = t$ . While this is undoubtedly a function that covaries with time, it is in no sense a quantum observable.

This is not intended as a proof that there is something wrong with the idea of a quantum clock, but rather an indication that we should be wary of blindly following a classical to quantum transcription procedure. In this case, there is a legitimate quantum clock in the vicinity, but it must be expressed in terms of expectation values rather than operators. Making use of Ehrenfest's theorem,

$$\frac{d}{dt}\langle A \rangle = -i\langle [A, H] \rangle,$$

we can express the quantum clock corresponding to  $\tau$  as follows:

$$\langle Q(t'') \rangle - \langle Q(t') \rangle = \langle \psi_{t''} | Q \psi_{t''} \rangle - \langle \psi_{t'} | Q \psi_{t'} \rangle \approx t'' - t',$$

where the accuracy of the clock will depend on the spread in position of the states  $\psi_{t'}$  and  $\psi_{t''}$ .

However, note that this clock works by taking the difference of two separate measurements of position at distinct times, which (being incompatible) must be performed on *different* systems in the ensemble. If a description in terms of a single system is required then the effect of the initial measurement of position on the later measurement must be taken into account. Considered as two separate measurements, the difference in the statistical averages of the two measurements of  $Q$  will provide an approximate indication of the elapsed time. But such a 'clock' is very little use in setting up a system of inertial co-ordinates, for example. For that purpose, we would prefer a clock observable that can be measured on a single system, at a particular time. Chapter 6 is devoted to the definition of such clock observables.

With the example of a free particle in mind, it is worth briefly discussing one of the earliest attempts to define a quantum clock by Aharonov & Bohm (1961). They consider the following observable of the free particle as a quantum clock,

$$T_{AB} = \frac{m}{2} (QP^{-1} + P^{-1}Q).$$

This differs from  $T_a$ , the time of arrival, only by a difference in overall sign. This is enough to ensure that  $T_{AB}$  covaries in the direction of a clock, but that does not suffice to make it a clock observable. Consider the corresponding classical expression,  $\tau_{AB} = \frac{m}{p}q(0)$ . This expression has no dependence on time, and so doesn't describe a clock. A closely related

classical clock is  $\tau_c = \frac{m}{p}q(t)$ , arrived at from our earlier expression by choosing  $q(0) = 0$ . But the quantum analogue of this expression is again  $T'_c$  (as above) since in quantum mechanics we cannot set an observable to zero by choosing a particular initial state.<sup>30</sup>

Finally, it has often been suggested that a particle falling freely in a uniform gravitational field supplies a good example of a clock variable in classical and quantum mechanics (Busch et al., 1994; Hilgevoord, 2002; Busch, 2007). However, the same argument as above shows that the formal similarities between the two cases are illusory, and that such a system does not provide a good example of a quantum clock, in the sense considered here.<sup>31</sup> The classical Hamiltonian is  $h_g = \frac{1}{2m}p^2 - mgq$ . The equations of motion, therefore, are  $p(t) = p(0) + mgt$  and  $q(t) = q(0) + \frac{p(0)}{m}t + \frac{1}{2}gt^2$ . The following function provides a good clock, irrespective of whether  $p(0) = 0$ ,

$$\tau_g(p(t); p(0)) = \frac{1}{mg}(p(t) - p(0)).$$

This has led some to suggest that, given a Hamiltonian  $H = \frac{1}{2m}P^2 - mgQ$ , the following operator provides a good quantum clock

$$T_g(t) = \frac{1}{mg}P(t).$$

However, this ignores the dependence of the classical expression on  $p(0)$ , which corresponds to the Schrödinger picture momentum operator  $P$ , i.e.  $P(0)$ . The correct expression is, therefore,

$$T_g(t) = \frac{1}{mg}(P(t) - P(0)).$$

But, as with the free particle clock, this involves incompatible measurements of  $P(0)$  and  $P(t)$  which (obviously) cannot be performed at the same time, nor on the same system (without taking into account the effect of the first measurement on the second). This is again an ensemble level expression, which should properly be expressed in terms of the expectation values  $\langle P(0) \rangle$ ,  $\langle P(t) \rangle$  for measurements performed at different times on distinct systems.

## 2.4 LOCAL AND GLOBAL COVARIANCE IN CLASSICAL MECHANICS

The expressions for clocks and event times we have met so far have had the characteristic that covariance with time holds for all values of  $t$  (if not all values of  $p$ ). This can be seen,

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<sup>30</sup>We may, however, set an expectation value to zero by this means.

<sup>31</sup>See Chapter 5 for further reasons to be suspicious of this purported clock observable.



therefore, as a *global* condition on such time functions. However, a periodic system, which may make for an excellent clock, will nonetheless fail to provide a time function that meets this global condition. For example, the phase of a simple harmonic oscillator provides a good *periodic* clock, whose value faithfully tracks time through a single period of oscillation. The same value of phase will, however, correspond to many values of the time parameter.

In more detail, the Hamiltonian of the simple harmonic oscillator is  $h_{osc} = \frac{1}{2m}(p^2 + m^2\omega_0^2q^2)$ , so that the dynamical motions of the particle describe a family of ellipses in phase space. For a given value of energy, the state of the system at a time is a point lying on the corresponding ellipse. The equations of motion yield  $q(t) = q(0) + \frac{p(0)}{m}t$ ,  $p(t) = p(0) - m\omega_0^2q(0)t$ , but in this form the periodic motion of the point about the ellipse is obscured. This problem is displayed prominently by the following candidate time function:

$$\tau(q(t), p(t); p(0), q(0)) = \tan^{-1} \left( \frac{mq(t)}{\omega_0 p(t)} \right).$$

We can see that  $\tau$  is linear in time,  $\dot{\tau} = \{\tau, h_{osc}\} = 1$ , but the range of this function is just  $-\pi/2 \leq \tau \leq \pi/2$ . There is, therefore, a sense in which this is a *local* time function, but not a *global* time function.

Since the system is conservative, the energy of the particle is fixed, and we should be able to describe the motion of the system state about the ellipse solely in terms of an angular coordinate. We seek, that is, action-angle co-ordinates  $J, v$  such that the energy is proportional to  $J$  and  $\dot{v} = \text{const}$ . With this in mind, we define  $J$  such that  $h_{osc} = \omega_0 J$ . Now define  $q, p$  in terms of  $J$  and  $v$  by setting  $q(J, v) = \sqrt{\frac{2J}{m\omega_0}} \sin v$  and  $p(J, v) = \sqrt{2m\omega_0 J} \cos v$ , thus ensuring that  $h_{osc} = \omega_0 J$ . This gives our canonical transformation, and we can see that the equations of motion in the new canonical co-ordinates  $(J, v)$  read  $\dot{J} = 0$ ,  $\dot{v} = \omega_0$ , as desired.

If we integrate these equations we have  $J(t) = J_0$ , a constant, while  $v$  increases linearly with time  $v(t) = v(0) + \omega_0 t$ . However, the linearity of this relationship is illusory since the position (or momentum) of the particle depends on  $v(t)$  through the cosine (or sine), which gives the motion its natural periodicity of  $2\pi/\omega_0$  seconds. Thus the same state  $(q(t), p(t))$  will be occupied by the system at all times  $t' = t + m2\pi/\omega_0$ , with  $m \in \mathbb{Z}$ . To make this periodicity explicit, we may introduce complex ‘quadrature’ variables

$$h_{osc} = \omega_0 J (\cos v + i \sin v)(\cos v + i \sin v) = \omega_0 J (P + iQ)(P - iQ) = |Ae^{i\phi}|^2$$

where  $Q = \sqrt{\frac{m\omega_0}{2J}}q$  and  $P = 1/\sqrt{2m\omega_0 J}p$ . This gives us a description of the motion in terms of a real amplitude  $A = \sqrt{\omega_0 J}$  and a complex phase  $e^{i\phi} = \cos v + i \sin v = P + iQ$ , so that  $\phi(t) = v_0 + \omega_0 t$ .

This suggests another candidate time function,  $\tau_{osc}(v(t); J_0, v(0)) = -i \ln(e^{i\phi(t)})/\omega_0$ . This function has as its range  $-\pi/\omega_0 \leq \tau \leq \pi/\omega_0$ , i.e. a full period of the motion (with units of time). As the time  $t$  increases the function  $\tau$  forms a characteristic sawtooth with gradient one and period  $2\pi/\omega_0$ . However, there is a sense in which the discontinuity in this time function is not matched by a discontinuity in phase. It is easy to see that at each time the rate of change of the phase parameter  $\phi$  is linear,  $\dot{\phi} = \omega_0$ . The Poisson bracket of  $\tau$  with  $h_{osc}$  provides the means to express this *local* covariance:  $\dot{\tau} = \{\tau, h_{osc}\} = 1$ . It also provides an expression of their conjugacy, analogous to the Heisenberg CCR in quantum mechanics.

This approach differs crucially from Hilgevoord's approach to defining classical "time variables" (Hilgevoord, 2002, 2005; Hilgevoord & Atkinson, 2011). The defining characteristic of Hilgevoord's time variables is a *global* covariance condition  $\tau_{Hil} \rightarrow \tau_{Hil} + t$ , for all  $t \in \mathbb{R}$ . This global covariance condition implies local covariance,  $\{\tau_{Hil}, h_{Hil}\} = 1$ , but the global condition is stronger since it is not satisfied by our  $\tau_{osc}$ , while the local condition is. It is, however, satisfied by  $\phi$ , considered as a physical variable rather than a phase. In fact, Hilgevoord (2005) suggests that we treat  $\phi$  and  $J$  as the position and momentum of a physical system (an ideal clock) instead of canonical co-ordinates that describe the periodic motion of a harmonic oscillator.

In that case, we have as before  $\{\phi, J\} = 1$ , but we can give this relation a global expression in terms of the Hamiltonian vector fields  $X_\phi, X_J$  associated with  $\phi$  and  $J$ .<sup>32</sup> First consider the Hamiltonian vector field of  $J$ ,  $X_J = IdJ$ . At each point  $(\phi, J)$  of our phase space we have  $\dot{\phi} = 1$ , while  $\dot{J} = 0$ . Evidently the dynamical motions of such a system are given by curves  $\phi(t) = \phi(0) + t$ , and the corresponding one-parameter group that takes one along the curves is just  $h^t : \phi \rightarrow \phi + t$ . The Hamiltonian vector field of  $\phi$ ,  $X_\phi = Id\phi$  yields  $\dot{\phi} = 0$  and  $\dot{J} = 1$ . The dynamical motions of this system are  $J(s) = J(0) + s$  and the corresponding one-parameter group is  $h^s : J \rightarrow J + s$ .

The Poisson bracket, considered as an operation on Hamiltonian vector fields, associates with two vector fields  $X_\phi, X_J$  a third vector field  $X_{\{\phi, J\}} = [X_\phi, X_J] = \omega(X_\phi, X_J)$ , where  $\omega$  is the canonical symplectic form. In this case, the vector field just associates the unit vector

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<sup>32</sup>See, e.g., Arnol'd (1989).

with each point  $(\phi, J)$ . This provides a global expression of conjugacy, which describes the mutual covariance of the two variables  $\phi$  and  $J$  under the transformations associated with their respective Hamiltonian vector fields. If we follow Hilgevoord in regarding  $J$  as a physical Hamiltonian, then it describes a system with a variable  $\phi$  whose value covaries with time shifts along the dynamical curves,  $\phi(t) = \phi(0) + t$ . But this ideal clock has an odd characteristic: its Hamiltonian is unbounded above and below.

As a conservative classical system, this is perhaps not too worrying a characteristic for an ideal clock. We have already met such a system: the freely falling particle. But the consequences of an unbounded quantum Hamiltonian are quite alarming, as was discussed in Section 2.1.3. What concerns us here is the fact that many systems we care about do not have an unbounded Hamiltonian, and yet we were able to define quite satisfactory time functions. In the case of the free particle (Section 2.3.1), this function varied with time linearly, but in the case of a harmonic oscillator the covariance was periodic. This is one sense in which Hilgevoord demands *global* time covariance (i.e. covariance with time shifts covering the whole real line, rather than just an interval). There is also a sense in which  $\phi$  is globally *defined* rather than locally: the Hamiltonian vector field generated by  $J$  is defined at every point  $(\phi, J)$  of the phase space.

Note how our time function  $\tau_{osc}$  differs from  $\phi$ , therefore, in two respects:  $\tau_{osc}$  is undefined at  $(q, p) = (0, 0)$ , and, although  $\tau_{osc}$  varies linearly with  $t$  at every other point,  $\tau_{osc}$  does not globally covary with  $t$  since its range is periodic. That is, as we go about a dynamical curve (a closed ellipse) there is no way to continuously assign values in  $\mathbb{R}$  to each point in the curve (without a periodic discontinuity). On the other hand, the time function  $\tau$  we defined for the free particle is undefined when  $p = 0$ , but covaries with time globally. Evidently a periodic time function cannot be said to covary with time globally, yet a globally covariant time function must therefore increase linearly with  $t$  at every point at which it is defined,  $\{\tau, h\} = 1$ . This is an important distinction, but the two time functions also have an important feature in common: they cannot be defined at every point of phase space.

This means that there is no Hamiltonian vector field associated with  $\tau$  or  $\tau_{osc}$ , where a Hamiltonian vector field  $X$  assigns to every point  $x \in \mathcal{Q}$  a vector in the tangent bundle  $X(x) \in T\mathcal{Q}_x$ . Clearly  $\tau$  does not associate a vector with a point when  $p = 0$ , since then the value of  $\tau(q(t)) = \frac{m}{p(0)}(q(t) - q(0))$  is undefined. The fact that  $\tau$  is not a smooth function  $f : \mathcal{Q} \rightarrow \mathbb{R}$  also disqualifies it from membership of the Hamiltonian functions, which are

uniquely associated with a Hamiltonian vector field. This characteristic smoothness may be taken to be the defining feature of an observable of classical mechanics, in which case our time functions are not time observables. Note how this is a consequence of the failure of global *definition*, not global covariance.

This seems to be a general characteristic of the time functions we have met so far: they are not Hamiltonian functions, i.e. smooth functions  $f : \mathcal{Q} \rightarrow \mathbb{R}$ . This suggests that this might be a generic feature of time functions associated with semi-bounded Hamiltonians. The following result of Roberts (2014) shows that there is a sense in which this is indeed the case.

**Proposition 2.1.** *(Roberts). If  $\tau$  and  $h$  are smooth functions  $f : \mathcal{Q} \rightarrow \mathbb{R}$  such that  $\{\tau, h\} = 1$  then either  $h$  is unbounded or  $\tau$  generates an incomplete vector field.*

This proposition has the consequence that if  $h$  is semi-bounded then either there is no such  $\tau$ , or  $\tau$  generates an incomplete vector field. A vector field is complete if every maximal integral curve has the entire real line as its domain, otherwise it is incomplete.

This last clause is required to avoid potential counterexamples of the following sort, first suggested by John D. Norton.<sup>33</sup> Let  $h = e^p$  and  $\tau = q/e^p$ . Then  $\tau$  and  $h$  are smooth functions with  $h$  semi-bounded and  $\{\tau, h\} = 1$ . In this case,  $\tau$  generates an incomplete vector field. Hamilton's equations return  $\dot{q} = -qe^{-p}$  and  $\dot{p} = -e^{-p}$ . The integral curves are everywhere tangent to this vector field, and the following curves satisfy this requirement:  $q(s) = q_0 - s/e^p$ ,  $p(s) = \ln(e^p - s)$ . We can see that the curve passing through  $(q, p) = 0$  is incomplete since then  $q(s) = 0$  and  $p(s) = \ln(1 - s)$ , which is defined for  $s < 1$ . More generally,  $p(s)$  is not defined for  $s < e^{-p}$ .

It is not clear what physical significance incompleteness could have since it concerns the Hamiltonian vector field generated by  $\tau$  rather than  $h$ . In addition, the semi-bounded Hamiltonians we have examined in detail thus far have not had smooth time functions. What the above proposition shows, then, is that cases where smooth time functions can be associated with semi-bounded Hamiltonians are exceptional. To return to the case of the free particle, the time function  $\tau$  fails to be smooth as there is a stationary point at  $p = 0$ . At such a point the Hamiltonian vector field of  $h$  vanishes, and an integral curve passing through this point is constant,  $q(t) = q(0)$ . It is this feature which prevents us from inverting the equation of the curve to obtain a time function. This is a generic feature of stationary

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<sup>33</sup>Personal communication.

points, which rule out the possibility of defining a (monotonically increasing) time function along such a curve.

The Hamiltonian  $h = e^p$  was chosen so as to avoid the existence of stationary points, and thus led to a smooth time function. The Hamiltonians we encounter in practice, however, tend to involve the familiar  $p^2$  squared term associated with kinetic energy. A generic Hamiltonian featuring such a term along with a finite potential term that depends linearly on  $q$ ,  $h(p, q) = (1/2m)p^2 + V(q)$ , is guaranteed to have stationary points, and therefore does not possess an (everywhere defined) smooth time function. In the context of classical mechanics, this is no obstacle to using the motion of such a system as a clock. So long as the particular dynamical curve picked out by the motion of the system is a curve that does not pass through a stationary point, the equations of motion will provide a time function.<sup>34</sup>

In classical mechanics, in fact, the existence of a *locally* valid time function in such circumstances is essentially guaranteed. Consider a classical particle with a potential,  $h = (1/2m)p^2 + V(q)$ . In this example energy is conserved since  $\{h, h\} = 0$ , and thus the Hamiltonian is time independent.<sup>35</sup> Therefore, at every point on a dynamical curve the energy is the same  $h(q(0), p(0)) = h(q(t), p(t)) = E$ . Inverting the Hamiltonian, we have  $p = \pm\sqrt{2m(E - V(q))}$ . Hamilton's equations yield  $\dot{p} = dV/dq$ ,  $\dot{p} = p/m$ . Since we have a definition of  $p$  in terms of  $E$  and  $V(q)$ , we may integrate  $\dot{p}$  to obtain

$$\tau(q(t)) = C \pm \int \frac{dq}{\sqrt{2m(E - V(q))}}.$$

Even if this integral cannot be performed, this serves as an implicit definition of a time function which covaries along the dynamical curves.

This idea applies quite generally to systems in classical mechanics. Roberts (2014) argues that the existence and uniqueness of solutions to ordinary differential equations imply that any such Hamiltonian yields a locally defined time function, at least in the neighborhood of a non-stationary point.

**Proposition 2.2.** (*Roberts*). *If the integral curve generated by  $h$  passes through a point  $x \in \mathcal{Q}$  that is not a stationary point, then there exists a function  $\tau : B \rightarrow \mathbb{R}$  such that  $\tau(c_t) = \tau(c_0) + t$  in a neighborhood  $B$  of  $x$ .*

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<sup>34</sup>In quantum mechanics, however, these problems will become considerably more acute since in general a system will not possess a definite value of momentum.

<sup>35</sup>Note that a smooth function  $f$  is associated with a first integral, and so a conservation law, if  $\{f, h\} = 0$ .

Similarly, one could use Liouville's Theorem (Arnol'd, 1989, p. 283) to demonstrate that, locally at least, there is always a transformation available to action-angle co-ordinates (so long as the conditions of the theorem are met).<sup>36</sup> The defining characteristic of an angle variable is its linear evolution with time,  $\dot{\phi} = \omega_0$ , and so these are good local time functions. This, therefore, provides a generalization appropriate to our earlier discussion of the harmonic oscillator.

This discussion of the situation regarding time functions in classical mechanics has served to illustrate some of the subtleties that arise in quantum mechanics, without the additional conceptual difficulties that the quantum transition brings. We will see that the distinction between local time covariance (in terms of the Poisson bracket of two functions on phase space) and global time covariance (in terms of the Poisson bracket of Hamiltonian vector fields) is replicated in the distinction between the Heisenberg and Weyl Canonical Commutation Relations. If classical observables are required to be smooth functions  $f : \mathcal{Q} \rightarrow \mathbb{R}$  then Proposition 2.1 can be thought of in analogy to Pauli's Theorem. There are also methods of generating generic symmetric quantum time operators that can be thought of as analogous to Proposition 2.2. However, the interpretative and formal differences that arise in quantum theory are important, and reasoning by classical analogy will only take one so far.

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<sup>36</sup>To apply Liouville's Theorem requires that a system defined on a  $2n$ -dimensional phase space has  $n$  independent first integrals in involution, i.e. associated with (independent) smooth functions  $f_n$  such that  $\{f_n, f_m\} = 0$  for  $m \neq n$ . In case the phase space is of the form  $\mathbb{R}^{2n}$ , the dynamics of the system are confined to a sub-manifold of the phase space where the dynamical motions describe a torus invariant under the Hamiltonian flow. In that case, the angle variables provide ideal periodic time functions. More generally, there always exists a canonical transformation to action-angle variables in the region of a point on the sub-manifold.

### 3.0 ORDINARY QUANTUM MECHANICS, AND BEYOND

This chapter introduces some crucial elements of the formalism of modern quantum mechanics, within which our further investigations into time and time operators will take place. This presentation is not intended to be complete, and its purpose is rather to highlight some of the more specialized formal and interpretative apparatus that successive chapters will depend on, or bring into question.<sup>1</sup> The particular focus is the Spectral Theorem, and its role in securing a probabilistic interpretation of the state of a quantum system. The interest for us here lies in the use of the Spectral Theorem to justify the common restriction of the observables of the system to just the set of self-adjoint operators.

Put roughly, Pauli's Theorem (the topic of Chapter 4) prohibits the definition of self-adjoint time operators. However, as Chapters 5 and onward will show, it does not thereby prevent one from defining time operators that fail to be self-adjoint while having other desirable properties. The extent to which Pauli's Theorem serves as a prohibition on the existence of time *observables*, then, will depend on the extent to which the set of observables of a system must be limited to operators that are self-adjoint. I will argue here for the emerging consensus that this restriction is unnecessary, i.e. that the grounds for this in principle limitation on the observables of a system are less firm than was once thought, and that there is good reason to suppose that a full account of the empirical consequences of quantum mechanics requires a richer set of operators.

With this restriction removed, Pauli's Theorem is less of a prohibition on the existence of time observables and more of a restriction of their form, in which case there is reason to believe that the time operators of later chapters may have empirical significance despite their non-self-adjoint status. In turn, evidence of their empirical significance would also underline the necessity of expanding the set of observables to include these operators which,

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<sup>1</sup>The sort of textbook treatment that lies in the background here is exemplified by the classics of Jauch (1968) and Jordan (1969), or more recently, say, Sakurai (1994) and Ballentine (1998).

of necessity (i.e. as a consequence of Pauli's Theorem), cannot be self-adjoint. This concludes this brief preview of upcoming attractions.

Let us now turn to the quantum mechanics immortalized in textbooks of the recent past, what I will term Ordinary Quantum Mechanics (often called the Dirac-von Neumann formalism). Ordinary Quantum Mechanics (hereafter Ordinary QM) is characterized by the following commitments:

**State Space** The pure states are the vector states of unit norm in a (complex, separable) Hilbert space  $\mathcal{H}$  (equivalently, the elements of the projective Hilbert space  $\mathcal{PH}$ , i.e., the one-dimensional subspaces (rays) of  $\mathcal{H}$ .) The state of a system can be mixed, in which case the state is given by a density operator  $\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i|$ .<sup>2</sup>

**Observables** The observables of a system are the self-adjoint operators on  $\mathcal{H}$ , or equivalently the set of Projection Valued Measures (PVMs). By the Spectral Theorem (below) every self-adjoint operator  $A$  uniquely corresponds to a PVM  $P^A : \Delta \mapsto P^A(\Delta)$ , where  $\Delta \in \mathcal{B}(\mathbb{R})$  are the Borel subsets of  $\mathbb{R}$ .

**Dynamics** The dynamics of a system are given by a self-adjoint operator  $H$ .  $H$  is the unique generator of a unitary group  $U_t = e^{iHt}$  parameterized by time  $t$ .<sup>3</sup> If the state of the system is  $\rho$  at time  $t = 0$  then the state at any time  $t$  is  $\rho_t = U_t \rho U_t^\dagger$  (*Schrödinger Picture*). Equivalently, if an observable at time  $t = 0$  is given by the operator  $A$  then the observable at any time is  $A_t = U_t^\dagger A U_t$  (*Heisenberg Picture*).

**Predictions** The expectation value of an observable  $A$  in the state  $\rho$  is given by the trace  $\langle A \rangle_\rho = \text{tr}[\rho A]$ , which in the case of a pure state reduces to  $\langle A \rangle = \langle \psi | A \psi \rangle = |A\psi|^2$ . The probability measure  $p_\rho^A = \text{tr}[P^A(\Delta)\rho]$  describes the results of an experiment in which  $A$  is measured.

**State Update** In situations where an observable  $A$  is known to take an eigenvalue  $a_k$ , the state is updated by means of the corresponding projection  $P_k$  according to the von-Neumann-Lüders Rule,  $\rho \rightarrow \tilde{\rho} = P_k \rho P_k / (\text{tr}[\rho P_k])$ .

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<sup>2</sup>Here  $|\psi\rangle\langle\psi|$  is the one-dimensional projection onto the ray spanned by  $\psi$ . A state is pure if and only if it is not mixed, which is to say that a pure state cannot be written as sum of projections  $\lambda|\psi_1\rangle\langle\psi_1| + (1-\lambda)|\psi_2\rangle\langle\psi_2|$  with  $0 \leq \lambda \leq 1$ .

<sup>3</sup>Stone's Theorem guarantees the converse, such that the specification of a one-parameter unitary group  $U_t$  serves to uniquely define a corresponding self-adjoint operator  $H$ .



These commitments are meant to distill the essence of the consensus that emerged from the construction of Dirac's Transformation Theory, following the work of von Neumann that grounded those results in a Hilbert space setting, and thus functional analysis. These formal commitments are not without their empirical, mathematical, and theoretical justifications, and they do fit together into a more or less coherent whole (bracketing the usual concerns about their ability to return a consistent description of the measurement process).

However, there is a sense in which quantum mechanics provides the most compelling example of a case of theory choice that is underdetermined by the empirical evidence: competing interpretations of non-relativistic quantum theory differ radically in their metaphysical commitments while accommodating the same empirical evidence and supplying the same empirical predictions, at least up to current experimental capabilities and interests. Determining the precise theoretical referent of "quantum mechanics" is thus somewhat problematic, and in making a particular choice one is apt to take a stance on issues of interpretation that remain controversial.

In starting with Ordinary QM I am attempting to keep matters as interpretation neutral as one could hope, but there remains some degree of arbitrariness in this choice. In its favor, there is generally an asymmetric relationship between other formalisms for quantum theory and the Hilbert space formalism of Ordinary QM, in that rival interpretations are required to recover the results of Ordinary QM but not vice versa. There is also a sense in which Ordinary QM fails to take a stand on the metaphysical issues with which other formulations are concerned. This may be thought to be a defect, philosophically speaking, but it is also an asset since it thereby remains interpretation neutral (to some extent).

### 3.1 WHY (NOT) ORDINARY QUANTUM MECHANICS?

The point of view I will take in this chapter is that quantum mechanics is first and foremost a theory of experimental results, and that before discussing matters of interpretation it is necessary to arrive a formalism that can provide detailed predictions for any possible experimental determinable quantity. This means that quantum mechanics will give predictions relative to a particular experimental arrangement, which minimally consists of a *state preparation* and a *registration* device. Ideally, the state preparation serves to produce a unique

state that characterizes the relevant properties of the identically prepared individual systems which together constitute the experimental ensemble. The registration device provides some means to discriminate between various outcomes of the experiment by means of a permanent record.

Minimally, then, quantum mechanics should provide the means to describe the results of these experiments in terms of: (i) a language containing propositions that describe the possible outcomes of an experiment, (ii) a set of probability measures over those outcomes, and (iii) laws that serve to derive those probability measures from the system state. Since the theory is inherently probabilistic, the validity of its predictions is to be evaluated in terms of the statistics of the results of an ensemble, i.e., by consideration of the limit of the relative frequencies of experimental outcomes.<sup>4</sup> Nonetheless, we will take it as axiomatic that the system state applies to the description of individual systems, and thus describes single case probabilities, although this is itself somewhat controversial.<sup>5</sup>

Mathematically, we define the possible experimental outcomes in terms of a measurable space  $(\Sigma, \mathcal{X})$ , where  $\mathcal{X}$  is a  $\sigma$ -algebra defined on  $\Sigma$ , a nonempty set. An assignment of probabilities to each outcome  $X \in \mathcal{X}$  is given by a map  $p : \mathcal{X} \rightarrow \mathbb{R}$ . This leads to the following definition.

**Definition 3.1.** *The triple  $(\Sigma, \mathcal{X}, p)$  is a probability space and  $p$  is a (generalized) probability measure if the following conditions are met:*

$$1. p(X) \geq 0 \text{ for all } X \in \mathcal{X} \quad (\text{positivity})$$

$$2. p(\Sigma) = 1 \quad (\text{unity})$$

$$3. p(\cup_i X_i) = \sum_i p(X_i) \text{ for countable, mutually disjoint families } X_i \quad (\sigma\text{-additivity})$$

These probabilities are to be determined empirically by conducting an appropriate series of experiments, holding the state preparation and the entire experimental arrangement constant. If quantum mechanics is to be an empirically adequate probabilistic theory, then it needs to provide the correct probabilities in any given experimental situation.<sup>6</sup>

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<sup>4</sup>This is the minimal or operational interpretation of (e.g.) Busch et al. (1996, p. 4).

<sup>5</sup>This should be read as an attempt to provide as an account of the sort of empirical description that, minimally, any interpretation of the theory must ultimately provide. An interpretation of quantum mechanics being an account of the theory that goes beyond the mere prediction of experimental results.

<sup>6</sup>The treatment here owes much to that of Busch et al. (1995a).

The way that this is done in Ordinary QM is through the trace prescription. The trace prescription says that the quantum state provides an assignment of probabilities to the experimental outcomes (represented by an appropriate set of operators) through the trace. That is,

$$p_\rho(X) = \text{tr}[E(X)\rho], \quad (3.1)$$

where  $\rho$  is the density operator representing the state (a trace-class operator of unit trace), and  $E(X)$  is the operator representing the experimental outcome  $X$ .<sup>7</sup> To ensure that  $p_\rho$  is a valid probability measure, we need to lay down some conditions that the operators  $E(X)$  must satisfy.

First, since  $\rho$  has unit trace, we require that  $E(\Sigma) = \mathbb{I}$ , which ensures that  $\text{Pr}(\Sigma) = 1$ . Similarly, to ensure that  $\text{Pr}(X) \geq 0$  we require that  $E(X)$  is a positive operator,  $E(X) \geq O$ .<sup>8</sup> The final condition we need to ensure is  $\sigma$ -additivity, which is achieved as below, which provides the definition of a (normalized) Positive Operator Valued Measure (POVM).

**Definition 3.2.** *A normalized Positive Operator Valued Measure (POVM)  $E$  is a map from a measurable space  $(\Sigma, \mathcal{X})$  to the bounded operators on a Hilbert space  $E : \mathcal{X} \rightarrow \mathcal{B}(\mathcal{H})$  such that*

1.  $E(X) \geq O$  for all  $X \in \mathcal{X}$  (positivity)
2.  $E(\Sigma) = \mathbb{I}$  (unity)
3.  $E(\bigcup_i X_i) = \sum_i E(X_i)$  (in the weak operator topology) for countable, mutually disjoint families of  $X_i$  (weak  $\sigma$ -additivity)

So long as the  $E(X)$  are elements of a POVM, then, the trace prescription provides a probability measure. Considered together as a POVM, the  $E(X)$  provide a description of the experimental outcomes in terms of the Hilbert space associated with the system. The requirements suffice to determine that  $E(X)$  is a self-adjoint operator with spectrum  $[0, 1]$  since, therefore,  $\mathbb{I} \geq E(X) \geq O$ .

Ordinary QM associates experimental arrangements with the measurement of an *observable*, represented by a self-adjoint operator on  $\mathcal{H}$ . Historically, this association was established by von Neumann's Spectral Theorem, which established that unbounded operators

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<sup>7</sup>The trace of an operator  $F$  in  $\mathcal{H}$  is  $\text{tr}[F] = \sum_k \langle e_k | (F^*F)^{1/2} e_k \rangle$ , where  $\{e_k\}$  is an orthonormal basis for  $\mathcal{H}$ . Essentially, an operator is of trace class if its trace is finite.

<sup>8</sup>An operator  $A$  is positive,  $A \geq O$  if  $\langle \psi | A \psi \rangle \geq 0$  for all  $\psi \in \mathcal{H}$ . If bounded,  $A$  is therefore self-adjoint. This defines a partial order on (bounded) self-adjoint operators  $A, B$ , i.e.  $A \geq B$  if and only if  $A - B \geq O$ .

(such as position and momentum) could be given a so-called spectral resolution in terms of projections on  $\mathcal{H}$ .<sup>9</sup> A spectral resolution is a POVM, but it is a POVM of a special sort known as a Projection Valued Measure (PVM).

**Definition 3.3.** *A Projection Valued Measure (PVM)  $P$  is a POVM,  $P : \mathcal{X} \rightarrow \mathcal{L}(\mathcal{H})$ , where  $\mathcal{L}(\mathcal{H})$  is the lattice of projections of a Hilbert space  $\mathcal{H}$ , such that*

$$4. E(X)E(Y) = E(Y)E(X) = 0 \text{ for all } X, Y \text{ such that } X \cap Y = \emptyset \quad (\text{orthogonality})$$

This extra condition is enough to ensure that if  $E(X)$  is an element of a PVM then it is an orthogonal projection,  $E(X)^2 = E(X)$ .<sup>10</sup> The key distinction is that the elements of a PVM must associate disjoint measurable sets with mutually orthogonal subspaces of  $\mathcal{H}$ . This distinction is important since it separates POVMs that have spectral resolutions (corresponding to a self-adjoint operator) and those that do not. It also underlies the common interpretation of the  $E(X)$  as potential *properties* of a system, which relies on just those characteristics that separate a PVM from a POVM. Namely, (i) repeatability (from  $E(X)^2 = E(X)$ ), and (ii) mutual exclusivity (from  $E(X)E(Y) = E(Y)E(X) = 0$  if  $X \cap Y = \emptyset$ ).<sup>11</sup>

**Theorem 3.1.** [*Spectral Theorem (von Neumann)*] *Let  $A$  be an (unbounded) operator self-adjoint on a dense domain  $\mathcal{D}_A \subseteq \mathcal{H}$ . Then there exists a unique Projection Valued Measure (PVM) from the Borel subsets of  $\mathbb{R}$  to the lattice of projections of  $\mathcal{H}$ ,  $P^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ , such that*

$$\langle \phi | A\psi \rangle = \int_{\mathbb{R}} \lambda d\langle \phi | P_{\lambda}^A \psi \rangle,$$

for all  $\phi, \psi \in \mathcal{D}_A$ , where  $P_{\lambda}^A = P^A((-\infty, \lambda])$ ,  $\lambda \in \mathbb{R}$ .

Conversely, given a PVM  $P^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$  there exists a unique self-adjoint operator  $A$  whose domain is the set of all  $\psi \in \mathcal{H}$  such that

$$\langle A\psi | A\psi \rangle = \int_{\mathbb{R}} \lambda^2 d\langle \psi | P_{\lambda}^A \psi \rangle < \infty.$$

The Spectral Theorem, then, shows that if the observables of the theory are restricted to the set of self-adjoint operators, as by Ordinary QM, that this amounts to a restriction of the allowed POVMs to just the set of PVMs  $P^A$ .<sup>12</sup> This means that the trace prescription

<sup>9</sup>See Duncan & Janssen (2013) for some historical background.

<sup>10</sup>This is because bounded positive operators are automatically self-adjoint.

<sup>11</sup>Note again that these are logically equivalent if  $E(X)$  is an element of a POVM.

<sup>12</sup>See Teschl (2009, Theorem 3.7, p. 97) for a proof.

serves to associate a probability measure to a measurement of an observable  $A$  (a self-adjoint operator) in any state  $\rho$  (a density operator) as follows

$$p_\rho^A(\Delta) = \text{tr} [P^A(\Delta)\rho],$$

where  $\Delta \in \mathcal{B}(\mathbb{R})$  is a Borel subset.

The empirical adequacy of Ordinary Quantum Mechanics thus amounts to the following claim: for every experimental situation there exists an observable  $A$  (a self-adjoint operator) such that  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), p_\rho^A)$  correctly describes the probabilities for the experimental outcomes  $\Delta \in \mathcal{B}(\mathbb{R})$  for a system in the state  $\rho$ .

One might well wonder to what extent some other rule could serve to provide these probabilities from the state  $\rho$ , and thus provide a competitor to Ordinary QM. Assuming that the experimental outcomes are to correspond to elements of the lattice of projections,  $P \in \mathcal{L}(\mathcal{H})$ , it turns out that the trace prescription provides a unique assignment of probabilities (for a Hilbert space of dimension greater than 2).

**Theorem 3.2** (Gleason). *Let  $\mu : \mathcal{L}(\mathcal{H}) \rightarrow [0, 1]$  be a (generalized) probability measure such that*

1.  $0 = \mu_\rho(0) \leq \mu(P)$  for all  $P \in \mathcal{L}(\mathcal{H})$ ,
2.  $\mu(\mathbb{I}) = 1$ ,
3.  $\mu(\sum_i P_i) = \sum_i \mu(P_i)$  for mutually orthogonal  $P_i$

*Then, so long as  $\dim(\mathcal{H}) \geq 3$ , every such probability measure  $\mu$  defines a positive trace class operator  $\rho$  with unit trace such that  $\mu_\rho(P) = \text{tr} [\rho P]$  and, conversely, every such operator  $\rho$  uniquely determines a (generalized) probability measure on  $\mathcal{L}(\mathcal{H})$ ,  $\mu_\rho(P) = \text{tr} [\rho P]$ .*

Gleason's Theorem has rightly been regarded as the foundation underlying the use of the trace prescription in Ordinary QM. It says that, through the trace prescription (and so long as the Hilbert space has dimensionality greater than two), any probability measure on the lattice of projections of a system uniquely determines the state of the system, and every state of the system uniquely determines a (generalized) probability measure on the lattice of projections.

The trace prescription, then, serves to determine the possible states of a quantum system. The trace  $A \mapsto \text{tr} [A] \in \mathbb{R}$  is a positive linear functional on  $\mathcal{H}$ , the trace functional. Let us

denote the set of trace class operators by  $\mathcal{T}(\mathcal{H})$  and the positive trace class operators with trace one by  $\mathcal{S}(\mathcal{H})$ . The one-dimensional projections  $|\psi\rangle\langle\psi|$  are positive operators of trace one which are the extremal elements of  $\mathcal{S}(\mathcal{H})$ , a convex set. Any element of  $\mathcal{S}(\mathcal{H})$  may be expressed as a convex combination of these extremal elements, i.e. as a mixed state, so that the elements of  $\mathcal{S}(\mathcal{H})$  are precisely the density operators  $\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i|$ , with  $0 \leq w_i \leq 1$  and  $\sum_i w_i = 1$ .

From here, we are in easy reach of the idea that a pure state  $|\psi\rangle\langle\psi|$  (which cannot be written as a linear combination  $\rho = (1 - \lambda)|\psi_1\rangle\langle\psi_1| + \lambda|\psi_2\rangle\langle\psi_2|$ ) must correspond to a ray, the one-dimensional subspace onto which  $|\psi\rangle\langle\psi|$  projects. From there the association with a unit vector  $\psi \in \mathcal{H}$  such that  $\langle\psi|\psi\rangle = 1$  quickly follows, and thence to the Born rule  $\Pr_\psi(\Delta) = \langle\psi|P(\Delta)\psi\rangle$ . There is thus a clear sense in which the probabilistic interpretation of quantum mechanics according to Ordinary QM corresponds to the first clear interpretive rule of quantum mechanics, as it arose in the late 1920s. But Gleason's Theorem also provides a sense in which the minimal interpretation of quantum mechanics as a probabilistic theory, combined with the restriction of state valuations to the projections, appears to *compel* the use of both the trace prescription.

Nonetheless, one can still question whether or not the trace prescription suffices to determine that the experimental outcomes must be described by elements of the lattice of projections  $\mathcal{L}(\mathcal{H})$ . That is, one can ask: is there a sense in which the operator that appears in the trace prescription alongside  $\rho$  *must* be a projection if it is to provide an appropriate probability measure? Remarkably, the answer to this question is *no*, since (as we have seen) POVMs as well as PVMs suffice to associate a state  $\rho$  with a probability measure through the trace prescription. Correspondingly, Busch (2003) shows that a state valuation on the *effects*  $\mathcal{E}(\mathcal{H})$  (the set of positive operators  $E$  such that  $\mathbb{I} \geq E \geq O$ ) suffices to determine a (further generalized) probability measure through the trace prescription, which in turn suffices to uniquely determine a density operator.

**Theorem 3.3** (Busch). *Let  $\nu : \mathcal{E}(\mathcal{H}) \rightarrow [0, 1]$  be a (generalized) probability measure such that*

1.  $0 = \nu_\rho(O) \leq \nu(E)$  for all  $E \in \mathcal{E}(\mathcal{H})$ ,
2.  $\nu(\mathbb{I}) = 1$ ,

3.  $\nu(\sum_i E_i) = \sum_i \nu(E_i)$  for countable families of  $E_i$  such that  $\sum_i E_i \leq \mathbb{I}$ .<sup>13</sup>

Then each probability measure  $\nu_\rho$  determines a unique state  $\rho \in \mathcal{S}(\mathcal{H})$  such that  $\nu(A) = \text{tr}[\rho A]$  for all  $A \in \mathcal{E}(\mathcal{H})$  and, conversely, every state  $\rho$  determines a unique probability measure  $\nu(E) = \text{tr}[\rho E]$ .

Whereas a projection  $P$  has spectrum  $\{0, 1\}$ , an effect  $E$  is a positive operator with spectrum  $[0, 1]$ . A positive operator  $A \geq O$  is an operator such that  $\langle \phi | A \phi \rangle \geq 0$  for all  $\phi \in \mathcal{A}$ . The (bounded) self-adjoint operators on  $\mathcal{H}$  are partially ordered by the relation  $\geq$ , where  $A \geq B \leftrightarrow A - B \geq O$ . The projection operators are also self-adjoint, and are thus partially ordered by  $\geq$ . For projections, this partial order coincides with the partial order of subspace inclusion, defined by the range of the projection. (Note that an effect is a projection exactly when it is idempotent,  $E^2 = E$ .) However, positive operators need not be defined by their range, and the set of effects  $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : O \leq A \leq \mathbb{I}\}$  does not have the structure of a lattice.

Nonetheless, the set of effects provides a straightforward generalization of the lattice of projections, in the sense the state space  $\mathcal{S}(\mathcal{H})$  naturally defines probability measures  $\nu : \mathcal{E}(\mathcal{H}) \rightarrow [0, 1]$  through the trace prescription via Busch's Theorem. Given the availability of this conservative extension of Ordinary QM, we are apt to ask: why should Ordinary QM restrict the observables that can be measured in an experiment to the set of PVMs rather than the POVMs? This question can be made more pressing by considering that the empirical adequacy of Ordinary QM (as stated above) requires that every experimental arrangement can be described in terms of PVMs.

That is, there is a sense in which Ordinary QM is empirically adequate only if the PVMs suffice to describe *every* experimental arrangement by an assignment of projections to possible experimental outcomes. If state valuations on the effects required to secure the empirical adequacy of the theory, then this is evidence that the observables of Ordinary QM are overly restricted. This result would not be too alarming, however, since in that case the theory could be made empirically adequate simply by expanding the definition

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<sup>13</sup>Note that in case  $E^2 = E$  for all  $E_i$  then these conditions reduce to those of Gleason's Theorem, as they must. Also note that there is some subtlety required here in understanding the relationship of the probability measure given here to that of Gleason's Theorem. While Gleason's Theorem returns a probability measure defined on some subset of  $\mathcal{E}(\mathcal{H})$  (i.e.  $\mathcal{L}(\mathcal{H})$ ) the probability measure supplied by Busch's Theorem is defined on all of  $\mathcal{E}(\mathcal{H})$ . Further note that, unlike Gleason's Theorem, Busch's Theorem applies to a Hilbert space of two dimensions. It is also considerably easier to prove.

of the observables of the theory to include the effects, i.e. the POVMs. Nonetheless, this observation should serve to illustrate to the philosopher of science just what is at stake here.

The recent consensus in the foundations of physics community is, it seems, that POVMs are required to reach a description of all possible quantum experiments. Following the development of the POVM formalism by Davies & Lewis (1970), Holevo (1973), Kraus (1983), Ludwig (1983) and others in response to various concerns in quantum measurement theory, it has been convincingly argued that even canonical quantum experiments, such as the original demonstration of discrete electron spin by Stern and Gerlach, require the use of POVMs for their description.<sup>14</sup> This state of affairs does not obtain so widely among philosophers of physics, who have perhaps preferred to idealize away such difficulties.<sup>15</sup> But some of the concerns that might lead one to the importance of POVMs have less to do with the messiness of empirical fact and more to do with the abstract characterization of the possible quantum states of a quantum system.

The key observation here is that the space of states  $\mathcal{S}(\mathcal{H})$  is convex, whereas the lattice of projections  $\mathcal{L}(\mathcal{H})$  does not form a convex set. However, the set of effects  $\mathcal{E}(\mathcal{H})$  is convex, in the sense that if  $E_1, E_2$  are effects then  $E_3 = \lambda E_1 + (1 - \lambda)E_2$ , with  $0 \leq \lambda \leq 1$ , is another effect. This makes the set of effects the natural partner of the state space  $\mathcal{S}(\mathcal{H})$  in a sense that the lattice of projections is not. The ‘naturalness’ of this partnership can be given a precise mathematical characterization, by observing the set of effects is *dual* to the state space in the following sense.

To see this, we consider the state space in a more abstract setting. The trace class operators  $\mathcal{T}(\mathcal{H})$  form a real Banach space  $V$  under the trace norm.<sup>16</sup> The states  $\mathcal{S}(\mathcal{H})$  form a positive norm closed cone  $K$  in  $V$ . The dual of  $V$  is a Banach space  $V^*$  which is isomorphic to  $\mathcal{B}(\mathcal{H})$ , and the dual cone  $K^*$  is isomorphic to the set of positive operators  $\mathfrak{B}(\mathcal{H})^+$ .<sup>17</sup> This duality leads again to an association of probabilities with POVMs rather than PVMs.

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<sup>14</sup>See (Busch et al., 1996, pp. 165–174) for this example.

<sup>15</sup>Here I cannot help but quote Ruetsche’s memorable aside on the sociology of philosophers of physics in the late twentieth century, whose “treatments (generally) address foundational questions that can be raised in the context of finite dimensional Hilbert spaces . . . Indeed, for a dark and macho while, the main game in philosophy of quantum mechanics was to find the smallest Hilbert space in which a certain foundational point—a “no go” result—could be made.” (Ruetsche, 2011, §3.1). In this connection, it is worthy of note that the routine use of POVMs in quantum information theory takes place in the finite-dimensional context. However, there is evidence of a recent groundswell of recognition that the use of POVMs in measurement contexts may have implications for the philosophy of physics, see, e.g., (Dickson, 2006; Wallace, 2007).

<sup>16</sup>See (Holevo, 2011, pp. 63–68). The trace class operators can also be thought of as products of Hilbert-Schmidt operators, which form a Hilbert space.

<sup>17</sup>See Davies & Lewis (1970) for details.



Consider the outcome space  $(\Sigma, \mathcal{X})$  belonging to a particular experiment. The trace prescription associates every  $X \in \mathcal{X}$  with a positive number  $p_\rho(X) \in [0, 1]$  representing the probability of outcome  $X$  in the state  $\rho$ . Thus  $X \mapsto p_\rho(X)$  corresponds to a linear functional on  $K$  which represents the state valuation for the experiment to which the outcome space corresponds. Assuming that  $p_\rho$  is convex linear in the state space, so that  $p_{\lambda\rho_1+(1-\lambda)\rho_2} = \lambda p_{\rho_1} + (1-\lambda)p_{\rho_2}$ , then  $p_\rho$  may be extended to a linear functional on  $\mathcal{T}(\mathcal{H})$  which lies in  $V^*$  and which is of the form  $p_\rho(X) = \text{tr}[\rho E(X)]$ , where  $E(X)$  is a positive operator  $0 \leq E(X) \leq \mathbb{I}$ , i.e.  $E(X) \in \mathcal{E}(\mathcal{H})$ .<sup>18</sup> This leads to the following result.

**Theorem 3.4** (Davies & Lewis). *Let  $p_\rho \in V^*$  be a state valuation on  $(\Sigma, \mathcal{X})$  such that*

1.  $0 \leq p_\rho(X)$ ,
2.  $p_\rho(\Sigma) = 1$ ,
3.  $p_\rho(\bigcup_i X_i) = \sum_i p_\rho(X_i)$  for countable families of disjoint  $X_i$  with convergence in the weak\* topology of  $V^*$ .

*Then  $X \mapsto E(X)$  is a normalized POVM.*<sup>19</sup>

What this shows is that while every PVM uniquely determines a state valuation  $p_\rho$  through the trace, the converse does not hold: a state valuation in general determines a POVM rather than a PVM. This is, in effect, a restricted form of Busch's Theorem that serves to associate a particular experiment with outcome space  $(\Sigma, \mathcal{X})$  to a POVM  $X \mapsto E(X)$ . Again, this strongly suggests that the entire set of effects is required to fully spell out the empirical consequences of quantum mechanics rather than its restriction to the lattice of projections  $\mathcal{L}(\mathcal{H})$ . Since the set of PVMs is a proper subset of the POVMs, the existence of experiments whose outcome space does not correspond to a PVM also demonstrates (through the Spectral Theorem) that there are experiments that do not correspond to the measurement of an operator self-adjoint on the system's Hilbert space.<sup>20</sup>

<sup>18</sup>See (Kraus, 1983, pp. 21–30) for details of this procedure.

<sup>19</sup>See (Davies & Lewis, 1970, Thm. 1) or (Holevo, 1973, Prop. 3.2).

<sup>20</sup>The question remains, however, of whether those experiments really do involve the measurement of an operator that is not self-adjoint. They could correspond to the joint or repeated measurement of a collection of self-adjoint operators, or an operator self-adjoint on a distinct Hilbert space (of, e.g., a supersystem), or of something else. In Chapter 7 I argue that Pauli's Theorem creates difficulties for the interpretation of event time measurements in this manner.

### 3.2 POVMS AND THE SPECTRAL THEOREM

While von Neumann's Spectral Theorem provided a correspondence between the set of self-adjoint operators and the projection operators (to which he had given a probability interpretation through the trace) he was quick to note that not all symmetric operators could be given a spectral representation.<sup>21</sup> This problem cannot arise in a finite dimensional Hilbert space since there every symmetric operator has a spectral resolution. Let's briefly see how this works. We need a  $n$ -dimensional complex vector space  $\mathcal{V}$  equipped with an inner product (this is also a Hilbert space since it is therefore complete). A linear operator  $A : \mathcal{V} \rightarrow \mathcal{V}$  is defined by an  $n \times n$  matrix of scalars  $a_{jk}$ , which are in general complex. Taking the adjoint  $A^\dagger$  of the operator results in a new matrix with elements  $a_{kj}^*$ .  $A$  is self-adjoint (or Hermitian or symmetric) if  $A = A^\dagger$ , that is if  $a_{jk} = a_{kj}^*$ . Every Hermitian operator is normal,  $A^\dagger A = AA^\dagger$ , and a unitary operator  $U$  is one whose adjoint is equal to its inverse  $U^\dagger U = \mathbb{I} = UU^\dagger$ ,  $u_{kj}^* u_{jk} = u_{jk} u_{kj}^* = 1$ . In this context, we have the following 'baby' spectral theorem.

**Proposition 3.1** (Spectral Theorem (Finite Dimensional)<sup>22</sup>). *Let  $\mathcal{V}$  be an  $n$ -dimensional complex vector space with an inner product  $(\phi, \psi)$ . Let  $A$  be an  $n \times n$  self-adjoint (normal) matrix. Then there exists an orthonormal basis for  $\mathcal{V}$  consisting of eigenvectors of  $A$  with real (complex) eigenvalues  $a_i$ . Equivalently,  $A$  is unitarily diagonalizable, i.e., there exists a unitary matrix  $U$  such that  $U^\dagger A U$  is a diagonal matrix with  $a_i$  as its diagonal entries.*

Let's put this result into bra-ket notation. If we denote the eigenvectors of an operator  $A$  by kets  $|a_i\rangle$ , Theorem 1 tells us that if  $A$  is a normal operator, then we can write

$$A = \sum_{i=1}^n a_i |a_i\rangle \langle a_i|; \quad \mathbb{I} = \sum_{i=1}^n |a_i\rangle \langle a_i|.$$

Where on the right the orthonormal basis of eigenvectors form a *resolution of the identity* such that any vector can be decomposed as a sum of eigenvectors of  $A$  with complex coefficients,

$$|\psi\rangle = \left( \sum_{i=1}^n |a_i\rangle \langle a_i| \right) |\psi\rangle = \sum_{i=1}^n \langle a_i | \psi \rangle |a_i\rangle.$$

This resolution of the identity also defines a Projection Valued Measure, essentially just the map between the eigenvalues  $a_i$  and the projections  $|a_i\rangle \langle a_i|$ . In an introductory physics class,

<sup>21</sup>See (Von Neumann, 1955, ft. 105).

<sup>22</sup>See, e.g., (Riesz & Sz.-Nagy, 1990, p. 260).

this may be as far as exposition of the spectral theorem goes. This attitude is encapsulated by the following passage, taken from the notes of a contemporary graduate quantum mechanics course:

We are, as usual, employing the rules for finite-dimensional Hilbert spaces, relying on the experts to assure us that they will (usually, anyway) work for the infinite-dimensional cases we may be interested in.<sup>23</sup>

The cases of interest here will be, unfortunately, just those cases where the rules for finite-dimensional Hilbert spaces fail to work out.

In a finite-dimensional (complex) Hilbert space, there is essentially no distinction to be drawn between the set of self-adjoint operators and the symmetric operators; they amount to precisely the same thing. In an infinite-dimensional Hilbert space, however, the two notions come apart. This is because of the existence of unbounded operators, which are often of vital importance for quantum theory. An operator  $B$  is bounded on a domain  $\mathcal{D}_B \subseteq \mathcal{H}$  if

$$\|B\psi\|^2 = \langle B\psi|B\psi\rangle < k\|\psi\|^2 \text{ for some } k \in \mathbb{R}_+ \text{ for all } \psi \in \mathcal{D}_B.$$

The domain of a bounded operator can always be extended to the entire Hilbert space. That is, if  $B$  is an operator bounded by  $k$  on domain  $\mathcal{D}_B$  then there exists an extension  $B' \supseteq B$  bounded by  $k$  with domain  $\mathcal{H}$  such that  $B'$  agrees with  $B$  on  $\mathcal{D}_B \subseteq \mathcal{H}$ . This means that there is no need to speak of a domain of a bounded operator, since it can always be extended to the whole space.

Of particular interest to us are the symmetric operators. An operator is symmetric on a domain  $\mathcal{D}_A \subseteq \mathcal{H}$  if  $\langle \psi|A\phi\rangle = \langle A\phi|\psi\rangle$  for all  $\phi, \psi \in \mathcal{D}_A$ . If  $A$  is bounded and symmetric then there exists an interval  $[m, M] \subset \mathbb{R}$  such that  $m$  is the greatest lower bound  $m\|\psi\| < \|A\psi\|$  and  $M$  is the least upper bound  $M\|\psi\| > \|A\psi\|$ , for all  $\psi \in \mathcal{H}$ . Then  $A$  has a spectral representation as follows

**Theorem 3.5** (Spectral Theorem (Bounded)<sup>24</sup>). *Corresponding to a bounded symmetric operator  $B$  in a Hilbert space  $\mathcal{H}$  there is a spectral family of projections  $P_\lambda^A$  on the interval  $[m, M]$  such that*

$$\langle \phi|A\psi\rangle = \int_m^M \lambda d\langle \phi|P_\lambda^A\psi\rangle,$$

for all  $\phi, \psi \in \mathcal{H}$ .

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<sup>23</sup>The author will remain nameless.

<sup>24</sup>See, e.g., (Riesz & Sz.-Nagy, 1990, p. 275). This theorem was first proved by Hilbert, thus justifying the appellation ‘Hilbert space’.

The Spectral Theorem of von Neumann achieves the same feat for unbounded self-adjoint operators, but the properties of being symmetric and being self-adjoint are not coextensive. To see why, first observe the following remarkable fact that only bounded symmetric operators can be defined on the entire Hilbert space.

**Theorem 3.6** (Heillinger & Toeplitz<sup>25</sup>). *Every symmetric operator  $B$  in  $\mathcal{H}$  such that  $B\psi \in \mathcal{H}$  for all  $\psi \in \mathcal{H}$  is necessarily bounded.*

This implies that any unbounded symmetric operator  $A$  cannot be defined on the entire Hilbert space, i.e.  $\mathcal{D}_A \subset \mathcal{H}$ , and so there can be no extension of an unbounded symmetric operator to another such operator, everywhere defined.

For a brief insight into the difficulties that ensue, consider the fact that every closed symmetric operator  $A$  has a matrix representation  $a_{ij} = a_{ji}^*$  on an orthonormal (countably infinite) set of basis vectors taken from its domain. So far, so much like the finite dimensional case. But if  $A$  is unbounded then there is no guarantee that the application of a unitary matrix  $u_{ij}$  will result in an operator in  $\mathcal{H}$ . That is, the matrix  $a'_{rs} = u_{ri}^* a_{ij} u_{js}$  will not necessarily meet the necessary condition for defining a closed symmetric operator,  $\sum_{i=1}^{\infty} |a_{ij}|^2 < \infty$ . And even if it does, the symmetric operator  $A'$  that results may have a domain  $\mathcal{D}_{A'}$  such that  $\mathcal{D}_{A'} \cap \mathcal{D}_A = \emptyset$  (Akheizer & Glazman, 1993, p. 101–102).

I introduce these concerns here to highlight the following issue: given an operator  $A$  symmetric and unbounded on a domain  $\mathcal{D}_A \subset \mathcal{H}$  it is often a non-trivial matter to determine whether or not  $A$  has an extension to a self-adjoint operator on  $\mathcal{H}$  (and, indeed, it may not). For an unbounded symmetric operator  $A$  to be self-adjoint, its domain must be equal to the domain of its adjoint,  $\mathcal{D}_A = \mathcal{D}_{A^\dagger}$ . If the domain of  $A$  is dense in  $\mathcal{H}$ ,<sup>26</sup> then it has an adjoint operator  $A^\dagger$  whose domain is closed,  $\overline{\mathcal{D}_{A^\dagger}} = \mathcal{D}_{A^\dagger}$ , but it is not therefore self-adjoint,  $A \neq A^\dagger$ , since it may be the case that  $\mathcal{D}_A \subset \mathcal{D}_{A^\dagger}$ .<sup>27</sup>

The reason that this issue becomes particularly pressing is that, in general, a POVM determines a symmetric operator, which thus does not necessarily have a self-adjoint extension.

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<sup>25</sup>See, e.g., (Riesz & Sz.-Nagy, 1990, p. 296).

<sup>26</sup>A domain  $\mathcal{D} \subset \mathcal{H}$  is dense if for all  $\psi \in \mathcal{H}$  there exists a sequence of vectors  $\psi_n \in \mathcal{D}$  such that  $\psi_n \rightarrow \psi$ .

<sup>27</sup>See, e.g., (Jordan, 1969, pp. 30–31).

<sup>28</sup>See Werner (1986).

**Proposition 3.2** (Werner<sup>28</sup>). *Let  $E^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  be a POVM. Then  $E^A$  defines a symmetric operator  $A$  whose domain is the set of all  $\psi \in \mathcal{H}$  such that*

$$\langle A\psi | A\psi \rangle = \int_{\mathbb{R}} \lambda^2 d\langle \psi | E_{\lambda}^A \psi \rangle < \infty,$$

where  $E_{\lambda}^A = E^A((-\infty, \lambda])$ .

The problem of deciding whether or not an unbounded symmetric operator has a self-adjoint extension (or many) was a central concern of von Neumann's original proof of the spectral theorem. Roughly, von Neumann's method of proof was to first demonstrate that a unitary operator always has a spectral representation, and then show that a self-adjoint operator  $A$  always corresponds to a unitary operator  $U_A$ , defined by the Cayley transform

$$U_A = (A - i\mathbb{I})(A + i\mathbb{I})^{-1}.$$

We can see that if  $A$  is symmetric then  $U_A$  is an isometry,  $U_A^\dagger U_A = \mathbb{I}$ , but if  $A$  is in addition self-adjoint then  $U_A^\dagger = U_A^{-1}$ , i.e.  $U_A$  is unitary. This provides a necessary and sufficient condition for  $A$  to be self-adjoint since the converse is also true: if  $A$  is a symmetric operator and  $U_A$  is unitary (and thus defined on the whole of  $\mathcal{H}$ ) then  $A$  is self-adjoint (Riesz & Sz.-Nagy, 1990, pp. 321–326). But not all symmetric operators are self-adjoint. This can be seen as follows.

Let  $A$  be a closed symmetric operator with dense domain  $\mathcal{D}_A \subset \mathcal{H}$  and range  $\mathcal{R}(A) = \{A\psi \in \mathcal{H} : \psi \in \mathcal{D}_A\}$ . Then  $\mathcal{R}(A - i)$  and  $\mathcal{R}(A + i)$  are closed linear subspaces of  $\mathcal{H}$ . The dimensions of their orthogonal complements,  $m = \dim \mathcal{R}(A - i)^\perp$  and  $n = \dim \mathcal{R}(A + i)^\perp$ , are called the *deficiency indices* of  $A$ . Evidently  $m, n$  may take any non-negative integer value, including  $+\infty$ . Since  $U_A$  is unitary only if its domain is  $\mathcal{H}$ , and  $U_A$  is isometric if  $A$  is symmetric, then for  $A$  to be self-adjoint we must have  $\mathcal{R}(A + i) = \mathcal{R}(A - i) = \mathcal{H}$ , i.e.  $m = n = 0$ . That is, a closed symmetric operator  $A$  is self-adjoint if and only if its deficiency indices are  $(0, 0)$ . In case  $A$  is symmetric but not self-adjoint  $U_A$  will be isometric but not unitary.

We now consider extensions of  $A$ ; closed symmetric operators  $A' \supseteq A$ . The inverse Cayley transform defines a symmetric operator,

$$A = i(\mathbb{I} + U_A)(\mathbb{I} - U_A)^{-1},$$

and thus the question of whether  $A$  has a self-adjoint extension (or many) reduces to the question of whether  $U_A$  has isometric extensions that are unitary (Riesz & Sz.-Nagy, 1990,

p. 327). If  $A$  is not self-adjoint, then at least one deficiency subspace  $\mathcal{R}(A - i)^\perp$  or  $\mathcal{R}(A + i)^\perp$  has dimensionality greater or equal to one,  $(m \geq 1) \vee (n \geq 1)$ .

To reduce the deficiency index  $m$  (or  $n$ ) we can introduce an isometry from  $\mathcal{R}(A - i)^\perp$  to  $\mathcal{R}(A + i)^\perp$  (and vice versa). If the deficiency indices are equal,  $m = n$ , then we will be able to find isometries that reduce each deficiency index correspondingly. These isometries are in one-to-one correspondence with the closed symmetric extensions of  $A$  (Reed & Simon, 1975, p. 140). Since an extension  $A' \supset A$  is self-adjoint if and only if both of its deficiency indices are zero, it follows that  $A$  will have a self-adjoint extension if and only if  $A$  has equal deficiency indices. Moreover, the number of self-adjoint extensions is given by the value of the deficiency indices of  $A$  (when  $m = n$ ). Thus an operator with unequal deficiency indices has no self-adjoint extensions. If, in addition, either  $n = 0$  or  $m = 0$  then there can be no symmetric extension at all. This provides the following standard classification.

**Proposition 3.3** (von Neumann<sup>29</sup>). *Let  $A$  be a symmetric operator on a dense domain  $\mathcal{D}_A$  then*

1.  *$A$  is essentially self-adjoint if and only if  $m = 0 = n$ .*
2.  *$A$  has self-adjoint extensions if and only if  $m = n$ , corresponding bijectively to the isometries between the deficiency subspaces of  $A$ .*
3. *If either  $m = 0 \neq n$  or  $n = 0 \neq m$  then  $A$  has no nontrivial symmetric extensions and  $A$  is maximal symmetric.*

Combined with the Spectral Theorem, this already gives some important insights into the relationship between symmetric operators and POVMs. First, a symmetric operator uniquely corresponds to a PVM if and only if it is essentially self-adjoint. Second, if a symmetric operator has self-adjoint extensions then it corresponds non-uniquely to a set of PVMs, one for each extension. Note that a symmetric operator is maximal symmetric if it cannot be extended to a larger domain. Every essentially self-adjoint operator is thus maximal symmetric but, evidently, not every maximal symmetric operator is self-adjoint. The relation of the maximal symmetric operator to POVMs and PVMs is given by the following theorem.

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<sup>29</sup>See, e.g., (Reed & Simon, 1975, p. 142).

**Theorem 3.7** (Alkeizer & Glazman<sup>30</sup>). *A symmetric operator  $A$  corresponds to a unique POVM  $E^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  if and only if it is maximal symmetric. In addition,  $E^A$  is a PVM,  $E^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ , if and only if  $A$  is essentially self-adjoint.*

This entails that a symmetric operator that is maximal symmetric but not essentially self-adjoint (i.e. which has deficiency indices  $m = 0 \neq n$  or  $n = 0 \neq m$ ) corresponds to a POVM that is not a PVM. We will call such operators maximally symmetric. This suggests the following generalization of the spectral theorem, adapted to include all maximal symmetric operators.

**Theorem 3.8** (Generalized Spectral Theorem). *Let  $A$  be a maximal symmetric operator on  $\mathcal{H}$  with domain  $\mathcal{D}_A$ . Then  $A$  uniquely defines a POVM  $E^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  such that*

$$\langle \phi | A \psi \rangle = \int_{\mathbb{R}} \mu d\langle \phi | E_{\mu}^A \psi \rangle$$

for all  $\phi, \psi \in \mathcal{D}_A$ . Conversely, given a POVM  $E^A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  there exists a unique symmetric operator  $A$  whose domain is the set of all  $\psi \in \mathcal{H}$  such that<sup>31</sup>

$$\langle A \phi | A \psi \rangle = \int_{\mathbb{R}} \mu^2 d\langle \psi | E_{\mu}^A \psi \rangle < \infty.$$

One of the interesting characteristics of a POVM corresponding to a (non-self-adjoint) maximally symmetric  $B$  is that the second moment of the POVM is not equal to  $B^2$ , that is

$$B^2 \neq \int_{\mathbb{R}} \mu^2 dE_{\mu}^B.$$

We will call  $B$ , the first moment of the POVM  $E^B$ , the *expectation operator* of  $E^B$  since  $\langle B \rangle_{\rho} = \text{tr}[B\rho]$  gives the expectation value of  $E^B$  in the state  $\rho$  in the same sense that  $\langle A \rangle_{\rho} = \text{tr}[A\rho]$  gives the expectation value of  $P^A$  for  $A$  self-adjoint. However, whereas  $\langle A^2 \rangle_{\rho} = \text{tr}[A^2\rho]$  is the expectation value of the second moment of  $P^A$ , the expectation value of  $B^2$  concerns the probability measure  $p_{\rho}^{B^2}(X) = \text{tr}[\rho E^{B^2}(X)]$  which is evidently not the same thing in this case. The following quantity, called the noise operator,  $\Delta_B$ , measures this departure

$$\Delta_B = \int_{\mathbb{R}} \mu^2 \langle \phi | dE_{\mu}^B \psi \rangle - \langle B \phi | B \psi \rangle^2,$$

<sup>30</sup>See (Akheizer & Glazman, 1993, p. 135).

<sup>31</sup>The slightly restricted form of the converse is required to acknowledge the fact that a POVM might define a symmetric operator with (e.g.) many self-adjoint extensions.

for  $\phi, \psi$  in the domain of  $B$ . Note that if  $E^B$  is a PVM then  $\Delta_B = 0$  for all  $\phi, \psi \in \mathcal{D}_B$ , conversely if  $\Delta_B > 0$  then  $B$  corresponds to a POVM rather than a PVM.

One of the better known results concerning POVMs of this type is the Naimark Dilation Theorem. Naimark demonstrated that although a maximally symmetric operator  $B$  may not have a self-adjoint extension on  $\mathcal{H}$ , it always has an extension to an operator  $B'$  self-adjoint on a ‘larger’ Hilbert space  $\tilde{\mathcal{H}} \supset \mathcal{H}$ .<sup>32</sup> By the Spectral Theorem there exists a unique PVM  $P^{B'} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\tilde{\mathcal{H}})$ , and the POVM  $E^B$  is obtained again by projection onto  $\mathcal{H}$ .

**Theorem 3.9** (Naimark’s Dilation Theorem<sup>33</sup>). *Let  $E^B : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  be a POVM on  $\mathcal{H}$ . Then there exists a unique (up to unitary equivalence) minimal Hilbert space  $\mathcal{K} \supseteq \mathcal{H}$  and a PVM  $P^{B'} : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{K})$  and a projection  $\mathbf{P}$  onto  $\mathcal{H}$  such that*

$$E^B(X)\psi = \mathbf{P}P^{B'}(X)\psi \tag{3.2}$$

for all  $\psi \in \mathcal{H}$  and all  $X \in \mathcal{B}(\mathbb{R})$ . The Hilbert space  $\mathcal{K}$  is minimal if it is the smallest such Hilbert space containing the union of the closed subspaces  $E^B(X)\mathcal{H}$ .

We can see that a PVM provides a trivial case of this theorem, since the minimal Hilbert space is identical to the original and the projection is thus the identity. Considering the converse of this theorem, we see that a PVM  $P^A$  defined on the whole of  $\mathcal{H}$  becomes a POVM when projected to a subspace of  $\mathcal{H}$ . This may have a physical interpretation, e.g., if we perform a measurement on an object system entangled with its environment then we calculate the results via the reduced density matrix obtained by ‘tracing out’ the environmental degrees of freedom. In that case the correlations between the system and environment remain at the level of the combined states and manifest in a lack of orthogonality of these ‘smaller’ subspaces, and thus an operator self-adjoint on the entire Hilbert space will correspond to a POVM rather than a PVM on the subspace of the object system.

### 3.2.1 Technical Interlude: Abstract Algebraic Insights

Before concluding our discussion of Naimark’s Theorem, it is worth remarking on its generalization to Stinespring’s Theorem (following Landsman (1999)), particularly since this theorem will be of interest to us later on. This will lead to a discussion of the relation of

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<sup>32</sup>The scare quotes are intended to indicate that both Hilbert spaces will often both have the same (infinite) dimensionality.

<sup>33</sup>See, e.g., (Busch et al., 1995a, p. 32).



Stinespring's Theorem to the Spectral Theorem. However, this subsection can be readily skipped by the reader without losing the thread of the overall argument. By defining the terms appearing in Stinespring's Theorem, and observing that the results of Naimark's Dilation Theorem fall under those definitions, we will see that Naimark's Theorem follows as a special case of Stinespring's Theorem.

**Theorem 3.10** (Stinespring<sup>34</sup>). *A mapping  $\phi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$  of  $\mathfrak{A}$ , a  $C^*$ -algebra with identity, into  $\mathfrak{B}(\mathcal{H})$ , the bounded operators of a Hilbert space  $\mathcal{H}$ , is completely positive if and only if there exist a Hilbert space  $\tilde{\mathcal{H}}$ , a unitary isomorphism  $U : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ , a Hilbert space  $\mathcal{K} \supset \tilde{\mathcal{H}}$ , a projection  $P : \mathcal{K} \rightarrow \tilde{\mathcal{H}}$  and a representation  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{K})$  of  $\mathfrak{A}$  on  $\mathcal{K}$  such that*

$$\pi : A \in \mathfrak{A} \rightarrow U\phi(A)U^{-1} = P\pi(A)P$$

for all  $A \in \mathfrak{A}$ .

An abstract  $C^*$ -algebra  $\mathfrak{A}$  is a  $*$ -algebra over the complex numbers  $\mathbb{C}$ , complete with respect to a norm  $\|\cdot\|$  such that  $\|A^*A\| = \|A\|^2$  and  $\|AB\| \leq \|A\|\|B\|$  for all  $A, B \in \mathfrak{A}$ .<sup>35</sup> For our purposes, we will need to consider two examples of a  $C^*$ -algebra. First, and probably the most familiar to philosophers of physics, is  $\mathfrak{B}(\mathcal{H})$ , the bounded operators on a (separable) Hilbert space  $\mathcal{H}$ . The involution is provided by the adjoint operation on  $\mathfrak{B}(\mathcal{H})$  and the norm is the Hilbert space operator norm,  $\sup |\langle A\psi|A\psi \rangle|^{1/2}$ , where  $A \in \mathfrak{B}(\mathcal{H})$  and  $\psi$  ranges over the unit vectors. Since operators  $A, B \in \mathfrak{B}(\mathcal{H})$  need not commute (i.e. in general  $AB - BA \neq 0$ )  $\mathfrak{B}(\mathcal{H})$  is a non-Abelian  $C^*$ -algebra.

The second example of a  $C^*$ -algebra we will consider is the algebra of continuous complex-valued functions which vanish at infinity ( $C_0(X)$  on a topological space<sup>36</sup>  $X$  equipped with the supremum norm  $\|f\|_\infty = \sup_{x \in X} |f(x)|$ , with addition and multiplication defined pointwise<sup>37</sup> and involution given by (pointwise) complex conjugation,  $f^* = \bar{f}$ . This algebra is thus Abelian (i.e. commutative). A topological space  $X$  defines a Borel  $\sigma$ -algebra  $\mathcal{X}$ , which is

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<sup>34</sup>See Stinespring (1955).

<sup>35</sup>A  $*$ -algebra  $\mathfrak{A}$  is an algebra closed under an involution  $*$  :  $\mathfrak{A} \rightarrow \mathfrak{A}$ , and an algebra over  $\mathbb{C}$  is a collection of elements on which the operations of binary addition, binary multiplication and multiplication by complex numbers are defined, which is closed with respect to those operations. See Chapter 3 of Ruetsche (2011) for an accessible introduction to these notions.

<sup>36</sup>A topological space is a non-empty set  $\Sigma$  given an open set topology, i.e. a collection of open sets (subsets of  $\Sigma$ ) such that (i) the empty set and  $\Sigma$  is open (ii) Any union of open sets is open (iii) the intersection of any finite number of open sets is open. We will consider topological spaces that are locally compact and Hausdorff.

<sup>37</sup>Such that  $(f + g)(x) = f(x) + g(x)$  and  $(fg)(x) = f(x)g(x)$  for  $x \in X$ .

the  $\sigma$ -algebra generated by the open sets of  $X$ . Equipped with this  $\sigma$ -algebra, it is a measurable space.

With these examples in hand, we can see that a POVM  $E : \mathcal{X} \rightarrow \mathcal{E}(\mathcal{H})$  defines a map  $\omega : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$ . The properties of a POVM ensure that  $\mu_\psi := \langle \psi | E(\Delta) \psi \rangle$  is a probability measure for each unit vector  $\psi \in \mathcal{H}$ . By integrating a function  $f \in C_0(X)$  over the entire space  $X$  with respect to this measure we obtain

$$\langle \psi | \omega(f) \psi \rangle := \int_X d\mu_\psi f(x)$$

which defines a bounded positive operator  $\omega(f) \in \mathfrak{B}(\mathcal{H})$ .<sup>38</sup> That is,  $E : \mathcal{X} \rightarrow \mathcal{E}(\mathcal{H})$  defines a positive map  $\omega : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$ .<sup>39</sup> Furthermore, it is a completely positive map, since any positive map from an Abelian  $C^*$ -algebra to any  $C^*$ -algebra is automatically completely positive.

Since the map provided by any POVM thus satisfies the conditions of Stinespring's Theorem, there exists a Hilbert space  $\mathcal{K}$  and a representation<sup>40</sup>  $\pi : C_0(X) \rightarrow \mathfrak{B}(\mathcal{K})$  such that  $U\omega(A)U^{-1} = P\pi(A)P$ . In order to recover Naimark's Theorem, we need to show that this representation defines a PVM on  $\mathcal{K}$ , related to the POVM on  $\mathcal{H}$  by projection.

Characterized abstractly, a (normalized) POVM  $E : \mathcal{X} \rightarrow \mathfrak{B}(\mathcal{H})$  is equivalent to a positive map  $\omega : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$  which takes sequences of functions  $f_i \rightarrow 1_X$  to sequences of operators  $\omega(f_i) \rightarrow \mathbb{I}$ , where  $1_X$  is the function  $x \mapsto 1$ , for all  $x \in X$ . Given such a map the functional  $\langle \psi | \omega(f) \psi \rangle$  defines a probability measure  $\mu_\psi$  for each  $\psi \in \mathcal{H}$  (for functions of  $X$  with compact support). The positive operators of the corresponding POVM  $\Delta \mapsto E(\Delta)$  are defined by setting  $\langle \psi | E(\Delta) \psi \rangle := \mu_\psi(\Delta)$  for each  $\psi \in \mathcal{H}$ . If, in addition,  $\omega$  is a representation then  $\omega(\Delta_1 \Delta_2) = \omega(\Delta_1) \omega(\Delta_2) = \omega(\Delta_2) \omega(\Delta_1)$ , in which case  $\omega$  defines a PVM on  $\mathcal{H}$ . Thus we can see that Stinespring's Theorem also serves to provide the PVM promised by Naimark's Theorem through the representation  $\pi : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$ .

This more general point of view also gives a perspective from which to see more fully the correspondence between PVMs and self-adjoint operators announced by the Spectral Theorem. First note that an important use of the spectral theorem is in defining the spectral

<sup>38</sup>See Pedersen (1989, pp. 165–166) for more details of this procedure.

<sup>39</sup>A linear map  $\phi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$  is positive if  $A \geq 0$  implies  $\phi(A) \geq 0$ , for all  $A \in \mathfrak{A}$ . An element  $A \in \mathfrak{A}$  is positive,  $A \geq 0$ , if  $A = B^*B$ , for some  $B \in \mathfrak{A}$ . For each positive operator  $B \in \mathfrak{B}(\mathcal{H})^+$  there exists a unique positive operator  $B^{1/2} \in \mathfrak{B}(\mathcal{H})^+$  such that  $(B^{1/2})^2 = B$ .

<sup>40</sup>A linear map  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$  is a  $*$ -homomorphism or representation if it preserves the algebraic operations and involution, i.e.  $\pi(A + B) = \pi(A) + \pi(B)$ ,  $\pi(AB) = \pi(A)\pi(B)$ , and  $\pi(A^*) = \pi(A)^*$ . A representation is always a positive map since  $\pi(A^*A) = \pi(A)^*\pi(A)$ .

calculus for measurable functions  $f : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}$  which, in terms of the spectral family for a self-adjoint operator  $A$ , can be written as  $f(A) = \int_{\mathbb{R}} f(\lambda) dP_{\lambda}^A$ . A bounded operator  $B \in \mathfrak{B}(\mathcal{H})$  is a function of  $A$ ,  $B = f(A)$ , if and only if  $B$  commutes with every bounded operator that commutes with  $A$  (Jordan, 1969, p. 70). The set of all such operators is a von Neumann algebra  $W^*(A) \subset \mathfrak{B}(\mathcal{H})$ , reached by taking the double commutant  $W^*(A) = \{\{A\}'\}'$ , i.e.  $W^*(A)$  is the set of all bounded operators that commute with all the bounded operators that commute with  $A$ . Therefore,  $B \in W^*(A)$  if and only if  $B = f(A)$  for some  $f$ . Evidently, then,  $W^*(A)$  is a commutative subalgebra of  $\mathfrak{B}(\mathcal{H})$ ,  $W^*(A) \subset W^*(A)'$ .

We can now show that the Spectral Theorem can be formulated as a special case of Stinespring's theorem, in the following sense. First we define the multiplication operators for a topological space  $X$ , which are operators  $M_f \in \mathfrak{B}(L^2[X])$  such that

$$M_f g = fg, \quad g \in L^2[X].$$

Consider the set of  $M_f$  such that  $f$  is a function with compact support,  $\{M_f : f \in C_c(X)\}$ . The strong closure of this set is a maximal commutative subalgebra  $\mathfrak{A} \subset \mathfrak{B}(L^2[X])$ , and there exists a \*-isomorphism  $\phi : C_0(X) \rightarrow \mathfrak{A}$  (Pedersen, 1989, Thm. 4.7.6). A PVM defines a completely positive map  $\omega : C_0(X) \rightarrow \mathfrak{B}(\mathcal{H})$  into a subalgebra of commuting operators  $\mathfrak{D} \subset \mathfrak{B}(\mathcal{H})$ . If this subalgebra is maximal commutative, i.e.  $\mathfrak{D} = \mathfrak{D}'$ , then there exists an isometry  $U : L^2[X] \rightarrow \mathcal{H}$  such that  $f : C_0(X) \rightarrow \mathfrak{D}$  is a representation, where  $f(D) = UM_fU^{-1}$  (Pedersen, 1989, Thm. 4.7.7). This is thus a special case of Stinespring's Theorem in which the projection is the identity.

This result is closely related to the Halmos (1963) form of the spectral theorem that says, briefly, that every self-adjoint operator is unitarily equivalent to a multiplication. In more detail,

**Theorem 3.11** (Spectral Representation (Halmos)). *If  $A$  is an unbounded self-adjoint operator on  $\mathcal{H}$  then there exists a (real-valued) bounded measurable function  $M \in L^2[\mathcal{X}, d\mu]$  on a measurable space  $X$  with measure  $\mu$  and an isometry  $U$  from  $L^2[\mathcal{X}, d\mu]$  onto  $\mathcal{H}$  such that*

$$(U^{-1}AUf)(x) = M(x)f(x) \text{ for } x \in X$$

for each  $f \in L^2[\mathcal{X}, d\mu]$ .

From this version it is possible to prove the von Neumann form (Theorem 3.1), but it is noteworthy that the existence of a spectral representation does not uniquely determine a multiplication operator  $M$ .<sup>41</sup>

It remains for us to show that we can set up a correspondence between the self-adjoint operators and the PVMs by the means above. First we note that if  $S \in \mathfrak{B}(\mathcal{H})$  is self-adjoint then there exists a  $*$ -homomorphism  $f : \mathcal{B}_b(X) \rightarrow \mathfrak{B}(\mathcal{H})$  from the bounded Borel functions of  $X$  into  $W^*(S)$  (Pedersen, 1989, Thm. 4.5.4). Since we are also interested in unbounded operators, we need to consider the operators affiliated with  $W^*(S)$ , where an operator  $A$  is affiliated with  $W^*(S)$  if and only if  $W^*(A) \subset W^*(S)$ . Note that neither  $A$  nor  $S$  need be bounded for this relation to hold. If  $W^*(S)$  is maximal commutative<sup>42</sup> in  $\mathfrak{B}(\mathcal{H})$  then there is a  $*$ -isomorphism  $f : L(X) \rightarrow \mathfrak{B}(\mathcal{H})$  between the measurable functions on  $X$  and the class of normal operators in  $L^2[X]$  affiliated with  $\mathfrak{A} = \{M_f : f \in C_0(X)\}$  (Pedersen, 1989, Thm. 5.3.2). In addition, there exists an isometry  $U : L^2[sp(S)] \rightarrow \mathcal{H}$  such that  $f \rightarrow f(S) = UM_fU^{-1}$  is a  $*$ -isomorphism between the measurable functions on the spectrum of  $S$  and the class of normal operators affiliated with  $W^*(S)$  (Pedersen, 1989, Thm. 5.3.3).<sup>43</sup> This highlights the importance of the spectrum of  $S$  in setting up a spectral representation. The Spectral Theorem, in von Neumann form, can then be stated as follows:

**Theorem 3.12** (Spectral Theorem (Pedersen)<sup>44</sup>). *Every self-adjoint operator  $S$  on  $\mathcal{H}$  defines a  $*$ -isomorphism from  $f \in L(sp(S))$  onto the  $*$ -algebra  $\mathfrak{A}$  of normal operators affiliated with  $W^*(S)$ , such that*

$$f(S) = \int_{sp(S)} f(\lambda)dP(\lambda); \quad S = id(S) = \int_{sp(S)} \lambda dP(\lambda); \quad \mathbb{I} = 1(S) = \int_{sp(S)} dP(\lambda),$$

where  $L(sp(S))$  is the  $C^*$ -algebra of measurable functions on the spectrum of  $S$ , and  $W^*(S) = \{\{S\}'\}'$  is the double commutant of  $S$  (and thus a von Neumann algebra). This defines a PVM  $P^S : \mathcal{B}(sp(S)) \rightarrow W^*(S)$ , such that  $P(\Delta) \in W^*(S)$  is a projection, for all  $\Delta \in \mathcal{B}(sp(S))$ .

<sup>41</sup>See the Appendix of Berezin & Shubin (1991) for a detailed proof and discussion.

<sup>42</sup>This is equivalent to the requirement that  $S$  be multiplicity free. A self-adjoint operator in  $\mathcal{H}$  is multiplicity free if there is a vector  $\psi \in \cap_n \mathcal{D}_{S^n}$  for  $n \geq 0$  such that the vectors  $\{S^n\psi : n \geq 0\}$  are dense in  $\mathcal{H}$  (Pedersen, 1989, p. 211).

<sup>43</sup>These operators are normal rather than self-adjoint since they will include (e.g.) unitary operators.

<sup>44</sup>See for this Pedersen (1989) Theorems 4.5.7, 5.3.8, and 5.3.10.

Conversely, to a PVM  $P^S : \mathcal{B}(X) \rightarrow \mathcal{L}(\mathcal{H})$  there corresponds a  $*$ -algebra of normal operators affiliated with  $W^*(S)$  given by  $\langle \phi | f(S) \psi \rangle = \int_X f(\lambda) d\mu_{\phi, \psi}(\lambda)$ , i.e.

$$f(S) = \int_X f(\lambda) dP(\lambda) \quad \text{with} \quad S = \int_X \lambda dP(\lambda); \quad \mathbb{I} = 1_X(S) = \int_X dP(\lambda).$$

This appears to be the most general form in which the correspondence between PVMs and self-adjoint operators can be given.

### 3.3 CONDITIONAL PROBABILITIES AND MEASUREMENTS

Many of the initial results concerning POVMs arose out of the application to quantum mechanics of the operational (i.e. algebraic) approach to quantum field theory of Haag & Kastler (1964). This emphasis on operations gave the approach a greater generality than just the consideration of projective measurements, covered by the Projection Postulate of the Dirac-von Neumann formalism. However, projections have a special role to play in defined conditional probabilities, it seems. The form of the expression for a conditional probability on the lattice of projections  $\mathcal{L}(\mathcal{H})$  is uniquely determined by a generalization of Gleason's Theorem.

**Theorem 3.13** (Cassinelli & Zanghi<sup>45</sup>). *Let  $\mu(A) = \text{tr}[A\rho]$  be a probability measure on  $\mathcal{L}(\mathcal{H})$ , and let  $E \in \mathcal{L}(\mathcal{H})$  be a projection such that  $\mu(E) \neq 0$ . Define a functional  $P_\mu(\cdot|E)$  on  $\mathcal{L}(\mathcal{H})$  such that*

1.  $P_\mu(\cdot|E)$  is a probability measure on  $\mathcal{L}(\mathcal{H})$  and,
2. for all  $F \in \mathcal{L}(\mathcal{H})$  such that  $F \leq E$

$$P_\mu(F|E) = \frac{\mu(F)}{\mu(E)}$$

Then the functional is uniquely given by

$$P_\mu(F|E) = \frac{\text{tr}[EFE\rho]}{\text{tr}[E\rho]}.$$

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<sup>45</sup>See (Cassinelli & Zanghi, 1983, Thm. 2).

This provides a compelling justification for Lüders' Rule,  $\rho \rightarrow \tilde{\rho} = P_k \rho P_k / (\text{tr} [\rho P_k])$ , since the trace prescription applied to  $\tilde{\rho}$  takes the form

$$P_\mu(F|P_k) = \frac{\text{tr} [P_k F P_k \rho]}{\text{tr} [P_k \rho]},$$

using the cyclic property of the trace. This clearly has an interpretation as a conditional probability: given that the result of the previous measurement corresponds to  $P_k$ , this expression provides the probability for  $F$ .

The extension of the observables to the POVMs suggested above stands in need of a interpretation for conditional probability on the effects  $\mathcal{E}(\mathcal{H})$ . The fact that a positive operator  $B \geq O$  has a unique positive square root  $(B^{1/2})^2 = B$  leads to the standard generalization of Lüders' Rule to effects, the so-called *Lüders operation*<sup>46</sup>

$$\rho \rightarrow \bar{\rho} = \frac{B^{1/2} \rho B^{1/2}}{\text{tr} [B \rho]}, \quad (3.3)$$

which reduces to Lüders' rule when  $B^2 = B$ . By following the example above we would be led to define

$$W(A|B) = \text{tr} [A \bar{\rho}] = \frac{\text{tr} [B^{1/2} A B^{1/2} \rho]}{\text{tr} [B \rho]} \quad (3.4)$$

as a conditional probability for the effect  $A$  given the effect  $B$ . But it is easily seen that this does not have the form of a conditional probability (unless  $B$  is a projection) since if  $B^2 \neq B$  then  $W(B|B) \neq 1$ .

Effects, then, although apt to play the predictive role of a projection, are not apt to update the state (at least not in a way that leads to a meaningful conditional probability). There is nonetheless a close association between effects and state transitions, or operations. This is given by the Kraus representation, which associates effects with operations, maps  $\phi : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ . Evidently Lüders' Rule and the (aptly named) Lüders operation provide examples of an operation.

Lüders' Rule, conditionalized on the observation of a particular outcome  $P_k$ , is known as a selective operation. The closely related non-selective operation, known as a Lüders transformation, can be thought of as taking place at the ensemble level without any conditionalization on the outcome of a measurement that would remove a system from consideration as part of the ensemble. That is, the equivalent non-selective Lüders' transformation is  $\rho \mapsto \bar{\rho} = \sum_k P_k \rho P_k$ . Kraus' Representation Theorem shows that any operation at all can be put into that form.

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<sup>46</sup>See Busch et al. (1995a, p. 37).

**Theorem 3.14** (Kraus<sup>47</sup>). *For any operation  $\phi : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  there exist operators  $A_k \in \mathcal{B}(\mathcal{H})$  where  $k \in K$  is at most a countable index such that*

1.  $\sum_k A_k^* A_k \leq \mathbb{I}$  for all finite subsets of  $K$
2.  $\phi\rho = \sum_k A_k^* \rho A_k$  for all  $\rho \in \mathcal{S}(\mathcal{H})$
3.  $\phi^* B = \sum_k A_k^* B A_k$  for all  $B \in \mathcal{B}(\mathcal{H})$
4.  $E = \phi^* \mathbb{I} = \sum_k A_k^* A_k$  is an effect i.e.  $E \in \mathcal{E}(\mathcal{H})$ .

*Conversely, any such collection of operators  $A_k$  defines an operation  $\phi$  whose corresponding adjoint  $\phi^*$  and effect  $E$  are defined as above.*

This theorem demonstrates that to each effect  $E = (E^{1/2})^2$  there corresponds an operation which maps a state  $\rho$  to another state  $E^{1/2} \rho E^{1/2}$ . In fact, there exist many such operations, since  $E = A^* A$  does not uniquely correspond to an operator  $A$  unless  $A$  is positive, in which case  $A = E^{1/2}$ . Even then the application of a unitary transformation,  $A = U E^{1/2}$ , leads to a distinct operation associated the same effect  $E = A^* A$ . This may suggest that an effect is best thought of as being associated with an equivalence class of operations. What these operations correspond to physically, however, is less than clear.

To this end, however, Kraus (1983) provides another representation theorem which apparently demonstrates that every POVM on the object system can be measured by means of (a) an ancilla (a apparatus system to be coupled to the object system) initially in a pure state; (b) a unitary interaction between the ancilla and object system; (c) a projective measurement on the ancilla which leaves the object systems state unchanged. Before considering this complete description of the measurement situation provided by Kraus (1983), it will be instructive to address the interpretation of the preliminary result derived by Holevo (1973). Making use of Naimark's Theorem, Holevo was able to show that every POVM on the object system can be measured as a PVM on a larger system that includes the object system as a subsystem (i.e. by including an ancilla).

**Theorem 3.15** (Holevo<sup>48</sup>). *Let the Hilbert space of the object system be  $\mathcal{H}$ . Then any POVM  $E(X)$  on  $\mathcal{H}$  defines a Hilbert space  $\mathcal{H}_0$ , a pure state  $\psi_0 \in \mathcal{H}_0$  and a PVM  $P(X)$  on  $\mathcal{H} \otimes \mathcal{H}_0$*

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<sup>47</sup>See Kraus (1983).

<sup>48</sup>See (Holevo, 1973, Thm. 2.5.2).

such that

$$\mu_{\psi \otimes \psi_0}^P(X) = \mu_{\psi}^E(X) \quad (3.5)$$

for any state  $\psi \in \mathcal{H}$ . Conversely, any triple  $(\mathcal{H}_0, \psi_0, P)$  gives rise to the unique POVM in  $\mathcal{H}$  satisfying (3.5).

This shows how, in a sense, any measurement of a POVM on an object system can be thought of as the measurement of a PVM on a larger system, of which the object system is a subsystem. This measurement, however, must take place on the entire system. In contrast, Kraus (1983) was able to show that the measurement of the POVM on the object system could be completed by means of a projective measurement on the ancilla alone.

**Theorem 3.16.** [Kraus] *Let  $(\mathcal{H}, \rho)$  be the initial state of the object system, and let  $\phi : \rho \mapsto \varepsilon(\rho)$  be an operation on the object system. Then there exists an ancilla system with Hilbert space  $\mathcal{H}_0$ , a pure state  $\rho_0 \in \mathcal{H}_0$  of the ancilla, a unitary operator  $U$  on  $\mathcal{H} \otimes \mathcal{H}_0$ , and a projection  $Q_0$  on  $\mathcal{H}_0$  such that*

$$\varepsilon(\rho) = \text{tr}'[(\mathbb{I} \otimes Q_0)U(\rho \otimes \rho_0)U^{-1}(\mathbb{I} \otimes Q_0)] \text{ where } \text{tr}' \text{ is the partial trace on } \mathcal{H}_0. \quad (3.6)$$

*Conversely, any Hilbert space  $\mathcal{H}_0$ , pure state  $\rho_0 \in \mathcal{H}_0$ , unitary operator  $U$  on  $\mathcal{H} \otimes \mathcal{H}_0$ , and projection  $Q_0$  on  $\mathcal{H}_0$  defines an operation  $\phi : \rho \mapsto \varepsilon(\rho)$  with  $\varepsilon(\rho)$  as above.*

Since each effect is associated with an operation (or rather an equivalence class of them) this theorem appears to have the meaning that any effect on the object system can be measured with a projective measurement on the ancilla alone, by means of a correlation with the states of an ancilla set up by a unitary interaction. Kraus (1983) considers only complimentary effects  $F_1, F_2$  such that  $F_2 = \mathbb{I} - F_1$  but it is clear that the result applies to discrete sums of effects such that  $\sum_i F \leq \mathbb{I}$ , as before. This result, which underpins the so-called Standard Model of Measurement (Busch et al., 1996), apparently demonstrates that the inclusion of POVMs into the formalism of Ordinary QM requires very little interpretative extension.<sup>49</sup> In Chapter 7, I will argue that the measurement of event time observables (i.e. time shift covariant POVMs) cannot be so easily accommodated by these conventional means due to Pauli's Theorem, to which I now turn.

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<sup>49</sup>However, some valid concerns have been expressed regarding the interpretation of POVMs as joint measurements of incompatible observables by Uffink (1994).



## 4.0 ROUGH GUIDE TO PAULI'S THEOREM

We have already met Pauli's 'theorem' in its historical context in Chapter 2. There I mentioned that the argument which Pauli gave in his 1933 footnote is, strictly speaking, invalid since it admits a large class of counterexamples. But there are valid results in the vicinity of Pauli's proposition, and it is to these (I hope) that the phrase 'Pauli's Theorem' is taken to refer. These results were either stated and proved by later authors or follow as immediate corollaries from other well-known foundational results, which were not known to Pauli at the time (and often must be given in a further refined mathematical language). The purpose of this chapter is to provide a self-contained introduction to the correct statement and interpretation of 'Pauli's Theorem' by providing details of these putative counterexamples and the later results that avoid them.

We begin with a brief analysis of Pauli's original argument, which begins by positing the existence of a time observable  $T$ , canonically conjugate to the Hamiltonian  $H$ . Pauli succinctly stated his argument as follows:

It is generally not possible, however, to construct a Hermitian [i.e. symmetric] operator (e.g. as a function of  $p$  and  $q$ ) which satisfies this equation [ $HT - TH = i$ ]. This is so because ... it follows that  $H$  possesses continuously all eigenvalues from  $-\infty$  to  $+\infty$  (cf. Dirac, Quantum Mechanics, First edition (1930), 34 and 56) whereas on the other hand, discrete eigenvalues of  $H$  can be present. (Pauli, 1958, p. 63)<sup>1</sup>

Using the equalities found in Dirac's textbook one can easily write a formal argument that appears to establish Pauli's conclusion. That is, an argument given without regard to the domains of the operators involved and the validity of the operations on their domains and co-domains.<sup>2</sup>

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<sup>1</sup>I have dropped the factor of  $\hbar$  from the expression here, i.e. we are using natural units in which  $\hbar = 1$ . That will be the case from hereon in.

<sup>2</sup>Here I essentially follow the reconstruction offered by Galapon (2002a).

Using the putative time observable  $T$  we define the unitary shift operator  $e^{i\beta T}$  as an exponential expansion in  $T$ ,

$$e^{i\beta T} = \sum_{k=0}^{\infty} \frac{(i\beta)^k}{k!} T^k, \quad (4.1)$$

where  $\beta \in \mathbb{R}$ . Now iterate this relation to obtain the equality  $[H, T^n]\phi = -inT^{n-1}\phi$ . Thus we have<sup>3</sup>

$$[H, e^{i\beta T}] = \beta e^{i\beta T}. \quad (4.2)$$

Acting from the left with  $e^{-i\beta T}$  we obtain

$$e^{-i\beta T} H e^{i\beta T} = H + \beta, \quad (4.3)$$

which shows that  $e^{i\beta T}$  acts to shift the spectrum of  $H$  by  $\beta$ . But (4.3) also states that  $H$  and  $H + \beta I$  are unitarily equivalent operators, which implies that they have the same spectrum. If  $\beta$  can take any value in  $\mathbb{R}$ , it follows that the spectrum of  $H$  is  $\mathbb{R}$ . So Pauli's argument, as it appears in this footnote, takes the following form:

1. Assume that there exists a Hermitian (i.e. symmetric) operator  $T$  such that  $[H, T] = i$ .
2. From this relation it follows that  $e^{-i\beta T} H e^{i\beta T} = H + \beta$ , for  $\beta \in \mathbb{R}$ .
3. This implies that  $H$  has as its spectrum the entire real line.
4. But physical Hamiltonians do not have such a spectrum.
5. Therefore, the introduction of such an operator  $T$  is basically forbidden.

For *reductio* Pauli considered a Hamiltonian with discrete spectrum (e.g. a simple harmonic oscillator), but he might also have considered the condition that the spectrum of  $H$  be bounded from below (as did later authors).<sup>4</sup> However, no matter which assumption about the spectrum of  $H$  is made, there exist several classes of putative counterexamples to Pauli's 'theorem,' as stated.<sup>5</sup>

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<sup>3</sup>This also follows from the relation  $[H, f(T)] = f'(T)[H, T]$ .

<sup>4</sup>That he didn't may be explained by the prevalence and apparent success of Dirac's 'hole theory' at the time Pauli was writing. The Hamiltonian of Dirac's theory had a spectrum unbounded in both directions, and the negative energy states were filled by an uncountable infinity of negative energy electrons. This meant that if one of those electrons came to have a positive energy, it would leave vacated a 'hole' in the negative energy states, eventually interpreted by Dirac as an 'anti-electron' (i.e. a positron).

<sup>5</sup>See Galapon (2002a) for a detailed analysis of how Pauli's argument fails. For us, it suffices to see that it has counterexamples and is thus invalid.

These counterexamples arise from difficulties with the formal nature of the argument given above. By formal, we mean given without regard to the domain of definition of the operators involved. The problem with a purely formal argument is that concrete exemplifications of the formally defined operators may fail to obey the formal relations supposed to hold of them. As early as Garrison & Wong (1970) it was known that counterexamples to Pauli's Theorem could be found for discrete Hamiltonians with a semi-bounded spectrum.<sup>6</sup> Recently, Galapon (2002b) gives a method that applies to any  $H$  self-adjoint with a semi-bounded, discrete spectrum, resulting in a bounded, symmetric (and thus self-adjoint) operator  $T$  conjugate to  $H$ . That is, there exist counterexamples of the following sort:

1.  $H$  is a operator with a discrete, semi-bounded spectrum (i.e. non-empty point spectrum), self-adjoint on a dense domain of  $\mathcal{H}$ , a (separable) Hilbert space.
2.  $T$  is a bounded operator, self-adjoint on  $\mathcal{H}$ .
3. The commutation relation  $[H, T]\psi = i\psi$  holds for  $\psi \in \mathcal{H}$  such that the set of such  $\psi$  is dense in  $\mathcal{H}$ .

These time operators are periodic, and arise in familiar examples such as the quantum harmonic oscillator. In these examples the time operator  $T$  is bounded and symmetric, and thus self-adjoint. However, there is another class of examples in which  $T$  is unbounded and symmetric (or Hermitian), but fails to be self-adjoint.<sup>7</sup> That is, there exist pairs  $(H, T)$  such that<sup>8</sup>

1.  $H$  is a self-adjoint operator with a semi-bounded spectrum,  $H \geq c\mathbb{I}$ , self-adjoint on a dense domain of  $\mathcal{H}$ , a (separable) Hilbert space.
2.  $T$  is a unbounded operator with a continuous spectrum  $\mathbb{R}$ , symmetric on a dense domain of  $\mathcal{D}_T \subset \mathcal{H}$ .

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<sup>6</sup>See Galindo (1984) for the explicit construction of such pairs.

<sup>7</sup>One may be inclined to attribute to Pauli a usage of the word 'Hermitian' to mean self-adjoint rather than symmetric, as is common in the physics literature. However, Pauli's Handbuch article includes a detailed discussion of a case where a 'Hermitian operator' fails to be self-adjoint (Pauli, 1958, p. 65) so he was clearly aware of the distinction, yet used the word 'Hermitian' to mean 'symmetric.'

<sup>8</sup>The best known example is the time of arrival operator, which was already introduced in Section 2.3.4. Chapter 5 contains a detailed discussion of this operator. Hegerfeldt & Muga (2010) provide a recipe for generating such time operators, conjugate to an arbitrary Hamiltonian.

3. The commutation relation  $[H, T]\psi = i\psi$  holds for  $\psi \in \mathcal{D}_T$  such that the set of such  $\psi$  is dense in  $\mathcal{H}$ .

Pauli's argument, therefore, fails to secure his conclusion that 'the introduction of an operator  $T$  is basically forbidden.' The failure is not absolute, however, and there is an argument in the vicinity that serves to justify (at least in part) the received wisdom that 'time in quantum mechanics is not an observable but a parameter.' Before deciding how best to patch up Pauli's argument, it would be wise to first consider exactly how the premises were intended to justify Pauli's conclusion.

Recall that the Heisenberg equations of motion state that the rate of change of an observable is given by its commutator with the Hamiltonian,

$$\frac{dF}{dt} = i[F, H].$$

Thus, by requiring that  $T$  be canonically conjugate with  $H$ ,  $[T, H] = -i$ , the rate of change of  $T$  is determined to be 1. Integrating this formal relation, we obtain:

$$T(t) = T(0) + \int_0^t \left( \frac{dT}{dt} \right) dt = T(0) + t.$$

This is an expression of the covariance of  $T$  with time shifts, which we met in Chapter 2 as the defining property of a clock observable. But note how in integrating the expression we went from a local expression of covariance, referring to the rate of change at an instant, to a global expression, which holds for all  $t \in \mathbb{R}$ . These two formal expressions are not equivalent, as we saw in the case of classical mechanics in Section 2.4 (where the global covariance condition implied the local, but not vice versa). The same is true in quantum mechanics: the local covariance condition (given in terms of the Heisenberg equation of motion) can hold in the absence of the stronger global covariance condition.

In addition, recall that the requirement that a classical observable must be a smooth (everywhere defined) function on phase space was enough to disqualify the time functions that we did find from membership of the set of classical observables. The counterexamples to Pauli's argument mentioned here have a similar characteristic: they are associated with (locally) time covariant POVMs rather than PVMs, and thus according to Ordinary QM are not to be thought of as observables. As expressed above, Pauli's argument is blind to this distinction, and the proper expression of Pauli's Theorem will make this prohibition clear. That is, Pauli's Theorem proper will demonstrate the inconsistency of time covariance (both

local and global) with the requirement of Ordinary QM that an observable be associated with a PVM.

#### 4.1 FINDING PAULI'S THEOREM: GLOBAL COVARIANCE AND CANONICAL CONJUGACY

Before proceeding to give a version of Pauli's Theorem formulated in terms of global time covariance, we will first attempt to get straight on the distinction between local and global time covariance. We will say that  $T$  is locally time covariant if  $\frac{dT(t)}{dt} = k$  for all  $t$ , in which case the expectation value of  $T$  varies linearly with time,  $\frac{d}{dt}(\langle\psi|T(t)\psi\rangle) = k$ , for all  $\psi$  in the domain of  $T$ .<sup>9</sup> We will express the global covariance condition in terms of the expectation value of  $T$  as follows:

$$\langle\psi|T(t)\psi\rangle = \langle\psi|T(0)\psi\rangle + t,$$

for all  $t \in \mathbb{R}$  and  $\psi$  in the domain of  $T$ .

The critical operator expression that we are required to give a definite meaning to here is the time-dependent operator  $T(t)$ . To simplify matters, we will begin by assuming that the time operator  $T$  in question is self-adjoint with an unbounded spectrum. In that case, we can think of  $T(0) = T$  as a Schrödinger picture observable in the usual way, so that  $T(t)$  is just the Heisenberg picture family of observables reached by means of the one-parameter family of unitary operators generated by  $H$ ,  $U_t = e^{iHt}$ . That is,  $T(t) = U_{-t}TU_t$ , valid on the entire domain of  $T$ . This provides the following precise expression of global covariance,

$$T(t) = U_{-t}TU_t = T + \mathbb{I}t, \tag{4.4}$$

valid for all  $\psi$  in the domain of  $T$  and all  $t \in \mathbb{R}$ . First let us confirm that this global condition implies the local time covariance condition, i.e. the canonical conjugacy of  $H$  and  $T$ . The global covariance condition (4.4) can be written as a sum of nested commutators by means of the Baker-Campbell-Hausdorff identity, that is

$$e^{-iHt}Te^{iHt} = T - it[H, T] + \frac{(-it)^2}{2!}[H, [H, T]] + \frac{(-it)^3}{3!}[H, [H, [H, T]]] + \dots$$

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<sup>9</sup>Note that if  $T(t)$  is symmetric then this is the pointwise differentiation of a real valued function of  $t$ ,  $f(t) = \langle\psi|T(t)\psi\rangle$ .

Evidently, it is only the first commutator term that is non-zero in this case, so that  $t = -it[H, T]$ , i.e.,  $[H, T] = i$ . This is, however, a merely formal expression.

We can also achieve the same result with more rigor by an application of Stone's Theorem, which guarantees the existence of a unique self-adjoint generator for such a unitary family.

**Theorem 4.1** (Stone). *A one-parameter strongly continuous unitary group  $U_s$ , with  $s \in \mathbb{R}$ , has a unique self-adjoint generator  $A$  such that  $U_s = e^{iAs}$ . That is,*

$$\lim_{s \rightarrow 0} \frac{1}{is} (U_s - \mathbb{I})\psi = A\psi,$$

where  $\psi$  is in the domain of  $A$  if the limit exists.

At this juncture, it is worth remarking on the relation of the global to the local (i.e. instantaneous) form of the unitary dynamics, given to us by Stone's Theorem. Take the Schrödinger picture family of instantaneous states  $\psi_t = U_t\psi$ , where  $\psi \in \mathcal{H}$  is the state at time  $t = 0$ . The Schrödinger equation can be seen to result from Stone's Theorem in the following manner:<sup>10</sup>

$$\begin{aligned} HU_t\psi &= \lim_{\delta t \rightarrow 0} \frac{1}{i\delta t} (U_{\delta t} - \mathbb{I}) U_t\psi \\ &= \lim_{\delta t \rightarrow 0} \frac{1}{i\delta t} (U_{\delta t+t} - U_t)\psi = -i \frac{d}{dt} U_t\psi. \end{aligned}$$

Likewise, the Heisenberg equations of motion can be obtained by considering the instantaneous form of the Heisenberg picture evolution of an observable  $A(t) = U_{-t}AU_t$ . That is,

$$\begin{aligned} \frac{d}{dt}A(t) &= \frac{d}{dt}(U_{-t})AU_t + U_{-t}\frac{d}{dt}(AU_t) \\ &= i(A(t)H - HA(t)) \\ &= i[A(t), H] \end{aligned}$$

using the Schrödinger equation to get the second line from the first.

Now we will derive local covariance of  $T$ , expressed by the commutation relation  $[H, T] = i$ , from global covariance (4.4). First we apply  $U_t$  from the left to obtain (4.4) as a commutation relation,  $[T, U_t] = U_t t$ . Using this relation, along with an application of Stone's

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<sup>10</sup>This difference in sign here from, e.g., (Jordan, 1969, p. 98) is just due to a choice of convention for the exponent of the unitary group  $U_t$ .

Theorem to  $U_t$ , we have

$$\begin{aligned} [H, T] &= \lim_{t \rightarrow 0} \frac{1}{it} (U_t - \mathbb{I}) T - T \lim_{t \rightarrow 0} \frac{1}{it} (U_t - \mathbb{I}) \\ &= \lim_{t \rightarrow 0} \frac{1}{it} (U_t T - T U_t) \\ &= \lim_{t \rightarrow 0} i U_t = i \mathbb{I}. \end{aligned}$$

Thus global covariance implies local covariance of  $T$  through the Heisenberg equation of motion. The use of Stone's Theorem here shows the precise sense in which the local covariance condition is just the instantaneous form of the global condition.

In the presence of the assumption that  $T$  is self-adjoint, we can also use the global covariance of  $T$  with the unitary group  $U_t$  to derive a relationship between  $H$ , the generator of  $U_t$ , and  $U_s$ , the unitary group uniquely generated by  $T$ . This allows us to recover the last line of Pauli's argument, (4.3) above, and the last steps of the proof go through as he intended. (See below.) This provides us the global form of Pauli's Theorem, as given by Butterfield (2013).<sup>11</sup>

**Theorem 4.2** (Pauli's Theorem (Global)). *Let  $H$  and  $T$  be self-adjoint operators on  $\mathcal{H}$ , a separable Hilbert space. If  $T$  obeys a global covariance relation with  $U_t = e^{iHt}$ , the one-parameter unitary group uniquely generated by  $H$ , that is*

$$T(t) = U_{-t} T U_t = T + \mathbb{I}t$$

for all  $t \in \mathbb{R}$ , then  $H$  and  $T$  have the spectrum of the reals,  $\mathbb{R} = sp(H) = sp(T)$ .

Although this conclusion may be reached directly, using only Stone's Theorem and the Spectral Theorem, there is an illuminating route to the conclusion via the Stone-von Neumann Theorem, which concerns the Weyl form of the Canonical Commutation Relations (CCRs).

Any pair of self-adjoint operators that together obey a Weyl CCR we will call a Weyl pair, defined as follows.

**Definition 4.1.** *Weyl pair. A pair of self-adjoint operators  $(A, B)$  on a (separable) Hilbert space  $\mathcal{H}$  are a Weyl pair if and only if the one-parameter unitary groups they generate,  $U_\alpha = e^{iA\alpha}$  and  $U_\beta = e^{iB\beta}$ , obey the Weyl CCR*

$$U_\alpha U_\beta = e^{i\alpha\beta} U_\beta U_\alpha \tag{4.5}$$

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<sup>11</sup>It is hard to find an explicit statement of this result in the literature elsewhere, perhaps because it almost goes without saying. Exceptions include Srinivas & Vijayalakshmi (1981) and Galapon (2002a).

for all  $\alpha, \beta \in \mathbb{R}$ .

Every Weyl pair is also a Heisenberg pair, which we define as a pair of symmetric, densely defined operators that obey the Heisenberg CCR.

**Definition 4.2.** *Heisenberg pair.* A pair of symmetric operators  $(A, B)$  densely defined on a (separable) Hilbert space  $\mathcal{H}$  are a Heisenberg pair if and only if they obey the Heisenberg CCR

$$[A, B]\psi = i\psi \quad (4.6)$$

on a dense domain  $\psi \in \mathcal{D}_{AB} \cap \mathcal{D}_{BA} \subseteq \mathcal{H}$ .

However, a Heisenberg pair will often not be a Weyl pair: all the counterexamples to Pauli's original argument given in the previous section are Heisenberg pairs that are not Weyl pairs.<sup>12</sup>

It turns out that if  $T$  is self-adjoint then obeying the global time covariance condition (4.4) is sufficient to ensure that  $T$  forms a Weyl pair with  $H$ . Here is an argument to that effect. Since  $T$  is self-adjoint, by the Spectral Theorem there is a unique spectral family of projections  $P_\lambda$  such that  $T = \int_{\mathbb{R}} \lambda dP_\lambda$ , with  $P_{-\infty} = 0$  and  $P_\infty = \mathbb{I}$ . Thus we may write the global covariance condition (4.4) as follows

$$\begin{aligned} U_{-t} \int_{\mathbb{R}} \lambda dP_\lambda U_t &= \int_{\mathbb{R}} \lambda dP_\lambda + \int_{\mathbb{R}} t dP_\lambda \\ \int_{\mathbb{R}} \lambda d(U_{-t}P_\lambda U_t) &= \int_{\mathbb{R}} \lambda dP_{\lambda-t} \end{aligned}$$

where the last line involves a change of integration variable to  $\lambda + t$ . This equality implies that the spectral measures defined by  $d(U_{-t}P_\lambda U_t)$  and  $dP_{\lambda-t}$  are identical. (If they were not, then by the Spectral Theorem the two sides would define distinct self-adjoint operators, in contradiction with (4.4).) This implies that the PVM associated with  $T$  by the Spectral Theorem,  $P^T : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ , is globally covariant in the following sense:

$$U_{-t}P^T(\Delta)U_t = \int_{\Delta} d(U_{-t}P_\lambda U_t) = \int_{\Delta} dP_{\lambda-t} = \int_{\Delta+t} dP_\lambda = P^T(\Delta + t),$$

for all  $t \in \mathbb{R}$ , where  $\Delta + t$  is the set  $\{\lambda : \lambda - t \in \Delta\}$ . This covariance condition, applied directly to a PVM  $P^T$ , suffices to ensure that the corresponding self-adjoint operator  $T$  is

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<sup>12</sup>The significance of this fact for avoiding Pauli's Theorem was first noted by Garrison & Wong (1970), from whom this terminology is taken.



globally covariant. In that case, we define  $P_\lambda = P^T(-\infty, \lambda)$  as usual and find that

$$U_{-t}TU_t = \int_{\mathbb{R}} \lambda d(U_{-t}P_\lambda U_t) = \int_{\mathbb{R}} \lambda dP_{\lambda-t} = \int_{\mathbb{R}} (\lambda + t) dP_{\lambda'} = T + \mathbb{I}t.$$

This equality also demonstrates that  $(H, T)$  form a Weyl pair. Let  $U_s = e^{iT_s}$  be the unitary group uniquely generated by  $T$ . Then we have

$$\begin{aligned} U_t U_s U_t &= \int_{\mathbb{R}} e^{is\lambda} d(U_t^\dagger P_\lambda^T U_t) = \int_{\mathbb{R}} e^{is\lambda} dP_{\lambda-t}^T \quad (\text{by above}) \\ &= \int_{\mathbb{R}} e^{is(\tilde{\lambda}+t)} dP_{\tilde{\lambda}}^T \\ &= e^{ist} \int_{\mathbb{R}} e^{is\tilde{\lambda}} dP_{\tilde{\lambda}}^T = e^{ist} U_s \\ &\Rightarrow U_s U_t = e^{ist} U_t U_s \end{aligned}$$

which is the definition of a Weyl pair. To go back the other way we just apply Stone's Theorem and the definition of a Weyl pair to the original expression, to obtain:

$$\begin{aligned} U_{-t}TU_t &= U_t^\dagger \lim_{h \rightarrow 0} \frac{1}{is} (U_s - I)U_t \\ &= \lim_{s \rightarrow 0} \frac{1}{is} (U_t^\dagger U_s U_t - \mathbb{I}) \\ &= \lim_{t \rightarrow 0} \frac{1}{is} (e^{its} U_s - \mathbb{I}) \\ &= \lim_{t \rightarrow 0} \frac{1}{is} (e^{is(T+t)} - \mathbb{I}) \\ &= T + t\mathbb{I}. \end{aligned}$$

This discussion serves as a proof of the next proposition.

**Proposition 4.1.** *The following conditions are equivalent:*

1. Global Covariance (Self-adjoint).  $T, H$  are self-adjoint operators such that  $U_{-t}TU_t = T + \mathbb{I}t$ , with  $U_t = e^{iHt}$ , for all  $t \in \mathbb{R}$ .
2. Global Covariance (PVM).  $P^T(\Delta)$  is a time shift covariant PVM such that for every  $P^T(\Delta)$  and every  $t \in \mathbb{R}$  there exists a unique projection  $P^T(\Delta + t) = U_{-t}P^T(\Delta)U_t$ .<sup>13</sup>
3. Conjugate (Weyl).  $(H, T)$  are a Weyl pair.

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<sup>13</sup>This uniqueness requirement excludes periodic PVMs from consideration.

So we can see that, so long as  $T$  is self-adjoint, the condition of global covariance with shifts in  $t$ , generated by  $H$ , leads inevitably to the requirement that  $T$  and  $H$  are a Weyl pair. But the Stone-von Neumann Theorem shows that every Weyl pair is unitarily equivalent to a Schrödinger representation, i.e. to the usual position and momentum operators.

**Theorem 4.3** (Stone-von Neumann<sup>14</sup>). *Let  $T, H$  be a Weyl pair, then there exists a unitary transformation  $U$  such that  $U^\dagger T U = Q$  and  $U^\dagger H U = P$  form Schrödinger representation.*

The unitary equivalence of two operators implies that they have the same spectrum, and so  $H$  and  $T$  must both have the spectrum of the reals if  $Q$  and  $P$  do (which they do). This argument lies behind the common wisdom that Pauli's Theorem rules out the existence of time observables in quantum mechanics.

To wrap up this section, and demonstrate that Pauli had the right idea, at least, I provide a self-contained proof of the global version of Pauli's Theorem, Theorem 4.2. This proof bypasses the Stone-von Neumann Theorem, sticking closely to Pauli's original argument instead.<sup>15</sup>

*Proof.* By the above Proposition,  $(H, T)$  is a Weyl pair. Thus we may use  $U_s U_t = e^{ist} U_t U_s$  and Stone's Theorem to argue that

$$\begin{aligned} U_{-s} H U_s &= U_{-s} \lim_{t \rightarrow 0} \frac{1}{it} (U_t - \mathbb{I}) U_s \\ &= \lim_{t \rightarrow 0} \frac{1}{it} (U_{-s} U_t U_s - \mathbb{I}) \\ &= \lim_{t \rightarrow 0} \frac{1}{it} (e^{-ist} U_t - \mathbb{I}) \\ &= \lim_{t \rightarrow 0} \frac{1}{it} (e^{i(H-s)t} - \mathbb{I}) \\ &= H - s\mathbb{I}. \end{aligned}$$

Since  $H$  and  $T$  are a Weyl pair, they have a common dense domain (Putnam, 1967, Thm, 4.8.1) on which this equality holds. Thus  $H$  and  $H - s\mathbb{I}$  are unitarily equivalent, and hence have the same spectrum (Blank et al., 2008, 4.4.6c). Let  $\lambda$  be in the spectrum of  $H$ , then  $H - \lambda\mathbb{I}$  has no inverse. But since  $H - \lambda\mathbb{I} = (H - s\mathbb{I}) - (\lambda - s)\mathbb{I}$  it follows that  $\lambda - s$  is in the spectrum of  $H - s\mathbb{I}$ , and thus also in the spectrum of  $H$ . This holds for any  $s \in \mathbb{R}$  and so the spectrum of  $H$  is  $\mathbb{R}$ . □

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<sup>14</sup>See, e.g., Putnam (1967).

<sup>15</sup>It is my pleasure to acknowledge the assistance of Bryan Roberts through many helpful conversations and much correspondence about this version of Pauli's Theorem. This section as a whole has greatly benefited in many ways from this collaboration.

Since this theorem is valid, the counterexamples to Pauli's original argument must fail to satisfy the premises. When  $T$  is symmetric but not self-adjoint,  $T$  may satisfy covariance but avoid the conclusion. In this case,  $T$  and  $H$  do not form a Weyl pair. We have already met such an operator, the quantum time of arrival, and we will investigate its properties in due course. When  $T$  is symmetric and bounded, and thus self-adjoint, global covariance must fail. As indeed it does in cases that concern *periodic* time operators.

## 4.2 REFINING PAULI'S THEOREM: PERIODIC COVARIANCE CONDITIONS

The failure of this result to apply to Heisenberg pairs involving discrete Hamiltonians and periodic, self-adjoint time observables leads one to suspect that there is something special about such pairs, aside from their failure to form a Weyl pair. It is an interesting fact that if a PVM  $P^T$  covaries in a periodic manner then the Spectral Theorem fails to secure a unique correspondence with a self-adjoint operator. While the spectrum of an operator  $T$  with periodicity  $\tau$  will be an interval  $[\varphi, \tau + \varphi)$ , involving an arbitrary choice of phase  $\varphi$ , the Spectral Theorem (Thm. 3.1) supplies a PVM defined on the Borel subsets of the real line,  $\mathcal{B}(\mathbb{R})$ .

Let  $T_0$  be the operator given by the Spectral Theorem for a periodic PVM  $P^T$ , and let  $0 < \varphi < \tau$  be an arbitrary phase. Then  $U_{-\varphi}T_0U_\varphi = T_\varphi$  is a distinct self-adjoint periodic time observable associated with a PVM  $P^{T_\varphi}$  whose projections  $P^{T_\varphi}(\Delta) = U_{-\varphi}P^T(\Delta)U_\varphi$  also covary with time translations  $U_t$  in the sense above. That is, for  $\varphi = t \pmod{\tau}$ , with  $t \in \mathbb{R}$ , we have

$$\begin{aligned} U_{-t}T_0U_t &= \int_0^\tau \lambda d(U_{-t}P_\lambda^T U_t) = \int_0^\tau \lambda dP_{\lambda+t}^T \\ &= \int_\varphi^\tau (\lambda - \varphi) dP_\lambda^T + \int_0^\varphi [(\lambda - \varphi) + \tau] dP_\lambda^T \\ &= \int_0^\tau (\lambda - \varphi) dP_\lambda^T + \tau \int_0^\varphi dP_\lambda^T \\ &= T_0 - \varphi\mathbb{I} + \tau P^T([0, \varphi]) := T_\varphi. \end{aligned}$$

This resembles the arbitrary choice of phase for a periodic time function in classical mechanics, and seems to be characteristic of a periodic time observable in quantum mechanics.

This demonstrates how a periodic time observable can escape the global version of Pauli's Theorem by failing to satisfy the global covariance condition. However, there is a closely related periodic covariance condition reached by allowing non-uniqueness of the elements of the PVM related by a unitary transformation. This more permissive condition leads to a version of Pauli's Theorem which suffices to place severe restrictions on the existence of such periodic PVMs, despite the self-adjointness of their corresponding expectation operators.

Furthermore, this version relies on the less restrictive (and so stronger) spectral condition, which only requires that the Hamiltonian spectrum is semi-bounded, so including Hamiltonians with discrete spectra in its scope. (In contrast to the results of the previous section.) That is, the following version of Pauli's Theorem directly addresses periodic time observables.<sup>16</sup>

**Theorem 4.4** (Srinivas & Vijayalakshmi (1981)). *Let  $P^T(\Delta)$  be elements of a PVM and let  $U_t = e^{iHt}$  be the unitary group of time translations generated by  $H$ , such that*

1. *Spectrum. The Hamiltonian  $H$  is a self-adjoint operator bounded from below i.e. there exists a number  $c > -\infty$  such that  $H \geq c\mathbb{I}$ .*
2. *Covariance.  $P^T(\Delta)$  covaries with time translations. That is, for every time  $t \in \mathbb{R}$  there exists a projection  $P^T(\Delta + t)$  such that*

$$U_t P^T(\Delta) U_t^\dagger = P^T(\Delta + t).$$

*Then  $P^T(\Delta) = 0$  for all  $\Delta$ .*

Through the Spectral Theorem, then, this result suffices to show that there is no self-adjoint operator  $T$  obeying the global covariance condition with  $H$  semi-bounded. The result also applies to periodic time observables, however, since this covariance condition allows for identifications of projections corresponding to distinct periods. That is, if  $\tau$  is the period of a periodic time observable then projections  $P^T(\Delta + n\tau)$ , with  $n \in \mathbb{Z}$ , are identical. But this covariance condition is met by these projections, regardless. Moreover, Stone's Theorem guarantees that every self-adjoint Hamiltonian generates a unitary group  $U_t$ , with  $t \in \mathbb{R}$ ,

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<sup>16</sup>This version of the theorem is closely related to the recent 'no go' results of Malament (1996) and Halvorson & Clifton (2002) concerning relativistic locality. There are two methods of proof: Srinivas & Vijayalakshmi (1981) make use of Borcher's Lemma (like Malament), whereas Halvorson (2010) uses Hegerfeldt's Lemma (like Halvorson & Clifton). See Section 4.4 for details of the proofs.

regardless of the periodicity of the system. Therefore, this result also suffices to show that there is no self-adjoint operator  $T$  associated with a PVM whose projections obey a periodic covariance condition with the unitary group generated by  $H$ , semi-bounded.<sup>17</sup>

This extension of Pauli’s Theorem to cover periodic time observables is significant because it captures the counterexamples to Pauli’s original argument that escaped the global form of the theorem. If  $H$  has a discrete spectrum and forms a Heisenberg pair with  $T$ , periodic and bounded, they can both be self-adjoint and together form a Heisenberg pair without forming a Weyl pair. This might lead one to believe that these self-adjoint time operators correspond to a periodically time covarying PVM. The refined version of Pauli’s Theorem shows that this cannot be the case: a counterexample to Pauli’s original argument involving time covariance of a time operator must concern time covariant POVMs rather than PVMs, even though the Spectral Theorem associates such an operator with a PVM.<sup>18</sup>

This reveals all the counterexamples referred to previously to have a very similar form, despite the fact that some time operators forming a Heisenberg pair with the semi-bounded self-adjoint Hamiltonian are maximally symmetric while some are self-adjoint. As we have seen, a maximally symmetric operator associated with a POVM rather than a PVM. But since no bounded symmetric operator can be the expectation operator of a PVM that covaries periodically with a unitary group generated by a self-adjoint Hamiltonian with a semi-bounded spectrum, such operators must arise as the expectation operator of a POVM instead. This does not stop such operators from being covariant, however. Take the time of arrival,  $T_a$ , a maximally symmetric operator corresponding to a POVM  $E^{T_a}$ .<sup>19</sup> The covariance of the underlying POVM  $E^{T_a}(\Delta + t) = U_{-t}T_aU_t$  still yields the global covariance condition through the generalized spectral resolution  $E_\lambda = E^{T_a}(-\infty, \lambda]$  as follows

$$U_{-t}T_aU_t = \int_{\mathbb{R}} \lambda d(U_{-t}E_\lambda U_t) = \int_{\mathbb{R}} \lambda dE_{\lambda+t} = T_a - \mathbb{I}t$$

valid for all  $\psi$  in the domain of  $T_a$ .

Given the abundance of these time covariant symmetric operators, it is tempting to see Pauli’s Theorem as less of a “no go” theorem and more of a “go on then, if you must” theorem. That is, instead of preventing the introduction of time observables into quantum mechanics, Pauli’s Theorem really just tells us something interesting about the time observables that

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<sup>17</sup>Crucially, though, such an operator may also be associated with a time shift covariant POVM. See Section 5.3.

<sup>18</sup>This issue is addressed more fully in the discussion of phase in Section 5.3.

<sup>19</sup>See Section 5.4 for further details.

we can define: they will correspond to time covariant POVMs that are not PVMs. Given the expansion of the set of observables of quantum mechanics to include such POVMs that I argued for in Chapter 3, the question one needs to answer is interpretative: what is the empirical relevance of these time covariant POVMs? That is, what experimental situations do they correspond to (if any)?

Most of the rest of this dissertation can be seen as an attempt to answer this question positively. Specifically, in Chapter 6 I advocate the use of periodic time covariant POVMs as clock observables, and in Chapter 7 I argue that time covariant POVMs may be used to provide event time distributions. In Chapter 5, I present a detailed examination of some famous examples of such operators and how they differ from their PVM yielding brethren. First, though, I consider some interpretative morals that may already be drawn from this discussion of Pauli's Theorem.

### 4.3 HOW (NOT) TO INTERPRET PAULI'S THEOREM

We have seen that there is a considerable degree of subtlety involved in stating and proving 'Pauli's Theorem.' Now that our hunt for the results that lie behind Pauli's skeptical conclusion is complete, the stage is set for some preliminary interpretative remarks. The first definitive result that we obtained, Theorem 4.2, demonstrates that the following propositions form an inconsistent trilemma:

1.  $T$  is a self-adjoint operator with spectrum  $sp(T) = \mathbb{R}$ .
2.  $T$  covaries with time translations generated by  $H$ ,  $U_{-t}TU_t + T + \mathbb{I}t$ , for all  $t \in \mathbb{R}$ .
3.  $H$  is a self-adjoint operator with spectrum  $sp(H) \neq \mathbb{R}$

This certainly goes some way to justifying the conventional wisdom that time is a parameter rather than an operator in quantum theory. I will claim here that this rules out the existence of (what I will term) a canonical time observable corresponding to external time. Let us first consider the plausibility of these propositions in turn.

Since the dynamics of the theory is given by a one-parameter unitary group  $U_t$ , there is certainly a sense in which the (external) time should be allowed to range over the reals. This suggests that a canonical time observable  $T$  should be allowed to take any value in the

reals, and thus have an unbounded spectrum. It is also hard to see how an operator  $T$  whose expectation value doesn't covary with time shifts could be regarded as a time observable. To deny this would be (in effect) to deny the homogeneity of time.<sup>20</sup>

This is not to say that the homogeneity of time could not conceivably be denied, of course. Time could indeed be discrete, or have a beginning (or an end).<sup>21</sup> But it is hard to see how the failure of quantum mechanics to allow the definition of a canonical time observable could prejudice answers to metaphysical questions regarding the topology of time. Finally, since Stone's Theorem guarantees that a self-adjoint operator lies behind the unitary group giving the dynamics of the theory it is hard to deny that  $H$  must be self-adjoint. But could it plausibly have the spectrum of the reals?

First note that this requirement on the spectrum is really quite severe—as we have seen, Pauli considered the mere existence of Hamiltonians with discrete spectra to be sufficient to rule out the existence of a canonical time observable. It is straightforward to argue from the fact that most (if not all) quantum systems have Hamiltonians that prevent a canonical time observable for being defined to the conclusion that there is no canonical time observable. All we need is the premise that if a canonical time observable exists then it can be defined for every quantum system. Of course, one man's modus tollens is another man's modus ponens: there is a perversion of this argument which instead takes the existence of time observables as a premise and argues that, therefore, there exist quantum systems with unbounded Hamiltonians. But while this argument may hold water in specific cases (although I will contend that it doesn't), the existence of one or two such operators cannot compensate for the fact that the quantum systems we know and love feature Hamiltonians which prevent the definition of a canonical time observable.

So let us take it as given that there is no such thing as a canonical time observable (whatever that would be). What follows from this fact? Arguably, very little. Here my sympathies are with Hilgevoord (2005), who attributes demands for a canonical time observable to an unfortunate confusion between the spectral values of the position observable  $(q_x, q_y, q_z)$  and space  $(x, y, z)$ . If one were the victim of such a confusion, one could easily be led from the apparent symmetry of time and space in a relativistic space-time to the desire to introduce

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<sup>20</sup>See (Jauch, 1968, p. 197) for a parallel discussion regarding position and the homogeneity of space.

<sup>21</sup>If time itself is semi-bounded then  $U_t$  could be defined as a semi-group, i.e. a family of isometries rather than unitary operators. In that case  $H$ , the generator of the semi-group, would not be self-adjoint.

a time canonical operator into quantum mechanics. Despite my reservations about Hilgevoord’s attribution of such a confusion to the authors of quantum mechanics, (see Section 2.2), I have no reservations about the confusion itself: this would be a poor reason to want a canonical time observable.

But there is anyway something quite strange about the idea of a canonical time observable, (e.g.) thought of in the same way as position according to the conventional ‘Copenhagen’ interpretation. In that case, one would be led to expect that measuring a projection of the spectral resolution of  $T$  (say, the projection associated with the time interval corresponding to the specious present) would correspond to the experimental question, ‘is the system to be found in the present?’ Applying the usual interpretative rules, the possible answers are binary alternatives corresponding to (a) the system existing at this time or (b) existing at some other time. Either alternative is problematic. If (a), then the system is in ‘an eigenstate of existing at this time,’ which suggests that it will cease to exist very soon. If (b), then where has it gone? In the conventional account, measuring such projections would bring about gross non-conservation of probabilities.<sup>22</sup>

This brings me to Halvorson’s (2010) contention that a closely related no go result provides an argument against the *existence* of time. Admittedly, this paper of Halvorson’s is a work in progress. But is worth remarking that, as it stands, Halvorson’s claim that “insofar as quantities are represented by operators, time is not a quantity at all— not even an unobservable quantity” (1) is baseless since counterexamples to the claim that the quantity “amount of time” cannot be represented by an *operator* are fairly plentiful. Such operators are symmetric, and may even be self-adjoint. Charitably, we should read this instead as an argument that the non-existence of a canonical time operator implies that there is no such thing as the “amount of time” in quantum theory.

An obvious rejoinder to such an argument would be the claim that some quantities, such as space and time, are represented by parameters rather than operators in quantum mechanics, but are real nonetheless. This is (roughly) the perspective offered by Hilgevoord

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<sup>22</sup>Alternatively, one could explore an Everett-esque ‘no collapse’ account, in which case (by virtue of our relative existence in the present) the relative state of the system changes with time. Or perhaps there is a presentist-friendly ‘dynamical collapse’ account, whereby there is a continual collapse of the wavefunction (in the ‘time-representation’) into eigenstates of the present moment. This would require an A-series account of time, leading to (it seems) an extra time parameter with respect to which the time-dependent ‘moving spotlight’ of this dynamical collapse could be defined, in which case infinite regress beckons (McTaggart, 1908). My point here, however, is that Pauli’s Theorem saves us from the mindless exploration of these metaphysical rabbit-holes.



(2005). Halvorson, however, preemptively dismisses this response as resting on a “merely verbal” distinction. If we are taking this argument against the existence of time seriously, however, the rejoinder should be considered with more care. In favor of the claim that being represented by a parameter is sufficient to be taken seriously as a real quantity, consider the problems surrounding the definition of a satisfactory position operator in relativistic quantum mechanics. When philosophers of physics (including Halvorson himself) have used the apparent non-existence of such position operators as an opportunity for existential modus tollens, the chosen target has been the existence of a satisfactory quantum theory of relativistic particles set in space-time, rather than space-time itself.<sup>23</sup>

This suggests that there may be considerable leeway to avoid the conclusion of this eliminativist argument. That is, it does not follow from the fact that time fails to be represented by an operator with desirable properties that time *itself* does not exist (at least not without further argument). Now, this may be to take Halvorson’s argument sketch all too seriously, after all he concludes the short note on a note of puzzlement: “What is the difference between quantities that can be represented by operators, and those—such as ‘amount of time’—that cannot? And why is time the only such parametric quantity? What is special about time?” My inclination is to view his claim that “time [is] the only such parametric quantity” as an indication that Halvorson has here fallen victim to the kind of confusion outlined by Hilgevoord. By which I mean to say that time is not the only parametric quantity in quantum theory, since there are directly analogous parametric quantities concerning space.

Reflecting carefully on the role of the time parameter in quantum mechanics, we see that it enters into the theory through its appearance in the unitary group of time shifts,  $U_t$ . From there it comes to parameterize the states of the system (in the Schrödinger picture) through the Schrödinger equation, which is just the instantaneous form of the equality  $\psi_t = U_t\psi$  that determines the successive states of the system (where  $\psi$  is the state at  $t = 0$ ). But there is, of course, a directly analogous unitary group of spatial shifts  $U_a$ , generated by the (total) momentum observable. One could equally write a family of spatially shifted states  $\psi_a = U_a\psi$ , and thereby derive a ‘spatial Schrödinger equation’ which concerns the infinitesimal variation of the  $\psi_a$  (see Section 4.4). From this point of view, the additional significance of the group of temporal symmetries derives from the fact that we (as observers confined to the present)

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<sup>23</sup>See Halvorson & Clifton (2002) and in particular Malament (1996). It should be clear that the move to quantum field theory won’t help here.

are ourselves subject to an active transformation that takes us from our present to our future (at a rate of one second per second, as it were).

That is, while one could argue that the temporal symmetries  $U_t$  are special in virtue of their dynamical significance, they are certainly not special in virtue of involving a one-parameter (strongly continuous) unitary family of symmetries. However, Pauli's Theorem does pick out the group  $U_t$  as special in the following sense: if  $H$  obeys the spectrum condition then there is no self-adjoint operator  $T$  conjugate to  $H$  (in the Weyl sense) which could be the generator in shifts of the spectrum of  $H$ , the generator of  $U_t$ . So what Pauli's Theorem demonstrates is that there is no one-parameter unitary group parameterized by energy. It is not clear how this fact could bear on the existence of time.

However, it does entail that the time-energy uncertainty relation must be formally distinct from the position-momentum uncertainty relation, despite the sterling work of Hilgevoord (1996); Uffink (1990) to put them on the same footing. That is, Hilgevoord and Uffink's approach allows them to define two sets of relations for position and momentum: one set in which variation in position under spatial shifts is considered, with momentum playing the role of generator, and one set in which variation in momentum under shifts in momentum (Galilean boosts) is considered, with position playing the role of generator. On the other hand, when it comes to time and energy, only one set of relations may be defined (with energy playing the role of generator) since, according to Pauli's Theorem, there is no self-adjoint generator of shifts in energy.

This is, it seems, the most significant fact that follows from the non-existence of a canonical time observable. But it is also the least unexpected since Pauli's Theorem (global version) rests on the fact that the existence of a canonical time observable is logically equivalent to the existence of a unitary group of shifts in energy. Indeed, what we will see is that the cases where time observables can be defined in quantum theory are precisely those where they generate no such unitary group. For example, a maximally symmetric operator  $T_{ms}$  is the generator of a semigroup of isometries, parameterized by  $s \geq 0$ , rather than a unitary group parametrized by  $s \in \mathbb{R}$ . The existence of such a semigroup is consistent with the energy being bounded from below while the existence of a unitary group is not.

This observation, which also serves as a preview of upcoming attractions, concludes this initial discussion of interpretative matters. I have argued for the deflationary conclusion that the failure of quantum mechanics to allow for a canonical time observable has no special

significance for the metaphysics of time. That is not to say that Pauli’s Theorem (particularly in the refined form) has no metaphysical significance, however. For instance, in Pashby (2013) I argue that there are implications for metaphysical views of persistence in time. Moreover, in Chapter 6 I will argue that has an important application as a ‘no go’ result for the existence of an ideal clock. In Section 8.1.2 I consider an interpretation of this result as concerning the indeterminacy of the time of an event—the interpretation it was given by Srinivas & Vijayalakshmi (1981).

#### 4.4 APPENDIX: PROOFS OF PAULI’S THEOREM

This section contains details of the proof of the main result of this Chapter, Theorem 4.4. There are two methods of proof, which we will consider in turn. The first method of proof makes use of Borchers’ Lemma, which can be stated as follows:

**Lemma 4.1** (Borchers). *Let  $E$  and  $F$  be projection operators such that*

$$FU_tEU_t^\dagger = U_tEU_t^\dagger F$$

*for  $|t| < \alpha$  where  $\alpha > 0$ . Then  $FE = 0$  implies that*

$$FU_tEU_t^\dagger = 0 \text{ for all } t \in \mathbb{R}$$

The proof of Theorem 4.4 proceeds as follows.

*Proof.* Let  $\Delta$  be a bounded Borel subset  $\Delta \subset \mathbb{R}$  with diameter  $|\Delta| < \alpha$ . Since  $E^T$  is a PVM  $E^T(\Delta)$  and  $E^T(\Delta + \alpha)$  project onto orthogonal subspaces of  $\mathcal{H}$  so that  $E^T(\Delta)E^T(\Delta + \alpha) = 0$ . It also follows that these projections commute, and that  $E^T(\Delta + \alpha)$  commutes with  $U_tE^T(\Delta)U_t^\dagger$ . Therefore the conditions of Borchers’ Lemma are satisfied, and so  $E^T(\Delta + \alpha)U_tE^T(\Delta)U_t^\dagger = 0$  for all  $t \in \mathbb{R}$ . From *Covariance* we have  $E^T(\Delta) = U_{-\alpha}E^T(\Delta + \alpha)U_{-\alpha}^\dagger$  and so this implies that  $E^T(\Delta)^2 = 0$  i.e.  $E^T(\Delta) = 0$  for every bounded  $\Delta \subset \mathbb{R}$ . By taking the limit  $|\Delta| \rightarrow \infty$  we obtain  $E^T(\mathbb{R}) = 0$ , which demonstrates that  $E^T$  is not a PVM, for which  $E^T(\mathbb{R}) = \mathbb{I}$ . □

Roughly, if  $T$  is self-adjoint then it admits a PVM such that projection operators corresponding to disjoint spectral subsets are orthogonal, while the covariance of the PVM with time shifts entails that any projection  $P^T(\Delta)$  is unitarily equivalent to its time translate  $P^T(\Delta + t)$ . These two properties are not in contradiction by themselves: the PVM of a position observable  $P^Q$  covaries with shifts in position generated by momentum; position and momentum are a Weyl pair. It is the semi-boundedness of  $H$  that entails (by Borchers' Lemma) that the product of such a projection with its time translate must be zero if they are to correspond to orthogonal projections. But since the time shift is arbitrary, the only way to ensure this is if  $P^T(\Delta)$  is zero everywhere.

At this juncture, it is worth remarking on the relation of this proof to that of Malament (1996), where it is demonstrated that there is no relativistic localization scheme for a single quantum mechanical particle. There it is  $P^Q$  which is under consideration (the PVM associated with the position operator  $Q$ ), and which is shown to equal zero, but the method of proof relies again essentially on Borchers' Lemma and the same spectrum condition. This provides a hint that these two problems—the failure of quantum mechanics to provide a time observable and the impossibility of providing a sharp relativistic localization scheme—are related. We can think of Pauli's Theorem as a no go result for a sharp temporal localization scheme. Pauli's Theorem is, then, essentially a time-like version of Malament's theorem, concerning time intervals rather than (instantaneous) spatial regions.

The second method of proof makes use of the following lemma, which plays the role of Borchers' Lemma in these proofs.

**Lemma 4.2.** *[Hegerfeldt (1998)] Let  $H$  be semi-bounded and let  $U_t$  be the strongly continuous group of unitary operators generated by  $H$ , where  $t \in \mathbb{R}$ , and let  $P$  be a positive operator. Then for all  $\psi \in \mathcal{H}$  either*

1.  $\langle \psi | U_t^\dagger P U_t | \psi \rangle = 0$  for all  $t$ , or
2.  $\langle \psi | U_t^\dagger P U_t | \psi \rangle \neq 0$  for almost all  $t$  such that these  $t$  form a dense open set.

The lemma appears in more or less this form in Allcock (1969), Srinivas & Vijayalakshmi (1981), Unruh & Wald (1989), and is used explicitly by Halvorson (2010). Since it often features as an aside or a casual argument, it will be worthwhile to establish here a reasonably careful proof of the Lemma, based on the proof supplied by Hegerfeldt (1998).

*Proof.* The semi-boundedness of  $H$  implies that in its spectral resolution according to a spectral family  $E_\lambda$  there is no contribution to the integral for the range  $\lambda < C$ . Without loss of generality we may set  $C = 0$  and write

$$H = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_0^{\infty} \lambda dE_\lambda.$$

Since the spectral resolution of  $H$  in terms of the spectral family  $E_\lambda$  also appears in our definition of the  $U_t$ , we may write this integral in the same way

$$U_t = \int_0^{\infty} e^{i\lambda t} dE_\lambda.$$

Since  $P$  is a positive operator it has a unique positive square root  $P = (P^{1/2})^2$ . Define  $p(t) = \langle \psi | U_t^\dagger P U_t | \psi \rangle = \|P^{1/2} \psi_t\|^2 \geq 0$ . Since  $P^{1/2}$  is positive, if  $p(t) = 0$  then  $P^{1/2} \psi_t = 0$ . As  $U_t$  is a strongly continuous unitary group,  $p(t)$  is a continuous function and so the set of zeros  $\mathcal{N}_0$  is closed. Its complement  $\mathcal{N}_0^c$  is thus open.

For fixed  $\phi \in \mathcal{H}$  consider the function  $F_\phi(t)$  defined by the following inner product,

$$F_\phi(t) = \langle \phi | P^{1/2} \psi_t \rangle = \langle \phi | P^{1/2} U_t \psi \rangle = \int_0^{\infty} e^{i\lambda t} d\langle \phi | P^{1/2} E_\lambda \psi \rangle.$$

By the above argument, if  $p(t) = 0$  then  $F_\phi(t) = 0$ . Now we consider the same integral over a complex variable  $z = t + iy$  which defines a complex function  $F_\phi(z)$  which equals  $F_\phi(t)$  when  $y = 0$ .

$$F_\phi(z) = \int_0^{\infty} e^{i\lambda z} d\langle \phi | P^{1/2} E_\lambda \psi \rangle = \int_0^{\infty} e^{i\lambda t} e^{-\lambda y} d\langle \phi | P^{1/2} E_\lambda \psi \rangle$$

If  $P^{1/2}$  is bounded the integral will converge so long as  $|e^{i\lambda z}| = |e^{-\lambda y}|$ , which is the case for  $y \geq 0$  only (since  $U_t = e^{iHt}$ ). Therefore  $F_\psi(t)$  is the boundary function of a complex function  $F_\psi(z)$  defined in the upper half of the complex plane which is continuous for  $y \geq 0$  and analytic for  $y > 0$  (Rudin, 1987, p. 372).<sup>24</sup>

By the ‘‘edge of the wedge’’ theorem (Streater & Wightman, 1964, p. 74) there is a unique analytic extension to the lower half of the complex plane.  $F_\phi(t)$  is therefore the boundary function of a unique, everywhere analytic extension  $F_\psi(z)$  defined on the entire complex plane.<sup>25</sup>

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<sup>24</sup>Note how the semi-boundedness of  $H$  enters the argument here: if  $H$  had an unbounded spectrum, then this integral would not exist.

<sup>25</sup>This follows also from the Riemann-Schwarz reflection principle, see (Titchmarsh, 1952, p. 155).

Let the complement  $\mathcal{N}_0^c$  not be dense. Then there exists an open interval  $I$  such that  $F_\phi(t) = 0$  for all  $t \in I$ . Then  $F_\psi(z)$  vanishes on the whole of its domain and so  $F_\phi(t) = 0$  for all  $t$ .<sup>26</sup> This applies to any  $\phi \in \mathcal{H}$  and thus  $p(t)$ .

So far we have shown that either  $F_\phi(t)$  is identically zero, or it does not vanish on any open interval. It remains to show that 1. is the case unless  $\mathcal{N}_0$  is a set of measure zero. Since  $F_\phi(t)$  is the boundary value of a bounded analytic function it satisfies the following inequality unless it vanishes identically

$$\int_{-\infty}^{\infty} dt \frac{\ln F_\phi(t)}{1+t^2} > -\infty$$

If the set of zeros had non-zero measure then this integral would be  $-\infty$  (the natural logarithm being evaluated as the positive limit). Thus if  $F_\phi(t)$  vanishes for a set of positive measure it vanishes everywhere.  $\square$

There is also a noteworthy corollary that follows from the proof:

**Corollary 4.1.** *(No Successive Orthogonal States) Let  $P^{1/2} = \mathbb{I}$  and define the function  $F_\psi(t) = \langle \psi_{t'} | \psi_t \rangle$  then  $\langle \psi_{t'} | \psi_t \rangle \neq 0$  for almost all  $t$ , i.e., two dynamically allowed states belonging to the same history of a unitarily evolving quantum mechanical system cannot be mutually orthogonal, except for (at most) a set of instants of measure zero.*

Making use of this lemma we have the following proof of Theorem 4.4 due to Halvorson (2010), closely related to an argument given by Srinivas & Vijayalakshi (1981).<sup>27</sup>

*Proof.* Let  $f(t) = \langle \phi | P_{\Delta+t} \phi \rangle = \langle \phi | U_t^\dagger P_\Delta U_t | \phi \rangle$  (by 1.), where  $\phi$  is in range of  $P_\Delta$  and  $\Delta + t$  is the subset  $\Delta$  shifted by  $t$ . Clearly  $f(t)$  satisfies the premises Hegerfeldt's Lemma. From *Covariance* we have  $\langle \phi | P_{\Delta+t} | P_\Delta | \phi \rangle = 0$  for all  $t > |\Delta|$ . Since  $P_\Delta \phi = \phi$  we have that  $f(t) = \langle \phi | P_{\Delta+t} \phi \rangle = \langle \phi | P_{\Delta+t} P_\Delta \phi \rangle = 0$  for all  $t > |\Delta|$ , so that  $f(t)$  vanishes on a dense open interval. Therefore, case 1. applies and  $f(t) = 0$  for all  $t$ . Since we have  $0 = f(0) = \langle \phi | P_\Delta \phi \rangle = \langle \phi | \phi \rangle$  it follows that the range of  $P_\Delta$  is empty. The same argument applies for any  $\Delta$ , thus  $P_\Delta = 0$  for all  $\Delta$ .  $\square$

<sup>26</sup>This follows from the availability of power series representation for analytic functions. See (Rudin, 1987, p. 208).

<sup>27</sup>It should be noted that Allcock (1969) appears to have been the first to make (essentially) this argument. However, the argument he offers is invalid as he considers improper eigenstates of a time operator, which lie outside of the Hilbert space and so fail to meet the conditions of the lemma.

## 5.0 SOME CANDIDATE TIME OBSERVABLES

In Chapter 4 I introduced an investigation of Pauli’s Theorem by mentioning that the original argument given by Pauli admits a significant class of counterexamples. This chapter provides a detailed analysis of some of the better known examples of such pairs of operators. That is, pairs which consist of a self-adjoint operator  $H$  with spectrum bounded from below, and a symmetric operator  $T$  canonically conjugate to  $H$ ,  $[H, T]\psi = i\psi$ , on a dense domain. These are, therefore, Heisenberg pairs that are not Weyl pairs (i.e. pairs of operators that are canonically conjugate in terms of their commutator without exponentiated unitary groups that are canonically conjugate in the Weyl sense).

These operators have formed the basis of several attempts to introduce time observables into quantum theory. For the most part, I will remain silent on the question of their interpretation as time observables, and thus this chapter just contains an exposition of the necessary technical details of the individual cases. Although these details will be referred back to as needed, the reader who wishes only to engage with interpretative matters having to do with time in quantum mechanics may skip this chapter with few ill effects. There are, however, interpretative issues that arise which are of interest in their own right. For example, in Section 5.3 I address the vexed issue of the phase observable of the quantum harmonic oscillator.

I begin the chapter with an examination of the Schrödinger representation, which serves as a model against which the atypical cases that follow can be judged. I then proceed to consider periodic observables. First the ‘ideal clock’ of Hilgevoord, which involves an unbounded Hamiltonian (modeled on angle and angular momentum) and then the phase observables of the harmonic oscillator, which involves a semi-bounded Hamiltonian. In Section 5.4 I provide a detailed analysis of the time of arrival operator  $T_a$ , which has already come under discussion in Chapters 2 and 4. I conclude by offering a critical analysis of a system that has often been proposed as providing a model of a linear quantum clock—a freely

falling quantum particle. In Section 5.5 make use of the Equivalence Principle to provide an alternative quantization of the classical system with a semi-bounded Hamiltonian, and thus in which a candidate time observable does not arise.

## 5.1 COMMUTATION RELATIONS: THE SCHRÖDINGER REPRESENTATION

We will consider a system in one dimension, so the relevant space of functions is the square integrable functions on the real line  $\psi(x) \in L^2[\mathbb{R}]$ , a Hilbert space with inner product  $\langle \phi | \psi \rangle = \int_{-\infty}^{\infty} \phi^*(x)\psi(x)dx$ .<sup>1</sup> We define operators for position  $Q$  and momentum  $P$  by, respectively, multiplication and differentiation:

$$Q\psi(x) = x\psi(x) \quad P\psi(x) = -i\frac{d}{dx}\psi(x),$$

acting on square integrable functions on the the real line  $\psi(x) \in L^2[\mathbb{R}]$ . The domain of  $Q$  is  $\mathcal{D}_Q = \{\psi : x\psi(x) \in L^2[\mathbb{R}]\}$ , and the domain of  $P$  is  $\mathcal{D}_P = \{\psi : \psi(x) \text{ is differentiable a. e.}\}$ . It is easily seen that  $Q$  and  $P$  are symmetric on their respective domains.<sup>2</sup> But since  $Q$  and  $P$  are unbounded, their domains cannot be extended to the entire Hilbert space.<sup>3</sup> This makes the definition of their adjoints a somewhat delicate matter. For now, however, note that  $Q$  and  $P$  form a Heisenberg pair since

$$\begin{aligned} [Q, P]\psi &= QP\psi(x) - PQ\psi(x) = x\left(-i\frac{d}{dx}\psi(x)\right) - -i\frac{d}{dx}x\psi(x) \\ &= -ix\psi'(x) - (-i\psi(x) + -ix\psi'(x)) \\ &= i\psi(x) \end{aligned}$$

on the domain  $\mathcal{D}_{[Q,P]} = \{\psi \in \mathcal{D}_Q : \psi(x) \text{ is differentiable a. e.; } \psi'(x) \in \mathcal{D}_Q\}$ . This is the Schrödinger representation of the Heisenberg CCR, to which every irreducible representation of the Weyl CCR is unitarily equivalent (by the Stone-von Neumann Theorem). This, then, provides the necessary and sufficient conditions for a pair of self-adjoint operators to form a Schrödinger representation.

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<sup>1</sup>In terms of the inner product, the condition for inclusion of a function  $\psi$  in  $L^2[\mathbb{R}]$  is  $\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty$ .

<sup>2</sup>Recall an operator  $A$  is symmetric on a domain  $\mathcal{D}_A$  when  $\langle \phi | A\psi \rangle = \langle A\psi | \phi \rangle$  for all  $\phi, \psi \in \mathcal{D}_A$ .

<sup>3</sup>An operator  $A$  is bounded if  $\|A\psi\| = \langle A\psi | A\psi \rangle < k\|\psi\|$  for some  $k \in \mathbb{R}_+$ , for all  $\psi \in \mathcal{D}_A \subseteq \mathcal{H}$ . If an operator is defined everywhere and symmetric on all  $\phi, \psi \in \mathcal{H}$ , then it is bounded (Riesz and Nagy, p. 297).



However, this does not tell us what it takes for a Heisenberg pair to form a Schrödinger representation, since two densely defined symmetric operators  $A, B$  for which  $[A, B]\psi = i\psi$  holds on a dense domain are a Heisenberg pair, but may nonetheless fail to form a Weyl pair. The following theorem, due to Dixmier, gives the necessary and sufficient conditions for a Heisenberg pair to form, in addition, a Weyl pair.<sup>4</sup>

**Theorem 5.1** (Dixmier). *On a Hilbert space  $\mathcal{H}$  suppose that  $A$  and  $B$  are densely defined symmetric operators with a common dense domain  $\mathcal{D}_A \cap \mathcal{D}_B$  such that  $[A, B] = i\mathbb{I}$  on a dense domain  $\mathcal{D}_{[A,B]} \subset \mathcal{D}_A \cap \mathcal{D}_B$  with the following properties:*

1.  $\mathcal{D}_{[A,B]}$  is invariant under  $A$  and  $B$ ;
2. The restrictions of  $A$ ,  $B$ , and  $A^2 + B^2$  to  $\mathcal{D}_{[A,B]}$  are essentially self-adjoint.

*Then  $A$  and  $B$  are self-adjoint and  $(A, B)$  is a Weyl pair (or a direct sum of such pairs).*

Jauch (1968, p. 205) complains that these conditions are without an intuitive physical basis, but the following physical interpretation is possible for  $Q$  and  $P$ . The Weyl CCR states that  $Q$  and  $P$  are mutually covariant, and is logically equivalent to assuming the covariance of  $Q$  under spatial shifts generated by  $P$ . In addition,  $Q$  is the generator of shifts in momentum (Galilean boosts) and so both  $Q$  and  $P$  are closely linked to kinematic symmetries arising from the homogeneity of the underlying space-time. Further,  $Q^2 + P^2$  is the generator of parity transformations  $\Pi = e^{i\pi(Q^2+P^2)}$ .<sup>5</sup> Therefore, Dixmier's conditions can be interpreted as saying that a position operator covaries with spatial translations if and only if it also covaries with spatial reflections (which is what the requirements on the domain of the commutator amount to).<sup>6</sup>

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<sup>4</sup>See Nelson (1959, pp. 602-3) Cor. 9.1 and 9.3 for a proof that these are necessary and sufficient, respectively.

<sup>5</sup>This suggestion is due to Bryan Roberts (personal communication). We can see that this is the case since  $Q^2 + P^2$  has a complete set of eigenfunctions  $|n\rangle$ , which are just the eigenstates of the number operator  $N$  of the harmonic oscillator. Since, therefore,  $\Pi|n\rangle = (-1)^n|n\rangle$ , the effect of  $\Pi$  is to flip the sign of the odd numbered eigenstates only, as expected. Moreover,  $\Pi$  is both unitary and self-adjoint as required (see, e.g., (Ballentine, 1998, p. 371)). Since  $Q = a^\dagger + a$  and  $P = -(a^\dagger - a)$ , where  $a^\dagger$  and  $a$  are the raising and lowering operators respectively, it follows by direct calculation in the number representation that  $\Pi Q \Pi = -Q$  and  $\Pi P \Pi = -P$ , as desired.

<sup>6</sup>This interpretation of these requirements is due to Bryan Roberts (personal communication). Since we are only considering one-dimension of space, these are all the symmetry transformations available. Note, however, that in a space of higher dimensionality we could compose spatial rotations from combinations of parity transformations about distinct axes.

We have seen that  $Q$  and  $P$  are self-adjoint in virtue of composing a Weyl pair, but we may also show that they compose a Weyl pair in virtue of being self-adjoint. For an unbounded symmetric operator  $A$  to be self-adjoint, its domain must be equal to the domain of its adjoint,  $\mathcal{D}_A = \mathcal{D}_{A^\dagger}$ . If the domain of  $A$  is dense in  $\mathcal{H}$ ,<sup>7</sup> then it has an adjoint operator  $A^\dagger$  whose domain is closed,  $\overline{\mathcal{D}_{A^\dagger}} = \mathcal{D}_{A^\dagger}$ , but it is not therefore self-adjoint,  $A = A^\dagger$ . A symmetric operator  $A$  defined on a dense domain can only fail to be self-adjoint if its adjoint  $A^\dagger$  is an extension of  $A$  to a larger domain,  $\mathcal{D}_A \subset \mathcal{D}_{A^\dagger}$ . If a symmetric operator cannot be extended to a larger domain then it is a maximal symmetric operator. Every self-adjoint operator is thus maximal symmetric. But every maximal symmetric operator is not self-adjoint, as we shall see. We will now consider this question in detail for  $Q$ .

As the domain of  $Q$  is dense in  $\mathcal{H}$ ,<sup>8</sup> it has an adjoint operator  $Q^\dagger$  defined for all vectors  $Q^\dagger\psi \in \mathcal{D}_{Q^\dagger} \subset \mathcal{H}$  such that  $\langle \phi | Q^\dagger\psi \rangle = \langle Q\phi | \psi \rangle$  for every  $\phi, \psi \in \mathcal{D}_Q$ .<sup>9</sup> Since  $Q$  is symmetric,  $\langle \phi | Q\psi \rangle = \langle Q\phi | \psi \rangle$ , we have  $Q^\dagger\psi = Q\psi$  for all  $\psi \in \mathcal{D}_Q$ , and so the domain of  $Q^\dagger$  is an extension of the domain of  $Q$ ,  $\mathcal{D}_Q \subseteq \mathcal{D}_{Q^\dagger}$ . Therefore,  $Q$  is self-adjoint if the domain of  $Q^\dagger$  is no larger than the domain of  $Q$ .

Consider a vector  $\psi \in \mathcal{D}_{Q^\dagger}$ . By the definition of  $Q^\dagger$ , there exists a vector  $\chi = Q^\dagger\psi$  such that  $\langle \phi | \chi \rangle = \langle Q\phi | \psi \rangle$  for all  $\phi, \psi \in \mathcal{D}_Q$ . Consider a sequence of vectors  $\chi_n$  defined so that  $\chi_n(x) = \chi(x)$  for  $-n \leq x \leq n$  and  $\chi_n(x) = 0$  for  $|x| > n$ , and the corresponding sequence of vectors  $\psi_n$  defined likewise. Since  $\psi_n$  is therefore in the domain of  $Q$ , for each  $\chi_n$  there is a vector  $\psi_n \in \mathcal{D}_Q$  such that  $\chi_n(x) = x\psi_n(x)$ . It follows that  $\chi(x) = x\psi(x)$ , which implies that  $x\psi(x)$  is square integrable. Therefore  $\psi$  is in the domain of  $Q$  and the domain of  $Q^\dagger$  is not larger than the domain of  $Q$ . Thus  $\mathcal{D}_Q = \mathcal{D}_{Q^\dagger}$ , and  $Q$  is self-adjoint.

Since  $Q$  is self-adjoint, we may apply the Spectral Theorem to obtain its unique spectral representation  $Q = \int_{-\infty}^{\infty} \lambda dP_\lambda^Q$ . We may thus obtain the one-parameter unitary group generated by  $Q$ ,

$$U_a = e^{iQa} = \int_{-\infty}^{\infty} e^{i\lambda a} dP_\lambda^Q.$$

<sup>7</sup>A domain  $\mathcal{D} \subset \mathcal{H}$  is dense if for all  $\psi \in \mathcal{H}$  there exists a sequence of vectors  $\psi_n \in \mathcal{D}$  such that  $\psi_n \rightarrow \psi$ .

<sup>8</sup>For each  $\psi \in \mathcal{H}$  define a sequence of vectors  $\psi_n$  such that  $\psi_n(x) = \psi(x)$  for  $-n \leq x \leq n$ ,  $\psi_n(x) = 0$  for  $|x| > n$ . Since  $x\psi_n(x)$  is square integrable for all  $n$ ,  $\psi_n$  is in  $\mathcal{D}_Q$ , and  $\psi_n \rightarrow \psi$  as  $n \rightarrow \infty$ , i.e.  $\|\psi - \psi_n\| = \int_{-\infty}^{-n} |\psi(x)|^2 dx + \int_n^{\infty} |\psi(x)|^2 dx \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $\mathcal{D}_Q$  is dense in  $\mathcal{H}$ .

<sup>9</sup>Note that Dirac bra-ket notation (introduced at the end of the section) can be misleading in this regard since it effectively collapses the distinction between being symmetric and having an adjoint. See (Jordan, 1969, pp. 30–31) for a helpful exposition of this matter, on which the present discussion is based.

Now we consider the commutator of these unitary operators with  $P$  in the Schrödinger representation,

$$\begin{aligned} [P, U_a]\psi(x) &= -i \frac{d}{dx}(e^{iax}\psi(x)) + ie^{iax} \frac{d}{dx}\psi(x) \\ &= -i(iae^{iax}\psi(x) + e^{iax}\psi'(x)) + ie^{iax}\psi'(x) \\ &= ae^{iax}\psi(x) = aU_a\psi(x), \end{aligned}$$

valid for all  $\psi$  in the domain of the commutator,  $\mathcal{D}_{[Q,P]}$ .<sup>10</sup>  $P$  is also self-adjoint, although we will not prove it here.<sup>11</sup> By Prop. 4.1 above, this suffices to show that  $(Q, P)$  is a Weyl pair since it follows from  $[P, U_a] = aU_a$  that  $U_{-a}PU_a = P + a\mathbb{I}$  for all  $a \in \mathbb{R}$ , on a dense domain.

We conclude our discussion of the Schrödinger representation with a brief account of the relation between  $Q$  and  $P$ . These operators are unitarily equivalent, in the sense that they are related by the Fourier-Plancherel transformation

$$U\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(p)e^{-ipx} dx = \psi(p),$$

where  $\psi(k)$  is the vector  $\psi$  written in the momentum representation. This is a unitary operation and, since  $U^\dagger QU\psi(x) = -i\psi'(x)$ , we have  $P = U^\dagger QU$ . Note that position in the momentum representation acts by differentiation,  $Q\psi(k) = -i\psi'(k)$ , and momentum acts by multiplication  $P\psi(k) = k\psi(k)$ . This also provides another means to see that  $e^{iQa}$  and  $e^{iPb}$  produce shifts in momentum and position, respectively, since  $Ue^{iQa}\psi(x) = \psi(k - a)$  and  $U^\dagger e^{iPb}\psi(k) = \psi(x + b)$ .

This also provides a useful juncture to discuss the use of bra-ket notation, introduced by Dirac.<sup>12</sup> For our purposes, we need to observe that the following provides a valid re-expression of the spectral resolution of  $Q$  in the concrete situation of an inner product

$$\langle \phi | Q \psi \rangle = \int_{\mathbb{R}} \lambda d\langle \phi | P_\lambda^Q \psi \rangle = \int_{\mathbb{R}} dx x \langle \phi | x \rangle \langle x | \psi \rangle = \int_{\mathbb{R}} x \phi^*(x) \psi(x) dx.$$

This justifies the common practice of writing the resolution of the identity associated with  $Q$  as

$$\mathbb{I} = \int_{\mathbb{R}} dP_\lambda^Q = \int_{\mathbb{R}} dx |x\rangle \langle x|.$$

<sup>10</sup>Since  $U_a$  is unitary and thus bounded, its domain can always be extended to the entire space. In that case,  $aU_a\psi(x)$  is in  $\mathcal{D}_Q$ .

<sup>11</sup>See (Von Neumann, 1955, §2.9).

<sup>12</sup>See (Jordan, 1969, p. 60) for a more complete discussion.

The kets  $|x\rangle$  are not elements of the Hilbert space, but  $\langle p|x\rangle = e^{-ipx}$  provides the crucial factor in the Fourier transformation from one representation to another. That is,

$$\langle p|\psi\rangle = \int_{\mathbb{R}} dx \langle p|x\rangle \langle x|\psi\rangle = \int_{\mathbb{R}} dx e^{-ipx} \psi(x) = \psi(p).$$

So long as the more sophisticated expressions are ‘sandwiched’ between two vectors, they form a valid inner product that may be evaluated in the usual way. The notorious Dirac- $\delta$  arises when we use these rules to define ‘matrix elements’ for operators with continuous spectra such as,

$$\langle x|Q|x'\rangle = x'\delta(x - x'),$$

or expressions such as,

$$\langle p|p'\rangle = \int_{\mathbb{R}} dx \langle p|x\rangle \langle x|p'\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} dx e^{-ix(p-p')} = \delta(p - p').$$

These are to be considered as mere formal devices to aid calculation of legitimate inner products.<sup>13</sup> In considering self-adjoint operators with discrete (purely point) spectra, however, the kets represent genuine eigenvectors that lie in a Hilbert space, and the matrix elements are genuinely the entries of an (infinite-dimensional) matrix, operating on a vector given with respect to the complete orthonormal basis formed by the eigenvectors.

## 5.2 BOUNDED TIME OBSERVABLE: THE IDEAL PERIODIC CLOCK

This proposal for a time operator avoids Pauli’s Theorem by restricting the spectrum of the operator to an interval  $[0, \tau]$ . The periodic time observable is thus closely related to the angle observable  $\Phi$  associated with the (azimuthal) angular momentum  $L$ , which can be defined on  $L^2[-\pi, \pi]$  as follows

$$\Phi\psi(\varphi) = \varphi\psi(\varphi) \quad L\psi = -i\frac{d}{d\varphi}\psi(\varphi)$$

where  $\psi(\varphi) \in L^2[-\pi, \pi]$ . In that case,  $L$  has genuine eigenvectors  $|m\rangle = e^{im\varphi}$ , with  $m \in \mathbb{Z}$ , since then

$$L_z|m\rangle = -i\frac{d}{d\varphi}e^{im\varphi} = m|m\rangle.$$

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<sup>13</sup>We will not be making use of the ‘rigged’ Hilbert space construction, therefore. See, e.g., Ballentine (1998). Nor will we consider these deltas in their proper mathematical setting, i.e. as distributions.

That is to say that the spectrum of  $L$  is discrete, i.e. purely point, and the eigenvectors  $\{|m\rangle\}$  form a complete orthonormal basis for  $L^2[-\pi, \pi]$ . The spectral resolution of  $L$ , as given by the Spectral Theorem, can thus be written

$$L = \int_{\mathbb{R}} \lambda dP_{\lambda}^L = \sum_{m=-\infty}^{\infty} m|m\rangle\langle m|,$$

where  $|m\rangle\langle m|$  is the one-dimensional projection onto  $|m\rangle$ . We may write an arbitrary vector  $\psi(\varphi) \in L^2[-\pi, \pi]$  as

$$\psi(\varphi) = \langle \varphi | \psi \rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \langle \varphi | m \rangle \langle m | \psi \rangle = \sum_{m=-\infty}^{\infty} e^{im\varphi} \psi(m) = U^{\dagger} \psi(m),$$

where  $U^{\dagger}$  is the inverse Fourier transform.

Being symmetric and bounded,  $\Phi$  is self-adjoint on the domain  $\mathcal{D}_{\Phi} = \{\psi(\varphi) \in L^2[-\pi, \pi] : \varphi\psi(\varphi) \in L^2[-\pi, \pi]\}$ . Its spectral resolution can be written

$$\Phi = \int_{-\pi}^{\pi} \lambda dP_{\lambda}^{\Phi} = \int_{-\pi}^{\pi} \varphi d\varphi |\varphi\rangle\langle \varphi|$$

where  $|\varphi\rangle \notin L^2[-\pi, \pi]$  is an ‘improper’ eigenvector which, properly speaking, belongs to the space of distributions  $\mathcal{S}^*$  provided by the ‘rigged’ Hilbert space  $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^*$ .<sup>14</sup> Nonetheless, the projections

$$P_{\Phi}(\Delta) = \int_{\Delta} dP_{\lambda}^{\Phi} = \int_{\Delta} d\varphi |\varphi\rangle\langle \varphi|$$

are well-defined on  $L^2[-\pi, \pi]$ . Thus writing the same vector  $\psi$  in the angular momentum representation we have

$$\begin{aligned} \psi(m) &= \langle m | \psi \rangle = \int_{-\pi}^{\pi} d\varphi \langle m | \varphi \rangle \langle \varphi | \psi \rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\varphi e^{-im\varphi} \langle \varphi | \psi \rangle = U\psi(\varphi), \end{aligned}$$

where  $U$  is the Fourier-Plancherel transformation.

This gives us a concrete expression for our improper angle eigenvectors in the angular momentum representation:

$$\langle m | \varphi \rangle = \sum_{m=-\infty}^{\infty} |m\rangle\langle m | \varphi \rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\varphi} |m\rangle,$$

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<sup>14</sup>See Ballentine (1998) for a discussion of this construction.

and the means to verify that they are indeed (improper) eigenvectors:

$$\langle m|\Phi|\varphi\rangle = \int_{-\pi}^{\pi} d\varphi' \varphi' \langle m|\varphi'\rangle \langle \varphi'|\varphi\rangle = \varphi \frac{1}{\sqrt{2\pi}} e^{-im\varphi} = \varphi \langle m|\varphi\rangle.$$

We conclude our initial analysis by examining how  $\Phi$  acts in the angular momentum representation. The matrix elements are:

$$\begin{aligned} \langle m|\Phi|n\rangle &= \int_{-\pi}^{\pi} d\varphi \varphi \langle m|\varphi\rangle \langle \varphi|n\rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \varphi e^{i(n-m)\varphi} = \frac{1}{2\pi} \left[ \frac{e^{i(n-m)\varphi} (1 - i(n-m)\varphi)}{(n-m)^2} \right]_{-\pi}^{\pi} \\ &= \begin{cases} \pi/2 & \text{if } n = m \\ \frac{i}{n-m} & \text{otherwise.} \end{cases} \end{aligned}$$

This leads to

$$\begin{aligned} \langle m|\Phi|\psi\rangle &= \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle m|l\rangle \langle l|\Phi|n\rangle \langle n|\psi\rangle \\ &= \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\varphi \langle m|l\rangle e^{i(n-l)\varphi} \varphi \langle n|\psi\rangle \\ &= \sum_{l,n=-\infty;l \neq n}^{\infty} \frac{i}{n-l} \langle m|l\rangle \langle n|\psi\rangle + \frac{\pi}{2} \langle m|\psi\rangle, \end{aligned}$$

which indicates that  $\Phi$  can be written as

$$\Phi = \sum_{m,n=-\infty;m \neq n}^{\infty} \frac{i}{(n-m)} |m\rangle \langle n| + \frac{\pi}{2} \mathbb{I}.$$

We now verify that  $(\Phi, L)$  are a Heisenberg pair. Evaluated in the angular momentum representation, we have

$$\begin{aligned} L\Phi &= \left( \sum_m m |m\rangle \langle m| \right) \left( \sum_{l \neq n} \frac{i}{(n-l)} |l\rangle \langle n| + \frac{\pi}{2} \sum_l |l\rangle \langle l| \right) \\ &= \sum_{m \neq n} \frac{im}{n-m} |m\rangle \langle n| + \frac{\pi}{2} L, \end{aligned}$$

and similarly for  $\Phi L$ ,

$$\begin{aligned} \Phi L &= \left( \sum_{l \neq m} \frac{i}{(l-m)} |m\rangle \langle l| + \frac{\pi}{2} \sum_l |l\rangle \langle l| \right) \left( \sum_n n |n\rangle \langle n| \right) \\ &= \sum_{m \neq n} \frac{in}{n-m} |m\rangle \langle n| + \frac{\pi}{2} L, \end{aligned}$$

so that the commutation relation returns

$$\begin{aligned} [\Phi, L] &= \Phi L - L\Phi \\ &= i \sum_{n \neq m} |m\rangle\langle n|, \end{aligned}$$

which evidently is not equal to  $i\mathbb{I}$  since it is undefined when  $m = n$ . Indeed, as Susskind & Glogower (1964) originally pointed out, this relation cannot hold on the entire domain of  $L$  since

$$\langle m|\Phi L - L\Phi|n\rangle = (n - m)\langle m|\Phi|n\rangle,$$

which for  $m = n$  returns zero. However, this commutation relation does hold on the dense domain

$$\mathcal{D}_{[\Phi, L]} = \left\{ \psi \in \mathcal{D}_L : \sum_{m=-\infty}^{\infty} |\langle m|\psi\rangle| = 0 \right\},$$

which demonstrates that  $\Phi, L$  are a Heisenberg pair. Evidently, though, they cannot form a Weyl pair since if they did the Stone-von Neumann Theorem would entail that they have the spectrum of the reals, which they do not. Furthermore, since they are not a Weyl pair, they do not form a Schrödinger representation.

Nonetheless, since  $\Phi$  and  $L$  are self-adjoint, they can be exponentiated through the spectral calculus to form appropriate families of unitary operators,

$$e^{i\Phi k} = \int_{-\pi}^{\pi} e^{i\lambda k} dP_{\lambda}^{\Phi} = \int_{-\pi}^{\pi} d\varphi e^{i\varphi k} |\varphi\rangle\langle\varphi|; \quad e^{iL\phi} = \int_{\mathbb{R}} e^{i\lambda\phi} dP_{\lambda}^L = \sum_{m=-\infty}^{\infty} e^{im\phi} |m\rangle\langle m|.$$

where  $k \in \mathbb{Z}$  and  $\varphi \in \mathbb{R}$ . Moreover, these are indeed families of shift operators, as can be seen from the following:

$$\begin{aligned} \langle m|e^{i\Phi k}|n\rangle &= \int_{-\pi}^{\pi} d\varphi e^{i\varphi k} \langle m|\varphi\rangle\langle\varphi|n\rangle \\ &= \int_{-\pi}^{\pi} d\varphi e^{im\varphi} e^{-i\varphi(n-k)} \\ &= \langle m|n - k\rangle, \end{aligned}$$

from which it follows that

$$\begin{aligned} \langle m|e^{i\Phi k}|\psi\rangle &= \sum_{l, n} \langle m|l\rangle\langle l|e^{i\Phi k}|n\rangle\langle n|\psi\rangle \\ &= \sum_{l, n} \langle m|l\rangle\langle l|n - k\rangle\langle n|\psi\rangle \\ &= \langle m + k|\psi\rangle = \psi(m + k). \end{aligned}$$

This justifies writing  $e^{i\Phi k}$  as

$$e^{i\Phi k} = \sum_m |m+k\rangle\langle m|.$$

Similarly, for  $e^{iL\phi}$  we have

$$\begin{aligned} \langle\varphi|e^{iL\phi}\psi\rangle &= \sum_{l,m} e^{il\phi}\langle\varphi|m\rangle\langle m|l\rangle\langle l|\psi\rangle \\ &= \frac{1}{\sqrt{2\pi}} \sum_m e^{-im(\varphi-\phi)}\langle m|\psi\rangle \\ &= \langle\varphi-\phi|\psi\rangle = \psi(\varphi-\phi). \end{aligned}$$

where it should be noted that the shift is periodic,  $(\varphi-\phi) \bmod 2\pi$ , since it appears as a complex phase. Had we treated  $(\Phi, L)$  as a Schrödinger representation and attempted to reason accordingly we would have been erroneously led to conclude that  $e^{iL\phi}\psi \notin L^2[-\pi, \pi]$  since  $U^\dagger e^{iP a}\psi(k) = \psi(x+b)$ . By working in the angular momentum representation, we can see that this is not the case since functions of angle are periodic.

Although they do not form a Schrödinger representation, it seems reasonable to suppose that angle and angular momentum are ‘true’ conjugates. In particular, their exponential groups can be seen to obey a relation that appropriately reflects their mutual covariance in much the same way as the Weyl CCR. That is,

$$\begin{aligned} e^{i\Phi k} e^{iL\phi} &= \left( \sum_m |m+k\rangle\langle m| \right) \left( \sum_l e^{il\phi} |l\rangle\langle l| \right) \\ &= \sum_m e^{im\phi} |m+k\rangle\langle m| \\ &= \sum_l e^{i(l-k)\phi} |l\rangle\langle l-k| = e^{-ik\phi} \left( \sum_l e^{il\phi} |l\rangle\langle l| \right) \left( \sum_m |m+k\rangle\langle m| \right) \\ &= e^{-ik\phi} e^{iL\phi} e^{i\Phi k}. \end{aligned}$$

The only distinction between this Weyl-like relation and the real thing is that this relation only holds for  $k \in \mathbb{Z}$ , with  $\phi \bmod 2\pi$ . But this is exactly what one would expect from this pair of conjugate operators, one of which has a discrete spectrum and one of which has a periodic spectrum.



The resulting covariance of  $\Phi$  and  $L$  is interestingly distinct from that of the Schrödinger representation. First,  $L$  covaries with shifts generated by  $\Phi$  in the sense that, for integer  $k$ ,

$$\begin{aligned}
e^{-ik\Phi} L e^{ik\Phi} &= e^{-ik\Phi} \left( \sum_m m |m\rangle \langle m| \right) \left( \sum_n |n+k\rangle \langle n| \right) \\
&= \left( \sum_l |l-k\rangle \langle l| \right) \left( \sum_m m |m\rangle \langle m-k| \right) \\
&= \sum_l l |l-k\rangle \langle l-k| = \sum_m (m+k) |m\rangle \langle m| \\
&= L + k\mathbb{I}.
\end{aligned}$$

So far so good. But things are less straightforward for angle. First, we evaluate the ‘matrix elements’

$$\langle m | e^{iL\phi} \Phi e^{-iL\phi} | n \rangle = e^{i(m-n)\phi} \langle m | \Phi | n \rangle,$$

which allows us to conclude that,

$$\begin{aligned}
\langle m | e^{iL\phi} \Phi e^{-iL\phi} | \psi \rangle &= \sum_n e^{i(m-n)\phi} \langle m | \Phi | n \rangle \langle n | \psi \rangle \\
&= \sum_{l,n} \int_{-\pi}^{\pi} d\varphi \varphi e^{i(l-n)(\varphi+\phi)} \langle m | l \rangle \langle n | \psi \rangle,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\langle \chi | e^{iL\phi} \Phi e^{-iL\phi} | \psi \rangle &= \sum_{m,n} \int_{-\pi}^{\pi} d\varphi \varphi e^{i(m-n)(\varphi+\phi)} \langle \chi | m \rangle \langle n | \psi \rangle \\
&= \sum_{m,n} \int_{-\pi+\phi}^{\pi+\phi} d\varphi' (\varphi' - \phi) e^{i(m-n)\varphi'} \langle \chi | m \rangle \langle n | \psi \rangle \\
&= \sum_{m,n} \int_{-\pi+\phi}^{\pi+\phi} d\varphi' \varphi' e^{i(m-n)\varphi'} \langle \chi | m \rangle \langle n | \psi \rangle - \phi \sum_{m,n} \int_{-\pi+\phi}^{\pi+\phi} d\varphi' e^{i(m-n)\varphi'} \langle \chi | m \rangle \langle n | \psi \rangle.
\end{aligned}$$

Comparing this to

$$\langle \chi | \Phi | \psi \rangle = \sum_{m,n} \int_{-\pi}^{\pi} d\varphi \varphi e^{i(m-n)\varphi} \langle \chi | m \rangle \langle n | \psi \rangle$$

it would be tempting to read it as saying that  $\Phi$  is covariant under shifts in angle in the same way as  $Q$  is covariant under shifts in position. That would be a mistake, however, since this is a definite integral, for which a change of variables results in a change of limits. We can

see this most clearly by writing the same expression in terms of the spectral calculus:

$$\begin{aligned}
\langle \chi | e^{iL\phi} \Phi e^{-iL\phi} \psi \rangle &= \int_{-\pi+\phi}^{\pi+\phi} (\lambda - \phi) d\langle \chi | P_\lambda^\Phi \psi \rangle \\
&= \int_{-\pi+\phi}^{\pi} (\lambda - \phi) d\langle \chi | P_\lambda^\Phi \psi \rangle + \int_{-\pi}^{-\pi+\phi} ((\lambda - \phi) + \pi) d\langle \chi | P_\lambda^\Phi \psi \rangle \\
&= \int_{-\pi}^{\pi} \lambda d\langle \chi | P_\lambda^\Phi \psi \rangle + \phi \int_{-\pi}^{\pi} d\langle \chi | P_\lambda^\Phi \psi \rangle + \pi \int_{-\pi}^{-\pi+\phi} d\langle \chi | P_\lambda^\Phi \psi \rangle \\
&= \langle \chi | \Phi + \phi \mathbb{I} + \pi P^\Phi([-\pi, -\pi + \phi]) | \psi \rangle := \langle \chi | \Phi_\phi \psi \rangle.
\end{aligned}$$

This demonstrates that the behavior of  $\Phi$  under shifts generated by  $L$  is not strictly covariance, since  $e^{iL\phi} \Phi e^{-iL\phi}$  defines a distinct operator,  $e^{iL\phi} \Phi e^{-iL\phi} = \Phi_\phi$ . This indicates that the phase observable  $\Phi$  is only unique up to a phase factor  $\phi \pmod{2\pi}$ . This is yet another indication that we are not dealing here with a Schrödinger representation.

This kind of behavior led Levy-Leblond (1976) to propose that the unitary operator  $e^{i\Phi}$  provides a better candidate for the angle observable than  $\Phi$ . Although a unitary operator is not self-adjoint, its periodicity in  $\varphi$  (identified by Susskind & Glogower as the cause of the difficulties with  $\Phi$ ) serves to provide an effective parameterization of phase, despite its complex spectrum. As Uffink (1990) contends, the supposed unobservability of a complex number—often cited as a reason to restrict the set of observables to the self-adjoint operators<sup>15</sup>—is no bar to viewing the spectral values of  $e^{i\varphi}$  (the set  $\{z \in \mathbb{C} : |z| = 1\}$ ), as a natural parameterization of angle since they serve as an effective co-ordinatization of the circle  $S_1$ .

It is easily seen from the Weyl-like relation above that  $e^{i\Phi}$  has the desired covariance property

$$e^{iL\phi} e^{i\Phi} e^{-iL\phi} = e^{-i\phi} e^{i\Phi} = e^{i(\Phi-\phi)},$$

and we may easily show that

$$[e^{i\Phi}, L] = \sum_m m |m+1\rangle \langle m| - (m+1) |m+1\rangle \langle m| = -e^{i\Phi}.$$

However, since similar relations are obeyed by the entire family of operators  $e^{ik\Phi}$ , the following question naturally arises: *which* of the operators  $e^{ik\Phi}$  is the angle observable (canonically conjugate to  $L$ )? Furthermore, the physical significance of the operators  $e^{ik\Phi}$  lies in their being members of a unitary group of angular momentum shift operators, and so it would

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<sup>15</sup>Note that this is also in itself poor reasoning, since a symmetric operator has a real spectrum, but may not be self-adjoint.

be odd to view one (or all) of them as themselves representing angle itself. So what does represent angle?

I contend that these difficulties suggest that the covariant quantity ‘angle’ is best represented by the corresponding PVM, given by the Spectral Theorem as a map from the Borel subsets of  $\mathbb{R}$  to the projections  $P^\Phi : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ . The non-uniqueness of this family simply reflects the fact that the spectrum of  $\Phi$  is periodic, and any mapping from  $S_1$  to  $\mathbb{R}$  is non-unique up to a phase. This explains the non-uniqueness of the resulting self-adjoint angle operators  $\Phi_\phi$ . But, better, we can regard the PVM associated with angle as a map from the Borel subsets of the circle  $Z \in \mathcal{B}(S_1)$ , which can be defined as the intersection of  $S_1$  with the Borel subsets of the plane  $\mathcal{B}(\mathbb{R}^2)$ . The PVM  $P(Z)$  is thus defined

$$P(Z) = \int_Z dP_\varphi = \int_Z d\varphi |\varphi\rangle\langle\varphi| = \sum_{m,n} \int_Z d\varphi e^{i(m-n)\varphi} |m\rangle\langle n|,$$

where  $\varphi$  is to be understood as the angle around a circle, the points of which are parameterized by  $e^{i\varphi}$ . The covariance of  $P(Z)$  under shifts in angle can be seen directly since

$$\begin{aligned} e^{iL\phi} P(Z) e^{-iL\phi} &= \sum_{m,n} \int_Z d\varphi e^{i(m-n)(\varphi+\phi)} |m\rangle\langle n| \\ &= \sum_{m,n} \int_{Z-\phi} d\varphi' e^{i(m-n)\varphi'} |m\rangle\langle n| \\ &= P(Z - \phi) \end{aligned}$$

where  $\phi$  is evidently modulo  $2\pi$ . This has the result that projections  $P(Z)$  related by a shift of  $\phi = 2n\pi$  are identified,  $P(Z + 2n\pi) = P(Z)$ . The operators  $e^{ik\varphi}$  are given in terms of  $P(Z)$  as follows

$$e^{i\Phi k} = \int_0^{2\pi} e^{i\varphi k} dP_\varphi,$$

where  $P_\varphi = P([0, \varphi])$  involves an arbitrary choice of origin. The uniqueness of these operators is not threatened by that arbitrary choice. Thus the covariance of  $e^{i\varphi}$  and its commutation relation with  $L$  should be seen as a *consequence* of the covariance of angle (in the form of the PVM  $P(Z)$ ) rather than constituting it. On the other hand, the phase operators

$$\Phi_\phi = \int_\phi^{\phi+2\pi} \lambda dP_\lambda$$

clearly depend on the choice of the phase, in the sense that  $P_\lambda = P([\phi, \phi + \lambda])$ . This explains their non-uniqueness. Thus, likewise, the failure of  $\Phi$  to covary with shifts in angle is no bar to viewing angle—represented by the PVM  $P(Z)$ —as a covariant quantity.

### 5.3 BOUNDED TIME OBSERVABLE: HARMONIC OSCILLATOR PHASE

The original suggestion that the phase of a quantum harmonic oscillator would function as a quantum clock is due to Susskind & Glogower (1964). But the correct definition of a phase observable is a topic of considerable interest in its own right, and has a venerable history that begins with Dirac’s (1927) famous paper on quantum electrodynamics. In that paper Dirac introduced the use of creation and annihilation or “ladder” operators to provide a representation of the dynamics of the electromagnetic field. These operators were derived from number and phase operators by aid of an analogy with the representation of a classical single mode field by amplitude and phase. However, although the ladder operators he posited could be given a firm theoretical basis, the assumption he made that number and phase were a canonical pair (like position and momentum) turned out to be mistaken.

This was first pointed out by Susskind & Glogower (1964) who suggested replacing the problematic phase operator (resembling the argument of a complex number) with a pair of operators resembling the Euler decomposition into trigonometric functions, which Levy-Leblond (1976) advocated combining into a single unitary operator. Garrison & Wong (1970) and Galindo (1984) were able to construct self-adjoint operators conjugate to phase, providing a counterexample to Pauli’s original argument that did not depend on the Hamiltonian being unbounded (see Chapter 4). These developments often displayed an implicit understanding of the fact that the underlying structure was that of a phase POVM (rather than a PVM) but this connection was made explicit by Busch et al. (1995b) as a means of displaying the connections between various approaches to the quantum phase observable. The following discussion aims to fill in some details of the account of phase that thus emerged.

The Hamiltonian of the (one-dimensional) quantum harmonic oscillator has eigenvalues  $E_n = (n + \frac{1}{2})\hbar\omega$  with  $n \geq 0$ , and displays the characteristically quantum ‘zero-point’ energy of the lowest energy state  $E_0 = 1/2\hbar\omega$ . This suggests that the Hamiltonian can be written in units of  $\hbar\omega$  as  $H = N + \frac{1}{2}$ , where  $N = a^\dagger a$  is the number operator, which is self-adjoint with spectrum  $\mathbb{N}_0$ .<sup>16</sup> By letting  $a^\dagger = 1/\sqrt{2}(Q - iP)$  and  $a = 1/\sqrt{2}(Q + iP)$  we see that  $H = N + 1/2 = Q^2 + P^2$ , which (up to some constants) is the Hamiltonian in the Schrödinger representation. The number eigenstates  $N|n\rangle = n|n\rangle$  are obtained by repeated application

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<sup>16</sup>Thus we have again adopted natural units, in which  $\omega = \hbar = 1$ .

of  $a^\dagger$  to the ground state  $N|0\rangle = 0$ ,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle,$$

and form an complete orthonormal basis for  $\mathcal{H}$ .<sup>17</sup>

This indicates that there is an interesting similarity to the case of angle and angular momentum, with the number eigenstates  $|n\rangle$  playing the role of the angular momentum eigenstates  $|m\rangle$ . The difference lies in the fact that  $L$  has the spectrum  $\mathbb{Z}$  whereas  $N$  has spectrum  $\mathbb{N}_0$ . Another indication comes from the way that the creation and annihilation operators act to raise and lower the number eigenstates

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle; \quad a|n\rangle = \sqrt{n}|n-1\rangle \text{ with } a|0\rangle = 0.$$

By analogy with  $e^{i\Phi}$  from the previous section, we might expect the following operators to be conjugate unitary operators related to the phase

$$V = \sum_{n=0}^{\infty} |n\rangle\langle n+1|; \quad V^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|.$$

To deepen the analogy consider the following ‘generalized phase states,’

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle.$$

It is easily seen that these are (improper) eigenstates of  $V$ , in the sense that  $V|\phi\rangle = e^{i\phi}|\phi\rangle$ .<sup>18</sup> This analogy with  $e^{i\Phi}$  might lead one to believe that  $V$  is a unitary operator, but this would be a mistake.<sup>19</sup>  $V$  is an isometry but not a unitary transformation since  $V^\dagger V \neq \mathbb{I}$ . We can see this by direct calculation in the number basis:

$$V^\dagger V = \sum_{n,m \geq 0} |n+1\rangle\langle n|m\rangle\langle m+1| = \mathbb{I} - |0\rangle\langle 0|.$$

Nonetheless,  $V$  is an isometry (Levy-Leblond (1976) calls it ‘one-sided unitary’) as we see from the fact that

$$VV^\dagger = \sum_{n,m \geq 0} |n\rangle\langle n+1|m+1\rangle\langle m| = \mathbb{I}.$$

<sup>17</sup>See, e.g., (Jauch, 1968, p. 211-215) for a detailed derivation of these results.

<sup>18</sup>They are not, however, eigenstates of  $V^\dagger$  since  $V^\dagger|\phi\rangle = e^{-i\phi}(|\phi\rangle - |0\rangle)$ .

<sup>19</sup>Carruthers & Nieto (1968) provide a quick reductio argument for this conclusion as follows. If  $V^*V = VV^* = \mathbb{I}$  then we have  $[a, a^\dagger] = (VN^{1/2}(VN^{1/2})^\dagger - (VN^{1/2})^\dagger VN^{1/2}) = VNV^\dagger - N = \mathbb{I}$ . Thus if  $V$  is unitary then  $N$  and  $(N + \mathbb{I})$  are unitarily equivalent. However, this can’t be the case since the spectrum of the latter is  $\mathbb{N}_1$  rather than  $\mathbb{N}_0$ .

So whereas angle  $\Phi$  was a self-adjoint operator associated with a family of unitary operators  $e^{ik\Phi}$ , with  $k \in \mathbb{Z}$ , there is no such unitary group in the case of phase. There is, however, a semigroup  $\{V^l\}$ , with  $l \geq 0$ , defined as follows:

$$V^l = \sum_{n=0}^{\infty} |n\rangle\langle n+l|.$$

It is easily confirmed that  $V^l(V^l)^\dagger = \mathbb{I}$ , but  $(V^l)^\dagger V^l = \mathbb{I} - \sum_{n=0}^l |n\rangle\langle n|$ . This behavior is characteristic of  $\{V^l\}$  forming a semigroup rather than a group.<sup>20</sup>

Were  $V^l$  a group rather than a semigroup, we would be able to apply Stone's Theorem to find its unique self-adjoint generator. Nonetheless, a semigroup does have a generator, by which we mean a symmetric operator whose (generalized) spectral resolution defines a series of exponential operators. The distinction here is that the generator of a semigroup is associated with a POVM that is not a PVM. There is, in fact, a similar result to Naimark's Dilation Theorem (Thm. 3.9) that applies here:

**Theorem 5.2** (Cooper<sup>21</sup>). *If  $\{V_s\}$  is a semigroup of isometric transformations of  $\mathcal{H}$  then there exists a larger Hilbert space  $\mathcal{K} \supset \mathcal{H}$  in which there exists a one-parameter unitary group  $\{U_s\}$  and a projection  $\mathbf{P}$  onto  $\mathcal{H}$  such that*

$$V_s\psi = \mathbf{P}U_s\psi \tag{5.1}$$

for all  $\psi \in \mathcal{H}$  and all  $s \geq 0$ .

Once we obtain the unitary group  $\{U_s\}$  by dilation in this manner, we may apply Stone's Theorem to obtain the self-adjoint generator of that group. In the present case we have

$$V^l = \sum_{n=0}^{\infty} |n\rangle\langle n+l| = \mathbf{P}U^k = \sum_{n=0}^{\infty} |n\rangle\langle n+l|.$$

The generator of the group  $U_k$  (operators on  $\mathcal{K} \supset \mathcal{H}$ ) will be a self-adjoint operator  $\Phi'$  on  $\mathcal{K}$  such that  $U^k = e^{ik\Phi'}$  for all  $k \in \mathbb{Z}$ . This operator closely resembles the angle observable of the previous section and, since it is self-adjoint, we can write  $U^k$  in terms of its spectral resolution,

$$U^k = e^{ik\Phi'} = \int_{-\pi}^{\pi} e^{ik\lambda} dP_{\lambda}^{\Phi'}.$$

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<sup>20</sup>A semigroup is a family of transformations  $\{T_s\}$  defined only for  $s \geq 0$  such that  $T_0 = \mathbb{I}$  and  $T_r T_s = T_{r+s}$  for all  $r \geq 0, s \geq 0$ . See (Riesz & Sz.-Nagy, 1990, p. 393).

<sup>21</sup>See (Riesz & Sz.-Nagy, 1990, p. 396).

This suggests that there is a symmetric operator  $\Phi$  on  $\mathcal{H}$  with a generalized spectral resolution  $E_\lambda^\Phi$  such that

$$V^l = \int_{-\pi}^{\pi} e^{il\lambda} dE_\lambda^\Phi = \mathbf{P} \int_{-\pi}^{\pi} e^{il\lambda} dP_\lambda^{\Phi'}$$

for all  $l \geq 0$ . This relationship, in fact, serves to define the POVM  $E^\Phi$  uniquely (up to unitarity) (Busch et al., 1995b).

Another indication that phase is represented by a POVM rather than a PVM is the fact that the phase ‘eigenstates’  $|\phi\rangle$  are not mutually orthogonal,

$$\langle\phi'|\phi\rangle = \frac{1}{2\pi} \left[ \frac{1}{2} + \pi\delta(\phi - \phi') + \frac{i}{2} \cot(1/2)(\phi - \phi') \right].$$

However, the  $|\phi\rangle$  form an overcomplete (i.e. non-orthogonal) basis and so provide a (generalized) resolution of the identity, which we can write as

$$\mathbb{I} = \int_{-\pi}^{\pi} d\phi |\phi\rangle\langle\phi| = \sum_{n,m \geq 0} \int_{-\pi}^{\pi} d\phi e^{i(n-m)\phi} |m\rangle\langle n|.$$

We can confirm the covariance of  $E^\Phi$  with shifts generated by number, in analogy with covariance of angle under shifts in angular momentum, as follows

$$e^{-iN\theta} E^\Phi(Z) e^{iN\theta} = \sum_{m,n \geq 0} \int_Z d\phi e^{i(n-m)(\phi+\theta)} |m\rangle\langle n| = E^\Phi(Z + \theta).$$

This also implies that  $\Phi$ , the expectation operator of the POVM  $E^\Phi$ , can be given as follows, in direct analogy with the angle operator,

$$\begin{aligned} \Phi &= \int_{-\pi}^{\pi} d\phi \phi |\phi\rangle\langle\phi| = \sum_{m,n \geq 0} \int_{-\pi}^{\pi} d\phi \phi e^{i(n-m)\phi} |m\rangle\langle n| \\ &= \sum_{m \neq n, m, n \geq 0} \frac{i}{(n-m)} |m\rangle\langle n| + \frac{\pi}{2} \mathbb{I}. \end{aligned}$$

This operator is symmetric and bounded, therefore it is self-adjoint.<sup>22</sup> This leads to an apparent conflict with the Spectral Theorem, which appears to state that every self-adjoint operator has a spectral resolution.

We can see, however, that this conflict is indeed only apparent. We have in effect defined the POVM  $E^\Phi$  through its covariance properties under the unitary group generated by  $N$  (an operator with a spectrum bounded from below). Therefore, Pauli’s Theorem entails that

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<sup>22</sup>See Galapon (2002b) for a painstakingly direct proof of this fact.

$E^\Phi$  is not a PVM. Nonetheless,  $\Phi$ , a self-adjoint operator, has a unique spectral resolution corresponding to a PVM. This spectral resolution, however, is not covariant under shifts in phase generated by  $N$ .<sup>23</sup> Note also that the periodicity of phase also creates the same difficulty for the covariance of  $\Phi$  as for the angle observable, so that

$$e^{-iN\theta}\Phi e^{iN\theta} = \Phi + \theta\mathbb{I} + \pi E^\Phi([-\pi, -\pi + \phi]) = \Phi_\theta. \quad (5.2)$$

There is thus a sense in which requiring phase shift covariance alone is insufficient to pick out a phase operator uniquely. Lahti & Pellonpää (2000) show that requiring (i) phase-shift covariance, (ii) that the number observable  $N$  generates phase shifts, and (iii) that a phase observable generates number shifts, is sufficient to pick out  $E^\Phi$  uniquely (up to unitary equivalence). We have made each of these assumptions above. In that case, it seems we have just as good a reason regard  $E^\Phi$  as a canonical representation of phase as we did to regard  $P(Z)$  as the canonical representation of angle. It is only at this level of abstraction that the relation between a measurement of phase and a single covariant quantity defined on the physical Hilbert space becomes apparent.

## 5.4 LINEAR TIME OBSERVABLE: TIME OF ARRIVAL

### 5.4.1 The Classical Distribution of Arrival Times

To consider the simplest possible case, take a classical free particle in one dimension, with the event in question being arrival at the origin. The position of the particle is  $q(t) = q_0 + p/mt$  and on a space-time diagram the possible trajectories will appear as diagonal lines with gradient  $p/m$ , each trajectory crossing the origin  $q = 0$  exactly once. Setting  $q(t) = 0$  and rearranging we find that  $t = -(mq_0/p)$ , which gives the time  $t$  at which the particle “arrives” at the origin. (Similarly we could have considered the time at which the particle arrives at any point  $q(t) = x$ .) So long as  $p \neq 0$  (in which case this expression is ill-defined) if we evolve the system forward by  $t'$ , then the particle arrives at the origin  $t'$  seconds earlier.

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<sup>23</sup>See the discussion of Lahti & Pellonpää (2000).



Perhaps a more telling comparison arises from analysis of the classical phase space distribution, defined for an ensemble of non-interacting particles by<sup>24</sup>

$$\int f(q, p, t) dq dp = 1.$$

The probability density for finding a particle at a point  $x$  at time  $t$  is found by integrating this quantity over  $p$ ,

$$\rho(x, t) = \int f(x, p, t) dp.$$

The time at which a particle crosses a point  $X$  is  $T = (X - q_0)m/p_0$ , where  $q_0, p_0$  are the initial data. Consider the following distribution, defined for  $f(q, p \geq 0, t)$  i.e. for an ensemble with only positive momenta

$$\Pi(T; X) = \rho(X, T) = \int_0^\infty f(X, p, T) \frac{p}{m} dp = \int_0^\infty f(X, p, T) \frac{p}{m} \delta(q - X) dp.$$

Since  $\rho(X, T)dt$  corresponds to the fraction of particles that cross  $X$  between  $T$  and  $T + \Delta t$ , we have

$$\int_{-\infty}^\infty \Pi(T; X) t dt = \int_{-\infty}^\infty dq_0 \int_0^\infty dp_0 f(q_0, p_0, 0) \frac{(x - q_0)m}{p_0}.$$

This quantity corresponds to the average arrival time for the ensemble. Note that there is a singularity at  $p_0$ , which corresponds to a particle failing to cross the point  $x$  at all, but the integral converges so long as it is canceled somewhat by  $f$ .

In order to accommodate leftward moving particles also, we extend the integral to all of  $p$ ,

$$\Pi[T; X] = \int_{-\infty}^\infty dp f(X, p, T) \frac{|p|}{m} = \Pi_+[T; X] + \Pi_-[T; X].$$

A crucial property possessed by this distribution is covariance under time shifts. This can be seen by defining the time co-ordinate  $t = t_0 + T$  relative to a reference instant  $t_0$ , where  $f(t_0)$  is the distribution associated with  $t_0$ . We have

$$\Pi[T; X; f(t_0)] = \int_{-\infty}^\infty dq f(q, p, T + t_0) \frac{|p|}{m} \delta(q - X)$$

so that

$$\Pi[T; X; f(t_0)] = \Pi[T + t_0; X; f(0)].$$

The quantum analog of the distribution  $f$  is the Wigner function  $W$ , but the key difference is that  $W$  is not a true probability distribution but only a ‘quasi-probability’ distribution

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<sup>24</sup>In what follows I draw heavily from the excellent review article by Muga & Leavens (2000).

since it may take negative values. The extent to which this is the case for a given quantum state is taken to indicate the ‘non-classicality’ of the state. It is a somewhat surprising fact that even for rightward directed states whose momentum is *solely* positive, the Wigner function need not be positive. That is, there exist states (and times) for which the direction of probability current flows leftward, even though the momentum has only positive support. This is known as the *backflow effect*, first noted by Allcock (1969), and is related to the non-commutativity of the experimental question “Is the particle in the region  $x \geq 0$ ?” with the requirement that the state be one of only positive momentum.<sup>25</sup>

This indicates that the relation between the classical and quantum distributions cannot be so simple. However, Kijowski (1974) demonstrated that the classical time of arrival distribution  $f(t)$  may be recovered uniquely from the following axioms.

1.  $\Pi \geq 0$
2.  $\int_{-\infty}^{\infty} \Pi dt = 1$
3. If  $f' = (q + x, p, t) = f(x - q, p, t)$  then  $\Pi(F') = \Pi(F)$
4.  $\Pi$  has the minimum variance  $(\Delta t)^2$ .

This result demonstrates (at least) that there exists a set of necessary conditions that define the arrival time in classical mechanics (for the case of free motion). Remarkably, Kijowski also showed that these axioms serve to define uniquely a corresponding *quantum* distribution. The distribution obtained by Kijowski as a result of imposing his axioms on the quantum case takes the following form for a state with only positive momentum

$$\Pi_K[T; X; \psi(t_0)] = \left| \int_0^{\infty} dp \left( \frac{p}{mh} \right)^{\frac{1}{2}} \langle p | \psi(t_0) \rangle e^{-ip^2 T / (2mh)} e^{ipX/\hbar} \right|^2$$

This gives the probability of a free particle arriving at  $X$  at an instant  $t = t_0 + T$ , where  $t$  is also the parameter that appears in the unitary evolution of the state  $\psi(t)$ . The general case (with positive and negative momentum) is obtained by considering the sum of the two separate integrals, that is

$$\Pi_K[T; X; \psi(t_0)] = \sum_{\alpha} \left| \int_0^{\alpha\infty} dp \left( \frac{p}{mh} \right)^{\frac{1}{2}} \langle p | \psi(t_0) \rangle e^{-ip^2 T / (2mh)} e^{ipX/\hbar} \right|^2,$$

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<sup>25</sup>Perhaps the easiest way of explaining this comes from considering the Bohmian picture: the backflow effect is associated with Bohmian particle trajectories which change the sign of their momentum under the influence of the quantum potential *even when the particle is free*.

where  $\alpha = \pm 1$ . Since the dependence on  $t$  arises only through the state  $\psi(t_0)$ , the distribution is covariant under time shifts,

$$\Pi_K[T; X; \psi(t_0)] = \Pi[T + t_0; X; \psi(0)].$$

This is precisely the behavior one would expect of an event time. The distribution this defines was the same as that derived by Allcock (1969) as an approximation, and has as its first moment the time of arrival  $T_a$ . This was the subject a detailed study in Chapter 5. Furthermore, the Weyl quantization of the classical expression above results in much the same operator  $T_a$ —not to mention other quantization schemes as well (Muga & Leavens, 2000, p. 390). Since  $T_a$  is maximal symmetric but not self-adjoint, the quantum distribution is associated with a POVM and so higher moment operators are not obtained in the usual manner, i.e.  $T_a^2$  is *not* the second moment of the POVM  $E^{T_a}$ .

#### 5.4.2 The Quantum Distribution

Here we investigate in more detail the properties of  $T_a$ , which, given its similarity to the operator of Aharonov and Bohm (1961), is often referred to as the Aharonov-Bohm time of arrival operator (Muga & Leavens, 2000). However, the difference of overall sign signals their original intent to regard their operator as a quantum clock: the classical expression corresponding to their operator returns  $q(t) = vt$ , which entails that  $q(0) = 0$  so that the position of the particle is proportional to the elapsed time, as required of a clock. We will be interested here in the time of arrival operator, interpreted as returning the time of an event (the particle's crossing the point  $x = 0$ ) rather than a clock observable, measurement of which would return an estimate of the external parameter time  $t$ .<sup>26</sup> This operator  $T_a$ , in symmetrized form, is

$$T_a = -\frac{m}{2} (QP^{-1} + P^{-1}Q).$$

It is easily seen that, formally at least,  $T_a$  is canonically conjugate to  $H = (1/2m)P^2$  in the sense that

$$[T_a, H] = \frac{1}{2} (P^{-1}[Q, P^2] + [Q, P^2]P^{-1}) = i.$$

Therefore,  $(T_a, H)$  form a Heisenberg pair. In accord with Pauli's Theorem, since  $H$  is semi-bounded  $T_a$  is not self-adjoint.

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<sup>26</sup>See Section 2.3.4 for a critique of the idea that a quantum free particle admits such a clock observable.

Making the substitution  $q = i \frac{d}{dp}$  we obtain the momentum representation of the operator,

$$T_a = \frac{im}{2} \left( \frac{d}{dp} \frac{1}{p} + \frac{1}{p} \frac{d}{dp} \right) = \frac{im}{p} \left( \frac{d}{dp} - \frac{1}{2p} \right).$$

Since this involves a factor of  $(1/p)$  there will be a singular point at  $p = 0$ , and this is the source of the difficulties which prevent  $T_a$  from being self-adjoint. However, we can define a suitable dense domain on which  $T_a$  is symmetric. The problem will be restricting the functions  $\psi(p) \in \mathcal{H}$  so that  $|T_a\psi|^2 < \infty$ , with the problematic part of the integral being the region around 0, i.e. we require

$$\int_{-a}^a (T_a\psi)^*(p)(T_a\psi)(p)dp < \infty,$$

with  $a > 0$ . If  $\psi(p) \sim p^{1/2}$  as  $p \rightarrow 0$  then this part of the integral vanishes, and if  $\psi(p) \sim p^{3/2}$  then the contribution is finite for any  $a$ .<sup>27</sup> Splitting the domain of integration into two parts from  $-\infty$  to  $-a$  and  $a$  to  $\infty$ , the dependence on  $a$  appears as<sup>28</sup>

$$\langle \psi | T_a \psi \rangle - \langle T_a \psi | \psi \rangle = -i \lim_{a \rightarrow +0} \left( \frac{|\psi(a)|^2 + |\psi(-a)|^2}{a} \right).$$

Since  $T_a$  is formally symmetric, we chose the domain so that this limit vanishes, which it won't if  $\psi(p) \sim p^{1/2}$  as  $p \rightarrow 0$ , so instead we require  $\psi(p)/p^{3/2} \rightarrow 0$  as  $p \rightarrow 0$ . Thus  $T_a$  is symmetric on the domain

$$\mathcal{D}_{T_a} = \left\{ \psi \in \mathcal{H} : \lim_{p \rightarrow 0} \frac{\psi(p)}{p^{3/2}} \rightarrow 0; T_a\psi \in \mathcal{H} \right\}.$$

This restricted domain for  $T_a$  leads Paul (1962) to doubt the physical meaningfulness of this operator. Indeed, Heathcote (1990) argues similarly that there is a problem in general for the foundations of quantum mechanics involving unbounded operators, since functions not in the domain of an unbounded operator represent vector states for which the expectation value of the operator is not defined. But in that case it is not clear that the difficulties faced by  $T_a$  are different in kind from those faced by the unbounded symmetric operators such as  $Q$  and  $P$  that Heathcote considers, which are generally taken to be physically meaningful.

It is just an unfortunate fact of life that unbounded operators will not be defined for all vectors in  $\mathcal{H}$ . But, since the domain is dense, a sequence of vectors in the domain can be found which approximates a problematic vector state arbitrarily well. And in the case of time of

<sup>27</sup>Since  $T_a p^{3/2} = 2i/\sqrt{p}$  the integral evaluates as  $4(\log(a) - \log(-a)) = i4\pi$ .

<sup>28</sup>See Paul (1962) for details.

arrival, there is a reasonable expectation that some states will have an ill-defined expectation value for arrival at the origin anyway, since the classical time of arrival is undefined for states with  $p = 0$ . Furthermore, as Heathcote notes, although the expectation value of a symmetric unbounded operator is undefined for states not in its domain, the probability distribution for the outcomes of measurements associated with any state at all can be obtained by means of the associated PVM (or POVM in this case).

Our interest now is in whether  $T_a$  is self-adjoint. First, since it is densely defined it does possess an adjoint  $T_a^\dagger$  whose domain is closed.<sup>29</sup> To see this, we use von Neumann's formula for essential self-adjointness, according to which a symmetric operator has a unique self-adjoint extension (i.e. is essentially self-adjoint) if and only if its deficiency indices are  $(0, 0)$  (See Section 3.2). That is, we want to compare the number of eigenvectors of  $T_a^\dagger$  with eigenvalue  $i$  with the number of eigenvectors with eigenvalue  $-i$ . The critical calculation involves seeing whether there exists some  $\psi \in \mathcal{H}$  such that

$$\langle \psi | (T_a \pm i) \phi \rangle = 0$$

for all  $\phi \in \mathcal{D}_{T_a}$ . For  $T_a^\dagger$ , then, the following are eigenfunctions corresponding to  $i$

$$\psi_\pm(p) = \Theta(\pm) \sqrt{\pm p} e^{-p^2/2m},$$

where  $\Theta$  is the Heaviside step function. There are no eigenvectors with eigenvalue  $-i$ . The deficiency indices are thus  $(2, 0)$ , demonstrating that  $T_a$  is maximal symmetric but not self-adjoint.<sup>30</sup>

In contrast to the case of the harmonic oscillator, where we could compare number and phase to the angular momentum and angle pair and so see how to 'double' the energy spectrum, it is not immediately obvious how one should go about constructing such a representation for the free particle. The guiding principle here is that, as realized by Dirac back in 1926, time, being (in principle) conjugate to energy, should take the representation  $-i\partial/\partial E$  in the energy representation. The key, then, is to find the energy representation, which we can do in terms of the momentum representation  $\mathcal{H}(\mathbb{R}, dp)$ . Egusquiza & Muga (1999) suggest the isomorphism

$$\mathcal{H}(\mathbb{R}, dp) \equiv \mathcal{H}(\mathbb{R}^+, dE) \oplus \mathcal{H}(\mathbb{R}^+, dE)$$

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<sup>29</sup>The following analysis follows that of Egusquiza & Muga (1999) rather closely.

<sup>30</sup>See the more detailed general discussion of Section 3.2.

which decomposes  $\mathcal{H}$  into a direct sum of orthogonal subspaces corresponding to positive and negative momenta (so that the inner product is given as a sum). In essence, the idea is that a vector can be written in the energy representation as follows

$$\langle E|\psi\rangle = \left(\frac{m}{2E}\right)^{1/4} \left[\psi^+(\sqrt{2mE}) + \psi^(-\sqrt{2mE})\right] = \left(\frac{m}{|p|}\right)^{1/2} [\Theta(p)\psi(p) + \Theta(-p)\psi(p)].$$

Or equally in the momentum representation

$$\langle p|\psi\rangle = \left(\frac{|p|}{m}\right)^{1/2} \left[\Theta(p)\psi^+\left(\frac{p^2}{2m}\right) + \Theta(-p)\psi^-\left(\frac{p^2}{2m}\right)\right] = \left(\frac{2E}{m}\right)^{1/4} [\psi^+(E) + \psi^-(E)].$$

Under this isomorphism, the time of arrival has the representation  $T_a = (-i\partial_E) \oplus (-i\partial_E)$  where each of these operators is isomorphic to momentum on the *half* line, due to the semi-bounded spectrum of  $H$ .

Interestingly, the study of momentum on the half line goes back to von Neumann, who used it to illustrate the distinction between self-adjoint and maximally symmetric operators in 1929. Rather than consider a free particle on the real line, with Hilbert space  $\mathcal{H} = L^2(-\infty, \infty)$ , we consider a free particle on the positive half line  $\mathcal{H}_{1/2} = L^2[0, \infty)$ . Although the position and momentum operators are defined in the usual manner, with  $Q'$  acting by multiplication and  $P' = -i\frac{d}{dx}$ , the domain of the momentum operator must be restricted so that functions vanish at the origin,

$$\mathcal{D}_{P'} = \{\psi \in \mathcal{H}_{1/2} : \psi' \in \mathcal{H}_{1/2} : \psi(0) = 0\}.$$

This leads to the fact that  $e^{ix}$  is an eigenfunction of  $P'^{\dagger}$  with eigenvalue  $i$ , while there is no eigenfunction with eigenvalue  $-i$ . Thus  $P'$  has deficiency indices  $(1, 0)$ .

To find the Naimark dilation of  $P'$  we merely reintroduce the other half of the real line. Let us first write down an explicit expression for the POVM associated with  $P'$ ,

$$F([a, b]) = \int_a^b dp' |p'\rangle\langle p'|,$$

where  $|p'\rangle = (1/\sqrt{2\pi})e^{ip'x}$ . These improper generalized eigenvectors form an overcomplete basis for  $\mathcal{H}_{1/2}$ . They are not orthogonal, which is indicative of  $P'$  being maximally symmetric. Now let's consider momentum on the full line  $P$ , associated with a PVM

$$E([a, b]) = \int_a^b dp |p\rangle\langle p|,$$

where again  $|p\rangle = (1/\sqrt{2\pi})e^{ipx}$ , but these form a complete basis for  $\mathcal{H} = L^2(-\infty, \infty)$ . Consider the following inner product for  $\psi, \phi \in \mathcal{H}$ :

$$\langle \phi | E([a, b]) \psi \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_a^b dp, \langle \phi | x \rangle \langle x | p \rangle \langle p | y \rangle \langle y | \psi \rangle.$$

It is clear that if we define a projection  $P$  such that  $(P\psi)(x) = \Theta(x)\psi(x)$ , for all  $\psi \in \mathcal{H}$ , that the effect of  $P$  on  $E([a, b])$  will be to return  $F([a, b])$ . That is,

$$\langle \phi | PE([a, b]) \psi \rangle = \int_0^{\infty} dx \int_0^{\infty} dy \int_a^b dp \langle \phi | x \rangle \langle x | p \rangle \langle p | y \rangle \langle y | \psi \rangle = \langle \phi | F([a, b]) \psi \rangle.$$

Thus  $E([a, b])$  is the PVM of our Naimark dilation, defined on  $\mathcal{H} \supset \mathcal{H}_{1/2}$ . For future reference, it will be useful to write the associated probability density  $\Pi_\psi(p)$  for a state  $\psi \in \mathcal{H}'$ , given as follows

$$\langle \psi | F(dp) \psi \rangle = \frac{1}{2\pi} \int_0^{\infty} dx \int_0^{\infty} dy \langle \psi | x \rangle e^{ip(x-y)} \langle y | \psi \rangle dp = \Pi_\psi(p) dp.$$

The same procedure applies for  $T_a$ , where we will find its Naimark dilation in the ‘doubled’ energy representation. Momentum on the half line corresponds to the time of arrival, whereas position is self-adjoint on the half line, and so corresponds to the energy of the free particle. In that case, to follow the analogy we express  $T_a$  in the energy representation as  $T_a = -i\frac{d}{dE}$ . Reintroducing the decomposition into positive and negative momenta, we have  $T_a = (-i\partial_E) \oplus (-i\partial_E) = T_a^+ \oplus T_a^-$  acting on  $\mathcal{H}(\mathbb{R}^+, dE) \oplus \mathcal{H}(\mathbb{R}^+, dE)$ , where  $T_a^\pm$  are isomorphic to momentum on the half line. The Naimark dilation is to be found in  $\mathcal{H}^\pm = \mathcal{H}(\mathbb{R}, dE) \oplus \mathcal{H}(\mathbb{R}, dE)$ , which extends the energy representation to functions of negative energy.

Consider the operator  $\epsilon$  that acts by multiplication on this extended space,  $(\epsilon\psi)(x) = x\psi(x)$ . This operator is self-adjoint, and has a corresponding PVM

$$P([a, b]) = \int_a^b dE |E\rangle \langle E|,$$

where  $|E\rangle = (1/\sqrt{2\pi})e^{iEt}$ . Correspondingly, there is a self-adjoint operator  $\tau$  which acts by differentiation in the ‘energy basis’ and has a PVM

$$T([a, b]) = \int_a^b dt |t\rangle \langle t|,$$

where  $|t\rangle = (1/\sqrt{2\pi})e^{-iEt}$ . If so, we have a Schrödinger representation, formed by  $\epsilon$  and  $\tau$ . That is, an ‘energy-time representation’; in this ‘larger’ Hilbert space of the Naimark

dilation, ‘energy’ and ‘time’ are true conjugates.<sup>31</sup> Consider the following inner product for  $\phi, \psi \in \mathcal{H}^\pm$ ,

$$\langle \phi | E^T([a, b]) \psi \rangle = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \int_a^b dt \langle \phi | E \rangle \langle E | t \rangle \langle t | E' \rangle \langle E' | \psi \rangle.$$

As before, the ‘probability density’ is introduced as follows

$$\langle \psi | E^T(dt) \psi \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \langle \psi | E \rangle e^{it(E-E')} \langle E' | \psi \rangle dt = \Pi_\psi^T(t) dt.$$

This can be written in terms of the direct sum introduced above,

$$\Pi_\psi^{E^T}(t) = \left| \int_{-\infty}^{\infty} dE \frac{e^{-iEt}}{\sqrt{2\pi}} \psi^+(E) \right|^2 + \left| \int_{-\infty}^{\infty} dE \frac{e^{-iEt}}{\sqrt{2\pi}} \psi^-(E) \right|^2.$$

As before, if we introduce a projection  $P$  such that  $(P\psi)(E) = \Theta(E)\psi(E)$  then we obtain a POVM by projection  $PE^T([a, b]) = F^{T_a}([a, b])$ . This leads to a probability density for  $\psi \in \mathcal{H}$  of

$$\Pi_\psi^{F^{T_a}}(t) = \left| \int_0^\infty dE \frac{e^{-iEt}}{\sqrt{2\pi}} \psi^+(E) \right|^2 + \left| \int_0^\infty dE \frac{e^{-iEt}}{\sqrt{2\pi}} \psi^-(E) \right|^2,$$

which is the probability density associated with the time of arrival  $T_a$ . Therefore the POVM corresponding to  $T_a$  is  $PE^T([a, b]) = F^{T_a}([a, b])$ .

A function  $\psi \in \mathcal{H}^\pm$  has the momentum representation

$$\psi(p) = \left( \frac{|p|}{m} \right)^{1/2} \left[ \psi^+ \left( \frac{p^2}{2m} \right) + \psi^- \left( \frac{p^2}{2m} \right) \right],$$

in terms of which the effect of the projection  $P$  is as follows

$$P\psi(p) = \left( \frac{|p|}{m} \right)^{1/2} \left[ \Theta(p)\psi^+ \left( \frac{p^2}{2m} \right) + \Theta(-p)\psi^- \left( \frac{p^2}{2m} \right) \right].$$

This gives the probability density for  $F^{T_a}$  in terms of momentum,

$$\Pi_\psi^{F^{T_a}} = \left| \int_0^\infty dp \left( \frac{p}{2\pi m} \right)^{1/2} e^{-ip^2 t/2m} \psi^+(p) \right|^2 + \left| \int_{-\infty}^0 dp \left( \frac{-p}{2\pi m} \right)^{1/2} e^{-ip^2 t/2m} \psi^-(p) \right|^2.$$

This is all we need to explicitly calculate expectation values and probabilities for the time of arrival via its POVM. The time translation covariance of the the PVM  $E^T([a, b])$  is preserved

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<sup>31</sup>The scare quotes indicate the interpretative hazards of regarding the Naimark dilation as a providing a representation of a physical quantity of a physical system here: since no quantum systems have unbounded energy spectra, this cannot be a representation of a quantum system.



when projected back down to the POVM, but  $T_a$  is not the generator of shifts in energy;  $T_a$  is the generator of a semigroup of shifts in energy. There is a sense, then, in which  $T_a$  cannot be ‘the’ canonical time observable. This is as it should be, since  $T_a$  is a case specific observable, in the sense that it is one of a family of time of arrival observables:  $T_a$  corresponds to arrival at the origin,  $x = 0$ , and so forms one of a one-parameter family of ‘arrival at  $x$  operators,’ reached by spatial translation of  $F^{T_a}([a, b])$  under the usual unitary group.

## 5.5 LINEAR TIME OBSERVABLE: FREELY FALLING PARTICLE

The Hamiltonian here consists of the usual kinetic term and a potential term,  $H = (1/2m)P^2 + mgQ$ , where  $g$  is the acceleration due to the potential field (assumed here to be gravitational). Since  $H$  is self-adjoint (being the sum of two self-adjoint operators) there exists a unitary group  $U_t = e^{iHt}$ . Formally, we have  $[Q, H] = (i/m)P$  and  $[P, H] = -img$ . Thus  $[Q, U_t] = -(1/m)U_tPt$  and  $[P, U_t] = mgtU_t$ , and the Heisenberg equations of motion yield

$$Q(t) = U_{-t}QU_t = Q - (P/m)t - \frac{1}{2}gt^2 \quad P(t) = U_{-t}PU_t = P + mgt.$$

This suggests that the operator  $T_g = P/(mg)$  has commutator  $[T, H] = -i$  and increases linearly with time  $T_g(t) = U_{-t}T_gU_t = T_g + t$ . Since  $P$  is self-adjoint,  $T_g$  is self-adjoint, and  $(T_g, H)$  forms a Weyl pair. As Pauli’s Theorem demands,  $H$  has an unbounded spectrum.

There is good reason to doubt that this is the correct quantization of the classical system, however, since I here propose an alternative quantization scheme in which the spectrum of the Hamiltonian is bounded from below. The form of the classical equation of motion indicates that the gravitational potential can be removed by means of a time dependent co-ordinate transformation into the freely falling frame,  $q(t) \rightarrow q'(t) = q(t) + (1/2)gt^2$ . I will now show that this transformation implements Einstein’s original Equivalence Principle: no experiment can differentiate between the effects of a homogeneous gravitational field and the effects of constant acceleration. The acceleration experienced by the classical particle due to the gravitational field is  $\ddot{q} = -g$ , and the acceleration relative to the primed co-ordinates is  $\ddot{q}' = 0$ .

Since there are no other forces acting on the system in the primed frame, Newton’s Second Law returns  $F' = m\ddot{q}' = 0$  and the Lagrangian is  $L' = (1/2)m\dot{q}'^2$ . The Hamiltonian in the primed frame is thus  $h' = \frac{1}{2m}p'^2$ . When we solve Hamilton’s Equations for

$h', q', p'$  we find that  $q'(t) = q'(0) + p'(t)/m$  and  $p'(t) = p'(0)$ . The initial data are the same in each frame  $p'(0) = p(0)$ ,  $q'(0) = q(0)$ . Performing the inverse transformation  $q'(t) \rightarrow q(t) = q'(t) - (1/2)gt^2$  we find that  $q(t) = q(0) + p(0)/m - (1/2)gt^2$ , as before. We also have Newton's Second Law in the unprimed frame,  $F = m\ddot{q} = -mgt$ , which when integrated indicates that the unprimed momentum is  $p(t) = p(0) - (1/2)gt^2$ . There is thus no observable difference between the behavior of the accelerating classical particle described in the unprimed co-ordinates and the classical particle with potential. However, the latter Hamiltonian function involves a potential term and has no lower bound on its energy, whereas the primed Hamiltonian does not.

Now we consider the quantization of the primed system,  $q' \rightarrow Q'$ ,  $p' \rightarrow P'$  with  $[Q', P'] = i$ . The Hamiltonian is  $H' = (1/2)P'^2$ . Since  $P'$  commutes with  $H'$  we have  $P'(t) = P'$ . The dynamics is given by the group  $U_t = e^{iH't}$ . Using the fact that  $[Q', U_t] = (t/m)P'U_t$ , we have the equation of motion for a free particle,

$$Q'(t) = U_t^\dagger Q' U_t = U_t^\dagger \left( \frac{t}{m} P' U_t + U_t Q' \right) = Q' + \frac{t}{m} P'.$$

To return to the time-dependent frame, we need to apply a general (time dependent) passive Galilei transformation  $U_G(t)$ . This will consist of two parts: a boost, generated by  $mQ$ , and a spatial transformation, generated by  $P$ , which ensures that the wavefunction is invariant under the transformation. Writing the (improper) primed position eigenstates in terms of the unprimed eigenstates we have

$$|x'\rangle = |x - (1/2)gt^2\rangle = e^{-(1/2)iPgt^2} |x\rangle.$$

The time dependent velocity is, therefore,  $gt$  and so we have  $e^{iQmgt}$  for the boost. Putting these together (see Brown & Holland (1999)) we have

$$U_G(t) = e^{i[(1/2)Pgt^2 - mQgt]} = e^{i(mg^2t^3)/4} e^{(igt^2P)/2} e^{-imQgt},$$

where the equality follows from the identity  $e^{A+B} = e^A e^B e^{[A,B]/2}$ , valid for operators that commute with their commutator. Thus the Heisenberg picture momentum operator in the unprimed frame is

$$P(t) = U_{-t} U_G P' U_G^{-1} U_t = P' + mgt\mathbb{I},$$

as before. The Heisenberg picture position operator in the unprimed frame is

$$Q(t) = U_{-t} U_G Q' U_G^{-1} U_t = Q'(t) - (1/2)gt^2 = Q' + \frac{t}{m} P' - (1/2)gt^2\mathbb{I},$$

in accord with the classical equation of motion. But now we find that  $T_g = P(t)/(mg) = P' + t\mathbb{I}$  which, although covarying with time as desired, commutes with  $H'$ . It thus fails to satisfy the premises of Pauli's Theorem. In this unprimed description of the system, the source of the covariance is merely the co-ordinate transformation  $q(t) \rightarrow q'(t)$ , represented in quantum mechanics by the unitary group  $U_G(t)$ .

So we have two inequivalent quantizations of what is ostensibly the same classical system, at least according to the Equivalence Principle. It is easy to see that there is no unitary transformation that implements the Equivalence Principle here, since if there were then  $H$  and  $H'$  would have the same spectrum. In response, one might nonetheless claim that they represent the same physical system, and thus assert that unitary equivalence is not necessary for physical equivalence. This would not be without precedent: Ruetsche (2011) considers cases where it is feasible to assert that unitary equivalence is sufficient but not necessary for physical equivalence. Certainly the identical expectation values given by the alternative quantization schemes (when considered in the same frame) are good reason to take them to be equivalent. If one would prefer to consider these as inequivalent systems then two questions arise: (i) which is the correct quantization? (ii) what, then, is the status of the Equivalence Principle in quantum mechanics? The latter of these questions, although fascinating, will have to await further examination. The former could be answered in favor of the alternative quantization I provide by means of the following argument.

When we measure the momentum of a quantum particle in the lab, in a (roughly) homogeneous gravitational field on the Earth's surface, we are measuring the momentum with respect to a frame at which the particle is not at rest. We have every reason to believe that there is a freely falling reference frame in which a quantum particle, subject only to a gravitational force in the lab frame, behaves as if it were free (at least in a local region). In that frame, a measurement of energy would return a result that is greater than or equal to zero. The difference between the energy measured in the freely falling frame and the lab frame can be entirely attributed to their relative acceleration. This means that there exists a frame, accelerating with respect to the freely falling frame, in which the (expectation value for) energy of the particle is greater than any given real number. But it is not true that there exists a frame in which the (expectation value for) energy is less than any given real number. The energy, as measured, is therefore bounded from below.

## 6.0 QUANTUM CLOCKS

Equipped with a detailed discussion of the technical apparatus of modern quantum mechanics and its application to some specific cases of putative time observables, we are now in a position to return to the questions raised by Schrödinger and Pauli regarding the existence of ideal clocks, discussed in Section 2.1.3. Let us review their exchange. Schrödinger defined an ideal clock as a system that evolves through a succession of orthogonal states such that its state at a time is a perfect indicator of the value of the time parameter that appears in the Schrödinger equation. He pointed out that the time-energy uncertainty relation would seem to imply that such a system (having a sharply defined state for each time) would have a completely uncertain value for energy. On these grounds, he rejected the existence of such states as “physically meaningless.”

This point of view will be backed up by the following discussion of Pauli’s Theorem. There is a sense (to be made precise) in which Pauli’s Theorem entails that such an ideal quantum clock must have an unbounded Hamiltonian spectrum, and thus allows states of infinite uncertainty in energy. However, in response to Schrödinger, Pauli countered that such a state of affairs is unproblematic since the situation regarding position in time is no more desperate than the situation regarding position in space. In both cases, he maintained, the ideal indicator is to be reached as the limit of sequence of physically possible systems. I will call this Pauli’s Strategy.

My proposal for a sequence of such clocks comes from consideration of the quantum harmonic oscillator, and in particular the suggestion of Garrison & Wong (1970) regarding the use of coherent states as a ‘pointer’. This is in conflict with Hilgevoord’s proposal for an ideal periodic clock, which has an unbounded Hamiltonian spectrum (Hilgevoord, 2002, 2005; Hilgevoord & Atkinson, 2011). In Section 6.2 I argue that this manifest unphysicality is unacceptable, even considered as the limiting behavior of a sequence of (potentially) physically allowed clocks. I explain how Hilgevoord’s sequence of clocks can be regarded as

an approximation to the quantum harmonic oscillator clock that I propose, rather than his unphysical ideal clock.

Before proceeding to discuss the details of the no go results and positive proposals for quantum clocks, I would like to sound a note of general caution regarding the analogy between the use of a point particle as an ideal indicator of position in space and the use of an ideal clock as an ideal indicator of location in time.<sup>1</sup> A real clock indicates the elapsed time by means of some non-temporal variable, for instance the angular position of the hand of a rotary clock. It is the position of the hand that covaries with the elapsed time, and the elapsed time is estimated from the relative position of the hand (compared to its position at an earlier time).

That is, a physical clock indicates the time not in virtue of its location in time, but by having some degree of freedom that covaries with time. Indeed, as we saw in Section 2.3.1, the position of a classical point particle in space can serve as an ideal clock. On the other hand, it is not clear just what the location of a point particle in space indicates about position in space. By itself, a location in space is a spatial point, and a location in time is a temporal instant. Just as a system is not a clock in virtue of being located in time, nor is a system an indicator of position in virtue of being located in space. Rather, it seems better to say that the position of a point particle can be used as a measure of relative distance in space: either distance from one particle to another or distance from the position of the particle at one time to its position at another time.

In the latter case, it is important that the positions in space at various times are attributed to the same thing: the point particle, considered as a persisting object. Thus the point particle serves as an indicator of location in space in virtue of its persistence through time. The danger of regarding the mere location of a particle in space at a time as an indicator of position, is that the temporal analogue would indicate the time merely by virtue of its location in time. Something that has a precise location in time, however, is not a clock—a persisting thing—but an event. This confusion is exhibited by the recent proposal of Hegerfeldt & Muga (2010) for quantum clock observables, which, as I argue in Section 6.4, are not clock observables but event observables.

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<sup>1</sup>This analogy features heavily in analysis of Hilgevoord (2002, 2005); Hilgevoord & Atkinson (2011).

## 6.1 NO IDEAL QUANTUM CLOCKS

Let us examine exactly what Pauli's Theorem manages to tell us about quantum clocks. Classically, a clock is a time function that covaries along the dynamical curves either locally or (in addition) globally. In Section 2.3, this was distinguished from an event time, which covaries with the initial data (and in the opposite direction). In Hamiltonian (analytical) classical mechanics, the existence of a clock function was sufficient to allow one to infer the value of the time parameter (up to periodicity) from the instantaneous state (in conjunction with the initial data). There is a sense in which quantum mechanics replicates this idea quite nicely, and a sense in which it makes it much more problematic.

First, the good news. Given a non-periodic quantum system whose Hamiltonian is exactly known, precise knowledge of the state at two times suffices to determine the time interval between those states. If one knows the Hamiltonian and the state  $\psi$  at  $t = 0$  then, since the Schrödinger equation is first order in time, one knows the state at all other times  $\psi_t = U_t\psi$ . This family of states parameterized by  $t$  associates with each instant of time  $t$  a unique state  $\psi_t$ , knowledge of which can be used to infer the elapsed time. This is analogous to the use of a classical time function to infer the elapsed time in terms of a parameterized curve in phase space. Where quantum mechanics complicates matters is in seemingly providing in principle limitations on the extent to which the state can be precisely known at a moment of time.

On the standard account offered by Ordinary QM, a system can be prepared in a precise state at  $t = 0$ , but knowledge of the values of variables at later times is mediated by the measurement of observables (self-adjoint operators or spectral projections thereof). This suggests the following model of a quantum clock: a system is prepared in a known state at  $t = 0$  and a clock observable is measured at time  $t'$ , the results of this measurement providing knowledge of the elapsed time  $t' - t$ . This is perhaps more suggestive of a stopwatch rather than a clock in the traditional sense: this quantum clock observable returns the time when measured, and only when measured. This is like a digital stopwatch that displays the elapsed time only when the 'lap time' button is pressed. One highly desirable characteristic for such a quantum observable to possess is an appropriate classical limit in which the time can be

known with arbitrary precision. This would thus mirror the situation of the infinitely massive particle, assumed to provide an ideal indicator of position.<sup>2</sup>

In the Schrödinger picture, the expectation value of an observable  $T_c$  measured at time  $t$  for a system prepared in state  $\psi$  at time  $t = 0$  is given by  $\langle \psi_t | T_c \psi_t \rangle$ . If  $T_c$  is to function as a quantum clock observable, then this expectation value had better covary with time, in the sense that

$$\langle \psi_{t'} | T_c \psi_{t'} \rangle - \langle \psi_t | T_c \psi_t \rangle = t' - t,$$

for at least some subset of times. This is a somewhat weaker requirement than the demand that  $T_c$  itself covary with time,  $T_c(t) = T_c(0) + \mathbb{I}t$ , which (as an operator identity) applies to all states of the system (or at least vectors in the domain of  $T_c$ ) and also for all times.

That requirement, as we have seen, is inconsistent with the Hamiltonian of the system having a spectrum of anything but  $\mathbb{R}$ , i.e. having a lower bound on its energy. This global requirement also suffices to exclude periodic clocks, whose time operators do not covary in this way. As I argued in Chapters 4 and 5, it is better to express the covariance requirement in terms of the underlying POVM  $E^{T_c}$ . That is, we require that, for every  $t \in \mathbb{R}$  and every  $E^{T_c}(\Delta)$ , there exists a projection  $E^{T_c}(\Delta + t)$  such that

$$U_{-t} E^{T_c}(\Delta) U_t = E^{T_c}(\Delta + t),$$

where  $\Delta \in \mathcal{B}(\mathbb{R})$  is a Borel subset of times. This can be satisfied by a periodic clock observable with period  $\tau$  since  $E^{T_c}(\Delta) = E^{T_c}(\Delta + n\tau)$  is consistent with this relation. Expressed in these terms, the refined version of Pauli's Theorem (Theorem 4.4) serves to exclude the possibility that  $E^{T_c}$  is a PVM, so long as the Hamiltonian is semi-bounded. There are thus no quantum clock observables corresponding to PVMs.

Unruh & Wald (1989, p. 2606) prove a similar result which they claim establishes that there can be no perfect quantum clock, which they take to be a system possessing a time operator whose observed values increase monotonically with parameter time  $t$ .

**Proposition 6.1** (Unruh & Wald). *If the Hamiltonian  $H$  is bounded from below, then there can be no operator  $T$  possessing eigenstates  $|T_0\rangle, |T_1\rangle, |T_2\rangle, \dots$  with the following properties:*

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<sup>2</sup>I will not explore the interesting issue of whether this is the most appropriate account of an ideal position indicator, other than providing the simple-minded observation that the Heisenberg picture position operator of a free particle is  $Q(t) = Q(0) + (1/m)Pt$ , which strongly suggests that a system lying at the limit of infinite mass will stay put.

1. Each  $|T_n\rangle$  is an eigenstate of the projection operator onto the (non-overlapping) spectral interval  $T_n$ , with  $T_0 < T_1 < T_2 < \dots$
2. For each  $n$  there is state  $a |T_m\rangle$  such that  $m > n$  and  $|\langle T_n|T_m\rangle|^2 > 0$  for some interval  $t > 0$ .
3. For each  $n$  there is no  $m < n$  such that  $|\langle T_m|T_n\rangle|^2 \neq 0$ .

I supply here my own proof based on the results of Section 4.4.

*Proof.* Let  $E^T(T_n)$  be the projection associated with the temporal interval  $T_n$  so that 1. is satisfied by the states  $E^T(T_n)|\psi\rangle = |T_n\rangle$ , for  $\psi \in \mathcal{H}$ . Condition 3. says that  $\langle T_m|T_n\rangle = \langle T_m|E^T(T_n)|\psi\rangle = 0$  for all  $n > m$ , which implies that  $\langle T_m|U_t^\dagger E^T(T_n)U_t|\psi\rangle = 0$  for all  $t > 0$ , which by Hegerfeldt's Lemma (Lemma 4.2) implies that  $\langle T_m|T_n\rangle = 0$  for all  $t$  and thus all  $n, m$ . We have  $|\langle T_n|T_m\rangle|^2 = \langle T_n|T_m\rangle^\dagger \langle T_n|T_m\rangle = \langle T_m|T_n\rangle \langle T_n|T_m\rangle = 0$ , in contradiction with 2. □

Roughly, then, there can be no quantum clock passing through successive states  $|T_n\rangle$  which has a non-zero probability of running forward in time while having no possibility of running backwards. This quantum clock would resemble less of a quantum stopwatch and more of a quantum ratchet.<sup>3</sup> However, this theorem provides a useful indication of what a non-ideal quantum clock would look like. Obviously, if we want our clock to run at all we cannot jettison condition 2., so it seems that condition 3. must be the one to go: real quantum clocks will have a finite probability of 'running backward in time.' This is not to say, however, that this clock will pass through a series of states 'out of temporal sequence,' as it were, since the succession of states is given by the Schrödinger equation. More precisely, if 3. fails then the successive states of such a clock will not be mutually orthogonal. In fact, Corollary 4.1 states that no dynamically allowed quantum mechanical system can have successive states that are mutually orthogonal for more than a set of instants of measure zero.

There is, therefore, a very strong case to be made against the existence of an ideal quantum clock, passing through a succession of orthogonal states. To the extent that all physical systems have a semi-bounded Hamiltonian, then, the existence of an ideal quantum clock is ruled out. A periodic clock can, however, have a Hamiltonian with a *bounded* spectrum, as

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<sup>3</sup>I owe this observation to Gordon Belot (in conversation).



we will see. The use of these finite-dimensional systems to model ideal clocks goes back to Salecker & Wigner (1958), and these clocks have become somewhat standard in the quantum time literature due to their simplicity (e.g. Peres (1980)). Recently Hilgevoord (2002; 2005) has advocated these finite-dimensional clocks as providing successive approximations to an ideal periodic quantum clock (with unbounded Hamiltonian). The idea is that an sequence of  $n$ -dimensional systems approaches the infinite-dimensional ideal clock in the limit as  $n$  goes to infinity. The unphysical ideal clock is thus promoted to an idealization, on his account, approximated as closely as one likes by physically allowed systems.

## 6.2 FOLLOWING PAULI'S STRATEGY

I have already discussed in Section 2.1.3 the sense in which Pauli's analogy between position and time fails to answer Schrödinger's worry: where the ideal indicator of position involves a system with infinite energy, the ideal indicator of time involves a state of infinite *uncertainty* in energy. Nonetheless, Pauli's overall strategy can reasonably be applied. In both cases, Pauli proposed, the remedy is the same: while a system that would serve as an ideal indicator is an idealization, it may nevertheless be approached as the limit of a sequence of physically allowed systems. Hilgevoord (2005) explicitly considers and endorses Pauli's strategy:

An ideal quantum clock is an idealization which can be arbitrarily approximated. In a sense, an ideal clock is the analogue for time of a point particle, an ideal indicator of position.”<sup>4</sup> (Hilgevoord, 2005, p. 50)

This suggests an analogy between the relation of the (unphysical) ideal clock to the systems that approximate it and the relation that a boundary point of an open interval of  $\mathbb{R}$  bears to the set of points in the interval: although the boundary point lies outside the set, it is the limit of a sequence of points that are in the set, and as such may be approximated arbitrarily while remaining within the set.<sup>5</sup>

But, as previously discussed, there is an important distinction between the infinitely massive particle and the maximally uncertain energy states of an ideal clock. To rehearse the

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<sup>4</sup>Recall here the discussion of the introduction of this chapter regarding the dangers of this analogy.

<sup>5</sup>My thanks to Mike Tamir for useful conversations on this topic.

arguments offered in Section 2.1.3, the problems with a system with a unbounded Hamiltonian are twofold: first, such a system would appear to provide a source of boundless energy; second, used as an ideal clock, this system will be in a state of complete uncertainty in energy, making interacting with it a risky business. Plus, of course, we know of no such systems in Nature.

In a recent article, Norton (2012) introduces a distinction between idealization and approximation that will be useful for us here. An idealization, he proposes, “is a real or fictitious system, distinct from the target system, some of whose properties provide an inexact description of some aspects of the target system.” An approximation, on the other hand, “is an inexact description of a target system” (Norton, 2012, p. 209). One of the illustrations he provides concerns a body falling through a resistive fluid. According to Newton’s Laws, this introduces a coefficient of friction  $k$  into the acceleration of the body  $\ddot{x} = g - k\dot{x}$ . We obtain a good approximation of the motion of the target system (where the friction coefficient is small) by setting  $\dot{x}(t) = gt$  and discarding the higher order terms in the power series expansion. Introducing a (fictitious) reference system whose behavior is perfectly described by this approximation, we may regard the reference system as an idealization of the target system (the system in the world whose behavior we intend to describe). Hence we have promoted an approximation to an idealization.

Alternatively, we might want to regard the limiting system itself as an idealization of the target system, in which case “we consider the target system to be one of a sequence of systems whose limit system is intended to be the idealizing system. The promotion fails if the sequence fails to have a limit system or has a limit system with unsuitable properties” (Norton, 2012, p. 211). For example, the system might allow for violations of energy conservation. This introduces a distinction between two types of sequences of systems: those that have a limit system which can function as an idealization and those which don’t. Thus, if Schrödinger is right, then the ideal quantum clock is problematic in a way that an infinitely massive particle is not: it cannot serve as an idealization of a target system (i.e. a real clock).

In more detail, if we consider the set of all possible quantum systems, i.e. systems with semi-bounded Hamiltonians, then the infinitely massive particle lies within that set. The Hamiltonian of a sequence of (otherwise identical) massive particles, after all, differs only with respect to the mass parameter (a  $c$ -number). In the limit of infinite mass, the expectation

value of the Hamiltonian becomes infinite and the change in the position of the particle over time goes to zero.<sup>6</sup> While we know of no particles of infinite mass, we do encounter many systems in the world of finite mass that are usefully described as having no uncertainty in position: macroscopic objects.<sup>7</sup>

These target systems lie in a sequence of systems whose limit is a particle of infinite mass. In taking this limit, one is not required to leave the set of physically allowed systems, and thus the limiting system itself may serve as an idealization which explains the corresponding property of the target system. To return to the analogy with an open interval of  $\mathbb{R}$ , we can always include the boundary points into the set by taking the closure of the interval without leaving  $\mathbb{R}$ . On the other hand, while a real clock is a system described by a semi-bounded Hamiltonian, Hilgevoord's ideal clock is not and thus it lies outside the set of physically possible systems. So even if the ideal clock can be reached as the limit of a sequence of systems belonging to the set of physically possible quantum systems, it cannot be included in that set. This is analogous to taking the limit of a sequence of rationals  $\{x_n\} \subset \mathbb{Q}$  that converges to an irrational number: although each element of the sequence lies in  $\mathbb{Q}$ , the limit does not.

Of course, one could object that the fact that the ideal clock lying at the limit is itself unphysical is irrelevant if it is only being used to represent some target system in some (unproblematic) respect, and, furthermore, there may exist a (physically allowed) system in the sequence that approximates the target quite well. This objection is fine, so far as it goes, but what it concedes is that inferences concerning the behavior of the limiting system must be made with great care. In particular, we had better be careful of inferences that rely in the one characteristic that Pauli's Theorem tells us not to expect in real systems: self-adjointness of a clock observable or, equivalently, orthogonality of successive states of a quantum clock.

However, the use of ideal clocks as objects of foundational study has often failed to exhibit such care. Consider the following discussion of ideal quantum clocks (possessing unbounded Hamiltonians) taken from a recent survey article:

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<sup>6</sup>One might object that this limit is also unphysical, since any attempt to bring about a concentration of limitless mass at a point results in the creation of a black hole (Gambini et al., 2007). This should fail to worry us here, however, since I am merely suggesting that this *fictitious* system can be used to describe some aspects of a target system.

<sup>7</sup>This recalls the exercise of calculating the de Broglie wavelength of a particle of macroscopic mass, commonly set in an introductory quantum mechanics class.

It is true that ideal clocks are ‘unphysical’ in a sense, but then so are point particles. Both are consistent with the formalism of quantum mechanics. Together they illustrate the similarity between the quantum mechanical treatment of indicators of position in space and indicators of position in time. . . . In a sense, an ideal clock is better behaved than a point particle, for an eigenstate of such a particle’s position spreads with infinite velocity. By contrast, the eigenstates of time operators do not spread at all, but rather transform into eigenstates belonging to a different eigenvalue[.] (Hilgevoord & Atkinson, 2011)

This exhibits precisely the sort of problematic inferences that one should be on guard against in cases of this sort. As I have argued at length, this ideal clock is disanalogous to an ideal indicator of position in the important respect that the ideal clock does not represent the behavior of a physical system at all, not even a fictitious one. In addition, the inference made here from the particular case (an ideal clock) to the general case (time operators) is deeply problematic since the properties of the ideal clock referred to here are possessed *in virtue* of its lying outside the set of physically possible systems. As consideration of Pauli’s Theorem reveals, a quantum system has a time operator with orthogonal eigenstates only if the Hamiltonian of the system fails to be semi-bounded.<sup>8</sup>

To be fair to Hilgevoord, he has often claimed that these ideal clocks *are* physically allowed systems, just ones with a special property: having states of completely uncertain energy. Hilgevoord also maintains that the unbounded spectrum of  $H$  need not be as problematic as is often thought

[F]or an isolated system, the demand that its Hamiltonian has a lower bound is not at all necessary. Thus, from the point of view of the quantum mechanical formalism, our [ideal] clock is a bona fide physical system. Hilgevoord (2002, p. 304)

That is, although the spectral condition is designed to prevent a system with an unbounded Hamiltonian from providing a source of unlimited energy, there is another way to prevent this from happening: make sure that all such systems are completely isolated, always.

But do we have reason to think that systems which are, as a matter of principle, non-interacting are physically possible? How could such a system be prepared in the lab? Moreover, since the whole point of a clock is to allow one to read off the time, a clock which can *in principle* never be read seems to constitute an uncomfortably Pickwickian solution to

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<sup>8</sup>This also serves to exclude systems with bounded Hamiltonians, which it seems from the context are also under consideration here. More on this later.

the problem posed by the no go results of Section 6.1. Perhaps anticipating this response, Hilgevoord & Atkinson (2011) add that the system could be coupled to another system, so long as that system cannot ‘take up’ an infinite amount of energy, i.e., a system with a Hamiltonian  $H_2$  whose spectrum possesses an upper bound.

The problem with this is that the joint system, with Hamiltonian  $H' = H + H_2 +$  interaction term, still has an unbounded energy spectrum, and so remains capable of supplying an indefinitely large amount of energy to its surroundings. If we adopt the semantic view, and thus regard a theory as a collection of models, then admitting even a single system with an unbounded Hamiltonian into the models allowed by the theory means that the theory now admits a whole class of models with a joint Hamiltonian whose spectrum is unbounded from below. Given that quantum mechanical systems of particles have no upper bound on their Hamiltonian, there is every reason to suppose that such a system coupled to Hilgevoord’s ideal clock would experience large and unpredictable fluctuations in energy.

On the contrary, there is good reason to build into our interpretative principles the idea that only systems with physically reasonable properties are physically possible, and that we should cast out unphysical models from inclusion in the theory. Thus it is controversial, to say the least, to claim that “the formalism of quantum mechanics allows for the existence of ideal clocks, and that the energy of such systems is unbounded” (Hilgevoord & Atkinson, 2011). Functional analysis may allow for the definition of collections of mathematical operators that resemble systems of quantum mechanics, but that does not imply that those collections of operators correspond to a physical system. On Hilgevoord’s view, it seems, Pauli’s Theorem is to be thought of as an informative account of the ideal clock: Pauli’s Theorem tells us that all ideal clocks have unbounded Hamiltonians. But these ideal clocks are not the sort of things that exist outside of Platonic Heaven. There can be no charge of logical inconsistency against those who would hold fast to the existence of such ideal clocks and embrace the consequences, come what may. But we need not join them; we may look elsewhere.

### 6.3 HOW TO BUILD A QUANTUM CLOCK

This section presents an alternative account of how one could go about the definition of a suitable ideal periodic clock observable, which respects the spectrum condition. I will

first review the case against the ideal quantum clock, outline some potential responses, and propose a particular series of physically allowed systems as representing the best hope of fulfilling Pauli's Strategy. This account mirrors Hilgevoord's attempt to follow Pauli's Strategy quite closely, which leads to the claim that each system lying in the sequence approaching Hilgevoord's ideal clock corresponds to a system with the same formal structure in a sequence approaching a clock with an semi-bounded Hamiltonian. This latter sequence provides the best hope of satisfying Pauli's Strategy.

### 6.3.1 Action, Angle as Ideal Quantum Clock

Hilgevoord's suggestion for the ideal periodic clock follows the logic of Pauli's Theorem to a fault: ideal clocks exist, and, therefore, possess unbounded energy spectra. The concrete Hamiltonian he suggests for an ideal (periodic) quantum clock amounts to a trivial redefinition of the angular momentum operator  $L$  considered in Section 5.2. That is, in order to have units of energy Hilgevoord suggests we consider the operator  $H = \omega L$ , where  $L$  is the angular momentum operator and  $\omega$  is a constant of appropriate units. This gives the Hamiltonian of the system a superficial resemblance to the action-angle form of the (classical) harmonic oscillator. But this is not the Hamiltonian of a quantum harmonic oscillator, since the spectrum of  $L$  is unbounded in both directions. This gives the operator  $L$  a quite different role, in which the energy of the system is proportional to the value of the angular momentum (rather than how it couples to the  $B$ -field in the Pauli Equation).

In fact, Hilgevoord (2005) reaches this ideal clock Hamiltonian by considering the direct quantization of a classically conjugate pair of action and angle variables. This is a questionable move on conceptual grounds, as I pointed out in Section 2.4. The action-angle form of Hamiltonian classical mechanics is reached by a canonical transformation from a description of the system in appropriate configuration and momentum variables. It is particularly useful for systems with periodic motions. A canonical transformation preserves relations of conjugacy since it leaves the Poisson bracket invariant. On the other hand, the equivalent transformation in quantum mechanics is a unitary transformation, in which case the Hamiltonian in the quantum 'action-angle form' will be unitarily equivalent to its transformed version, and thus have the same spectrum. Hilgevoord exploits this fact to define a system whose energy is unbounded above and below.

In the case of a quantum harmonic oscillator, this unitary transformation leads to the number-phase representation, in which the number operator retains the (discrete and semi-bounded) spectrum of the original Hamiltonian (see Section 5.3). What Hilgevoord suggests instead, in effect, is a (classical) canonical transformation to the action-angle form followed by promotion of action and angle to a pair of quantum operators. This provides an inequivalent quantization in which the ‘Hamiltonian’ comes to have an unbounded spectrum, and evidently does not have a unitary transformation relating it to the quantum harmonic oscillator. The classical covariance of action and angle is retained in the quantum description, but it is not clear how those variables relate to the original configurational variables of the classical system (i.e. position and momentum). Essentially, the complaint I have here is that the procedure of canonical transformation followed by quantization is not equivalent to quantization followed by unitary transformation.<sup>9</sup>

Regardless of the means by which it is reached, the Hamiltonian  $H = \omega L$  has a conjugate operator  $\tau = \Phi/\omega$  with units of time, as desired. By the relation derived earlier for  $\Phi, L$ , these are form a Heisenberg pair  $[\tau, H] = i$  (on a dense domain excluding the momentum eigenstates). The improper angle ‘eigenstates’ become the improper clock ‘eigenstates’

$$|\tau\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im\tau/\omega} |m\rangle.$$

The unitary time evolution of which is linear (but periodic), in the sense that

$$e^{i\omega L t} |\tau\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im(\tau/\omega + \omega t)} |m\rangle = |\tau + \omega t\rangle.$$

Were one able to prepare the system in such a state (which is debatable since these are improper eigenstates that lie outside of Hilbert space), comparing the states at different times would give a measure of elapsed time. More feasibly, a state highly peaked about  $|\tau\rangle$  could be prepared. The effect of  $e^{i\omega L t}$  on such a state is akin to the effect of a spatial translation of a wavepacket highly peaked in space, i.e., it does not change anything but its location in time.

This evolution is essentially, then, just a change of complex phase—not the behavior we would expect from a typical Hamiltonian operator. However, the quantum harmonic oscillator does behave much like that when the initial state is a coherent state, i.e., a state

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<sup>9</sup>See also the discussion of Section 5.5.

which minimizes the product of uncertainties in position and momentum, and then the corresponding phase is a physical quantity representing the angular location of the peak. As Schrödinger first observed, the Hamiltonian of the quantum harmonic oscillator acts to preserve this special property a coherent state, unlike the free particle Hamiltonian, for example. It is unclear, however, just how one is supposed to reach a Schrödinger representation of Hilgevoord’s ideal clock, in which such state could be defined.<sup>10</sup> Hilgevoord (2005) does note that if “ $m$  is confined to values  $m \geq 0$ , the system becomes [formally] equivalent to a harmonic oscillator,” but he bemoans the fact that “the resulting mutilated [phase] states are no longer orthogonal and there is no ideal phase or time operator in this case.” (p. 59).

In the previous section I introduced my doubts that such an ideal clock could be regarded as an idealization at all, and it is difficult to see just what kind of physical system is being described here: what target system could be usefully described in these terms? The lack of a Schrödinger representation is particularly distressing given that it also makes the classical limit difficult to describe. It is all very well saying that the corresponding classical system is one with action-angle variables, but that is akin to saying that a given curve in three-dimensional space can be described in spherical co-ordinates: no information about the Hamiltonian function is thus conferred. Worse than this, the system is not one with a *description* in terms of action-angle co-ordinates, but rather a system whose Hamiltonian is just the action.<sup>11</sup> Given its manifest unphysicality, I would rather maintain that this so-called ideal quantum clock results more from an unfortunate mutilation of the phase observable of the quantum harmonic oscillator (i.e. its Naimark dilation), rather than vice versa.

### 6.3.2 Coherent States As the Hands of a Quantum Clock

The suggestion that the phase of a quantum harmonic oscillator could serve as a clock observable goes back (at least) to Susskind & Glogower (1964), with Garrison & Wong (1970) providing a concrete suggestion for a physical realization of this system as a quantum clock. The great advantage of considering states of a quantum harmonic oscillator is their experimental accessibility. The collective behavior of such systems is described by quantum optics,

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<sup>10</sup>Note that the angular momentum eigenstates can be given in the position representation, where  $L$  appears as the  $z$ -component of orbital angular momentum in a system with a central potential, i.e., a Hydrogen atom. Defined on the usual infinite dimensional (separable) Hilbert space  $L^2[\mathbb{R}^3]$ , the  $L_i$  are written in terms of the position and momentum operators  $Q_i, P_i$ , where  $i = x, y, z$ . In particular,  $L = L_z := Q_x P_y - P_x Q_y$ . It is not at all clear what Hilgevoord’s suggested Hamiltonian corresponds to in this context.

<sup>11</sup>See the discussion of Section 2.4.



which is essentially the study of low energy quantum electrodynamic phenomena. Since the energies are low, the Galilean form of quantum theory provides a good approximation of these phenomena by means of a Fock representation of the EM field, i.e. in terms of excitations of an infinite collection of quantum harmonic oscillators. This suggests a great deal about the expected behavior in the classical limit, and gives us confidence that the formal manipulations and relations will have physical counterparts accessible within a quantum optics lab.

First let us confirm that the phase states behave as promised. (Note that these are again improper eigenstates that are not vectors of the Hilbert space.) The Hamiltonian is  $H = N + 1/2$ , the eigenstates of which  $H|n\rangle = (n + 1/2)|n\rangle$  correspond to energies  $\hbar\omega(n + 1/2)$ , with  $n \geq 0$  (having earlier removed the factor of  $\hbar\omega$  from the expression of  $H$  for convenience). (See Section 5.3 for a more extensive discussion.) Evidently the spectrum of  $H$  is bounded from below. The unitary evolution of the system is given by  $U_t = e^{iHt}$ , and these phase states behave like the hand of a clock

$$U_t|\phi\rangle = U_t \sum_{n=0}^{\infty} e^{in\phi}|n\rangle = \sum_{n=0}^{\infty} e^{in(\phi+t)}|n\rangle = |\varphi + t\rangle,$$

with periodicity  $(\varphi + t) \bmod 2\pi$ . Although these improper eigenstates have a dubious claim to physicality, the suggestion of Garrison & Wong is to use Glauber states for the hand of our clock. These Gaussian states were given by Glauber (1963) as coherent states of the electromagnetic field, but were first considered by Schrödinger in 1926 as coherent states which minimize the product of the uncertainties in position and momentum,  $\Delta P \Delta Q$ .

For a single harmonic oscillator, the Glauber states  $|\alpha\rangle$  are formed from a superposition of number eigenstates as follows:

$$|\alpha\rangle := e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{\frac{1}{2}}} |n\rangle,$$

with  $\alpha = |\alpha|e^{-i\theta}$ . These are states with a Poisson distribution, characterized by a complex amplitude  $\alpha$  and phase  $\theta$ , which in the limit  $|\alpha| \rightarrow \infty$  approximate classical states of the electromagnetic field with definite amplitude and phase. They have the property of being unchanged by applications of the annihilation operator, up to a phase factor,

$$a^\dagger|\alpha\rangle = \alpha|\alpha\rangle.$$

Note that the label  $\alpha$  refers not to an ‘occupation number,’ but rather is related to the value of the expectation value,  $\langle \alpha | N | \alpha \rangle = |\alpha|^2$ . Setting  $\theta = \omega\tau$  and parameterizing a set of Glauber states by time,  $\Lambda_\tau$ , these have the desired covariance property

$$e^{itH} \Lambda_\tau = \Lambda_{\tau+t},$$

with obvious periodicity.

The complications introduced by the non-uniqueness of the phase observable<sup>12</sup> are not severe. The choice of a particular phase observable  $\Phi_\theta$  to measure just involves an arbitrary choice of phase, akin to choosing a particular position of the hand of a rotary clock to be zero. Therefore, as long as we are able to measure the same phase observable twice, the expectation values of the measurements will serve as a measure of elapsed time, up to the periodicity of the clock. The inaccuracy of the clock can be made as small as one likes, and disappears in the classical limit as the phase states  $\Lambda_\tau$  peaked about  $\tau$  approach a  $\delta$ -function.<sup>13</sup>

Physically, this essentially corresponds to choosing coherent states of a single mode electromagnetic with larger and larger (average) numbers of photons, and so energy  $\langle E_\alpha \rangle \approx |\alpha|^2$ . Thus we reach a limiting situation that closely resembles the infinitely massive particle, an ideal indicator of position. Moreover, as Garrison & Wong (1970) point out, the periodic restriction is (in principle at least) no restriction at all since two clocks with periods whose ratio is irrational would provide a perfect linear clock. In practice, even clocks with (known) rational ratios of period could provide a greatly expanded range and accuracy, so long as they can be measured at the same instant.

### 6.3.3 Approximating Phase and Angle

The use of finite-dimensional systems to approximate the behavior of an ideal quantum clock goes back to Salecker & Wigner (1958). Hilgevoord regards the Salecker-Wigner discrete periodic clock as a approximation of his proposal for an ideal periodic clock, as discussed in Section 6.3.1. These discrete clock operators can be seen as resulting from the restriction of the (improper) angle eigenstates  $|\phi\rangle$  to a finite sum over angular momentum eigenstates  $|m\rangle$

$$|\varphi_k\rangle := \frac{1}{\sqrt{2l+1}} \sum_{k=-l}^l e^{im\phi_k} |m\rangle,$$

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<sup>12</sup>See (5.2) of Section 5.3.

<sup>13</sup>See Lahti & Maczynski (1998) for a recent analysis of the uncertainty relations for such states.

where  $\phi_k = k\frac{2\pi}{2l+1}$ ,  $k = -l, \dots, l$ . These are the eigenstates of an operator  $\Phi$  self-adjoint on the  $2l + 1$ -dimensional Hilbert space spanned by the restricted angle states  $|\varphi_k\rangle$ ,

$$\Phi_k := \sum_{k=-l}^l \varphi_k |\varphi_k\rangle \langle \varphi_k| = \sum_{-l \leq m, n \leq l} \frac{1}{2l+1} \sum_{k=-l}^l \phi_k e^{i(m-n)\phi_k} |m\rangle \langle n|.$$

It is easily seen that the periodicity of angle will again prove problematic for the covariance of  $\Phi_k$  under shifts in angle, that is,  $e^{iL\varphi} \Phi_k e^{-iL\varphi} \neq \Phi_k + \mathbb{I}\varphi$ . Let us consider instead the covariance of the associated PVM  $E_l([a, b])$ , where  $0 \leq a \leq b \leq 2\pi$ ,

$$E_l([a, b]) = \sum_{\varphi_k \in [a, b]} |\varphi_k\rangle \langle \varphi_k|.$$

The effect of a shift in angle can be calculated as follows:

$$\begin{aligned} e^{iL\varphi} E_l([a, b]) e^{-iL\varphi} &= \sum_{-l \leq m, n \leq l} \frac{1}{2l+1} \sum_{\varphi_k \in [a, b]} e^{i(m-n)(\varphi_k + \varphi)} |m\rangle \langle n| \\ &= \sum_{-l \leq m, n \leq l} \frac{1}{2l+1} \sum_{(\theta_k - \varphi) \in [a, b]} e^{i(m-n)\theta_k} |m\rangle \langle n| \\ &= \sum_{\theta_k \in [a + \varphi, b + \varphi]} |\theta_k\rangle \langle \theta_k| = E([a + \varphi, b + \varphi]). \end{aligned}$$

The last equality holds so long as  $\phi \in \{\varphi_k\}$ , otherwise the effect of the transformation will be to define a distinct PVM which lies in an orthogonal  $2l + 1$ -dimensional subspace of  $\mathcal{H}$ .

The family of PVMs  $E_l$  can be considered as a series of approximations to the POVM  $P(Z)$ , which converges to  $P(Z)$  as  $l \rightarrow \infty$ , or equivalently we may say that  $\Phi_k \rightarrow \Phi$  as  $l \rightarrow \infty$ . This fact is used by Hilgevoord to justify his contention that his ideal periodic clock may be approximated by a discrete periodic clock with a bounded Hamiltonian spectrum. Now, if he regards this as serving to assuage worries about the unphysicality of his unbounded Hamiltonian, the following should serve to assuage his worries about the ‘mutilation’ of the non-orthogonal phase states.

The Pegg-Barnett phase operator is defined on an  $s$ -dimensional subspace  $\mathcal{H}_s \subset \mathcal{H}'$  spanned by the  $s \in \mathbb{N}$  number states  $|0\rangle, |1\rangle, \dots, |s\rangle$ . The Pegg-Barnett phase states  $|\theta_{s,k}\rangle$  are defined for  $\theta_{s,k} = k\frac{2\pi}{s+1}$ ,  $k = 0, 1, \dots, s$  by

$$|\phi_{PB}\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s e^{in\theta_{k,s}} |n\rangle.$$

Note the close similarity to the definition of the  $|\varphi_k\rangle$ . Since  $\mathcal{H}_s$  is finite-dimensional, the following defines a spectral measure

$$\begin{aligned} M_s([a, b]) &= \sum_{\theta_{k,s} \in [a, b]} |\theta_{k,s}\rangle \langle \theta_{k,s}| \quad (0 \leq a \leq b \leq 2\pi) \\ &= \sum_{m, n} \frac{1}{s+1} \sum_{\theta_{k,s} \in [a, b]} e^{i(n-m)\theta_{k,s}} |n\rangle \langle m|. \end{aligned}$$

The Pegg-Barnett phase operator (for dimension  $s$ ) is self-adjoint and is defined

$$\Phi_{PB,s} := \sum_{\theta_{k,s} \in [a, b]} \theta_k |\theta_{k,s}\rangle \langle \theta_{k,s}|.$$

As  $s \rightarrow \infty$ , we recover  $E^\Phi$ , the phase POVM of Section 5.3, as the weak limit of  $M_s$ . Thus we have that  $\Phi_{PB,s} \rightarrow \Phi$  as  $s \rightarrow \infty$ .<sup>14</sup>

Thus the construction of the Pegg-Barnett phase operators takes place in a directly analogous manner to the finite-dimensional Salecker-Wigner clocks that Hilgevoord regards as finite-dimensional approximations to his ideal clock. That means that there is a one-to-one correspondence between  $2l + 1$ -dimensional Salecker-Wigner clock operators and  $2l + 1$ -dimensional Pegg-Barnett phase operators. Since the interest of these clock operators lies just in their abstract characterization as covarying with time (i.e. forming a Heisenberg pair with the restricted Hamiltonian) there is no reason not to regard these structurally identical Pegg-Barnett phase observables (displaying the same time covariant behavior) as the proper objects of foundational study. But these finite-dimensional clocks converge to an clock observable of a infinite-dimensional system with a semi-bounded Hamiltonian, i.e., the phase POVM of the harmonic oscillator.

In other words, Hilgevoord views the Salecker-Wigner discrete clocks as converging to the Naimark dilation of the phase observable of the harmonic oscillator, rather than the phase observable itself. Why would one chose the sequence of otherwise identical approximations that converges to an observable of a manifestly unphysical system rather than a physically meaningful one? No matter; there is no need to make that choice. We may equally regard the the discrete approximations to an ideal clock as approaching a system in the limit that can serve as a valid idealization: the harmonic oscillator.

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<sup>14</sup>Note that this provides another way to pick out  $E^\Phi$  as the canonical phase POVM: Lahti & Pellonpää (2001) characterize  $E^\Phi$  as the only covariant measure that has a projection valued discretization in this sense. See this paper for rigorous derivations of the limits and relationships given here.

Moreover, note that in this sense the approximating sequence is not to be thought of as a series of physical systems, but as a series of approximate measurements (of finite-dimensional observables) that approach an ideal measurement of the phase observable of a single physical system, an observable defined on the infinite-dimensional state space of that system.<sup>15</sup> The approach to an ideal clock, therefore, involves an additional limit: the behavior of a quantum harmonic oscillator approaches that of an ideal clock in the limit of infinite energy of the system.<sup>16</sup>

### 6.3.4 Relation to Atomic Clocks

The reader may perhaps be wondering why, in advocating a particular schema for defining a quantum clock on the basis of its physical reasonableness, I have failed to describe the quantum clocks that are in common use as timekeeping devices: the atomic clocks based on Caesium, Rubidium, and the like. The essential idea behind the current generation of quantum clocks is to use an atomic transition of known energy (and thus frequency) to calibrate a single mode laser such that its frequency is stable at this known atomic frequency. The physics is simple: take a positive ion (or a collection) of (e.g.)  $\text{Ca}^+$  and tune the laser frequency to that which the electron in the outer shell requires to transition to a state of higher energy.

In order to ensure that the ions are in a known state, they first are filtered through a Stern-Gerlach apparatus, sensitive to the spin of the outer electron. The ions which undergo the transition are then put through another magnetic filter, and the strength of the resulting ionic current provides an indication of the closeness of the frequency of the laser to that of the atomic transition. Any variation in frequency will result in a decrease in the probability of the transition and so this maximum, once located, is easily maintained by tuning the laser accordingly. Since this transition energy is known to a high degree of accuracy, it provides a standard against which the frequency of the laser may be known precisely. The laser frequency, therefore, provides a clock in the sense of a frequency source, much like a quartz

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<sup>15</sup>In fact, every POVM of an infinite-dimensional Hilbert space can be approximated by a sequence of PVMs restricted to a finite-dimensional subspace in this manner. This is the content of Naimark's Approximation Theorem, a less well-known cousin of his Dilation Theorem. See Olkhovsky (2011). One can also regard the confined time of arrival operators of Galapon et al. (2004) (with discrete spectra) as approximations to the continuous time of arrival POVM in the same way. On this see Galapon et al. (2005).

<sup>16</sup>There is some indication that uncertainty in event time POVMs also depends on the expectation value of energy in a similar way. See Brunetti & Fredenhagen (2002a).

crystal inside a digital watch (which acts to pick out a preferred frequency through mechanical vibration, corresponding to a electronic resonance through the piezoelectric effect). The time is kept by a digital counter which counts the number of elapsed periods.

There are two things to say here. The first is that the atomic clocks of this kind seek to provide a *frequency* standard, whereas the clocks I have described so far are based upon the idea that an ideal quantum clock is one which possesses an observable which provides from its measured values an estimation of elapsed time. As such, the atomic clock does not itself tell the time but rather provides a known frequency so that the elapsed time follows from counting the number of periods of the source frequency. This extra apparatus is not the subject of quantum mechanics as conventionally understood, so the *entire* system does not, in this sense, provide a truly quantum clock.

The second thing to say is that the treatment of the harmonic oscillator is somewhat relevant to the theoretical basis on which the atomic clock operates. Its operation depends on the use of a single mode laser which naturally is described by the theory of quantum optics, whose foundation lies in the description of the quantum harmonic oscillator and the coherent states described above (which properly reside in Fock space). In fact, the existence of (a family of) phase observables indicates that there is another paradigm of atomic clock design—an atomic stopwatch, if you will.

The atom clocks I have described here can use laser cooled ions or even a single Aluminum ion as their frequency source. As clocks, they operate by the principle of counting periods of the tuned laser frequency, but the results of this chapter indicate that it should be possible to tune the frequency of a coherent optical state as described here, and then measure the phase of that state with an accuracy that depends on its energy. This provides an indication of the amount of time that has elapsed *within* a period. Thus we can see how the theoretical limits we have explored here are straightforwardly related to a potential physical realization.<sup>17</sup>

## 6.4 ON THE ALLEGED PREVALENCE OF QUANTUM CLOCKS

The quantum clocks of this chapter (call them quantum stopwatches if you will) have relied on the characteristic behavior of the system state over time when prepared in a basis that is

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<sup>17</sup>See also the discussion of (Muga et al., 2002, §§8–9), (Muga et al., 2010, §13).

stable over time. The preparation of the system in a known such state and subsequent measurement of a suitable observable provides an indication of the elapsed time. In terms of the dynamics of the theory, this is readily represented by the time evolution of the Schrödinger picture instantaneous state, followed by the instantaneous measurement of a Schrödinger picture clock observable. The covariance requirement is met due to the characteristic variation of the expectation value of such a Schrödinger picture observable with time.

This is to be contrasted with the behavior of an event time observable which, evaluated in the Heisenberg picture, returns a temporal distribution for the time of occurrence of some event. The characteristic covariance of this distribution results from the behavior of a Heisenberg state under time shifts, and has the opposite sense. Thus the expectation value of an event time observable does not change with time: it is a property of the temporal probability distribution derived from the Heisenberg state, which does not vary with time. Consider the time of arrival of a classical free particle. The time at which a particle arrives at the origin does not change as the particle moves along its path. The event time changes if the initial state is varied, but the initial state does not vary with time: it corresponds to an entire evolution of the system.

In this way, the conceptual difference between a clock observable and an event time observable is obscured by the observation that they covary with time in opposite directions. The event time distribution is calculated from the Heisenberg state, and, therefore, has no Schrödinger picture equivalent (which would have to correspond to an instantaneous measurement). There can be no instantaneous measurement of a temporal distribution, whatever that might mean. Moreover, it is easy to see that a Schrödinger picture operator corresponds to a time-indexed Heisenberg picture operator, i.e. something to be measured at a particular time. That is, measuring an arbitrary Schrödinger picture operator  $A$  in state  $\psi_t$  is the same thing as measuring a Heisenberg picture operator  $A(t)$ . But, although an event time observable does not vary in time, it certainly can be defined as a Heisenberg picture operator. (See Section 7.1.) These non-instantaneous Heisenberg operators simply do not correspond to a Schrödinger picture operator, which can be measured at an instant.

There is, therefore, no way to regard these event time operators as providing clock times, despite their superficial resemblance to genuine clock observables. I bring up this point to challenge the recent claim of Hegerfeldt & Muga (2010) to have provided the means to generate both event time observables and clock observables for arbitrary quantum systems.

In the next chapter, I will explore the sense in which the former claim is valid. Here, I expose the sense in which the latter is bogus.

The time operators in question are defined by means of a generalized spectral family of self-adjoint operators  $O \geq F_\tau \geq \mathbb{I}$  with the usual properties (see Chapter 3) so that they define a symmetric operator

$$T = \int_{\mathbb{R}} \tau dF_\tau.$$

The covariance of this operator is equivalent to the covariance of the operators  $F_\tau$ ,  $U_t F_\tau U_t = F_{\tau+t}$ , which are elements of the corresponding POVM, i.e.  $F_\tau = F(-\infty, \tau]$ . These operators are defined such that  $\langle \psi | F_{-\infty} \psi \rangle = 0$  and  $\langle \psi | F_\infty \psi \rangle = 1$ , where  $\psi \in \mathcal{H}$  is a vector of unit norm. This is achieved by requiring that

$$\int_{-\infty}^{\infty} d\tau \frac{d}{d\tau} \langle \psi | F_\tau \psi \rangle = 1,$$

for all  $\psi$ . But, by the covariance relation above, this is just to say that

$$\int_{-\infty}^{\infty} d\tau \frac{d}{d\tau} \langle U_t \psi | F_0 \tau U_t \psi \rangle = 1.$$

For an arbitrary Schrödinger picture observable  $A \geq O$  (a positive operator)  $\langle U_t \psi | A U_t \psi \rangle = f(t)$  is a monotonically increasing function of  $t$ . The requirement above establishes that  $f'(\tau) = \langle U_\tau \psi | A U_\tau \psi \rangle$  is bounded from above by 1. As a POVM,  $F(\Delta)$  assigns probabilities to time intervals  $\Delta$  through the usual means we have  $\text{Pr}(\Delta) = \langle \psi | F(\Delta) \psi \rangle$ . But the covariance property determines that  $\langle U_t \psi | F(\Delta) U_t \psi \rangle = \langle \psi | F(\Delta + t) \psi \rangle = \text{Pr}(\Delta + t)$ .

This is not a covarying probability that determines the time read by the moving hands of a clock, but a probability that assigns a number directly to a time interval. The resulting time operator is nothing but the expectation operator of that probability distribution, whose expectation value varies with the Heisenberg state. Its expectation value is given by the (generalized) spectral calculus as follows

$$\langle \psi | T \psi \rangle = \int_{-\infty}^{\infty} \tau d\langle \psi | F_\tau \psi \rangle = \int_{-\infty}^{\infty} \tau d\langle U_\tau \psi | F_0 U_\tau \psi \rangle.$$

The appearance of an entire history of Schrödinger picture states on the right hand side means that this operator cannot possibly be measured at an instant. A quantum clock observable provides a probability distribution for the value of a physical quantity (e.g. phase) when it is measured at a time, and the dynamics of the system determines how that quantity changes with time. When the probability distribution concerns *times* rather than values of a variable,



that cannot be a distribution that concerns a clock observable, which measures a physical quantity that covaries with the dynamical evolution of the system according to the external time parameter.

## 7.0 EVENT TIME OBSERVABLES

The topic of event time observables was introduced in Chapter 2 as a way to respond to Hilgevoord’s (2005) contention that the argument for the existence of time operators based on the analogy between position in space and position in time was based on a confusion. In Section 2.3.2, I countered that there is a valid analogy between location and space and time when events are concerned: a co-ordinate in space-time refers to the location in space and in time of an event. This idea arises naturally when explaining the physical interpretation of Minkowski space-time, for instance. Here is a recent account of space-time given in these terms (intended for a general audience):

The fundamental notion is that of an *event*, which we think of as a physical occurrence having negligibly small extension in both space and time. That is, an event is “small and quick,” such as the explosion of a firecracker or the snapping of your fingers. Now consider the collection of all possible events in the universe—all events that have ever happened, all that are happening now, and all that will ever happen; here and elsewhere. This collection is called space-time. It is the arena in which physics takes place in relativity. (Geroch, 2005, p. 201, original emphasis)

The guiding idea of an event time observable is that the position observable in quantum mechanics, when applied to a concrete measurement situation like a diffraction experiment, often predicts the spatial distribution of some set of events, say, the appearance of a dot on a luminescent screen. These events also have a *temporal* distribution, in terms of the time after emission at which they occur (i.e. the time of flight). An event time observable, then, predicts the distribution in time of these events, just as a position observable predicts their distribution in space.

## 7.1 CHARACTERIZING EVENT TIME OBSERVABLES

In modern quantum mechanics, to derive such a distribution from the system state we are required to define a suitable operator, customarily (in Ordinary QM) a self-adjoint operator. Observables are often defined by their properties under symmetry transformations, for example Jauch (1968, pp. 197–199) regards the covariance of the position operator with spatial translations (generated by the total momentum) as expressing the homogeneity of space. Essentially, Wightman (1962) shows that requiring this covariance is sufficient to uniquely define a self-adjoint position operator. One way of defining event time observables, then, is to do so through their characteristic covariance under time shifts of the initial state.

The first thing to note is that in the usual identification of observables with self-adjoint operators it is assumed that these operators act instantaneously. In the Schrödinger picture (in which the states vary with time) the expectation value of an observable  $A$  in the (pure) state  $\psi_t = U_t\psi$  is  $\langle A \rangle_t = \langle \psi_t | A \psi_t \rangle$ , whereas in the Heisenberg picture (in which the observables vary with time) the expectation value of an observable  $A(t)$  is given by  $\langle A(t) \rangle = \langle \psi | U_t^\dagger A U_t | \psi \rangle$ .<sup>1</sup> In the Schrödinger picture it makes very little sense to ask *when* a particular event occurs (in the sense of a probability for it occurring during some interval of time) since we may only interrogate the state at a instant of time. In contrast,

“event time measurements are extended in time, with sensitive detectors waiting to be triggered. The experimenter has no control over the time instant at which the detectors fire. This very instant constitutes the outcome of such a measurement. (Busch, 2007, p. 19)

However, in the Heisenberg picture we may define operators that involve more than one instant of time.

An event time observable is to be defined, therefore, by its behavior under temporal translations, generated by the Hamiltonian. In particular, we will require that such observables *covary* with time translations.<sup>2</sup> In effect, this means that the experimental results (depending on relative times) will be independent of the time at which the experiment begins, and so this expresses the homogeneity of time. This covariance depends on the time-independent (initial) state of the system, and hence is expressed in the Heisenberg picture. (See Section 2.3.2).

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<sup>1</sup>Evidently these return the same expectation value,  $\langle A \rangle_t = \langle A(t) \rangle$ , but they are not identical expressions.

<sup>2</sup>This is to be regarded as a necessary, not a sufficient condition.

If  $\psi$  is the Heisenberg picture state, then an event time operator  $T_e$  returns a distribution of times through its associated PVM  $E^{T_e}$  as follows,

$$W(I) = \langle \psi | E^{T_e}(I) \psi \rangle,$$

where  $I \in \mathcal{B}(\mathbb{R})$  is a Borel subset of times, and  $\psi_0$  is the Heisenberg state that equals the Schrödinger picture state  $\psi_0 = \psi$  at time  $t = 0$ , corresponding to the family  $\psi_t = U_t \psi$ . If we consider another Heisenberg state  $\psi_\tau$ , related to the first by a unitary transformation  $\psi_\tau = U_\tau \psi$  (a time shift), then the distribution of times in this state is given in the same way

$$W_\tau(I) = \langle \psi_\tau | E^{T_e}(I) \psi_\tau \rangle = \langle U_\tau \psi | E^{T_e}(I) U_\tau \psi \rangle.$$

The Heisenberg state  $\psi_\tau$  corresponds to the Schrödinger picture family  $\psi_{t'} = U_{t'} \psi_\tau = U_{t'+\tau} \psi$ , so that what is now the state at time  $t' = 0$  was the state at time  $t = \tau$ . Thus it is evident that the relationship between the distributions will be given by  $W_\tau(I) = W(I - \tau)$ , since the Heisenberg state  $\psi_\tau$  is just the state  $\psi$  prepared  $\tau$  seconds earlier, and thus any event that occurs at time  $t$  in state  $\psi$  will occur  $\tau$  seconds earlier in state  $\psi_\tau$ . Setting the two expressions to be equal we obtain

$$\langle U_\tau \psi | E^{T_e}(I) U_\tau \psi \rangle = \langle \psi | E^{T_e}(I - \tau) \psi \rangle.$$

Assuming that this equality holds for all  $\psi \in \mathcal{H}$  and all  $\tau \in \mathbb{R}$ , we have the operator equality

$$U_{-\tau} E^{T_e}(I) U_\tau = E^{T_e}(I - \tau). \quad (7.1)$$

This is the proper expression of time covariance for an event time POVM. The sense is opposite to the covariance of a clock observable, but also the interpretation is importantly distinct: an event time POVM returns a probability distribution for times (the time of an event) from a Heisenberg state. That distribution does not change with time. In contrast, a clock observable concerns the distribution of values of some physical quantity at a time, and is defined by the fact that this distribution covaries with the time parameter—i.e. with the dynamical evolution of the system state.

To take a simple example of an event time, consider an experiment consisting of a single radioactive atom and a Geiger counter that fully surrounds it. If the half-life of the atom is 1 hour, then the probability that the counter clicks in the first hour is  $1/2$ , the probability it clicks in the second hour  $1/4$ , the probability it clicks in the third hour is  $1/8$ , and so on. Thus

the probability that the counter clicks at some point in the future is given by an geometric series that tends to one as  $t$  tends to infinity. However, once the counter has been observed to click, the probability that it clicks in the future is essentially zero. This is evidently a experimentally meaningful situation, and we should expect that this phenomenological law may be derived from a detailed quantum mechanical description of the decay process. However, Pauli's Theorem shows that there is no way to implement this simple scheme in Ordinary QM.

This description would involve a (Heisenberg picture) quantum mechanical vector state  $\psi$  in a Hilbert space  $\mathcal{H}$ , a Hamiltonian  $H$  describing the time evolution of the system, and a series of projection operators  $T_i$  representing the proposition "the system decays during hour  $i$ " such that  $\langle \psi | T_1 \psi \rangle = 1/2$ ,  $\langle \psi | T_2 \psi \rangle = 1/4$ , and  $\langle \psi | T_3 \psi \rangle = 1/8$ , and so on. Requiring that the distribution of results respect the time translation symmetry implemented by the unitary group  $U_t = e^{-iHt}$ , we have that  $T_2 = U_t^\dagger T_1 U_t$ ,  $T_3 = U_t^\dagger T_2 U_t$ , and so on (where  $t$  is one hour). Even this bare bones sketch is enough to tell us something interesting about the operators  $T_i$ : if  $H$  is a self-adjoint operator with spectrum bounded from below then it follows that  $\langle \psi | T_{i+1} T_i | \psi \rangle \neq 0$  and so these operators  $T_i$  cannot be projections onto mutually orthogonal subspaces of  $\mathcal{H}$ .<sup>3</sup>

Thus there is no mixed state decomposition in terms of distinct eigenstates  $\psi_i$  of  $T_i$  (for which  $T_i \psi_i = \psi_i$ ) such that the  $\psi_i$  would correspond to the system decaying during distinct intervals of time, and neither can the  $T_i$  together serve to define a self-adjoint 'time of decay' operator. The former implication indicates that von Neumann's famous projection postulate cannot be applied to this situation; the latter than the identification of observables of the theory with self-adjoint operators is ill-suited to include the time of an event as an observable quantity.

Yet there seems every reason to suppose that the theory should be able to answer questions like, "When will the Geiger counter click?" or in a diffraction experiment, "When will a dot appear on the screen?" In failing to answer these questions, the theory would be fail to be empirically adequate; this failure would constitute a real 'problem of time' for the theory. But this problem can be overcome, and without modifying the dynamics: at first blush, the problem is not with the way that quantum mechanics defines the state of the system, but the way that probabilities are derived from the state.

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<sup>3</sup>This is closely related to Prop. 6.1. The proof is easily adapted to the current case if it is assumed that there is some  $k > 0$  such that for all  $i > k$  the probability of decay is zero.

## 7.2 DEFINING EVENT TIME OBSERVABLES

The history of attempts to define event time observables in quantum mechanics begins with Aharonov & Bohm's (1961) critique of the energy-time uncertainty relation, understood as a limitation on the possibility of measuring the energy of a system precisely in an arbitrarily short time. In essence, Aharonov & Bohm's objection amounts to the observation that the Hamiltonian is a self-adjoint operator like any other, and so can (according to the usual Dirac-von Neumann formalism) can be measured effectively at an instant. To combat what they see as a prevalence of somewhat heuristic arguments based on particular experimental setups adduced in support of the erroneous interpretation, they provide a detailed analysis of an experiment which serves as a counterexample, i.e. one in which the energy of the system under observation can be determined precisely in arbitrary short time. The 'apparatus' in this experiment is modeled by a free particle in one dimension, considered as providing a physical clock to measure the time at which the measurement interaction occurs.

However, the operator they suggest is closely related to the time of arrival of a classical free particle, and so it is plausible that the operator they defined is better understood as representing the time of an event rather than a clock. (See Section 2.3.4). In Chapter 5 the quantum time of arrival provided a case study to illustrate the mutually antagonistic relationship between the requirement of Ordinary QM that observables must correspond to self-adjoint operators and the idea that time covariance is constitutive of an event time observable, as expressed by Pauli's Theorem. That is, to the extent that event time observables are operators that covary with time shifts, Pauli's Theorem shows that event time operators are not observables of Ordinary QM. As I have suggested, the way out of this impasse is to allow for the experimentally observable quantity (the time of an event such as a detector firing) to be represented by a time covariant POVM.

The first steps towards this understanding of the quantum time of arrival as a time covariant POVM were taken by Allcock (1969) and Kijowski (1974).<sup>4</sup> Kijowski took an axiomatic approach to defining the quantum distribution of arrival times. The axioms he chose were based on axioms that were sufficient to derive the classical distribution. One of these axioms ensured its characteristic covariance under time translations. Corresponding to the corresponding quantum distribution he found our time of arrival operator  $T_a$ . Since

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<sup>4</sup>For details of these and other interesting episodes in the development of the quantum time of arrival see Muga & Leavens (2000).

every maximally symmetric operator defines a unique POVM (Akheizer & Glazman, 1993, Theorem 2, p. 135), this distribution can also be obtained from the POVM  $I \mapsto E^{T_a}(I)$  so defined.

Allcock, on the other hand, provided the first analysis of the difficulties of the time of arrival in terms of the positivity of  $H$ . By means of an analytic continuation argument he was able to show that the condition  $H \geq 0$  entails the non-orthogonality of the (generalized) eigenstates of  $T_a$ , characteristic of its failure to be self-adjoint.<sup>5</sup> He found that he could avoid this problem by ‘doubling’ the domain of definition of  $T_a$  to include negative eigenstates of  $H$  (i.e. negative energies), arriving at a new operator  $T_a^\pm$  that was self-adjoint, but defined on a ‘larger’ Hilbert space  $\mathcal{H}' \supset \mathcal{H}$ . This operator is canonically conjugate to the doubled Hamiltonian  $H^\pm$  and, so  $I \mapsto P^{T_a^\pm}(I)$  is time shift covariant PVM on  $\mathcal{H}'$ .

This provides another way to avoid Pauli’s theorem, since the spectrum of  $H^\pm$  is not bounded from below. However, the self-adjoint operator  $T_a^\pm$  he defined (and its corresponding PVM) is of negligible physical relevance since the Hamiltonian of a free particle evidently *does* have a spectrum bounded from below. Nonetheless, as is well-known (although apparently not by Allcock), by projecting the operators  $P^{T_a^\pm}(I)$  to the original Hilbert space via an orthogonal projection  $P_{\mathcal{H}}$  one obtains positive operators  $E^{T_a}(I) = P_{\mathcal{H}}P^{T_a^\pm}(I)$ , i.e. the positive operators that form the time of arrival POVM  $I \mapsto E^{T_a}(I)$  defined by  $T_a$ . This result is an instance of Naimark’s famous Dilation Theorem, which establishes that a POVM can always be extended to a PVM in a ‘larger’ Hilbert space in this way, to which it is related by a projection.<sup>6</sup>

Allcock and Kijowski can perhaps be forgiven for failing to make these connections, writing as they were at a time before the development of the POVM formalism by Holevo (2011), Kraus (1983), Ludwig (1983) and others (in response to various problems in quantum measurement theory). It was left to Werner (1986) to make the connection of the time of arrival to time covariant POVMs, which he did in some generality to arrive at a treatment of what he termed *screen observables*.<sup>7</sup> Making use of results from operator algebra and Mackey’s theory of imprimitivity, Werner obtained a general recipe for constructing POVMs covariant under a unitary group by means of their Naimark dilation. Screen observables are

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<sup>5</sup>See, again, Akheizer & Glazman (1993, Theorem 2, p. 135). The argument that Allcock gave is directly related to the ‘refined’ Pauli’s Theorem proof of Chapter 4, and can be seen as the immediate ancestor of that result.

<sup>6</sup>This summarizes the discussion of the time of arrival in Chapter 5.

<sup>7</sup>The results relied on by Werner are closely related to those given by Holevo (2011).

defined as POVMs covariant under the translation group of an arbitrarily chosen hyperplane. As such, they include the usual position observable (corresponding to an instantaneous hyperplane) and the time of arrival POVM just discussed (which, in three spatial dimensions, corresponds to arrival at a timelike hyperplane). Werner showed that the resulting POVM could be uniquely defined (subject to some other restrictions) for both non-relativistic and relativistic symmetry groups. I now proceed to summarize and motivate these developments, beginning from the distribution considered by Kijowski (1974), which was discussed earlier in Section 5.4.

### 7.2.1 Symmetries and Screen Observables

In considering the axiomatic definition of a suitable time of arrival distribution, the key input from classical mechanics was to provide the symmetries of the desired probability distribution. When Kijowski was writing, in 1974, the method of defining quantum observables through their behavior under a symmetry group was already well established. In particular, Wightman (1962) was able to uniquely define the position observable in Galilean and Lorentz invariant quantum mechanics by requiring that the probability distribution behave appropriately under Euclidean transformations. These symmetry requirements acted to constrain the form of the PVM since this gave a series of required covariance behaviors under transformations of the PVM (according to the corresponding unitary groups). Having fixed the form of the PVM, the self-adjoint position observable was uniquely given through the Spectral Theorem.

In the case of a PVM (such as position) the covariance of a probability distribution under a one-parameter family of unitary symmetries implies the existence of a self-adjoint generator of that family (through Stone's Theorem). More generally, Mackey's Imprimitivity Theorem showed that an irreducible unitary representation of a symmetry group gives rise to a System of Imprimitivity, in which the property of mutual covariance implies the existence of a pair of self-adjoint operators that form a Schödinger representation.<sup>8</sup> This is, in effect, a generalized version of the Stone-von Neumann Theorem.

The characteristic behavior of a probability distribution for the time of an event is covariance under shifts in time, but Pauli's Theorem entails there can be no PVM corresponding to

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<sup>8</sup>To state this important and powerful theorem, and give the required group theoretic definitions, would require rather more build up than befits the limited use that I will put it to. See (Dickson, 2006, §3.3.2.1) for a recent discussion by a philosopher of physics, and references therein.



such a distribution. As we have seen, this difficulty can be obviated through consideration of POVMs, and Kijowski’s seminal work in fixing the form of the probability distribution also served to fix a corresponding POVM, which led back to the Aharonov-Bohm time operator. The question naturally arose: could these symmetry requirements be used more generally to determine the form of an arbitrary time covariant POVM? This question was answered in the affirmative through the use of Naimark’s Theorem in the construction of *generalized* systems of imprimitivity by Holevo (2011), Cattaneo (1979), and others.

The idea behind this construction is quite simple. In the case of a time covariant POVM, the first moment of the POVM defines a maximal symmetric operator that is not self-adjoint, and thus cannot be the generator of a one-parameter family of unitary symmetries. As Pauli’s Theorem establishes, such a family can only exist if the spectrum of the Hamiltonian is the entire real line. However, when one considers the Naimark dilation of the POVM, a time covariant PVM, the covariance of this PVM under time shifts *is* equivalent to the existence of a one-parameter unitary group of shifts in energy, generated by the self-adjoint operator defined by the dilation. In projecting this family of unitary operators back to the original Hilbert space they become a semi-group of isometries, which correspond to the positive shifts in energy.

The idea of a generalized system of imprimitivity, then, is to form a genuine system of imprimitivity in the Naimark extension which retains its covariance properties when projected back to the physical Hilbert space. Using this means, Werner (1986) was able to generalize Wightman’s construction of PVMs covariant under spatial transformations (corresponding to a position observable) to POVMs covariant under (spatio-)temporal transformations. Thinking of the situation relativistically, Wightman’s construction of a position observable from a PVM is limited to the symmetry group of a space-like hyperplane, whereas Werner is free to consider “position” in a space-time hyperplane. Whereas the conventional position observable corresponds to a detection experiment performed everywhere in space at an instant, detection in a space-time hyperplane corresponds to an idealized screen detector, sensitive for all time in a particular spatial plane.

### 7.2.2 On Detection and Idealization

Arguably, the idealization involved in modeling any real detector as instantaneously sensitive but (maximally) spatially extended is considerably more severe than the idealization involved

in regarding a screen detector as having zero (spatial) width but unbounded spatio-temporal extent. Picture a typical diffraction experiment which involves a source emitting a beam of particles, a diffraction grating through which the beam passes, and a luminescent screen. The source of quanta (electrons or photons, say) emits a single quantum particle at a time, at a frequency such that only a single particle is ever in the apparatus. Some time after a particle is emitted, a dot appears on the screen, and, repeating the experiment many times, the relative intensity of these discrete events comes to form a characteristic spatial interference pattern.

The outcome of such an experiment is a detection event lying in the plane of the detector, occurring at some time after emission. The screen is sensitive over the *entire* course of the experiment, and an individual experiment ends only when a particle is detected. This means, in effect, that the probability of detection (at some time) is one, and so the probabilities given by the theory can be thought of as applying under the condition that the particle is, with certainty, detected at the screen (at some time). There is little harm, then, in extending the description of the screen such that the plane is unbounded, and the detector is sensitive over all time.

The idealization required to reach the normalization condition of the instantaneous position observable (measured at an instant  $t$ ) is much more severe, since it presumes that the particle is detected *somewhere* at  $t$ . Since in a typical experiment a particle is detected at most a handful of times, the attribution of probability one to the occurrence of an event to some particular time is problematic. The best interpretation of an instantaneous measurement of spatial position, it seems, is a detector spread throughout space, certain to fire when it switched on at an instant  $t$ . It is hard to imagine a detector we possess that has such characteristics. Far more usual is the experimental setup described above, in which a detector sensitive to some limited spatial region fires once in the course of an experiment. Consider here the advanced detectors in operation in particle physics experiments at CERN. These detectors contain banks of locally sensitive detector ‘elements,’ whose results are combined to give particle trajectories of the collision products.

Werner’s generalization of position observables to screen observables<sup>9</sup> corresponds to something like the following. Envision an empty universe containing a particle detector spread through space, sensitive for all time. The physics of this universe is simple: in this

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<sup>9</sup>Note again that the former is included in the definition of the latter.

universe there is some physical process that brings about detection at some time, at many times, or at no times. As it happens, the probability distribution for these detection events is given by a quantum state  $\psi$ , a vector in some Hilbert space  $\mathcal{H}$ . A Wightman position PVM provides the means to generate probabilities that answer the following sort of question: given that a detection occurs somewhere in this spacelike hyperplane (i.e. at time  $t$ ), what is the probability it occurs in  $\Delta$ ? A screen observable, concerning a space-time hyperplane, asks: given that a detection occurs on this two-dimensional spatial plane at *some* time, what is the probability it occurs at  $t$ ?

Each such observable picks out a different set of possible histories of this universe, i.e. a particular class of possible worlds. The probabilities supplied are *conditional* probabilities, since they apply to that class of possible worlds alone. So we ask: given that an event of a particular type occurs in this world, what is the probability it (the event) has these properties? In the case of Wightman position, we are constrained to conditions that assume an event occurs on some space-like hyperplane (and only there), and so we assign probabilities to its location in space (really an instantaneous space-time region).

In the case of a screen observable, we may consider any hyperplane we like and, assuming that an event occurs there, assign probabilities to a three-dimensional space-time region that lies in that hyperplane. But there are other classes of possible worlds we would like to assign probabilities to: what about the class of worlds in which an event occurs in a spatial region  $\Delta$  during a time interval  $I$ ? And what about an event given in terms other than its spatial location? Furthermore, what about a possible world in which more than one event occurs? The rest of this dissertation will provide the means to answer such questions.

### 7.3 GENERAL EVENT TIME OBSERVABLES

Most recently, Brunetti & Fredenhagen (2002b) supply a general recipe for constructing event time observables from an instantaneous projection or positive operator (i.e. an effect) taken to correspond to the occurrence of an event at that time. Key to this construction is the technique of operator normalization, by which a time covariant POVM is normalized according to the condition that the event in question occurs exactly once (in analogy with

Werner’s screen observables). I will give the general technique first, then describe its application and interpretation in a particularly simple case: a straightforward example concerning the detection of a particle within a detector occupying a volume of space  $\Delta$  rather than a plane.

In general terms, the construction of Brunetti & Fredenhagen (2002b) proceeds as follows. Take any effect, i.e. a positive operator  $\mathbb{I} \geq A \geq O$ .<sup>10</sup> The Schrödinger picture state at a time is  $\psi_t = U_t\psi$ , and the expectation value of  $A$  at time  $t$  is  $\langle \psi_t | A \psi_t \rangle$ . We seek a unique time covariant POVM  $I \mapsto E_A(I)$ , which is not related to  $A$  by a (generalized) spectral resolution but rather as a probability distribution for the occurrence of  $A$ , an event, during  $I = (t_1, t_2)$ , a time interval.<sup>11</sup> First, we define the positive operator

$$\langle \phi | B_A(I) \psi \rangle = \int_{t_1}^{t_2} \langle \phi_t | A \psi_t \rangle dt = \int_{t_1}^{t_2} \alpha_A(t) dt,$$

where  $\alpha_A(t)$  is the function of  $t$  given by that inner product. Due to the properties of  $A$ ,  $0 \leq \alpha_A(t) \leq 1$  for all  $t$ ,  $\phi$  and  $\psi$ , and thus  $B(I)$  is positive. Furthermore, the expectation value of  $B(I)$  is bounded from above by  $|t_2 - t_1|$ .

However, if  $\langle \psi_t | A | \psi_t \rangle = 1$  for all  $t$  the expectation value of  $B(\mathbb{R})$  in (Heisenberg) state  $\psi$  will not be bounded, and if  $\langle \psi_t | A | \psi_t \rangle = 0$  for all  $t$  then the operator  $B(I)$  is undefined. If  $A$  corresponds to a detection probability at time  $t$ , say, such states would correspond to the system having probability of detection in  $\Delta$  of one at every time  $t$ , or having no probability of detection at any time. To exclude such states from the domain of definition of the event time POVM, Brunetti & Fredenhagen suggest the following means. First we form the operators  $(B_A(I) + \mathbb{I})$ . Since  $B_A(I)$  is positive,  $(B_A(I) + \mathbb{I})$  does not have  $-1$  as an eigenvalue and so these operators have an inverse  $C_A(I) = (B_A(I) + \mathbb{I})^{-1}$ . If  $C_A(I)\psi \rightarrow \psi$  as  $I$  approaches  $\mathbb{R}$  then this is a state for which  $\alpha_A(t) = 0$  for all  $t$ . If  $C_A(I)\psi \rightarrow 0$  as  $I$  approaches  $\mathbb{R}$  then this is a state for which  $\alpha_A(t) = 1$  for all  $t$ . To exclude these states from the domain of our POVM, we take the orthogonal complement of the eigenspaces of  $C_A$  corresponding to eigenvalues 0 and 1.

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<sup>10</sup>It is an open question as to whether an arbitrary effect can be interpreted as the occurrence of an event. While I suspect that not every effect represents an event, I also suspect that events may be defined more generally than considering only spatial projections, say, would allow.

<sup>11</sup>Compare this with the construction of time operators by Hegerfeldt & Muga (2010) considered in Section 6.4, where the idea was to find an instantaneous effect which, when integrated over time, would define a time covariant symmetric operator. The end result is essentially the same, but the instantaneous effect from which the time operator is constructed has a clear physical interpretation, retained throughout.

On this domain (of states with a finite expectation value for all  $B_A(I)$ ) the self-adjoint operators  $C_A(I)$  form a net, with  $C_A(I) \leq C_A(J)$  for  $I \subset J$ .<sup>12</sup> The net of operators  $C_A(I)$  has a unique greatest lower bound  $C_A \leq \mathbb{I}$ , and we define the operator  $B_A(\mathbb{R}) = C_A^{-1} - \mathbb{I}$ . The inverse of this operator  $B_A(\mathbb{R})^{-1}$  is positive and thus has a unique positive square root  $B_A(\mathbb{R})^{-1/2}$ . This allows us to define the desired operator normalized event time POVM as follows

$$E_A(I) = B_A(\mathbb{R})^{-1/2} B_A(I) B_A(\mathbb{R})^{-1/2}.$$

It is easy to see that (on this domain)  $E_A(\mathbb{R}) = \mathbb{I}$ . Of course, it may be that this domain is the entire space  $\mathcal{H}$ , but in any case we can by this means define an event time POVM corresponding to the occurrence of  $A$ , on all states for which it makes sense to do so. Also note how this procedure highlights the sense in which event times cannot be given independently of the dynamics of a system, since the definition of  $E_A(I)$  depends not just on  $A$  but on the unitary group  $U_t$ . This is not surprising, since the distribution of event times will vary not just with the (Heisenberg) state of the system, but also with the dynamics given by the Hamiltonian.

### 7.3.1 Example: Localization as an Event

Here I will provide a detailed analysis of a simple case where the event in question is given by the instantaneous spatial projection of a localization system  $P_\Delta(t)$ , which is associated with the localization of a physical system in the region  $\Delta$  at the time  $t$ . On the conventional (Wightman) account of localization,  $P_\Delta(t)$  corresponds to a property of the system: the property of being located in  $\Delta$  at  $t$ . If the Heisenberg state of the system is  $\psi$  then the proposition that the system is located in the region  $\Delta$  at time  $t$  is true if  $P_\Delta(t)\psi = \psi$ .<sup>13</sup> We are going to interpret this fact about the quantum state  $\psi$  in terms of the occurrence of an event in  $\Delta$  at  $t$ , rather than the possession of a property.

David Lewis (1986) gave an account of an event as a property (or class) of spatio-temporal regions as follows, which is easily adapted to the interpretation of spatial localization as an event (assumed to occur at  $t$ ).

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<sup>12</sup>A net is a set of self-adjoint operators completely ordered by the relation  $\leq$ , where  $A \leq B$  if and only if  $\langle \psi | (B - A) \psi \rangle \geq 0$  for all  $\psi \in \mathcal{H}$ , bounded below or above. An increasing (decreasing) net has a unique self-adjoint operator as its least upper (greatest lower) bound. See, e.g., (Pedersen, 1989, 4.5.2).

<sup>13</sup>This is the forward direction of the famous ‘eigenstate-eigenvalue link,’ which says a system has a property if and only if it is in an eigenstate of the corresponding projection (with eigenvalue one).

To any event there corresponds a property of regions: the property that belongs to all and only those spatio-temporal regions, of this or any other possible world, in which that event occurs. Such a property belongs to exactly one region of any world where the event occurs ... (p. 243)

This closely resembles the account of Wightman localization given above, but it will require a little modification. Note that Lewis distinguishes ‘occurring in’ a region from ‘occurring within’ a region. If an event occurs within a region  $\Delta$  then, according to Lewis, it occurs within every super-region  $\Sigma \supset \Delta$ . However, if an event occurs *in* a region  $\Delta$  then it occurs within that region (and thus every super-region) but occurs in no super-region (nor any subregion).

An instantaneous position projection  $P_\Delta$  is an element of a PVM, and thus  $P_\Delta P'_\Delta = 0$  if  $\Delta$  and  $\Delta'$  are disjoint spatial regions,  $\Delta \cap \Delta' = \emptyset$ . If  $\Sigma \supset \Delta$  is a super region of  $\Delta$  then  $P_\Delta \psi = \psi$  implies that  $P_\Sigma \psi = \psi$ .<sup>14</sup> This provides us with a concrete realization of Lewis’s relation of ‘occurring within’: an event occurs within a region  $\Delta$  at  $t$  if and only if the Heisenberg state is  $\psi = P_\Delta(t)\psi$ . However, according to Lewis, an event occurring *in* a region  $\Delta$  does not occur in any super-region, nor any subregion. The region in which an event occurs is, therefore, ‘just the right size.’ This can be achieved here by means of the following definition: *Given a state  $\psi$ , a localization event occurs in a region  $\Delta(t)$  if and only if  $P_\Delta(t)\psi = \psi$  and there is no subregion  $\Omega \subset \Delta$  such that  $P_\Omega(t)\psi = \psi$ .*

This ensures that a localization event cannot occur in  $\Delta$  and its super-region  $\Sigma$ , since if it occurs in  $\Delta$  (and thus obeys the first condition) then the latter condition is not satisfied. This also excludes the situation where there is some subregion in which the event (with certainty) did not occur.

Say that we have a particle detector set up in region  $\Delta$ , and so the events we are describing with the projections  $P_\Delta(t)$  correspond to the occurrence of a detection event in  $\Delta$  at  $t$ . Naively, one might expect the following positive operator to describe the event of detection during the time interval  $I = [t_1, t_2]$ ,

$$T_\Delta(I) := \int_{t_1}^{t_2} P_\Delta(t) dt. \quad (7.2)$$

However, there is no guarantee that the expectation value of  $T_\Delta$  will be less than one, and so these will not form a normalized POVM. Brunetti & Fredenhagen’s suggestion is to use

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<sup>14</sup>Let  $P_\Delta \psi = \psi$  then, since  $P_\Sigma P_\Delta = P_\Delta$ , we have  $P_\Sigma \psi = P_\Sigma P_\Delta \psi = P_\Delta \psi = \psi$ .

the maximal such operator, in this case

$$T_{\Delta}(\mathbb{R}) = \int_{-\infty}^{\infty} P_{\Delta}(t) dt, \quad (7.3)$$

known as the dwell time operator,<sup>15</sup> to provide a suitable *operator* normalization in the following way.

Being positive,  $T_{\Delta}(\mathbb{R})$  has a unique positive square root such that  $(T_{\Delta}(\mathbb{R})^{1/2})^2 = T_{\Delta}(\mathbb{R})$ . Using the inverse of this operator, we define

$$E_{\Delta}(I) = T_{\Delta}(\mathbb{R})^{-1/2} T_{\Delta}(I) T_{\Delta}(\mathbb{R})^{-1/2}.$$

It is easily verified that  $E_{\Delta}(I)$  is a time translation covariant POVM.<sup>16</sup> This operator normalized POVM returns  $E_{\Delta}(\mathbb{R}) = \mathbb{I}$  and so the probability distribution obtained reflects the condition that the event given by  $P_{\Delta}$  occurs at some  $t$ . This is analogous to the assumption made in defining a screen observable so that the elementary event (detection in a hyper-plane) is assigned probability one.

### 7.3.2 Against Measurement (of Event Time POVMs)

The difficulty with providing an interpretation of events as conditional probabilities, conditioned on the outcome of a measurement, is the following. As a time covariant POVM, Pauli's Theorem guarantees that  $E_{\Delta}(I)$  is not a PVM, and thus  $E_{\Delta}(I)^2 \neq E_{\Delta}(I)$ . As was remarked upon in Chapter 3, this fact rules out an interpretation of  $E_{\Delta}(I)$  as a conditional probability, given by the usual means. To recap, Lüders' Rule updates the state by projecting it onto the eigenspace corresponding to the measured eigenvalue (normalizing according to the trace). That is, for a projection  $P_k$  onto the eigenspace associated with  $k$ ,

$$\rho \rightarrow \tilde{\rho} = \frac{P_k \rho P_k}{\text{tr}[P_k \rho]},$$

which leads to the following expression for conditional probability

$$\Pr(F|E) = \frac{\text{tr}[EFE\rho]}{\text{tr}[E\rho]} = \text{tr}[F\tilde{\rho}], \quad (7.4)$$

<sup>15</sup>See Jose Munoz et al. (2010) for a recent discussion.

<sup>16</sup>In connection with the time of arrival, Hoge (2008) verifies that the POVM  $E^{T_a}(I)$  is returned from  $E_{\Delta}(I)$  in the limit  $|\Delta| \rightarrow 0$ , that is, as detection within a volume becomes detection at a plane (or point). This suggests that the time of arrival comes about as an illicit conditionalization on a zero probability event. As do screen observables in general, including instantaneous position.

where  $\rho$  is a density matrix (a trace-class operator with unit trace norm).

Where  $E$  is a positive operator that is not a projection, the Lüders operation provides the next best thing:

$$\rho \rightarrow \bar{\rho} = \frac{E^{1/2}\rho E^{1/2}}{\text{tr}[E^{1/2}\rho]},$$

which reduces to Lüders' Rule just in case  $E$  is a projection, i.e.  $E^2 = E = E^{1/2}$ . Pauli's Theorem, however, entails that  $E_\Delta(I)$  is not a projection, and so we must use the Lüders operation. By following the earlier example of a conditional probability (7.4) we would be led to define

$$W(F|E) = \text{tr}[F\bar{\rho}] = \frac{\text{tr}[E^{1/2}FE^{1/2}\rho]}{\text{tr}[E\rho]} \quad (7.5)$$

as a conditional probability for the effect  $F$  given the effect  $E$ . But it is easily seen that this does not have the form of a conditional probability (unless  $B$  is a projection), since if  $B^2 \neq B$  then  $W(B|B) \neq 1$ .

That is, we cannot interpret the state  $\bar{\rho} = E_\Delta(I)^{1/2}\rho E_\Delta(I)^{1/2}$  (i.e. the state  $\rho$  subjected to the Lüders operation corresponding to the effect  $E_\Delta(I)$ ) as the state  $\rho$  conditioned on the occurrence of an event during the time interval  $I$ . This throws into doubt the use of this expression (7.5) by Hegerfeldt & Muga (2010) to define "conditional distributions" for operator normalized event time POVMs. There is also reason to doubt the interpretation of the quantity  $\text{tr}[E_\Delta(I)\rho]$  as the probability for the occurrence of the event  $P_\Delta(t)$  during  $I$  in state  $\rho$ . By definition, the probability of detection at  $t$  is  $\text{tr}[P_\Delta(t)\rho]$ , which seems to entail that the probability of detection during  $I$  is rather the cumulative probability of detection at each instant, i.e.  $\text{tr}[T_\Delta(I)\rho]$ . But  $T_\Delta(I)$  does not provide a properly normalized probability for detection in  $\Delta$  during  $I$ .

Note also that the Standard Model of Measurement (Section 3.3) is of no help here. The idea of Theorem 3.15 was to use the Naimark dilation of the POVM under consideration to define a PVM on the state space of a larger system. But Pauli's Theorem entails that the larger system must possess a Hamiltonian with an unbounded Hamiltonian. Such a system does not have a physical interpretation. (Recall the discussion of Allcock's suggestion for the time of arrival in Section 7.2.) This larger system is condemned, therefore, to be regarded a mathematical artifact rather than a representation of the states of a physical system. There is no way to measure an observable of an imaginary system.

A similar problem applies to Theorem 3.16, Kraus' Theorem, which demonstrates that a POVM on the object system could be measured by means of a projective measurement on an



ancilla, correlated with the object system by means of a unitary evolution. Pauli's Theorem entails that if a projective measurement of the ancilla is to serve as a measurement of a time POVM of the object system then the one-parameter family of unitary operators in question must be generated by a Hamiltonian with an unbounded spectrum, corresponding again to the dynamics of an imaginary, unphysical mathematical artifact. This makes the accommodation of time covariant POVMs within the Standard Model a practical impossibility. There is, therefore, no self-adjoint observable waiting in the wings whose measurement (according to the conventional account) will serve as a measurement of an event time POVM. Thus we require an interpretation of these operators that does not rely on a prior understanding of what it means to perform a projective measurement.

## 8.0 EVENT TIMES AS CONDITIONAL PROBABILITIES

We have found a general recipe for finding an event time POVM  $E_A(I)$  associated with an arbitrary (instantaneous) effect  $A$ . We have also found that POVMs are particularly ill-suited to supply conditional probabilities through the trace, which are typically defined for a projection through Lüders' Rule. In this chapter I present my proposal for making sense of these event time POVMs as conditional probabilities: probabilities for the occurrence of  $A$  during time interval  $I$  given the occurrence of the event (once) at some  $t \in \mathbb{R}$ . I will argue for this proposal by making use of the case considered above, where the occurrence of a particular event was associated with an instantaneous space-time region  $\Delta(t)$  through the spatial projection  $P_\Delta(t)$ .

The argument proceeds by analogy with conditional probabilities for instantaneous space-time regions, which are given by Lüders' Rule. I will argue for the view that conditional probabilities for the occurrence of events in non-instantaneous space-time regions should be given in just the same way as those for instantaneous space-time regions. This can be achieved in two (numerically equivalent) ways: either by defining temporal projections, so that Lüders' Rule may be applied directly, or by allowing conditionalization on non-normalized positive operators representing occurrence at a time interval. The first option involves making use of an 'extended' Hilbert space of functions of time and space. I will consider this option in the next section. The second option involves the definition of non-instantaneous Heisenberg picture observables. That is the approach I will initially pursue here.

### 8.1 EVENTS AND SPACE-TIME LOCALIZATION

As above, we will interpret  $P_\Delta(t)$ , a Heisenberg picture projection operator, as corresponding to the occurrence of an event within  $\Delta$  at  $t$ . According to the Born rule, if the state of

the system is  $\psi$ , a vector of unit norm, then the probability for occurrence in  $\Delta$  at  $t$  is  $\langle \psi | P_\Delta(t) \psi \rangle = \langle \psi_t | P_\Delta \psi_t \rangle$ , where  $\psi_t$  is the equivalent Schrödinger picture state with  $\psi_0 = \psi$ . According to the Born rule this probability is conditioned only on the state of the system, otherwise it is unconditioned. Interpreted as a probability for the occurrence of an event, however, this probability is conditioned on the occurrence of the event in question at time  $t$ .

We can see this by examining the nature of the elementary event, which is assigned probability one. The instantaneous projections together form a PVM  $\Delta \mapsto P_\Delta(t)$  on  $\mathcal{H}$ , the expectation operator of which is the Heisenberg picture position observable,  $P(t)$ , a self-adjoint operator. (For simplicity's sake, we will consider just one spatial dimension.) As projections, the operators  $P_\Delta(t)$  are ordered by subspace inclusion,  $P_\Delta(t) \leq P_\Sigma(t)$ , according to  $\Delta \subset \Sigma$ . We may then apply Lüders rule to two such projections to obtain a well-formed conditional probability for detection in  $\Delta$  at  $t$  given detection in  $\Sigma$  at  $t$  as follows:

$$\Pr(\Delta(t)|\Sigma(t)) = \frac{\text{tr}[P_\Sigma(t)P_\Delta(t)P_\Sigma(t)\rho]}{\text{tr}[P_\Sigma(t)\rho]} = \frac{\text{tr}[P_\Delta(t)\rho]}{\text{tr}[P_\Sigma(t)\rho]}. \quad (8.1)$$

This expression assigns probability one to the event corresponding to  $P_\Sigma(t)$ , i.e. detection within  $\Sigma$  given detection within  $\Sigma$  (at  $t$ ).

Consider a sequence of such expressions, ordered by subspace inclusion. The maximal spatial region, i.e. all of space, corresponds to the entire Hilbert space, and the associated projection is the identity,  $\mathbb{I}$ . According to this interpretation, then, the event corresponding to detection at  $t$  (i.e. detection at  $t$  somewhere in space) is assigned probability one by the standard normalization of the (instantaneous) position PVM. The probabilities given by the PVM  $\Delta \mapsto P_\Delta(t)$  are, therefore, already conditional probabilities: they are conditioned on the event of detection at  $t$ .

The expression for conditional probability (8.1) provides the means to define another position PVM, however, conditioned on the occurrence of the event within  $\Sigma$ . Consider the Hilbert space  $\mathcal{H}_\Sigma \subset \mathcal{H}$ , obtained by projection onto the subspace corresponding to  $P_\Sigma(t)$ , i.e.  $\mathcal{H}_\Sigma = \{P_\Sigma(t)\phi : \phi \in \mathcal{H}\}$ . On this Hilbert space  $P_\Sigma(t)$  corresponds to the identity, since  $P_\Sigma(t)\psi = \psi$  for all  $\psi \in \mathcal{H}_\Sigma$ , and all projections  $P_\Delta(t)$  such that  $P_\Delta(t) \leq P_\Sigma(t)$  together form a normalized PVM in  $\mathcal{H}_\Sigma$ . This is a trivial example, but we will use similar methods in the more challenging case of occurrence within a four-dimensional region.

### 8.1.1 Non-Instantaneous Localization

In Section 7.3 I argued that the use of POVMs for event times was unavoidable due to Pauli's Theorem, which prevents the definition of time covariant PVMs. However, Pauli's Theorem—which relies on the time translation covariance of the desired POVM—is not the only obstacle to associating a projection with the probability for occurrence of an event in a four-dimensional space-time region. Consider the positive operator  $T_\Sigma(I)$  defined as per (7.2) above, that is

$$T_\Sigma(I) = \int_{t_1}^{t_2} P_\Sigma(t) dt.$$

Let this operator represent the occurrence of an event within  $\Sigma$  during  $I$ , and let the operator  $T_\Delta(I)$  represent the occurrence of that event within  $\Delta$  during  $I$ .

I propose that the following expression gives the conditional probability for the occurrence within  $\Delta$  given occurrence within  $\Sigma$ :

$$\Pr(\Delta(I)|\Sigma(I)) := \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\Sigma(I)\rho]} = \frac{\int_{t_1}^{t_2} \text{tr}[P_\Delta(t)\rho] dt}{\int_{t_1}^{t_2} \text{tr}[P_\Sigma(t)\rho] dt}.$$

Since we are in effect adding probabilities from different times, we are treating the occurrence of an event at those times as independent. This makes sense considering that the condition ensures that just one event occurs, at some specific time. Each different time of occurrence corresponds, therefore, to a different possibility; a distinct possible world.

Now we seek to define a POVM that covaries with spatial translations representing the probability of detection within  $\Delta$  during  $I$  given detection within  $\Sigma$  during  $I$ . Since  $T_\Sigma(I)$  is a positive operator, this suggests the use of the Lüders operation (7.5) to condition on detection within  $\Sigma$  as follows

$$\rho \rightarrow \bar{\rho} = T_\Sigma(I)^{1/2} \rho T_\Sigma(I)^{1/2}.$$

Now we seek a POVM  $\Delta \mapsto E_I(\Delta)$  such that the following expression gives the right association of conditional probabilities to regions:

$$\Pr(\Delta(I)|\Sigma(I)) = \text{tr}[E_I(\Delta)\bar{\rho}] = \frac{\text{tr}[T_\Sigma(I)^{1/2} E_I(\Delta) T_\Sigma(I)^{1/2} \rho]}{\text{tr}[T_\Sigma(I)\rho]}.$$

Remarkably, this desideratum is satisfied precisely by introducing the following operator normalized expression for  $E_I(\Delta)$ :

$$E_I(\Delta) = T_\Sigma(I)^{-1/2} T_\Delta(I) T_\Sigma(I)^{-1/2}.$$

With this definition in place, we have

$$\Pr(\Delta(I)|\Sigma(I)) = \text{tr}[E_I(\Delta)\bar{\rho}] = \frac{\text{tr}[T_\Sigma(I)^{1/2}E_I(\Delta)T_\Sigma(I)^{1/2}\rho]}{\text{tr}[T_\Sigma(I)\rho]} = \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\Sigma(I)\rho]},$$

as before. This suggests that the operator normalization technique of Brunetti & Fredenhagen does serve to define a conditional POVM corresponding to a conditional probability, so long as it is used in conjunction with the appropriate conditionalization, i.e. the Lüders operation of the positive operator that represents the occurrence of the elementary event (picked out by the condition under which the probabilities apply).

Note, however, that  $\Delta \mapsto E_I(\Delta)$  is not a time covariant POVM but a spatially covarying POVM. That is,  $E_I(\Delta)$  covaries with spatial translations  $U_a$  as follows:

$$U_{-a}E_I(\Delta)U_a = T_\Sigma(I)^{-1/2}T_{\Delta+a}(I)T_\Sigma(I)^{-1/2} = E_I(\Delta + a),$$

assuming that translation by  $a$  does not remove  $\Delta$  from  $\Sigma$ . In fact, we may allow  $\Delta$  to move arbitrarily in space by considering conditionalization of the occurrence of the event anywhere in space during  $I$ , in which case we have

$$\Pr(\Delta(I)|I) = \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\mathbb{R}(I)\rho]} = \frac{\int_{t_1}^{t_2} \text{tr}[P_\Delta(t)\rho]dt}{\int_{t_1}^{t_2} \text{tr}[\rho]dt} = \frac{\text{tr}[T_\Delta(I)\rho]}{t_2 - t_1}.$$

In that case, the ‘operator normalized’ POVM corresponding to the probability of detection in  $\Delta$  during  $I$  given detection somewhere during  $I$  is just

$$E_I(\Delta) = \frac{1}{t_2 - t_1}T_\Delta(I).$$

Note that, as a spatially covariant POVM,  $E_I(\Delta)$  does not fall under the ambit of Pauli’s Theorem. The relative simplicity of this operator makes an investigation of its spectrum a tractable problem (at least numerically). In recent work with James Yearsley (in manuscript) we examine this operator (which we call the ‘quantum curfew’) for the free particle in one-dimension, and find the expectation value of  $E_I(\Delta)$  to be bounded from above by  $b < 1$ . This leads to the suspicion that there are no unit vectors in  $\psi \in \mathcal{H}$  such that  $E_I(\Delta)\psi$  for any  $\Delta$ , for any time interval  $I = [t_1, t_2]$ . Equivalently: there are no states  $\psi \in \mathcal{H}$  such that  $\langle \psi|P_\Delta(t)\psi \rangle = 1$  for all  $t \in I$ . This conjecture seems to be borne out when considering the free particle Hamiltonian. In general it turns out that this conjecture is not quite right, since there are states for which this relationship holds. However, those states have a severe limitation when considered as dynamical solutions of the Schrödinger equation.

**Proposition 8.1.** *Let  $P \leq \mathbb{I}$  be a Schrödinger picture projection operator,  $\psi \in \mathcal{H}$  be a vector of unit norm in a separable Hilbert space, and let  $U_t = e^{iHt}$  be the unitary group uniquely generated by  $H$ , a self-adjoint operator with semi-bounded spectrum. Let  $P(\{t_k\})$  be the projection operator that corresponds to the possession of the property  $P$  at every time  $t \in \{t_k\}$ . That is,  $\psi$  is in the range of  $P(\{t_k\})$  if  $PU_t\psi = U_t\psi$  for all  $t \in \{t_k\}$ .*

*Let  $\psi$  be in the range of  $P(\{t_k\})$  then either:*

1.  $\{t_k\}$  is a set with zero Lebesgue measure, or
2.  $\psi$  is in the range of  $P(\mathbb{R})$ , i.e.  $P(\{t_k\}) = P(\mathbb{R})$ .

*Therefore, there is no projection  $P(I)$  that corresponds to the possession of a property at an open interval of instants  $I \subset \mathbb{R}$  and at no other time.*

*Proof.* Let  $P_c$  be the projector onto the orthogonal complement of  $P$ . At each time  $t \in \{t_k\}$  we have  $\langle \psi | U_{-t} P_c U_t \psi \rangle = 0$ . The premises of Hegerfeldt's Lemma are satisfied by  $\psi$ ,  $U_t$  and  $P_c$  (see Lemma 4.2). Therefore,  $\langle \psi | U_{-t} P_c U_t \psi \rangle = 0$  for all  $t$ , unless  $\{t_k\}$  is a set of zero Lebesgue measure. Assuming that  $\{t_k\}$  has non-zero Lebesgue measure, it follows that  $\langle \psi | U_{-t} P U_t \psi \rangle = 1$  for all  $t$ . Thus  $PU_t\psi = U_t\psi$  for all  $t \in \mathbb{R}$ . Therefore, if  $\psi$  is in the range of  $P(\{t_k\})$  then  $\psi$  is in the range of  $P(\mathbb{R})$ , i.e.  $P(\mathbb{R}) \geq P(\{t_k\})$ . But, by definition, if  $\psi$  is in the range  $P(\mathbb{R})$  then  $\psi$  is in the range of  $P(\{t_k\})$ , i.e.  $P(\{t_k\}) \geq P(\mathbb{R})$ . Thus  $P(\{t_k\}) = P(\mathbb{R})$ .  $\square$

If  $\{t_k\} = I$  is a time interval, then, the only states for which  $\langle \psi | P_\Delta(t) \psi \rangle = 1$  for every  $t \in I$  are those which are confined to the region  $\Delta$  for all  $t \in \mathbb{R}$ .<sup>1</sup> Such states will arise if the Hamiltonian  $H$  can be decomposed into a direct sum of operators on closed subdomains,  $H = H_1 \oplus H_2$ , where  $P_\Delta H_1 \psi = H_1 \psi$  for all  $\psi$  in the domain of  $H$ . (Blank et al., 2008, 14.6.1) demonstrate that the inclusion in  $H$  of an infinite potential term surrounding  $\Delta$  is sufficient for such a decomposition to exist; it is necessary in the sense that a finite potential term will not have this effect.

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<sup>1</sup>This resembles the Quantum Zeno (or Watchdog) Effect, where continuous measurement of a projection confines the evolution of the system to the orthogonal subspace of that projection (Misra & Sudarshan, 1977).

### 8.1.2 No Non-Instantaneous Properties

Excluding such Hamiltonians from consideration, we reach some interesting conclusions regarding the Wightman interpretation of localization as a property of a system.<sup>2</sup> If we have reason to believe that the system is not localized in some region for all time then the times at which the system is localized *anywhere* within that region are severely limited. This is because the result applies to subregions as well. That is, if  $\Sigma \supset \Delta$  is a larger region that includes  $\Delta$  then  $P_\Sigma(t)\psi \neq \psi$  implies that  $P_\Delta(t)\psi \neq \psi$ .<sup>3</sup> So the conclusion of the result restricts localization within a subregion of the region just as much as it does for the region itself, and the region under consideration could be as large as one likes: the Earth, the Solar System, or so on.

Moreover, the condition that a state must satisfy to be localized within a region for a time interval is so severe that a non-zero *probability* that the system is localized anywhere outside of the region at *any* time  $t$  entails that the set of times at which it is localized within the region has measure zero. The upshot of all this is that if we are to admit the mere possibility that the particle could be detected outside of the lab next week, then it cannot be localized within the lab today at more than a set of times with zero measure. Therefore, on the Wightman interpretation the spatial localization properties are temporally *sparse*, in this sense.

Returning to the question of non-instantaneous localization, we have a proof that (given some mild restrictions on the Hamiltonian) any such POVM  $\Delta \mapsto E_I(\Delta)$  cannot be a PVM, since the operators  $E_I(\Delta)$  are not projections. If projections are associated with properties (and vice versa) then it seems, therefore, that there is no such property as being located in a spatial region at a time interval. Interpreted in terms of a property of a system, this seems to say that no system can be located within a bounded region of space for more than an instant (or at most a set of measure zero). This is an odd feature for a physical thing—presumably envisioned as a persisting object—to display. One would think that a persisting thing must be located *somewhere* at every time it exists.

This result, however, seems to suggest otherwise. So long as the Hamiltonian is well behaved (with a semi-bounded spectrum and without an infinite potential well) a quantum

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<sup>2</sup>In his “notion of localizability in a region”  $\Delta$ , these instantaneous projections are “supposed to describe a *property* of the system, the property of being localized in  $\Delta$ ” (Wightman, 1962, p. 847, original emphasis).

<sup>3</sup>Let  $P_\Delta(t)\psi = \psi$  then, since  $P_\Sigma(t)P_\Delta(t) = P_\Delta(t)$ , we have  $P_\Sigma(t)\psi = P_\Sigma(t)P_\Delta(t)\psi = P_\Delta(t)\psi = \psi$ , in contradiction with the assumption made above.

system described by a Heisenberg state  $\psi \in \mathcal{H}$  cannot be said to be located within any bounded region of space for more than a set of times of measure zero.<sup>4</sup> This puts a severe pressure on the idea that the normalization condition of a unit vector entails that a (non-relativistic) quantum particle has a position at every time. On the contrary, it would seem that a free particle undergoing unitary evolution is (almost) always not located anywhere in particular. This leads to the following uncomfortable dilemma for the would-be quantum ontologist who believes that quantum states describe persisting objects—things that persist in virtue of existing in space and time—and, with Wightman, believes that localization is a property of the system described by the quantum state.

If having a definite location in space at every time it exists is necessary for an entity to be a persisting object then *either* unitarily evolving quantum systems are persisting objects that (almost) always don't exist *or* unitarily evolving quantum systems are not persisting objects.

The first horn is rather unpleasant, I would say. Let us dwell there no further. The second horn, on the other hand, opens up the possibility of interpreting the quantum state in terms other than the properties of a persisting thing (the 'quantum system'). In this vein, I have begun to develop here an interpretation of the quantum state as a means to predict the occurrence of events. On this view, the things that have the property of being located in some spatial region at some time are *events*, not 'quantum systems' (whatever such a thing may be). This perspective is developed further and compared to some other related interpretative approaches in Chapter 9.

In sum, what if the Schrödinger picture quantum state did not describe the instantaneous properties of a persisting object, but rather just the probabilities for events to occur at a time? In that case, the failure of a quantum system to have spatial properties at every time becomes a failure of some event to occur within some region at every time. But events are temporally localized by their very nature; one would expect an event to happen in—to be located at—some spatial region at some subset of times, rather than everywhere at all times. If an event happens just once, in some place at some time, we would not be at all surprised; that is the nature of events.

Taking this point of view, where  $P_{\Delta}(t)$  represents the occurrence of an event at  $t$ , gives Pauli's Theorem an interesting new significance as a result related to temporal indeterminacy.

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<sup>4</sup>If it could, there would be a corresponding PVM (of at least two elements) that includes such a projection.



As Srinivas & Vijayalakshmi (1981) argue, the use of POVMs to represent the time of an event entails that there can be no state preparation that guarantees the occurrence of an event during some finite time interval. This suggests that quantum processes are inherently indeterministic in the following sense: given the quantum state of a system and concern with the occurrence of some event, there is no way to predict with certainty when that event will occur. This fact deserves further attention as an indication that quantum mechanics describes processes that are stochastic in time. To see what I mean here, compare the time of an atomic decay with a coin toss (assumed to be indeterministic). The time of decay is inherently unpredictable, and we assign probabilities to times. For a coin toss, the time at which the event occurs is assumed to be well-defined, but the character of the events that follow are assumed to be inherently unpredictable. Atomic decay is thus a process that is stochastic in time.

## 8.2 EVENT POVMS AS CONDITIONAL PROBABILITIES

In Section 8.1.1 I explored the use of operator normalization to define conditionalized POVMs for the occurrence of an event within a four-dimensional spatio-temporal region. These POVMs returned well-formed conditional probabilities when conditioning on the occurrence of the elementary event by means of a (non-normalized) Lüders operation. We considered conditional probabilities that concerned the occurrence of the event in question in some region of space rather than another (given occurrence in some spatio-temporal super-region). We may use exactly the same technique to define the conditional probability for occurrence in a region  $\Delta \times I$  given occurrence in  $\Sigma \times J$ , a super region of  $\Delta \times I$ .

That is, I propose the following expression as providing that conditional probability

$$\Pr(\Delta(I)|\Sigma(J)) = \text{tr}[E_I(\Delta)\bar{\rho}] = \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\Sigma(J)\rho]} = \frac{\int_{t_1}^{t_2} \text{tr}[P_\Delta(t)\rho]dt}{\int_{t_3}^{t_4} \text{tr}[P_\Sigma(t)\rho]dt}, \quad (8.2)$$

where  $J = [t_3, t_4]$ . If this is accepted as a legitimate conditional probability with that interpretation, then we find that the event time observables that were the concern of Brunetti & Fredenhagen arise as a special case. In particular, consider our earlier choice of instantaneous effect  $P_\Delta$ , conditioned on the occurrence of an event in  $\Delta$  at some  $t \in \mathbb{R}$ . The probability of

occurrence during  $I$  thus corresponds to the conditional probability

$$\Pr(I|\Delta) = \lim_{\tau \rightarrow \infty} \frac{\int_{t_1}^{t_2} \text{tr}[P_\Delta(t)\rho]dt}{\int_{-\tau}^{\tau} \text{tr}[P_\Delta(t)\rho]dt} = \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\Delta(\mathbb{R})\rho]}. \quad (8.3)$$

This conditional probability is well-formed, in the sense that the elementary event of detection in  $\Delta$  at some  $t$  is assigned probability one. It also has the following expression in terms of the operator normalized POVM  $I \mapsto E_\Delta(I)$  defined by applying Brunetti & Fredenhagen's scheme (see Section 7.3),

$$\Pr(I|\Delta) = \frac{\text{tr}[T_\Delta(I)\rho]}{\text{tr}[T_\Delta(\mathbb{R})\rho]} = \frac{\text{tr}[T_\Delta(\mathbb{R})^{1/2}E_\Delta(I)T_\Delta(\mathbb{R})^{1/2}\rho]}{\text{tr}[T_\Delta(\mathbb{R})\rho]} = \text{tr}[E_\Delta(I)\bar{\rho}].$$

This suggests that the generalized event time POVMs they define may be generally understood as providing conditional probabilities, *when applied to the system state conditioned by the corresponding Lüders operation*. Although straightforward to apply, this is an important conceptual advance. It also suggests that the distribution over times obtained by applying an event time POVM to the unconditioned state cannot be interpreted as conditional probabilities, in which case it is not clear how they should be interpreted at all. Furthermore, the operator normalized POVM, although conceptually valuable, is not necessary for the purpose of calculating these probabilities. Since these POVMs are often difficult to calculate, this approach also has pragmatic benefits.

Conceptually, it is important not to think of this conditionalization as a dynamical process (as an operation is often conceptualized). First, the conditionalization in this case involves all of time. It also concerns the Heisenberg state, not the Schrödinger state, and we cannot expect there to be any unitary evolution that implements it. This is not a process that unfolds in time; by adopting this conditionalization, we are not changing the state in manner that could be attributed to any physical process *at all*. Rather, the condition corresponds to the conceptual selection of a particular set of histories as being particularly relevant, and it is to these that the conditional probabilities apply. The operator normalized POVM corresponds to a further selection of a subset of those histories, and the probability concerns the likelihood that one of those histories is the actual history observed (with respect to the relevant set of background assumptions).

Once it is acknowledged that the probabilities concern histories, their expression in terms of a state space that does not explicitly include time becomes somewhat limiting. Once we have adopted a conditionalization of the state, we have fixed the subset of histories

under consideration and further conditionalization is not feasible—our condition, remember, concerns all time. Furthermore, the use of positive operators rather than projections limits the applicability of these expressions to the single case: if we had projections at our disposal we could repeatedly conditionalize on the occurrence of an event and remain within the ambit of Lüders’ Rule. In short, we have gone somewhat beyond our means in defining non-instantaneous probabilities in this instantaneous setting.

But there is no reason why our condition could not be given the form of a projection. We are interested in (e.g.) the projection onto states  $\psi_\Delta$  such that  $P_\Delta(t)\psi_\Delta = \psi_\Delta$  at every  $t \in \mathbb{R}$ . The relevant subspace is the states for which  $P_\Delta(t)\psi_\Delta = \psi_\Delta$ . The difficulty we face is that every vector  $\psi \in \mathcal{H}$  uniquely corresponds to a history  $\psi_t = U_t\psi$ , but the histories we are interested in (corresponding to the occurrence of an event within some time interval, say) must be defined more generally. It is here that the limitations of the Heisenberg and Schrödinger pictures start to bite since each concerns (equivalent) instantaneous projections.

To free ourselves from this restriction, let us consider instead vector valued functions of  $t$ ,  $\Psi(t) = \psi(t)$ , with  $\psi(t) \in \mathcal{H}$ . Such a function represents an entire history of a system, i.e a possible world. We may thus define a history  $\Psi_\Delta(t)$  corresponding to the desired elementary event by the function  $\Psi_\Delta(t) = P_\Delta U_t\psi$ . But these functions of  $t$  cannot lie in the instantaneous Hilbert space  $\mathcal{H} = L^2[\mathbb{R}^3]$  of functions of (configuration) space. Instead, we must consider the *temporally extended* Hilbert space  $\mathcal{H}_+ = L^2[\mathbb{R}^3] \otimes L^2[\mathbb{R}] \cong L^2[\mathbb{R}^4]$  of functions of space and time. The conditional probabilities obtained here can be precisely replaced in that setting, but with projections onto time intervals, and—correspondingly—Lüders’ Rule proper.

### 8.3 THE EXTENDED HILBERT SPACE

Vectors in the extended Hilbert space  $\Psi \in \mathcal{H}_+$  (constructed on Dirac’s extended configuration space) are defined from vectors in  $\mathcal{H}$  by means of a vector valued function of  $t$ . That is, each vector  $\Psi$  can be written  $\Psi(t) = \psi(t)$ , where  $\psi(t) \in \mathcal{H}$ , the usual Hilbert space of the system. A Heisenberg state  $\psi \in \mathcal{H}$  uniquely corresponds to a family of Schrödinger states  $U_t\psi = \psi_t \in \mathcal{H}$ , but the function  $\Psi(t) = \psi_t$  will fail to be included in  $\mathcal{H}_+$  since this function is not square integrable. Nonetheless, we will be able to set up a close correspondence between

vectors  $\Psi \in \mathcal{H}_+$  and Heisenberg states  $\psi \in \mathcal{H}$  that will serve our purpose of defining valid conditional probabilities via Lüders' Rule.

Following Naimark & Fomin (1957), we will define  $\mathcal{H}_+$  as a continuous direct sum of instantaneous Hilbert spaces  $\mathcal{H}_t$ , each with inner product

$$\langle \phi | \psi \rangle_t = \sum_k \langle \phi(t) | e_k \rangle \langle e_k | \psi(t) \rangle,$$

where  $\{e_k\}$  is a fixed orthonormal basis for  $\mathcal{H}$ . First, we consider more general vector-valued functions  $t \mapsto \phi(t) \in \mathcal{H}_t$ , where instantaneous vectors  $\phi(t)$  at different times need not be related by  $U_t$ . Such a function is measurable if  $f(t) = \langle \psi | \phi_t \rangle$  is measurable (with respect to the usual Borel measure on  $\mathbb{R}$ ) for arbitrary  $\psi \in \mathcal{H}$ . If two such functions  $\Psi(t), \Phi(t)$  are measurable, then so is the numerical function of  $t$  defined by their instantaneous inner product  $F(t) = \langle \Psi(t) | \Phi(t) \rangle_t$ . The set of all such measurable functions  $\Psi$  is a Hilbert space,<sup>5</sup> which corresponds to the continuous direct sum of the spaces  $\mathcal{H}_t$ , that is an integral with respect to Lebesgue measure:

$$\mathcal{H}_+ := \int_{\mathbb{R}} \oplus \mathcal{H}_t d\sigma(\mathbb{R}).$$

The inner product on  $\mathcal{H}_+$  may now be defined as

$$\langle \Phi | \Psi \rangle_+ = \int_{\mathbb{R}} \langle \Phi(t) | \Psi(t) \rangle_t d\sigma(\mathbb{R}). \quad (8.4)$$

Note that whereas  $\Psi(t) = U_t\psi$  is not included in this space, *partial* dynamical evolutions of the system are, i.e. if  $\Psi(t) = U_t\psi$  for  $t \in (t_1, t_2)$ , 0 otherwise then  $\Psi \in \mathcal{H}_+$ . Note also that, since  $\mathcal{H}_+$  was defined as a direct sum of instantaneous spaces  $\mathcal{H}_t$ , instantaneous vectors from different times are orthogonal. Thus we can define a projection operator  $P^T([t_1, t_2])$  which has the effect of truncating an arbitrary history  $\Psi(t)$  as follows:

$$P^T([t_1, t_2])\Psi(t) = \begin{cases} \psi(t) & \text{if } t_1 \leq t \leq t_2 \\ 0 & \text{otherwise.} \end{cases}$$

Considered as a map from temporal intervals to projections,  $I \mapsto P^T(I)$  provides the means to define a PVM from Borel subsets on  $\mathbb{R}$  to projections on  $\mathcal{H}_+$ . By the Spectral Theorem, there is a corresponding observable  $T_+$ , self-adjoint on  $\mathcal{H}_+$ . The conjugate observable to  $T_+$  is not the Hamiltonian (an operator on  $\mathcal{H}$ ), and so Pauli's Theorem is avoided. This feature of the correspondence between  $\mathcal{H}$  and  $\mathcal{H}_+$  deserves further comment.

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<sup>5</sup>Identifying, as usual, functions that differ only on a set of measure zero.

Consider a bounded operator  $B \in \mathfrak{B}(\mathcal{H})$  which, operating on a vector  $\psi \in \mathcal{H}$ , returns a vector  $B\psi \in \mathcal{H}$ . This defines an operator  $B_+$  as follows

$$\langle \Phi | B_+ \Psi \rangle_+ = \int_{\mathbb{R}} dt \langle \Phi(t) | B \Psi(t) \rangle_t,$$

where  $B\Psi(t) \in \mathcal{H}$ . The domain of  $B_+$  will be all  $\Psi \in \mathcal{H}_+$  such that  $B_+\Psi \in \mathcal{H}_+$ . We can expect the partial dynamical evolutions mentioned above to be in this domain.

This provides the following expression for the conditional probabilities for detection events, as defined earlier. Let  $\Psi(t) = U_t\psi$  for a system in Heisenberg state  $\psi$ . The history corresponding to detection within  $\Delta$ , therefore,  $P_{\Delta+}\Psi(t)$ . Detection during  $I$  corresponds to the projection  $P^T(I)$ . Lüders' Rule returns

$$\Pr(I|\Delta) = \frac{\langle \Psi | P_{\Delta+} P^T(I) P_{\Delta+} \Psi \rangle_+}{\langle \Psi | P_{\Delta+} \Psi \rangle_+} = \frac{\int_{t_1}^{t_2} \langle P_{\Delta} \Psi(t) | P_{\Delta} \Psi(t) \rangle_t dt}{\int_{\mathbb{R}} \langle \Psi(t) | P_{\Delta} \Psi(t) \rangle_t dt} = \frac{\langle \psi | T_{\Delta}(I) \psi \rangle}{\langle \psi | T_{\Delta}(\mathbb{R}) \psi \rangle}. \quad (8.5)$$

This is just the same expression we obtained before, Equation 8.3.

Now consider the action of a unitary time shift operator  $U_{\tau}$  on a vector  $\Psi(t) \in \mathcal{H}$ . Only if  $\Psi(t) = \psi_t = U_t\psi$  will this have the effect  $U_{\tau}U_t\psi = \psi_{t+\tau}$ , otherwise the effect is to map the vector  $\Psi(t) \in \mathcal{H}$  to a vector  $U_{\tau}\Psi(t) \in \mathcal{H}$ . That is, a distinct vector assigned to the same time  $t$ . This means that the projection  $P^T(I + \tau)\Psi \neq (U_{-\tau}P^T(I)U_{\tau})\Psi$ , at least not in general. The time covariance of the projections  $P^T(I)$  takes on a distinct, more general form under a unitary group  $U(s)$  defined on  $\mathcal{H}_+$  rather than  $\mathcal{H}$ .

To reflect on this fact, consider the relationship between a Hilbert space of one-dimensional functions  $\psi^1 \in L^2[\mathbb{R}]$  and a space of two-dimensional functions  $\psi^2 \in L^2[\mathbb{R}^2]$ . Whereas  $\psi^2(x)$  is a function of  $x$ ,  $\psi^2(x, y)$  is a function of two independent parameters  $x$  and  $y$ . If  $x$  and  $y$  correspond to orthogonal axes in two-dimensional Euclidean space, to place a function of  $x$  in this space requires a specification of the  $y$  co-ordinate. This suggests that we can set up a one-to-one correspondence of the form  $\psi^1(x) \rightarrow \psi^2(x, 0)$ . However, it is not clear that this function  $\psi^2(x, 0)$  is an element of  $\psi^2 \in L^2[\mathbb{R}^2]$  if the function is left unspecified for other values of  $y$ . To remedy this defect, we could set  $\psi^2(x, y) = \psi^1(x)$  for all  $y$ , but it is still not clear that  $\psi^2 \in L^2[\mathbb{R}^2]$  since it could fail to be square integrable. However, by mapping  $\psi^1(x)$  to an interval of  $y$ , i.e.  $\psi^2_{[-a, a]}(x, y) = \psi^1(x)$  if  $-a \leq y \leq a$ , 0 otherwise, we obtain a function of  $x$  and  $y$  that is certainly square integrable if  $\psi^1$  is.

Now consider a translation by  $b$  of a function  $\psi^2_{[-a, a]}$  along the  $y$ -axis. Such a transformation has the effect of mapping the function  $\psi^2(x, y)$  to the function  $\psi^2(x, y + b)$ . For a

function  $\psi^2_{[-a,a]}$ , translation by  $b$  along the  $y$ -axis has the effect  $\psi^2_{[-a,a]} \rightarrow \psi^2_{[-a+b,a+b]}$ . Now define the operator  $Q_y$  by multiplication in  $y$ ,  $(Q_y\psi^2)(x, y) = (y\psi^2)(x, y)$ . This provides the means to set up a Schrödinger representation with  $P_y$ , where  $(P_y\psi^2)(x, y) = (i\frac{d}{dy}\psi^2)(x, y)$ . All the discussion of Section 5.1 applies here, and one can show by the same means that  $Q_y$  and  $P_y$  are self-adjoint operators forming a Weyl pair, with  $P_y$  the generator of a unitary group of shifts in  $y$ ,  $U(b)$ . That is,  $U(b)\psi^2(x, y) = \psi^2(x, y - b)$ . There is evidently no hope of finding a representation of this group in terms of unitary operators acting on functions  $\psi^1 \in L^2[\mathbb{R}]$ .

The situation in  $\mathcal{H}_+$  is directly analogous, despite the fact that there exists in  $\mathcal{H}$  a unitary group parameterized by  $t$ . In terms of its physical significance, this group of spatio-temporal symmetries is on a par with spatial translations and rotations, Galilean boosts etc.: it is one of a collection of one-parameter unitary groups that together provide an irreducible representation of the Galilei Group. When a vector in  $\mathcal{H}$  is acted on by a unitary operator  $U_t$ , the result is another vector in  $\mathcal{H}$ , taken to be related to the original by a time translation. This is often interpreted dynamically, as a process that changes the instantaneous state of a physical system over time. Evidently this interpretation is not without physical significance since quantum mechanics equipped with unitary evolution alone has (empirically, at least) done quite well. But the fact that this representation of time translations exists on  $\mathcal{H}$  does not entail that other representations cannot be found, nor that (if found) they should have no physical significance.

If the state of a system is taken to be a non-instantaneous, spatio-temporal object  $\Psi \in \mathcal{H}_+$  giving an entire history of a system then the time-translations of that state correspond to different histories. This is analogous to the effect of  $U_t$  on the Heisenberg state  $\psi \in \mathcal{H}$ : in the Heisenberg picture,  $U_t\psi$  is distinct time-independent state. Considering an arbitrary function of  $t$  in the extended Hilbert space  $\mathcal{H}_+$ , however, the group of unitary time translations  $U(s)\Psi(t) = \Psi(t + s)$  is generated by a self-adjoint operator on  $\mathcal{H}_+$ . This operator, call it  $W_+$ , is the operator conjugate to  $T_+$ , acting by differentiation in the representation where  $T_+$  acts by multiplication. Together,  $(W_+, T_+)$  are a Schrödinger representation, and form a Weyl pair.

This entails that the PVM corresponding to  $T_+$  covaries with these time translations,  $U(-s)P^T(I)U(s) = P^T(I - s)$ . The covariance of  $W_+$  with translations generated by  $T_+$ , however, need have no special physical significance. Certainly, neither  $T_+$  nor  $W_+$  are ob-

servables of Ordinary QM, and there is no interpretative rule in place to say that self-adjoint operators on  $\mathcal{H}_+$  can be measured at an instant of time. There is no reason to regard the PVM  $P^T(I)$  as anything but a convenient means to associate observables of  $\mathcal{H}$  (representing events) with intervals of times, and the existence of  $W_+$  as a logical consequence of the demand that those associations covary with time translations. The physical significance of this time translation symmetry lies in the fact the result of an experiment concerning event times (at the ensemble level) is a distribution of times, and that this distribution covaries with the time at which the experiment begins.

#### 8.4 DISCUSSION AND INTERPRETATION

For the situations considered thus far, the group  $U(s)$  has been somewhat surplus to requirements since the families of states (histories) under consideration have all had the form  $\Psi(t) = \psi_t$ , for some set of  $t$ , and so the unitary group  $U_t$  would have sufficed. Conceptually, however, there is an important distinction since the role of the group  $U(s)$  is not dynamical. In the Schrödinger picture of Ordinary QM, we tend to imagine the vectors  $\psi_t$  as describing the successive instantaneous states of a substantial thing, viz., ‘the quantum system,’ imagined as a persisting object. That is, despite the fact that the wavefunction of a system almost always resides in a higher dimensional space than three-dimensional physical space, the intuition remains that these instantaneous states  $\psi_t$  are states of the *same* thing persisting through time—the collection of quantum particles, or whatever. Underlying this mode of thought is the intuition that quantum phenomena are the manifestations of the properties of an underlying thing whose identity through time is grounded in its continuing existence as a physical object (of some stripe).

This overtly metaphysical assumption is, however, actively in conflict with the four-dimensional picture appropriate to the use of the extended state space  $\mathcal{H}_+$ . Minimally, a history  $\Psi \in \mathcal{H}_+$  can be thought of just as a means of attaching probabilities to experimental outcomes associated with distinct sets of times. Any metaphysical interpretation of  $\Psi$  should at least be consistent with this use. Recall that given a Heisenberg state  $\psi \in \mathcal{H}$  we were able to obtain an assignment of conditional probabilities to the occurrence of an event at different times, conditioned on the occurrence of that event at some time.

The Heisenberg state  $\psi$  does not correspond to a vector in  $\mathcal{H}_+$ , however. In terms of  $\mathcal{H}_+$ , the conditional probabilities for times of occurrence are obtained by an application of Lüders Rule to projections representing occurrence at a time interval, and the condition under which these probabilities apply corresponds to a conditionalization of the history  $\Psi(t) = \psi_t$ . The experimental outcomes themselves correspond to a further conditionalization of the relevant history. To what extent can we regard these conditionalizations of histories as descriptions of the actual dynamical evolution of some substantial thing? To do so would require that, following the observation of some particular outcome, we must be able to assign a dynamical evolution to the substantial thing ‘passing through’ the experimental apparatus which brings about the observed outcome. These conditionalized histories, however, are not apt to play this role.

A better view of these projections is provided by thinking of them as providing a specification of the worlds possible according to the theory. On this view, the conditionalization of the state corresponds to a narrowing down of the possibilities under consideration. We can usefully think of these possibilities as a specification of a class of possible worlds, and the probabilities that are assigned to them as (roughly) Lewisian chances, i.e. as objectified assignments of credence. According to Lewis (1981), we are to think of ourselves as world-bound individuals occupying a single possible world, which we can think of here as corresponding to a maximal specification of all the events that have happened or will happen. If, therefore, as a world-bound individual I were to learn exactly which world I was in, i.e. which world is actual, I would know the entire history of the world: past, present and future. As a world-bound individual whose knowledge of the future is uncertain, however, I can at best assign probabilities (i.e. credences) to future events.

That is, I assign probabilities to future events conditioned on the events in my past and according to my expectation that the world in which I reside will contain events to my future of a certain type. For example, my assignment of probabilities summing to one to the possible outcomes of a future coin toss,  $\{heads, tails\}$ , is based on the assumption that I am not in a world in which the coin lands on its edge, or is vaporized before it lands, or what have you. This provides a sense in which the objective validity of an assignment of probabilities to future events depends on not just what lies to my past, but also what lies to my future (and that of my relevant counterparts). While specific knowledge of future events is inadmissible, I cannot assign meaningful probabilities to future events without some



(imperfect) knowledge of the future. This knowledge takes the form of specifying the class of worlds to which I belong, i.e. worlds in which an event of a certain type lies to my future (or that of my counterparts).

The objective chances, given relative to a time, must be similarly conditioned by the class of possible worlds to which they apply: roughly, worlds that agree on the past and, to a certain extent, agree to the future. On this view, an assignment of probabilities to future events can be thought of a probability measure on the relevant class of possible worlds. In the context of a physical theory specifying the laws of our world, we can take the (initial) state of the system to contain all the relevant information about the past of this world. The relevant set of possible worlds will be those which have the same laws and the same history, i.e. which assign the same initial state and the same dynamics to this particular experimental setup.<sup>6</sup> The probabilities supplied by the theory will be the chances for this set of possible worlds.

Appropriately interpreted, this is precisely what the prescription for conditional probabilities supplied here allows us to do for quantum theory. Thinking just in terms of the time of an event, we have an assignment of experimental outcomes to projection operators  $P^T(I)$ , the initial state is the Heisenberg state  $\psi$ , and the elementary event (assigned probability one) is the occurrence of the event given by  $P$  at some time  $t$ . Without yet fixing the state, the possible worlds corresponding to the elementary event are given by the histories  $P_+\Psi$ . We fix the state by setting  $\Psi(t) = \psi_t$ . Conditional probabilities for the event  $P$  given this state exist only if

$$0 < \lim_{\tau \rightarrow \infty} \int_{-\tau}^{\tau} \langle \psi_t | P \psi_t \rangle dt < \infty.$$

This has the interpretation that  $\psi$  describes a set of worlds in which the event is neither impossible, nor certain to occur at every moment. In that case,  $\psi$  defines a probability measure on the times at which the event can occur. We can think of this as a probability measure on the set of relevant possible worlds, differing as to when this event occurs.

Thus the partial history  $P^T(I)\Psi$  describes a set of possible worlds (those in which the event occurs during  $I$ ) rather than the state of physical object. The covariance of  $P^T(I)$  under time translations, therefore, reflects the relationship between the probabilities assigned to sets of possible worlds rather than the changing state of some physical thing. This way

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<sup>6</sup>Note that there is no need to think of the dynamics as specifying more than just how the chances will change with time.

of thinking reflects my belief that the laws of the actual world (i.e. quantum mechanics) are probabilistic in a particularly strong sense. That is, I believe that these laws concern inherently indeterministic or stochastic temporal processes. What I mean by this is that there is a sense in which a complete specification of the actual world together with its laws cannot be pieced together into a dynamical story that provides a continuous function assigning states to instants of time.

Despite the existence of well-confirmed and well-known quantum phenomena such as radioactive decay, of which no such description appears possible, the idea that our world might nonetheless permit such a continuous description has proved quite popular. In my opinion, this is due to an unfortunate inclination to project the apparent continuity of macroscopic physical processes, with which we have direct experience, onto quantum processes that take place on vastly shorter time-scales.<sup>7</sup> Note that this inclination was (at least at first) thwarted by matrix mechanics, which led to serious consideration of the possibility that natural processes could be discontinuous.<sup>8</sup> However, von Neumann's reconciliation of matrix and wave mechanics within Hilbert space made it all too easy to attribute the existence of a (strongly) continuous family of unitary operators  $U_t$  describing time translation symmetries to the supposed fact that quantum mechanical processes themselves are (in the main at least) continuous and deterministic.

The series of instantaneous states resulting from the action of this group on the initial state of a system came to be conceived of as describing the successive states of a persisting physical object. This led to a worrying conflict with the apparently discontinuous change in state resulting from measurement, and so the 'measurement problem' as traditionally conceived. This is not the place for a detailed historical analysis of the difficulties into which interpreters of quantum theory have been led by the idea that quantum mechanics is a theory of physical objects submitted to continuous dynamical processes—the so-called 'unitary evolution of the wavefunction'. However, I do claim that there is a possibility of interpreting quantum theory realistically (as a theory of processes that are in fact discontinuous and stochastic) without this problematic metaphysical assumption.

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<sup>7</sup>See Chapter 1 of Ladyman & Ross (2007) for a recent attack on the use of intuitions honed at the comfortable level of the macroscopic. See also the discussion of Russell (1927, pp. 147–153), which is particularly pertinent for my purposes.

<sup>8</sup>See the introduction of Heisenberg (1927).

## 8.5 FURTHER PROSPECTS FOR THIS APPROACH

The obvious limitation of the approach to event times considered here is that the conditional probabilities apply to an event so long as it is assumed to occur exactly once, over all time. This problem is thrown into sharp relief by these considerations from  $\mathcal{H}_+$ , according to which Lüders rule associates conditional probabilities with the occurrence of a single event. That such a restrictive condition is required to apply Lüders' rule seems to suggest that this limitation is, in fact, inherent to this method of generating conditional probabilities.

I propose that the solution to this difficulty lies in considering instead the *extended* Schrödinger equation, defined for functions of time and space as Dirac (1926a) originally envisioned it,

$$(H_+ - W_+)\psi(x, t) = \left( H_+ - i\frac{\partial}{\partial t} \right) \Psi = \hat{H}\Psi = 0.^9$$

The problem with interpreting this equation is that the operator  $\hat{H}$  has a continuous spectrum, and so there is no vector  $\Psi \in \mathcal{H}_+$  which is an eigenvector with eigenvalue 0. Without going into the details, Brunetti et al. (2010) show how solutions to this equation may be written in terms of a linear functional rather than a vector. While these physical solutions don't form a Hilbert space, and define non-normalizable 'weights' on the algebra of observables of  $\mathcal{H}_+$  rather than algebraic states, there is a construction which, given an operator on  $\mathcal{H}_+$  representing the occurrence of an event, leads to a (GNS) Hilbert space representation giving the expectation values for (a subset of) the algebra of observables on  $\mathcal{H}_+$ —the algebra of *events*—on the condition that the event in question did occur. The vectors of this Hilbert space representation are not states but observables—events—to which the weight assigns a (conditional) probability through the inner product of the GNS representation.

This weight thus provides a replacement for the trace, such that by conditioning on these events one can calculate probabilities for subsequent events, e.g. a detector's firing again elsewhere. This theory has a good claim to be regarded as a straightforward, conservative extension of the usual Schrödinger dynamics, but the form that the theory takes is surprisingly distinct: the solutions of the extended Schrödinger equation do not form a Hilbert space, the dynamics are not given by a unitary operator, and there is no meaning to the phrase 'the state of the system' (i.e. no normalized algebraic state assigning expectation values to operators) without first specifying an event (the occurrence of which can

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<sup>9</sup>Note that  $W_+$  is here the operator conjugate to  $T_+$ .

be used to give a probability assignment to further events). In particular, such probability assignments appear to correspond to genuinely conditioned state, rather than the conditional expectations considered here. This holds the promise of a generalization of these conditional probabilities for events to conditions which do not refer to all times, and beyond the rather limited case of the occurrence of a single event.

## 9.0 PHILOSOPHICAL REFLECTIONS

The topics addressed in detail in the foregoing chapters have required a certain degree of technical sophistication to approach, and the development of the tools to do so has required several chapters. These topics are of interest not just for the foundations of physics, but also for philosophy. In this concluding chapter, I have attempted to draw out some implications of the present work for some traditional philosophical topics concerning ontology and metaphysics. In Section 9.1 I use the account of quantum theory supplied in Chapter 8 as a means to flesh out a conception of an ontology for matter as a collection of events: an event ontology. Beginning with consideration of two recent advocates of an event ontology for quantum theory (Carlo Rovelli and Rudolf Haag), I compare my account with two related views: the GRW ‘flash’ ontology, and the Everettian interpretation due to Simon Saunders. This section concludes with a brief survey of where my account might turn out to be superior in terms of accounting for certain experimental phenomena.

In Section 9.2 I consider some implications for the metaphysics of time of the no go results of Chapter 6 concerning quantum clocks. In Section 9.3 I defend Bertrand Russell’s relational theory of time against a recent attack by Ulrich Meyer (2013), based on the idea that Russell embraced a thoroughgoing event ontology that (unlike Whitehead) sought to eliminate objects (or at least reduce them to events). I conclude with a (very) brief proposal that the adoption of event ontology may assist the strong structuralist program in philosophy of science of James Ladyman and collaborators (Ontic Structural Realism).

## 9.1 EVENT ONTOLOGY AND QUANTUM MECHANICS

What is quantum mechanics about? That is, what is the intended domain of an interpretation of the theory? In the long history of attempts to interpret quantum theory a wide

variety of answers have been given to this question, including: observations, experiments, wavefunctions, the universe, point particles, information. I have presented here a way to think of quantum theory in terms of the occurrence of events, which suggests that quantum theory may be *about* events. On this view, the probabilities supplied by the quantum state are probabilities for the occurrence of events, and the observables of the theory are to be interpreted as supplying probabilities for some event to occur, at some time. The dynamics of the quantum state do not reflect some changing state of matter, but just the change in those probabilities. An event ontology supposes that there is no need to posit any other ontological category apart from events. On an event ontology, then, quantum theory is a theory of events and therefore a theory of matter, since matter itself is ‘composed of’ nothing but a series of events.

In twentieth century analytic philosophy, event ontologies were proposed by (variously) Whitehead (1925), Russell (1927), and Davidson (1980). The event view of quantum theory has had some recent interest from notable theoretical physicists such as Carlo Rovelli and Rudolf Haag, who express in different ways the core idea of this view. Rovelli (2005) contrasts the “wave function ontology,” which takes the state “as the ‘real’ entity which fully represents the actual state of affairs of the world,” with his proposal for an “ontology of quantum events.”

A better alternative is to take the observed values . . . as the actual elements of reality, and view [the state] as a mere bookkeeping device, determined by the actual values . . . that happened in [the] past. From this perspective, the real events of the world are the ‘realizations’ (the ‘coming to reality’, the ‘actualization’) of the values . . . in the course of the interaction between physical systems. These quantum events have an intrinsically discrete (quantized) granular structure. (p. 115)

The key idea is that the changing quantum state given by the dynamics of the theory does not describe the changing properties of some physical object.<sup>1</sup> Rather, the quantum state describes the probabilities for events to occur; events that often arise as the result of interactions between systems.

In turn, here is Haag’s (2013) recent critique of the conventional view:

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<sup>1</sup>In approvingly citing Rovelli’s commitment to an ontology of events I do not mean to endorse his accompanying interpretation of quantum theory, Relational Quantum Mechanics, about which there is much to criticize. I shall not do so here, however.

What do we detect? The presence of a particle? Or the occurrence of a microscopic event? We must decide for the latter... [The microscopic event] is almost always the ionization of some molecule in the detector. ... [T]he standard use of the term “observable” does not really correspond to the needs of collision theory in particle physics. We do not measure a “property of a microscopic system”, characterized by a spectral projector of a self-adjoint operator. Rather we are interested in the detection of a microscopic event. The first task is to characterize the mutually exclusive alternatives for such an event. (p. 1310)

So in practice, i.e. in the context of a particle detection experiment, the theory concerns—is about—microscopic events, such as the ionization of a molecule by a cosmic ray. The detector is expressly designed to amplify these micro-events such that they reliably lead to a macroscopic record of detection, by which we mean detection at a place, at a *time*.

In Chapter 8, I suggested a re-interpretation of Wightman localization in terms of the occurrence of localized events within extended spatio-temporal regions. I claimed that, interpreted in terms of events, there is a crucial further question concerning localization to which quantum mechanics can supply an answer: *When* does an event occur? Providing a satisfactory answer to this question, I contend, gives an informative account of Haag’s ‘principle of random realization’ and provides the means avoid Rovelli’s paradox that “the statement that a certain specific outcome ‘has happened’ can be true and not-true at the same time” (p. 115). I will address these issues through comparison with the account of events offered by two extant alternative interpretations of quantum mechanics: dynamical collapse and the Everett interpretation.

John S. Bell, writing about the Ghirardi-Rimini-Weber (GRW) dynamical collapse theory, proposed to regard the spontaneous localizations of the wavefunction about a space-time point as events, and those events as the ‘local beables’ of the theory.<sup>2</sup> He expressed the idea of an event ontology as follows:

These [events] are the mathematical counterparts in the theory to real events at definite places and times in the real world (as distinct from the many purely mathematical constructions that occur in the working out of physical theories, as distinct from things that may be real but not localized, and as distinct from the ‘observables’ of other formulations of quantum mechanics, for which we have no use here). A piece of matter then is a galaxy of such events. As a schematic psychophysical parallelism we can suppose that our personal experience is more

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<sup>2</sup>This discussion does not apply to the continuous mass distribution variant of the theory. It is important that the localizations (or ‘flashes’) are discretely positioned.

or less directly of events in particular pieces of matter, our brains, which events are in turn correlated with events in our bodies as a whole, and they in turn with events in the outer world. (Bell, 1987, p. 205)

This would be consistent with a Humean view of physical law, as recently considered in the context of the GRW theory by Frigg & Hoefer (2007). They consider the idea that the localization events (or ‘flashes’) form a Humean mosaic, which they liken to a ‘pointilist picture,’ and note that the Humean would approve of the introduction of only occurrent events without “reference to any hidden powers or mechanisms explaining these occurrences, which would be unacceptable from a Humean perspective” (p. 381). On this basis, they argue that the GRW theory deserves serious consideration as the best system of laws (in its domain at least), and that the probabilities supplied by the theory can be interpreted as Humean objective chances.

In the last chapter I took the point of view that the introduction of an additional stochastic process regarding the time of an event (as advocated by Ghirardi, Rimini and Weber) is spurious, and showed how the state of a quantum system returns a probability distribution for the occurrence of an event at a time (and spatial location) without it. I also introduced the idea that one can think of the probabilities supplied in this context as Lewisian chances (i.e. objectified credences). This conception of quantum probabilities is subtly at odds with the idea of quantum theory as a Mills-Ramsey-Lewis best system.

Were quantum theory such a system, its success would be judged against its fit with the Humean mosaic as a whole: the entire history of the actual world. In the conception of the quantum state that I introduced, however, the success of the theory would have to be judged against its performance in predicting the future course of events, given sufficient information about the past events. As future events become present, those events are to be included in the conditions under which the probabilities apply; there is no mechanism to bring about the occurrence of the events within the theory, which merely supplies the (conditional) probabilities for their occurrence.

There is, therefore, a subtle distinction to be drawn between the role played here by quantum mechanics as a theory *in use* by an agent in the world, whose knowledge of the future course of events is uncertain, and as a best system for the total mosaic of events, which is assumed to be given without respect to epistemic limitations. As the recent analysis of Esfeld & Gisin (2013) would have it, the Humean perspective on GRW is as follows:



There only is the mosaic of flashes in space-time. Everything else supervenes on that mosaic. Given an initial configuration of flashes, there is nothing in nature that determines the temporal development of the distribution of flashes in spacetime, or that, more generally speaking, poses constraints on how that distribution can develop. (p. 4)

But if one takes the GRW theory as given and attempts to apply it as a situated agent, the theory supplies a dynamical story concerning the evolution of the initial state which seems to constrain the future distribution of events quite strongly indeed. Moreover, at least in rhetoric, the role of the spontaneous localization process seems to depart rather severely from this Humean ideal:

the idea is that the space-time in which physical processes develop exhibits some fundamentally stochastic, random aspects, which induce precisely the spontaneous localizations of the microscopic constituents of the universe. . . . nature itself (Einstein's God?) chooses to induce such a process according to random choices but with precise probabilities. (Ghirardi, quoted in [p. 376] Frigg & Hoefer (2007))

Now it is hard to resist giving a dynamical story like this, which invokes some sort of agency or disposition to bring about these quantum events. However, the promise of an Humean event ontology for quantum theory lies in rejecting the idea that there is any kind of dynamical story to be told where microscopic processes are concerned.

One way of proceeding, then, is to take quite seriously the idea that the Humean mosaic of events provides the first and last word when it comes to matters of ontology, necessity and probability. In that case, the best theory for us (as situated agents in the actual world) is not the theory which best fits the overall totality of events according to a dynamically evolving (albeit stochastically localized) instantaneous state, but the theory that is most informative in the situations of imperfect knowledge in which we habitually find ourselves. In those situations, the most informative theory will be one that fits the probabilities to the details of the situation rather than blindly applying a fixed distribution derived for a generic situation. That may not be the theory that gives the 'Humean Laws' (whatever sort of thing those could be), but it may be the theory that best provides the objective chances.

In rejecting the need for a dynamical story that invokes a succession of instantaneous states, one must do without an account of a dynamical process by which events come about. This leads Esfeld & Gisin (2013) to worry that "even in the non-relativistic flash ontology, it

remains unclear how the occurrence of flashes can be triggered through interactions” (p. 12). It seems to me that this rather misses the point: in an event ontology (a Humean one, at least) ‘interaction’ cannot be a shorthand for ‘causally bring about.’ However, Haag (2013) offers a helpful attempt to capture a notion that might replace the explanation offered by a dynamical account.

The theory provides the description of possible alternatives (the probability space) and the probabilities for the different possibilities. But we are still left to explain the emergence of individual facts whose appearance is governed by a statistical law which is intrinsic i.e. not due to any ignorance of hidden variables. This implies that the step from the virtual realm of propensities to reality is governed by a principle of random realization. The step is neither determined by previous history nor is it entirely free. The principle says that the theoretically predicted pattern will be realized by a sequence of many individual events unpredictable in the individual case. (p. 1311)

While this talk of propensities will not be acceptable to the strict Humean, I think the idea of a principle of random realization nicely captures the relationship between the (probabilistic) predictions of the theory and the occurrence of the events to which the probabilities are assigned. The key point is that there is no dynamical process supposed to bring this realization about: we just assume it to be true (by posit) that the universe will realize some pattern of events (lying to our future).

Having addressed Haag’s event ontology through consideration of GRW, I now come to address Rovelli’s (2005) paradox that “the statement that a certain specific outcome ‘has happened’ can be true and not-true at the same time” (p. 115) in the context of the Everett interpretation. In the Everett interpretation, the proposition that a particular outcome obtains (e.g. spin ‘up’) can be both true (with respect to an observer regarded as confined to a branch of the Universal Wavefunction) and not-true (with respect to the description afforded by the Universal Wavefunction itself). The Everett interpretation could scarcely take the dynamical (unitary) evolution of the state of a quantum system more seriously, yet it too allows for the notion of an event.

Bacciagaluppi (2002) suggests that the Everett interpretation might provide an adequate expression of Haag’s (1996) earlier contention that quantum theory is, ultimately, about the observer-independent occurrence of events. Bacciagaluppi contends that a process of decoherence leads to a branching structure of decoherent histories which can be understood

in terms of events located in space-time. Bacciagaluppi addresses this idea in the context of a relativistic space-time, where the allowed decoherence interactions are, therefore, to be constrained by locality conditions.

The conjecture is thus the following. If the axioms [of Haag's Local Quantum Field Theory] characterize decoherence interactions, any decoherent histories will be made up of local projections (in fact local *effects* [i.e. elements of  $\mathcal{E}(\mathcal{H})$ ] ...). The 'events' in each history will be embeddable in a Minkowski space-time. Identifying histories with Everett worlds, Everett worlds will thus have a Minkowski space-time structure. ... Our current motivation is to consider the (branching) space-time we live in as the collection of events *created* by the process of decoherence. (p. 116, original emphasis)

Less speculatively, similar ideas arise in the context of non-relativistic Everettian quantum mechanics through the use of decoherent histories, introduced into this context by Saunders (1995).

Very roughly, the idea is that, outside of the carefully controlled conditions of a laboratory experiment, a quantum system is continually interacting with its environment, a system of vastly many more degrees of freedom. Any system undergoing continuous interactions of this sort is subject to a process of decoherence which very rapidly acts to correlate states of the system with states of the environment. Tracing out the environmental degrees of freedom, the reduced density matrix of the system is, in general, found to be approximately diagonalized with respect to some dynamically preferred basis—the decoherence basis. If one were to decompose that vector basis into a sum of projections, dynamically allowed states of the system acted on by those projections would be (approximately) orthogonal. If there exists at some time such a decoherence basis, then there exists a sum of Heisenberg picture (instantaneous) projections onto (approximately) orthogonal states that forms a resolution of the identity.

A decoherent history is a time-ordered string of these projections, one from each such instantaneous decomposition. Two decoherent histories are (approximately) orthogonal when they consist of strings of mutually orthogonal instantaneous projections (considered pairwise at the same instant). In this framework, an event corresponds to an instantaneous projection and a decoherent history specifies a time-ordered set of events. This time-ordering suggests that the set of such histories has a branching structure, but recent work has shown that an

interpretation in terms of non-overlapping divergent worlds may be feasible (Saunders, 2010) or even preferable (Wilson, 2012).

To differentiate two worlds according to *when* an event occurs in this framework would require a series of instantaneous projections each corresponding to the occurrence of the same event, normalized over an appropriate time period such that the probability of occurrence is unity. Consider a process of radioactive decay. Decay at a time  $t$  would be represented by a set of projections, and a world in which decay occurs at  $t_1$  would correspond to a history that includes the ‘decay at  $t_1$ ’ projection at  $t_1$  but no other ‘decay at  $t$ ’ projections. The temporal normalization condition would amount to an extra constraint on the allowed histories so that they do not include repeated occurrences of the same decay event.

However, it is not at all clear how decoherence could act to bring about this situation when what is at issue is a process occurring independently of decoherence, in this case radioactive decay. Now, it could well be that the Heisenberg state of the entire system of decaying atom and environment can be given in such a way that the occurrence of decay leads to environmental decoherence *once it occurs* (in the sense that the decay product may be detected within a Wilson cloud chamber, say, leading to a macroscopic effect). But the decay process itself is not subject to decoherence effects (or at least is stable with respect to them) since it unfolds independently of its environment. What this suggests is that the consideration of decoherent histories is not going to be of help when our concern is with predicting *when* an event occurs, since the relevant constraints on those probabilities cannot be given instantaneously.

This is not to say that decoherence effects are not relevant to when an event occurs (considered independently of the decoherent history approach) because, where decoherence effects are significant, the state from which the probabilities for the time of occurrence is to be derived will be that of the system *and* environment. That is, one should consider environmental interactions to be relevant in describing the outcome of an experiment to the extent that it becomes empirically problematic to treat the system of interest as an independent and isolated object. The actual occurrence of an event at a time, however—considered as an experimental outcome—results here from Haag’s principle of random realization, which is a process that cannot be described unitarily.<sup>3</sup>

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<sup>3</sup>I take this point to be a consequence of the various proofs for the insolubility of the measurement problem. See, for example, (Busch et al., 1996, p. 73–79).

The point at issue here is not the existence of non-actual possible worlds—everything I have said in Chapter 8 is consistent with modal realism in as strong a sense as one could want—but rather the adequacy of unitary quantum mechanics as an empirical theory, i.e. as a valid description of every quantum phenomenon and process as it occurs in Nature. I do not claim to have *proved* here that unitary quantum mechanics is empirically or descriptively inadequate, or even to be able to give a strong argument in favor of such a claim. Rather, in suggesting that there is an alternative way of formulating quantum theory within an extended Hilbert space of functions of time and space, there arises the possibility that this extended theory may account for phenomena that the usual formalism does not.

Aside for accounting for the time of occurrence, which (arguably) may be achieved within the usual unitary framework, the phenomena I have in mind here concern time-energy interference. In particular, there are several experiments that can be regarded as a demonstration of *diffraction in time*, i.e. as representing something like a two-slit experiment in the energy-time domain.<sup>4</sup> In particular, the recent experiment of Lindner et al. (2005) leads Horwitz (2006) to argue that “The standard nonrelativistic quantum theory cannot be used to predict interference in time” (p. 1). In essence, his argument is just that instantaneous states cannot be integrated over times in a manner that would allow one to introduce ‘temporal superpositions’ of instantaneous states. This problem is entirely obviated by my proposed move to the extended Hilbert space (and, eventually, the extended Schrödinger equation). There, the occurrence of an event at an indeterminate time can be represented by a projection corresponding to the union of disjoint temporal intervals in just the same way as an indeterminate location in space may be represented in Ordinary QM.

I have not carried out the analysis here, but the possibility offered here for empirical tests that might choose between various competing interpretations or formulations of quantum theory is worthy of note. That is, the use of a Bohmian or dynamical collapse theory to describe these phenomena (for instance) may result in distinctive predictions for phenomena in the time domain that serve to empirically distinguish an interpretation from its competitors, and may even pick out one uniquely. This provides a strong motivation for further research concerning the questions addressed in this dissertation

There are (at least) two further reasons why the extended formulation of quantum theory suggested here is worthy of further study. First, the importance for theories (or at least

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<sup>4</sup>See (Busch, 2007, §3.6.3) for a recent review.

toy models) of quantum gravity is immediately suggested by the quantization of Brunetti et al. (2010) of a Freidman-Robertson-Walker cosmological model. The interpretation I have suggested for the extended Hilbert space theory in terms of events may usefully be applied to quantum versions of constrained Hamiltonian systems, considered without the derivation of a ‘physical’ Hilbert space of states annihilated by the constraint operator. Second, formulating of relativistic quantum mechanics within the extended framework allows for consideration of relativistic localization within a setting appropriate for a theory set in space-time. It is my expectation that the alleged problems with non-locality (see, e.g., Malament (1996)) will prove to be an artifact of the consideration of an instantaneous formulation of the theory.

## 9.2 RELATIONAL TIME, CLOCKS, AND EVENTS

In Chapter 6 I provided a detailed study of the definition and construction of quantum clocks—quantum systems that could be used to ascertain the passage of time. The operational significance of clock variables in general lies in their use in setting up a system of co-ordinates in which the laws of motion hold. That is, we use the regular motions of some physical system to set up a time co-ordinate with which respect to which the motions of other physical systems may be defined. The philosophical significance accrued by proofs for the existence (or otherwise) of these physical systems that provide good clocks will depend on one’s background metaphysics of time. Here is a (very) rough characterization of three such metaphysical positions and the significance they afford to the existence of good clocks.

**Realist** External time is absolute and a good clock is one that measures the passage of absolute time.

**Constructivist** Time is nothing more than that which is read by a good clock.

**Relationist** Time is an order on events and nothing more.

For the realist, the existence of physical systems that act as good clocks is nothing but a pragmatic issue. Newton took a characteristically obdurate stance on this matter.

It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore this duration ought to be distinguished from what are only sensible measures thereof; and from which we deduce it, by means of the astronomical equation. (Newton, 1934)

The significance of the ‘astronomical equation,’ which concerns the motion of the entire solar system, is that (in Newton’s time) it provided the most effective means to correct the non-uniformity of measures of absolute time such as the diurnal rotation of the Earth. The equation of motion for the solar system provides a more stable standard against which the uniformity of other motions may be judged.

For Newton, this is just a pragmatic affair of finding the best available standard for temporal duration, defined as that which most closely approaches the intervals marked off between events by absolute time. Absolute time is, admittedly, unobservable in itself, but since absolute motions are given with respect to it there is some hope for using those motions to arrive at a good approximation of absolute time. No measure of time that we could possess is incorrigible, but we may approach absolute time through successively better approximations. The realist, it seems, can rest content with the analysis of quantum clocks offered here.

For the constructivist, however, the matter is of critical importance since her very notion of temporal structure is closely linked to the means by which it may be measured. To take a recent example, Brown (2005) argues that the significance that the realist attaches to the geometrical structure of space-time is unjustified. He contends that what space-time really represents is nothing more than a codification of the behavior of special types of physical systems—good clocks and rigid rods—under certain dynamical motions corresponding to symmetry transformations. On this view the use of a dynamical theory to define a good clock must take on a special significance, since the very idea of a space-time geometry is to be parasitic on the behavior of such systems. If such systems are not available, according to the dynamical laws, then the structure of space-time lacks a precise operational definition.

For the constructivist, then, the existence of systems that act as good clocks is critical for the definition of a notion of time. But there remains a sense in which the constructivist maintains that time is to be given independently of the contents of the world: space-time

geometry is to be thought of in terms of the behavior that a clock *would* display in such-and-such a situation. For the relationist, on the other hand, the very notion of time is to be constructed from the contents of the world as given, without reference to ideal instruments or operations. In particular, Leibniz viewed the external time co-ordinate as ideal not because it was to be measured by an ideal clock, but because it must be constructed from (and thus must go beyond) what is given in the physical world: the temporal order of events.

The idea behind relationism is the following: given a full account of reality, the spatio-temporal facts are fixed by the relations among the events and objects of the world.<sup>5</sup> Moreover, there are no other temporal facts to be had. As Leibniz puts it: “times, considered without the things *or events*, are nothing at all, and . . . they consist only in the successive order of things *and events*” (Leibniz, 1716, original emphasis). In particular, Leibniz denies that time has metrical properties, to which Clarke (in their famous correspondence) responds: “it is obvious that time is not merely the order of things succeeding each other, because the quantity of time may be greater or less while the order of events remains the same.”<sup>6</sup>

More recent accounts of relationism in the context of dynamical theories have sought to implement these ideas by stripping out any trace of the objectionable absolute relations from the theory. Barbour & Bertotti (1982) set out to construct a theory of classical mechanics whose notion of state made use only of relationist-friendly relative spatial distances. The instantaneous configuration of a system of particles was to be completely specified by the pair-wise relative distances, and the dynamics of such a theory consists merely in the order in which such configurations obtain. By their procedure of ‘best-matching’ Barbour & Bertotti were able to deliver a dynamical ordering that recovered a significant fragment of conventional classical dynamics without reliance on an illicit external time parameter.

This constructed ordering allows them to recovering a time parameter from the actual course of events which need not be given ontological significance. Indeed, Barbour (1999)

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<sup>5</sup>Witness Leibniz: “This shows us how we should understand the statement ‘God created things at what time he pleased’, for it’s just a matter of what things he chose to create. Once he had decided on the things, and on how they were to relate to one another, there was no further choice to make concerning the time and the place, which have no intrinsic reality, nothing that can distinguish them, nothing discernible.” (Leibniz, 1716)

<sup>6</sup>Leibniz’s response to this (Fifth Letter, para. 54) is somewhat fudged, since he asserts that “order also has its quantity; there’s what goes first and what follows; there’s distance or interval.” Does he mean to concede the point to Clarke and admit that time does indeed have metrical properties in addition to order? Or is this to be read as a claim that the order of events suffices to determine the metrical relations of events, but temporal order is nonetheless fundamental? I believe the latter is the correct interpretation.



argues for the elimination of time from our ontology. But there remains a sense in which events have a objective temporal order since the successive states of the universe (in the description he hopes to recover) serves to define an effective notion of time. A more radical eliminativist view has been suggested by Rovelli (1991, 2009) motivated by his theoretical concern in the field of quantum gravity: Loop Quantum Gravity (Rovelli, 2004). According to Rovelli, there is no unique temporal ordering to be had: a description of the world according to our best physics fails to provide a unique ordering of events.<sup>7</sup> In terms of Rovelli's physics, the reason for this is simple: at the level of fundamental physics there simply is no change. Given the mutual inter-dependence of time and change, if there is no change then there is no time. This view has been described perspicaciously as *Parmenidian* by Belot & Earman (2001).<sup>8</sup>

It is incumbent on Rovelli, nonetheless, to recover our experience of time and change, which he attempts to do through the introduction of a relative notion of temporal evolution (Rovelli, 2002). While, according to his physics, there can be no unique ordering of events, there is an opportunity to recover an effective notion of change in terms of the relative values of physical variables, treated as clocks. Thus our experience of time is to be cashed out in terms of a subsystem of the world, characterized by a physical variable treated as a clock. Suitably chosen, the values of this 'clock' provide a parameterization of the evolution of the physical variables pertaining to the remainder of the system. This description of evolution and change, however, is essentially illusory, and any such description according to a particular decomposition and choice of clock is on all fours, metaphysically speaking, with another conflicting description according to a distinct choice of clock.

Although Rovelli (1997) describes himself as a relationist (since he does not posit separate or substantial spatio-temporal structures) there is a sense in which Barbour is, and Rovelli is not, a relationist about time. Barbour provides a global temporal order, which satisfies the Leibnizian desire for time to be given by relations among the events. On Rovelli's view, however, it is not clear what there is that can be described as a temporal relation, and certainly there is no objective order on events (nor an objective partial order).<sup>9</sup> Thus it

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<sup>7</sup>Note that this is distinct from relativistic concerns regarding the failure of the structure of a relativistic space-time to determine a unique global temporal order but rather a partial order on timelike events.

<sup>8</sup>After the pre-Socratic philosopher-poet who first turned the problem of change into a *modus tollens* against the existence of time.

<sup>9</sup>There is also the 'thermal time hypothesis' of Connes & Rovelli (1994), according to which time at a macroscopic level may be recovered as a statistical or thermodynamic phenomenon. This is reminiscent of the account given by Heisenberg in 1927. See Section 2.1.2.

seems that one could argue that Rovelli provides here an expression of Mach's philosophy, since he regards spatio-temporal notions as redundant: according to Rovelli, a complete description of the universe need make no mention of time, space, or temporal order.

The physical space I have in mind (and which also includes time) is therefore nothing more than the dependence of the phenomena on one another. The perfect physics, which knew of this basic dependence, would have no need of separate concepts of time and space because they would be already exhausted [by this description]. (Mach, 1866, translated)

This is because Rovelli hopes to give such a description in terms of a (proposed) theory of quantum gravity, Loop Quantum Gravity, which appears to do without such notions. On the other hand, Barbour appears to believe that there are determinate facts concerning temporal order, although the facts in question depend on the distribution of matter (or, in general relativity, the relation of spatio-temporal geometries).

But note that none of these latter day relationist physicists have considered time according to Leibniz's definition: as an order on *events*.<sup>10</sup> The possibility I have offered for interpreting quantum theory in terms of an ontology of events suggests the possibility of a relationist account of time based on this Leibnizian notion of an order on events.

The dialectical situation is as follows: the realist (or substantialist) posits the features of time (and space) as she needs them, securing its existence by what is essentially an appeal to an inference to the best explanation. The relationist rejects the explanatory value of this schema, since she regards the independent existence of time as a highly dubious proposition. (And why should she not? An appeal to the existence of time in an answer to the question 'What is time?' is hardly explanatory.) In Russell's words:<sup>11</sup> "The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil." (Russell, 1993).

Since Russell's moral character was (more or less) unimpeachable, he was compelled to take up the challenge of recovering the necessary features of substantial time (assumed to have the structure of the real line) in terms of more fundamental relations among events. This would, therefore, provide Leibniz with the means to answer Clarke's complaint that

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<sup>10</sup>Against Barbour, I would argue that he has provided an order on instantaneous spatial configurations of point particles (or fields, or spatial geometries, or whatever). A instantaneous spatial configuration is not an event but rather a state of affairs.

<sup>11</sup>Used here out of context, admittedly.

order is not sufficient to ground metrical claims. Russell's last publication on the topic was 'On Order in Time' (1936), in which he investigated the recovery of instants of time from fundamental relations among events of temporal *precedence* and *overlap*. This has the spirit of a reductionist project, since certain features of time (such as metrical features) are to be recovered from more fundamental relations of order.<sup>12</sup> However, the reduction in question is partial at best since temporal precedence and overlap are themselves irreducibly temporal notions. This is not, therefore, a reductive account of time, but of *times* (i.e. instants of time).<sup>13</sup>

### 9.3 IN DEFENSE OF RUSSELL'S TEMPORAL RELATIONISM

Meyer's (2013) fascinating recent monograph *The Nature of Time* argues for a 'modal' account of time by introducing instants of time as logical constructs related by primitive tense operators. He begins the book by offering a novel negative argument against temporal relationism, the view that times are to be analysed in terms of relations between events. With temporal relationism out of the way, Meyer advocates his view as the best hope for an account of time that meets the three criteria laid out in his Chapter 1: (i) solves the Problem of Change; (ii) agrees with physics; (iii) agrees with the folk theory of time. He also places their importance, I believe, in that order.

Here I will argue that Meyer is wrong to dismiss Russell's relational account of time on the basis of the argument offered in Chapter 2 of *The Nature of Time*. I contend that Russell (1927) can escape the negative conclusion of Meyer's argument by sticking to the view he presents in *The Analysis of Matter*, according to which material objects are sets of events. If Meyer's negative argument is unsuccessful, as I claim, then Russell's relationism deserves to be considered as a genuine competitor to Meyer's view. Moreover, in this dissertation I have argued that fundamental physics can be understood in terms of an event ontology, which has a mutually supportive relationship with an account of time based on the relations among events.

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<sup>12</sup>For recent attempts to see Russell's program through to completion equipped with some more up to date mathematical machinery see Anderson (1989); Bostock (2010).

<sup>13</sup>Leibniz, it seems, sought to ground the relations of temporal order in causal relations (see Futch (2008)), but every explanation has to stop somewhere.

### 9.3.1 Russell's Analysis of Matter

Russell (1927) confronts the question of the constitution of matter in the following terms:

Assuming that the physical world consists of substances with qualities and relations, are these substances to be taken as permanent bits of matter, or as brief events? Common sense holds the former view, though its "things" are only quasi-permanent. But science has found means of resolving "things" into groups of electrons and protons . . . The question is: Are electrons and protons part of the ultimate stuff of the world, or are they groups of events, or causal laws of events?" (pp. 243–4)

Russell proposes a two-stage reduction of 'things' (material objects) to events. First, the material objects of ordinary experience may be reduced in terms of the material objects posited by fundamental physics (electrons and protons). Second, these fundamental objects are to be reduced to series of events. Here is Russell's account of that second reduction:

The substitution of space-time for space and time has made it much more natural than formerly to conceive a piece of matter as a group of events. . . . Instead of a permanent piece of matter, we have now the conception of a "world-line," which is a series of events connected with each other in a certain way. The parts of one light-ray are connected with each other in a manner which enables us to consider them as forming, together, one light-ray; but we do not conceive a light-ray as a substance moving with the velocity of light. Just the same kind of connection may be held to constitute the unity of an electron. We have a series of events connected together by causal laws; these may be taken to *be* the electron, since anything further is a rash inference which is theoretically useless. (p. 245, original emphasis)

So Russell's conception of matter in term of events essentially follows from relativity: in relativistic physics, a material body follows a timelike curve (its world-line) and so it can be thought of as nothing more than the events linked by that curve. In a later chapter, he gives a concise summary of his view as follows: "Electrons and protons, however, are not the stuff of the physical world: they are elaborate structures composed of events, and ultimately of particulars." (p. 386).

He also explicitly defines change in these terms, and rejects the idea than an electron is anything more than a series of events.

Strings of events exist which are connected with each other according to the laws of motion; one such string is called one piece of matter, and the transition from

one event in the string to another is called motion. ... it is prudent, in physics, to regard an electron as a group of events connected together in a certain way. An electron *may* be a “thing” but it is absolutely impossible to obtain any evidence for or against this possibility, which is scientifically unimportant, because the group of events has all the requisite properties. . . . There is every reason, from the standpoint of perception, to desire an interpretation of physics which dispenses with permanent substance. (p. 246)

It seems quite justified, therefore, to regard Russell as wanting to *eliminate* material objects qua ‘persisting things’ from our ontology. The apparently permanent (or quasi-permanent) material objects of perception are to be analyzed, without remainder, in terms of events. This is made particularly clear in his critical discussion of Whitehead, who views spatio-temporally located events as mere aspects of a persistent object or process (to put it very roughly). Russell objects to this view “on purely logical grounds” as follows:

Given a group of events, the evidence that they are “aspects” of one “thing” must be inductive evidence, derived from perception . . . in calling two events “aspects” of one “thing,” we imply that their likeness is more important than their difference; but for science both are facts, and of exactly the same importance. (p. 248)

The idea appears to be this: the series of events that we take to compose a material thing are taken by us to be sufficiently similar to regard as a single persisting thing, but they are nevertheless distinct and so cannot be regarded as the aspects of the *same* thing. This is, in effect, to take the Problem of Change as an argument against the claim there exist persisting objects, self-identical at every time they exist. Russell contends that if events compose a material object with no remainder, then there simply cannot be a single thing that possesses the properties of all the events. In this respect, Russell’s view is eliminativist about material objects: in his event ontology, there is nothing that satisfies the requisite criteria. This is perfectly consistent with the view of quantum theory I offered in Chapter 8, which proposed to do without the idea that quantum systems are persisting objects.

### 9.3.2 Avoiding Meyer’s Negative Argument

Meyer’s (2013) argument against temporal relationism begins with his characterisation of relationism as the view that “all there is to time are events that stand in the relations of temporal overlap and precedence to one another.” (p. 2).<sup>14</sup> He proceeds to argue that events

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<sup>14</sup>All references are to Meyer (2013).

must be metaphysically simple if they are to play this role (pp. 15-16). He then argues that events cannot be metaphysically simple if objects are to participate in events. He defines participation as follows: “a material object participates in event  $e$  if the event’s occurrence is conditional upon that object’s existence” (p. 18).

Meyer’s argument is valid, but I do not think Russell’s relationism is threatened by it. We can lay out Meyer’s argument against relationism as follows:

1. (*Relationism*) All there is to time are events that stand in the relations of temporal overlap and precedence to one another (p. 2)
2. Events cannot be both the primary occupants of the time series and be metaphysically complex (pp. 15-16)
3. Events are metaphysically simple (from 1. and 2.)
4. A material object participates in an event if the event’s occurrence is conditional upon that object’s existence (*Df.* p. 18)
5. Participants in past events are participants in present events (from 4.)
6. If 5. is false, then either 1. is false or objects do not participate in events.
7. If 1. is true then either no objects participate in events or events are complex (p. 19)
8. Events cannot be both the primary occupants of the time series and be metaphysically simple. (from 1., 3. & 7.)
9. Relationism is false. (1., 3. & 8. by RAA)

The reason why this argument would not have troubled Russell is that Russell did not believe that material objects participate in events. In *The Analysis of Matter* Russell adopts an eliminativist (or at least reductionist) view of material objects, according to which they are to be analyzed in terms of events. That is, according to Russell, material objects are not metaphysically simple, and objects depend on events for their existence rather than the other way around. So Russell would regard the conditional statement of 4. as true but vacuous, and he could have accepted the dilemma of 6. without difficulty as rather than being forced to consider events to be complex, he could have taken the other horn by concluding that objects do not participate in events.

This way out of the argument is considered by Meyer (although not on behalf of Russell) and is dismissed on the grounds that

if we have no account of the relation between material objects and time then we also have no account of what it is for an object to have a property at a time, and thus no solution to the Problem of Change (pp. 19–20).

But rejecting the idea that material objects participate in events does not entail that one cannot provide an alternative account of the relation between material objects and time, and Russell does offer such an account—an account, moreover, that offers a solution to the problem of change. The reductionist view of matter that Russell argues for in *The Analysis of Matter* defines a material object as just a group of events. But, given relationism about time, the relation of material objects to time is straightforward: it is given in terms of the temporal relations possessed by the group of events which together compose the object.

Meyer regards the Problem of Change as that of “reconciling the possibility of change with the Indiscernability of Identicals . . . [where] the object [*a*] experiences change if and only if the following are true: *a* is *K*; *a* is not *K*” (p. 4). But since Russell identifies a material object with a series of events his solution is straightforward: *a* is nothing but a series of events, temporally related, some of which are *K* and some of which are not *K*. As a simple example, take *a* to be identified with a series of events  $e_i$ , and *K* to be ‘located in spatial region *R*.’ Then *a* is located in spatial region *R* if some subset of  $\{e_i\}$  is located in *R*, and not located in spatial region *R* if some subset of  $\{e_i\}$  is not.<sup>15</sup> So long as the same  $e_i$  is not said to be both be located in *R* and not located in *R*, then there is no contradiction involved in saying that *a* is *K* and *a* is not *K*.

This establishes, then, that Russell (at least, the Russell of *The Analysis of Matter*) can embrace the conclusion that no material objects participate in events because, on his view, *there are no material objects*. Thus he can avoid Meyer’s argument against temporal relationism by accepting premise 6. and asserting that objects do not participate in events. This suggests that an event ontology allowing for a relational account of time will have to deny the independent existence of objects, considered as concrete individuals persisting through time. Another consideration that Meyer invokes is the consistency of relationism

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<sup>15</sup>It may be objected this example is misleading, since one is also owed an account of regions of space in terms of events from the relationist. If this is troubling you, then substitute some other less controversial predicate in the above analysis.

with physical science. Russell takes his analysis to be suggested by both relativity and quantum theory (at least, quantum theory as he understood it). I think Russell was right, and his analysis can be brought up to date with a current understanding of modern physics. This dissertation can be read as an attempt to provide a suitable understanding of quantum theory; the corresponding project for relativistic physics awaits another occasion.

#### 9.4 CODA: AN ONTOLOGY FOR SCIENTIFIC REALISM?

There are interesting echoes of Russell's proposed reduction of objects to events in the recent move of Ontic Structural Realists to deny the existence of objects, tout court. As an indication of their rhetorical intent, Ladyman & Ross' (2007) recent book is suggestively entitled *Every Thing Must Go*. Like Russell, according to the Ontic Structural Realist there are no objects (conceived as self-subsisting independent entities) but, unlike Russell, there are also no non-structural elements of reality at all: structure is all there is. But Russell's denial of the existence of objects came from his reduction of persisting objects to series of events, considered as particulars. This suggests that the two theses may be separated: on one hand we have the denial that (unanalyzed) concrete material objects exist, on the other the claim that there are no concrete particulars *at all*.

Where Ontic Structural Realism has come under attack from metaphysicians as incoherent, it is often the latter claim that has drawn their ire: how can one have structural relations without the existence of relata for them to relate? Russell's proposal for an event ontology neatly sidesteps that objection since the structural relations in question *do* relate concrete particulars: events. It is noteworthy here that Russell's event ontology provides the historical context for his adoption of structural realism.<sup>16</sup> Where Russell advised restricting belief to structural relations, these were relations among ontologically independent events rather than objects or theoretical entities.

This use of events as a platform for structural realism suggests a possible rapprochement of scientific realism and anti-realism, whereby reference to unobservable entities posited by science (electrons, positrons etc.) is to be understood in terms of events. The reference

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<sup>16</sup>This circumstance that is seldom reflected upon in later discussions of Russell's structuralism, where he is often taken to be an Epistemic Structural Realist, e.g. (Frigg & Votsis, 2011).



relation, therefore, picks out certain series of events with a particular set of characteristic properties rather than some set of independent entities given a definite description in theoretical terms. The refinement and replacement of scientific theories by theories which afford incompatible descriptions of those unobservable entities does not imply that the earlier theories were unable to secure reference, since the (types of) series of events to which they referred may be picked out by other means which do display historical continuity (e.g. experimental practice).

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