Quantum Mechanics for Event Ontologists

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What is quantum mechanics about? That is, what is the intended domain of an interpretation of the theory? In the long history of attempts to interpret quantum theory a wide variety of answers have been given to this question, including: observations, experiments, wavefunctions, the universe, point particles, information. In this paper I explore a particular view of quantum mechanics which maintains that it is a theory of *events*. On this event ontology view, the probabilities supplied by the quantum state are probabilities for the occurrence of events, and the observables of the theory are to be interpreted accordingly. In contrast, the standard (Dirac-von Neumann) view maintains that observables correspond to physical quantities which, when measured, come to have definite values—values that represent possessed properties of the system. In this vein, Wightman (1962) interprets position experiments in terms of localization: on measurement, a particle comes to be localized within a particular region of space; it has the property of existing *here* rather then somewhere else.

The event view has had some recent interest from notable theoretical physicists such as Carlo Rovelli and Rudolf Haag, who express in different ways the core idea of this view. Rovelli (2005) contrasts the "‘wave function ontology," which takes the state "as the ‘real' entity which fully represents the actual state of affairs of the world," with his proposal for an "ontology of quantum events."

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> A better alternative is to take the observed values . . . as the actual elements of reality, and view [the state] as a mere bookkeeping device, determined by the actual values . . . that happened in [the] past. From this perspective, the real events of the world are the 'realizations' (the 'coming to reality', the 'actualization') of the values . . . in the course of the interaction between physical systems. These quantum events have an intrinsically discrete (quantized) granular structure. (p. 115)

The key idea is that the changing quantum state given by the dynamics of the theory does not describe the changing properties of some physical object.[1] Rather, the quantum state describes the probabilities for events to occur; events that often arise as the result of interactions between systems.

In turn, here is Haag's (2013) recent critique of the conventional view:

> What do we detect? The presence of a particle? Or the occurrence of a microscopic event? We must decide for the latter.. . . [T]he standard use of the term "observable" does not really correspond to the needs of collision theory in particle physics. We do not measure a "property of a microscopic system", characterized by a spectral projector of a self-adjoint operator. Rather we are interested in the detection of a microscopic event. The first task is to characterize the mutually exclusive alternatives for such an event. (p. 1310)

So in practice, i.e. in the context of a particle detection experiment, the theory concerns—is about—microscopic events, such as the ionization of a molecule by a cosmic ray. The detector is

[1]In approvingly citing Rovelli's commitment to an ontology of events I do not mean to endorse his accompanying interpretation of quantum theory, Relational Quantum Mechanics, about which there is much to criticize. I shall not do so here, however.

expressly designed to amplify these micro-events such that they reliably lead to a macroscopic record of detection, by which we mean detection at a place, at a *time*.

My main concern in this paper is the task of re-interpreting the spectral projectors of position in terms of events, and constructing an appropriate probability space for these events. That is, my focus will be on the re-interpretation of Wightman localization in terms of the occurrence of localized events within extended spatio-temporal regions. I claim that, interpreted in terms of events, there is a crucial further question concerning localization to which quantum mechanics must supply an answer: *When* does an event occur? Providing a satisfactory answer to this question, I contend, gives an informative account of Haag's 'Principle of Random Realization' and thus avoid Rovelli's paradox that "the statement that a certain specific outcome 'has happened' can be true and not-true at the same time" (p. 115).

The paper is laid out as follows. In Section 1, I provide an account of Wightman localization. In Section 2, I present a result that makes trouble for the conventional account of localization in terms of the possession of a property. In Section 3, I explore the idea that a localization experiment can be interpreted as a conditional probability for an event to occur in some region of space, given that it occurs at some time. I give an expression for these conditional probabilities in terms of Lüders Rule by forming a temporally extended space of histories of a system. In conclusion, I propose a philosophical interpretation of these probabilities as Lewisian objective chances, conditioned on the future occurrence of an event in the possible worlds to which the chances apply, but maintain that there is no need to adopt a corresponding Everettian 'no collapse' interpretation of quantum mechanics to accommodate them.

1. Localization as a Property The position of a quantum particle is given by the position observable, Q. But, considered as an operator, Q returns the *expectation value* of position. That is, a system in state ψ (a unit vector in $\mathcal{H}$, a Hilbert space) has an expectation value for position $\langle\psi|Q\psi\rangle$. This does not refer to the position of a particular system, or a particular run of a position experiment, but rather a certain characteristic of the likely distribution for an ensemble of such systems. In the conventional interpretation, probabilistic statements about individual systems take the form: 'The value of position lies in the interval $[a, b]$.' This statement is associated with a spectral projector of Q, $P^Q_{[a,b]}$, through the Spectral Theorem, and the probability that, on measurement, the statement is found to be true is $\langle\psi|P_{[a,b]}\psi\rangle$. If such a statement is true, i.e. if $\langle\psi|P_{[a,b]}\psi\rangle = 1$, then this has been interpreted as saying that the system is located within the spatial region corresponding to $[a, b]$.

Wightman (1962) showed that directly associating a projector P_Δ with a spatial region Δ, and demanding of these projectors that they be appropriately related under symmetry transformations of the underlying space-time, suffices to determine uniquely the position operator Q. In his "notion of localizability in a region" Δ, these projectors are "supposed to describe a *property* of the system, the property of being localized in Δ" (p. 847, original emphasis). A localization experiment corresponds to a series of experimental questions[2] which ask "is the particle located in Δ?" One of the crucial features of such questions is that they must be asked at a particular instant of time so that quantum mechanics may provide answers via the Born Rule.

That is, a projector P_Δ associates with a state of the system ψ a vector $P_\Delta\psi \in \mathcal{H}$. According to the Born Rule, the probability of finding a system in state ψ to be located in the region Δ is given by the inner product $\langle\psi|P_\Delta\psi\rangle$. Thus if the probability is one, it must be the case that $P_\Delta\psi = \psi$. This is the eigenstate-eigenvalue link: if the system is located in Δ then the state ψ is an eigenstate of the projection P_Δ, and if the state is such an eigenstate of P_Δ then it is located in

[2]This term is due to Mackey who provided the more general notion of a System of Imprimitivity, of which Wightman's system of localization is an example.

Δ with probability one. Introducing time through the Schrödinger picture, we associate a time indexed state $\psi_t \in \mathcal{H}$ with a time t through the unitary group $U_t = e^{iHt}$ by setting $\psi_t = U_t\psi$, where H is the Hamiltonian operator and ψ is the state at $t = 0$.

Operating on these time-indexed states, the projector returns a vector $P_\Delta\psi_t \in \mathcal{H}$, which (properly normalized) is a state located in Δ at the time t. The probability of localization in Δ at a time t is, therefore, $\langle\psi_t|P_\Delta\psi_t\rangle$. However, in general a state such as this will fail to be localized in Δ at any other time. In fact, we can say something quite definitive about the character of the times at which a state can be localized in this way. To do so requires the Heisenberg picture, which is reached by considering the time evolution of the observables rather than the state. That is, the Heisenberg projector corresponding to localization in Δ at time t is $P_\Delta(t) = U_{-t}P_\Delta U_t$, and the Heisenberg state of the same system is $\psi = \psi_0$, for all t.

Thus the probabilities supplied by the Heisenberg picture are numerically identical to those of the Schrödinger picture since $\langle\psi|P_\Delta(t)\psi\rangle = \langle\psi|U_{-t}P_\Delta U_t\psi\rangle = \langle\psi_t|P_\Delta\psi_t\rangle$. Conceptually, however, it is now more straightforward to associate properties of the system possessed at distinct times with the state of the system described by a single vector $\psi \in \mathcal{H}$. A system localized in region Δ_1 at t_1 and Δ_2 at t_2 must be associated with a state ψ such that $P_{\Delta_1}(t_1)\psi = P_{\Delta_2}(t_2)\psi = \psi$.[3]

2. A Problematic Result I will now show that the times at which a system is localized in this way are severely limited, so long as the Hamiltonian of the system obeys a common requirement known as the Spectrum Condition, which requires the system to have lowest value of energy below which it cannot drop. If this condition holds, as it does for all physically reasonable

[3]This amounts to the assumption that $P_{\Delta_1}(t_1)P_{\Delta_2}(t_2)\psi = P_{\Delta_2}(t_2)P_{\Delta_1}(t_1)\psi$, which won't be true in general (Malament 1996). This difficulty is closely related to the problem I display below.

quantum systems, then there can be no time interval such that a system is localized in some region at every time in that interval, unless it is localized within that region at *all* times. This follows from the following proposition.

Proposition 1. *Let P be a projection operator associated with a property of the system, let $\psi \in \mathcal{H}$ be a vector of unit norm in a separable Hilbert space, and let $U_t = e^{iHt}$ be the one-parameter unitary group uniquely generated by H, a self-adjoint operator with semi-bounded spectrum. Let $P(\{t_k\})$ be the projection operator that corresponds to the possession of the property P at every time $t \in \{t_k\}$. That is, ψ is in the range of $P(\{t_k\})$ if $P(t)\psi = \psi$ for all $t \in \{t_k\}$. Let ψ be in the range of $P(\{t_k\})$ then* either*:*

1. *$\{t_k\}$ is a set with zero Lebesgue measure,* or
2. *ψ is in the range of $P(\mathbb{R})$, i.e. $P(\{t_k\}) = P(\mathbb{R})$.*

Therefore, there is no projection $P(I)$ that corresponds to the possession of a property at an open interval of instants $I \subset \mathbb{R}$ and at no other time.

Let us consider the implications of this result in the context of localization. Applied to the projector P_Δ, the result says that the states which possess the property of being localized in Δ at more than one time are severely limited. One way of interpreting the result would be to say that if a system is confined to a region of space for a time interval (i.e. a continuous set of instants) then it is confined to that region for all time. This resembles the Quantum Zeno (or Watchdog) Effect where continuous measurement of a projector confines the evolution of the system to a subspace of its state space (Misra & Sudarshan 1977).

If, on the other hand, we have reason to believe that the system is not localized in some region for all time, then the times at which the system is localized *anywhere* within that region are severely limited. This is because the result applies to subregions as well. That is, if $\Sigma \supset \Delta$ is a larger region that includes Δ then $P_\Sigma(t)\psi \neq \psi$ implies that $P_\Delta(t)\psi \neq \psi$.[4] So the conclusion of the result restricts localization within a subregion of the region just as much as it does for the region itself, and the region under consideration could be as large as one likes: the Earth, the Solar System, or so on. Moreover, the condition that a state must satisfy to be localized within a region for a time interval is so severe that a non-zero *probability* that the system is localized anywhere outside of the region at *any* time t entails that the set of times at which it is localized within the region has measure zero.

The upshot of all this is that if we are to admit the mere possibility that the particle could be detected outside of the lab next week, then it cannot be localized within the lab today at more than a set of times with zero measure. Therefore, on the Wightman interpretation, the properties of systems (or at least the properties that we can hope to have empirical access to) are temporally *sparse*, in this sense. But what sort of persisting physical object fails to have spatial properties (in the regions we care about) at the vast majority of times? This is, I claim, a further indication that this interpretation of the state, as describing the changing properties of a physical thing, is a mistake. In its place, I propose an account of localization in terms of the occurrence of spatio-temporally located events rather than possessed properties.

3. Localization as Occurrence of an Event Picture a typical diffraction experiment which involves a source emitting a beam of particles, a diffraction grating through which the beam passes, and a luminescent screen. The source of quanta (electrons or photons, say) emits a single quantum particle at a time, at a frequency such that only a single particle is ever in the apparatus.

[4] Let $P_\Delta(t)\psi = \psi$ then, since $P_\Sigma(t)P_\Delta(t) = P_\Delta(t)$, we have $P_\Sigma(t)\psi = P_\Sigma(t)P_\Delta(t)\psi = P_\Delta(t)\psi = \psi$, in contradiction with the assumption made above.

Some time after a particle is emitted, a dot appears on the screen, and, repeating the experiment many times, the relative intensity of these discrete events comes to form a characteristic spatial interference pattern. Some things to note: first, the outcome of the experiment is an event, i.e. a definite occurrence situated in space and time; second, the time interval after which the dot appears will vary; finally, the screen is sensitive over the entire course of the experiment, and an individual experiment ends only when the particle is detected. Taken together, these observations suffice to show that the usual interpretation of localization as a property arising from an instantaneous measurement of position cannot be right.

Following Haag's suggestion (above), our first task is to characterize the mutually exclusive alternatives for our detection event. Clearly, a single event occurs in just one comparatively small region of the screen at one time. So our outcome space must allow for variation in space and time. Furthermore, since every run of the experiment ends with a detection, the elementary event (to which probability one is assigned) must correspond to a dot appearing somewhere on the screen at some time after emission. None of these characteristics mesh well with the idea that spatial localization is a property resulting from an instantaneous measurement of the corresponding projector. In particular, the elementary event for a localization experiment is always localization *at a time*.

As a first step, I propose that we interpret $P_\Delta(t)$ not as an experimental question asked at time t, but rather as a proposition about an event: the proposition that an event occurs in spatial region Δ at time t. David Lewis (1986) gave an account of an event as a property (or class) of spatio-temporal regions as follows, which is easily adapted to the present case.

> To any event there corresponds a property of regions: the property that belongs to all and only those spatio-temporal regions, of this or any other possible world, in which that event occurs. Such a property belongs to exactly one region of any world where

the event occurs . . . (p. 243)

Note first that Lewis distinguishes ‘occurring in’ a region from ‘occurring within’ a region. If an event occurs within a region Δ then, according to Lewis, it occurs within every super-region $\Sigma \supset \Delta$. This closely resembles the account of Wightman localization given in the previous sections since if $P_\Delta \psi = \psi$ then $P_\Sigma \psi = \psi$ for all $\Sigma \supset \Delta$. However, according to Lewis, an event occurring *in* a region Δ does not occur in any super-region, nor any subregion. The region in which an event occurs is, therefore, ‘just the right size.’ This can be achieved here by means of the following definition: *Given a state ψ, a localization event occurs in a region Δ if and only if $P_\Delta \psi = \psi$ and there is no subregion $\Omega \subset \Delta$ such that $P_\Omega \psi = \psi$.* This ensures that a localization event cannot occur in Δ and its super-region Σ, since if it occurs in Δ (and thus obeys the first condition) then the latter condition (required for occurrence in Σ) is not satisfied.

We can think of a certain class of states as giving the relevant space of possible worlds. We are interested in worlds in which a system prepared in (Heisenberg) state ψ results in a detection event, i.e. an event occurring within the screen some time after emission from the source. Let the detector be sensitive in region Δ over all times.[5] Then the possible outcomes of the experiment correspond to worlds where $P_\Delta(t)\psi = \psi$, i.e. worlds in which a detection event occurs at time t. The elementary event, to which we must assign probability one, is the occurrence of an event within Δ at some time $t \in \mathbb{R}$. This event (corresponding to a class of possible worlds) is associated with the state ψ_Δ for which $P_\Delta(t)\psi_\Delta = \psi_\Delta$ at every $t \in \mathbb{R}$.

Quantum mechanics gives conditional probabilities through Lüders’ Rule, according to which

[5]Effectively, we assume here that emission occurs some time in the distant past, i.e. at $t = -\infty$.

the probability of outcome A given outcome B is[6]

$$\Pr(A|B) = \frac{\langle P_B\psi|P_A P_B\psi\rangle}{\langle\psi|P_B\psi\rangle},$$

where P_A and P_B are projectors representing outcomes A and B, respectively, such that $P_A \leq P_B$. But what *instantaneous* projector could correspond to our elementary event?

It is here that the limitations of the Heisenberg and Schrödinger pictures start to bite, since each considers only instantaneous projectors. But the relevant projectors here concern a continuous interval of time, e.g. the projector onto all states ψ_Δ such that $P_\Delta(t)\psi_\Delta = \psi_\Delta$ at every $t \in \mathbb{R}$.[7] The difficulty we face is that every vector $\psi \in \mathcal{H}$ uniquely corresponds to a history $\psi_t = U_t\psi$, but the histories we are interested in (corresponding to the occurrence of an event within some time interval) must be defined more generally.

To free ourselves from this restriction, let us consider instead vector valued functions of t, $\Psi(t) = \psi(t)$, with $\psi(t) \in \mathcal{H}$. Such a function represents an entire history of a system, i.e a possible world. We may thus define a history $\Psi_\Delta(t)$ corresponding to the desired elementary event by the function $\Psi_\Delta(t) = P_\Delta U_t\psi$. But these functions of t cannot lie in the instantaneous Hilbert space $\mathcal{H} = L^2[\mathbb{R}^3]$ of functions of (configuration) space. Instead, we must consider the *temporally extended* Hilbert space $\mathcal{H}_+ = L^2[\mathbb{R}^3] \times L^2[\mathbb{R}]$ of functions of space and time.[8] We can now define a projector P^+_Δ on $\mathcal{H}_+$ that operates on every instantaneous state, $P^+_\Delta\Psi(t) = P_\Delta\psi(t)$.

[6] Lüders' Rule is usually given in terms of the trace, but for simplicity's sake we will only consider pure states here.

[7] In one-dimension Δ is an interval $[a, b]$. The relevant subspace is the square integrable functions on the interval $[a, b]$, i.e. $\psi_\Delta \in L^2[a, b]$, states for which $P_\Delta(t)\psi_\Delta = \psi_\Delta$ (implying that the (sub-)domain of the Hamiltonian is closed in $L^2[a, b]$).

[8] See the Appendix for a rigorous definition of $\mathcal{H}_+$ as a continuous direct sum.

Moreover, we can define a projection operator $P^T([t_1, t_2])$ which has the effect of truncating an arbitrary history $\Psi(t)$ as follows:

$$P^T([t_1, t_2])\Psi(t) = \begin{cases} \psi(t) & \text{if } t_1 \leq t \leq t_2 \\ 0 & \text{otherwise.} \end{cases}$$

Armed with these projectors onto times, we may now associate the outcome space of our diffraction experiment with conditional probabilities through Lüders' Rule. Let $\psi \in \mathcal{H}$ be the Heisenberg state of the system in question and let $\Psi(t) = U_t\psi$. Then, making use of the inner product on $\mathcal{H}_+$ (see Appendix), Lüders' Rule returns probability one for the elementary event corresponding to detection within Δ at some t, as required,

$$\Pr(\Delta|\Delta) = \frac{\langle P_\Delta^+\Psi|P_\Delta^+ P_\Delta^+\Psi\rangle_+}{\langle\Psi|P_\Delta^+\Psi\rangle_+} = \frac{\lim_{\tau\to\infty}\int_{-\tau}^{\tau}\langle\psi(t)|P_\Delta\psi(t)\rangle dt}{\lim_{\tau\to\infty}\int_{-\tau}^{\tau}\langle\psi(t)|P_\Delta\psi(t)\rangle dt} = 1.$$

But we may also obtain the conditional probability for detection during $I = [t_1, t_2]$ by means of the projector $P^T([t_1, t_2])$ defined above,

$$\Pr(I|\Delta) = \frac{\langle P_\Delta^+\Psi|P^T(I)P_\Delta^+\Psi\rangle_+}{\langle\Psi|P_\Delta^+\Psi\rangle_+} = \frac{\int_{t_1}^{t_2}\langle\psi(t)|P_\Delta\psi(t)\rangle dt}{\lim_{\tau\to\infty}\int_{-\tau}^{\tau}\langle\psi(t)|P_\Delta\psi(t)\rangle dt} < 1.$$

Thus we obtain the means to associate probabilities with the occurrence of events localized in time and space. From this perspective, the Heisenberg and Schrödinger pictures are rather limiting since a unit vector $\psi(t) \in \mathcal{H}$ can only be associated with an instantaneous elementary event, in which case having unit norm says that, with certainty, an event will occur at t. On the contrary, in defining the conditional probability $\Pr(I|\Delta)$ by an integral, we treat $\langle\psi(t)|P_\Delta\psi(t)\rangle$ as a probability *density* rather than a probability and so the probability of occurrence during any particular instant t is zero. This latter result seems to correctly reflect the probability that a detector sensitive for a mere instant would fire at that instant—real detectors are sensitive over time intervals, not collections of instants (with measure zero).

Before concluding it should be acknowledged that there are close links of my proposal to that of Brunetti & Fredenhagen (2002) for event time Positive Operator Valued Measures (POVMs), taken up by (e.g.) Hegerfeldt & Muga (2010). The crucial distinction, however, is that their interpretation of these POVMs does not serve to define a valid conditional probability (which assigns probability one to the occurrence of the elementary event described by the condition). The relation is as follows. If $I \mapsto F(I)$ is the POVM in $\mathcal{H}$ that they associate with an event occurring in Δ during I then my conditional probability (above) can be written as:

$$\Pr(I|\Delta) = \frac{\langle P_\Delta^+ \Psi | P^T(I) P_\Delta^+ \Psi \rangle_+}{\langle \Psi | P_\Delta^+ \Psi \rangle_+} = \frac{\langle \psi | B(\mathbb{R})^{1/2} F(I) B(\mathbb{R})^{1/2} \psi \rangle}{\langle \psi | B(\mathbb{R}) \psi \rangle}$$

where $B(\mathbb{R})$ is the positive operator $B([-\infty, \infty]) = \int_{-\infty}^{\infty} P_\Delta(t)dt$, and $F(I) = B(\mathbb{R})^{-1/2} B(I) B(\mathbb{R})^{-1/2}$ with $B(I) = \int_{t_1}^{t_2} P_\Delta(t)dt$.

4. Conclusion The conditional probabilities obtained for these spatio-temporally localized events can be usefully thought of a Lewisian chances (Lewis 1981): the condition serves to pick out those possible worlds in which the event in question occurs at *some* t, and the chances of detection within a time interval I at each of these worlds are given by the means described above. Although each possible world to which the chances apply is a world in which the event occurs at a definite time, those times are inadmissible before the experiment begins (which is when these probabilities are assigned). The usefulness of 'possible worlds talk' here may suggest that these events could be characterized within an Everettian 'no collapse' interpretation of quantum mechanics, perhaps using the Lewis-friendly possible world semantics of Wilson (2012). This would be consistent with Rovelli's seemingly paradoxical claim that "the statement that a certain specific outcome 'has happened' can be true and not-true at the same time" (2005, p. 115). However, the modern Everettian regards the (world-bound) occurrence of an event as the result of a process of decoherence, no mention of which has been made here. Instead, the occurrence of these events may be regarded as an indeterministic stochastic process confined to a single world, which is presumably how Haag intends his Principle of Random Realization to be interpreted.

Appendix The proof of Proposition 1 makes use of the following lemma, due to Hegerfeldt (1998).

Lemma 1. *(Hegerfeldt) For any positive operator P, any vector $\psi \in \mathcal{H}$, and any unitary group $U_t = e^{iHt}$ generated by a self-adjoint Hamiltonian H whose spectrum is semi-bounded* either*:*

1. $\langle\psi|U_{-t}PU_t\psi\rangle = 0$ *for all t,* or
2. $\langle\psi|U_{-t}PU_t\psi\rangle \neq 0$ *for (almost) all t.*

Proof. (Of Proposition 1). Let P_c be the projector onto the orthogonal complement of P. At each time $t \in \{t_k\}$ we have $\langle\psi|U_{-t}P_cU_t\psi\rangle = 0$. The premises of Hegerfeldt's Lemma are satisfied by ψ, U_t and P_c. Therefore, $\langle\psi|U_{-t}P_cU_t\psi\rangle = 0$ for *all* t, unless $\{t_k\}$ is a set of zero Lebesgue measure. Assuming that $\{t_k\}$ has non-zero Lebesgue measure, it follows that $\langle\psi|U_{-t}PU_t\psi\rangle = 1$ for all t. Thus $PU_t\psi = U_t\psi$ for all $t \in \mathbb{R}$. Therefore, if ψ is in the range of $P(\{t_k\})$ then ψ is in the range of $P(\mathbb{R})$, i.e. $P(\mathbb{R}) \geq P(\{t_k\})$. But, by definition, if ψ is in the range $P(\mathbb{R})$ then ψ is in the range of $P(\{t_k\})$, i.e. $P(\{t_k\}) \geq P(\mathbb{R})$. Thus $P(\{t_k\}) = P(\mathbb{R})$. □

Inspired by Naimark & Fomin (1957), we define the *extended* Hilbert space $\mathcal{H}_+$ as a continuous direct sum of instantaneous Hilbert spaces $\mathcal{H}_t$, each with inner product

$$\langle\Phi|\Psi\rangle_t = \sum_k \langle\phi(t)|e_k\rangle\langle e_k|\psi(t)\rangle,$$

where $\{e_k\}$ is a fixed orthonormal basis for $\mathcal{H}$. A function $\Psi(t)$ is measurable if $f(t) = \langle\phi|\psi(t)\rangle$ is measurable (with respect to the usual Borel measure on $\mathbb{R}$) for all $\phi \in \mathcal{H}$. If two such functions $\Psi(t), \Phi(t)$ are measurable, then so is the numerical function of t defined by their instantaneous inner product $F(t) = \langle\Psi|\Phi\rangle_t$. The set of all such measurable functions is a Hilbert space,[9] which corresponds to the continuous direct sum of the spaces $\mathcal{H}_t$, that is, an integral with respect to Lebesgue measure:

$$\mathcal{H}_+ := \int_{\mathbb{R}} \oplus \mathcal{H}_t \, d\sigma(\mathbb{R}).$$

The inner product on $\mathcal{H}_+$ may now be defined as

$$\langle\Phi|\Psi\rangle_+ = \int_{\mathbb{R}} \langle\Phi|\Psi\rangle_t \, d\sigma(\mathbb{R}). \tag{1}$$

It may be verified that $\mathcal{H}_+$ is thus a Hilbert space, and the condition for inclusion of a function Ψ in $\mathcal{H}_+$ is $\langle\Psi|\Psi\rangle_+ < \infty$. This means that $\Psi(t) = U_t\psi$ with $t \in \mathbb{R}$ is not included in this space, but *partial* dynamical evolutions of the system are, i.e. if $\Psi(t) = U_t\psi$ for $t \in [t_1, t_2]$, 0 otherwise, then $\Psi \in \mathcal{H}_+$.

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[9] Identifying, as usual, functions that differ only on a set of measure zero.

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