Curie's hazard: From electromagnetism to symmetry violation

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Abstract. We explore the facts and fiction regarding Curie's own example of Curie's principle. Curie's claim is vindicated in his suggested example of the electrostatics of central fields, but fails in many others. Nevertheless, the failure of Curie's claim is still of special empirical interest, in that it can be seen to underpin the experimental discovery of parity violation and of CP violation in the 20th century.

1. Introduction

Curie (1894) wrote that, "when certain causes produce certain effects, the elements of symmetry of the causes must be found in the produced effects" (Curie 1894, pg. 394)[1]. This claim has received mixed reviews. Brading and Castellani (2013) have suggested that a common interpretation of the principle is faulty, and Norton (2014) has argued that it is an exercise in dubious causal metaphysics. Many have suggested that the principle fails for the phenomenon of spontaneous symmetry breaking in quantum field systems, although Castellani (2003) and Earman (2004) have each argued that this is not the case.

In this paper I'd like to do two things. First, I would like to discuss Curie's own example of his principle in electromagnetism. It is a deceptively simple example. My aim will be to draw out the particular physical facts that allow Curie's statement to succeed in this example, by formulating and proving a sense in which it succeeds, while isolating a sense in which it can also fail. This is the core of what I would like to say: the truth of Curie's statement is contingent on special physical facts, which obtain in some cases but not others.

Second, I would like to point out that one of the more useful applications of Curie's principle is the detection of its failure, which can provide evidence that the laws of nature are symmetry-violating. Many commentators have focused on the connection between Curie's principle and a different concept, that of spontaneous symmetry breaking[2]. Here I will instead point out how Curie's principle played crucial role in

[1] "lorsque certaines causes produisent certains effets, les éléments de symétrie des causes doivent se retrouver dans les effets produits"

[2] C.f. Castellani (2003) and Earman (2004).

the famous experimental detections of parity violation and CP-violation in the 20th century.

2. Curie's example

Curie's original statement is slightly different from the statement that philosophers and physicists have come to refer to as Curie's principle. I will discuss the latter in more detail in the next section. For now, to keep track of the difference, I will refer to Curie's original statement as:

> *Curie's Hazard.* A symmetry of the causes must be a symmetry of the effects.

Curie gave the following example of this hazardous conjecture. Consider two oppositely charged plates, placed close together and centered on an axis as in Figure 1. Think of the charges as a "cause" whose "effect" is to give rise to an electric field. That effect, Curie says, must exhibit all the symmetries of the cause. So, since the charges are rotationally symmetric, the electric field must be too.

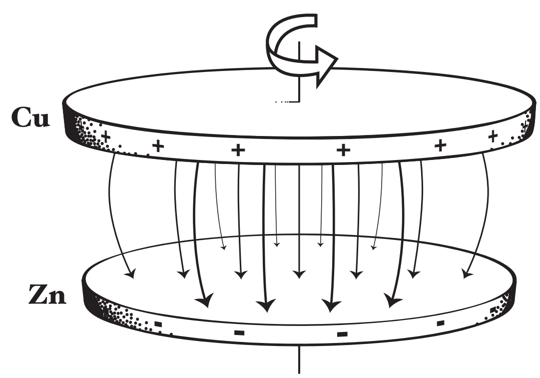

Figure 1. Curie's example: A symmetry of the charges is a symmetry of the electric field.

In Curie's own words:

> To establish the symmetry of the electric field, suppose that this field is produced by two circular plates of zinc and of copper placed one facing the other, like the plates of an air condenser. Considering a point on the common axis between the two plates, we see that this axis is an axis of isotropy and that every plane containing this axis is a plane of symmetry. *The elements of symmetry of the causes must be found in the produced effects; therefore the electric field is compatible with the symmetry* (Curie 1894, pg. 404, emphasis added)[3].

[3]My translation. The original reads: "Pour établir la symétrie du champ électrique, supposons que ce champ soit produit par deux plateaux circulaires de zinc et de cuivre placés en face l'un de

Curie's example is deceptively simple. In the case of two electric plates, it is true that a symmetry of the charge distribution is also a symmetry of the electric field. However, it is not true of Maxwell's equations more generally: a number of implicit assumptions are required in order for it to get off the ground. For example, if the particles in the plates were in motion it would of course cause the field lines to propagate asymmetrically, breaking the symmetry in the charge distribution.

Thus, an obvious implicit premise of Curie's argument must be an absence of motion. But even that is not enough. Suppose that there are no charges at all — that is, consider the vacuum. Maxwell's equations by themselves do not guarantee that an electric field will share the symmetries of the vacuum. On the contrary, there are plane wave solutions to the vacuum Maxwell equations in which the electric field propagates in any direction that one likes (Figure 2).

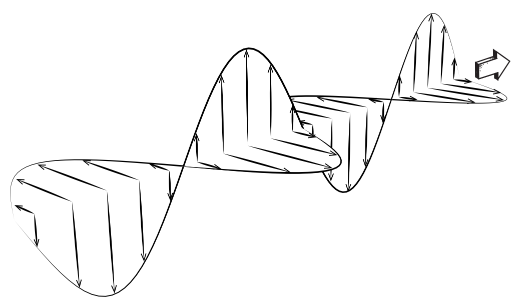

FIGURE 2. The vacuum has no charge or current, but is compatible with an electromagnetic plane wave propagating in any direction.

Getting Curie's example to work thus takes a little bit of care. In the next section, I'll explain how this can be done. If taken truly literally, Curie's hazard is simply wrong: the symmetries of a charge distribution are not necessarily symmetries of the electric field. However, if one presumes a certain amount of special physical facts about particular electromagnetic fields, then there are versions of Curie's hazard that are actually true.

l'autre, comme les armatures d'un condensateur à air. Considérons entre les deux plateaux un point de l'axe commun, nous voyons que cet axe est un axe d'isotropie et que tout plan passant par cet axe est un plan de symétrie. Les éléments de symétrie des cause doivent se retrouver dans les effets produits; donc le champ électrique est compatible avec la symétrie" (Curie 1894, pg. 404).

3. A symmetry theorem

Curie's hazard on electromagnetism can be made true given appropriate background assumptions. Let me begin with an informal discussion of the physics underlying Curie's argument, before turning to the more precise formulation of a symmetry principle for electromagnetism along these lines.

3.1. **Physics of Curie's example.** Curie's two-plate example can be characterized by the following facts.

(1) *Gauss' Law.* Electric fields are related to charge distributions by $\nabla \cdot \mathbf{E} = \rho$.
(2) *Electrostatics.* When there is no change in magnetic field, the electric field is roughly curl-free: $\nabla \times \mathbf{E} = 0$.
(3) *Central field.* The electric field "goes to zero" sufficiently quickly outside of some region, in a sense to be made precise in the next subsection.

These three statements express a divergence and a curl for a vector field that is subject to some appropriate boundary conditions. Given all this, it turns out that Curie's hazard about electric fields holds too. This result stems from two facts; I discuss their proof in the next subsection.

First, it turns out that all of the above relations are preserved by rotations. In particular we have (using $\mathbf{E}'$ and ρ' to represent the rotated field and charges),

(a) *Rotations preserve Gauss' law.* $\nabla \cdot \mathbf{E}' = \rho'$
(b) *Rotations preserve electrostatics.* $\nabla \times \mathbf{E}' = \mathbf{0}$
(c) *Rotations preserve centrality.* $\mathbf{E}' = \mathbf{0}$ on the boundary of and everywhere outside some region.

This kind of reservation does not hold of arbitrary smooth transformation, but does of rotations. We will see shortly that this stems from the fact that a rotation preserves the metric.

Second, an elementary result of vector analysis[4] shows that every central field $\mathbf{v}$ is uniquely determined by its divergence $(\nabla \cdot \mathbf{v})$ and its curl $(\nabla \times \mathbf{v})$. That is, if two such vector fields $\mathbf{v}$ and $\mathbf{v}'$ subject to these boundary conditions have the same divergence and curl, then $\mathbf{v} = \mathbf{v}'$.

These two facts allow one to say why Curie's hazard works in the example of the two plates. Suppose we have a charge distribution that is invariant under rotations:

$$\rho' = \rho.$$

[4]This result is a corollary of what is often called the Helmholtz-Hodge decomposition theorem.

Then by our first observation, the electric field E and its rotated counterpart E' have the same divergence and curl:

$$\nabla \cdot E = \rho = \rho' = \nabla \cdot E'$$
$$\nabla \times E = 0 = \nabla \times E'.$$

But the divergence and curl uniquely determine a vector field under these conditions, so $E' = E$. In other words, when the conditions (1)-(3) are satisfied, then a charge distribution ρ is invariant under rotations only if the electric field E is too, and Curie's hazard is correct.

3.2. **As a general theorem.** The argument above can be stated in more general and rigorous terms as follows. Let M be a smooth manifold, and let g_{ab} be a metric, assumed here to be a smooth symmetric invertible tensor field with inverse g^{ab}; I use Penrose abstract index notation for raising and lowering indices. Let ∇ be the derivative operator compatible with g_{ab} in the sense that $\nabla_a g_{bc} = \mathbf{0}$. A diffeomorphism $\varphi : M \to$ is called an *isometry* if $\varphi_* g_{ab} = g_{ab}$, and is the natural notion of a symmetry in this context.

We begin by collecting two facts about the derivative operator ∇; the proofs are included in an appendix. The first is that isometries "preserve" the derivative operator:

Proposition 1. *If $\varphi : M \to M$ is an isometry and λ^{bc}_{d} an arbitrary tensor field, then $\varphi_*(\nabla_a \lambda^{bc}_{d}) = \nabla_a \varphi_* \lambda^{bc}_{d}$.*

The second fact expresses the sense in which the divergence and the curl uniquely determine a vector field. I will state a geometric version of the standard result, which applies in many more geometries beyond the standard Euclidean metric on $\mathbb{R}^3$. To state this fact, we'll first need a general formulation of the divergence and curl of a vector ξ^a on a 3-dimensional manifold (a 3-manifold) M with volume element[5] ϵ^{abc}:

$$\mathrm{div}(\xi) = \nabla_a \xi^a$$
$$\mathrm{curl}(\xi) = (\nabla \times \xi)^c = \epsilon^{abc} \nabla_a \xi_b.$$

A few more definitions are needed. A metric g_{ab} on M is called *positive definite* if $\xi^a \xi_a \geq 0$, in which case (M, g_{ab}) is called a *Riemannian manifold.* Finally, we define what we mean for a vector field to be a "Central Field":

(Central Field) $E^a = \mathbf{0}$ on the boundary and outside of a region R.

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[5]A *volume element* for an n-dimensional Riemannian manifold (M, g_{ab}) is a smooth n-form that satisfies $\epsilon^{a_1 \ldots a_n} \epsilon_{a_1 \ldots a_n} = n!$. A manifold is *oriented* if it admits a volume element.

This formulation is slightly stronger than is necessary. In particular, central fields can be formulated for regions without boundary, such as $\mathbb{R}^3$, a version of our next Proposition still holds so long as the fields go to zero quickly enough (see e.g. Arfken 1985, §1.15). But this minor generalisation is considerably more complicated to state and prove, and the following result is sufficient for our purposes.

Proposition 2. *Let (M, g_{ab}) be an oriented simply-connected 3-dimensional Riemannian manifold, and let ξ^a and χ^b be two vector fields that each satisfy the Central Field assumption with respect to some (possibly different) region. If* $\mathrm{div}(\xi) = \mathrm{div}(\chi)$ *and* $\mathrm{curl}(\xi) = \mathrm{curl}(\chi)$, *then $\xi^a = \chi^a$.*

With these two facts in place, we can now finally state a theorem that captures some general conditions under which Curie's hazardous conjecture is true. The slightly stronger-than-necessary "Central Field" assumption will be adopted here too, as it is simpler and sufficient for our needs.

Theorem. *Let ρ be a scalar field, E^a a vector field, and let $\varphi : M \to M$ be an isometry on an oriented simply-connected 3-dimensional Riemannian manifold (M, g_{ab}). If,*

(1) *(Gauss' law)* $\rho = \nabla_a E^a$
(2) *(Electrostatics)* $(\nabla \times E)^a = \mathbf{0}$
(3) *(Central Field)* $E^a = \mathbf{0}$ *on the boundary and outside of a region R*

then $\varphi_* \rho = \rho$ (symmetric cause) only if $\varphi_* E^a = E^a$ (symmetric effect).

Proof. Let $\varphi_* \rho = \rho$. By Gauss' law,

$$\nabla_a E^a = \rho = \varphi_* \rho = \varphi_*(\nabla_a E^a) = \nabla_a \varphi_* E^a,$$

where the last equalty is an application of Proposition 1. Thus, E^a and $\varphi_* E^a$ have the same divergence. Moreover, by the assumption of electrostatics,

$$(\nabla \times E)^a = \mathbf{0} = \varphi_* \mathbf{0} = \varphi_* (\nabla \times E)^a.$$

Applying the definition of the curl to the right hand side now gives us,

$$\begin{aligned}(\nabla \times E)^a &= \varphi_* \epsilon^{bca} \nabla_b E_c = \pm \epsilon^{bca} \varphi_* \nabla_b E_c = \pm \epsilon^{bca} \nabla_b \varphi_* E_c \\ &= \pm (\nabla \times \varphi_* E)^a.\end{aligned}$$

where the second equality applies the fact that isometries preserve volume elements up to a sign[6]. But $(\nabla \times E)^a = \mathbf{0}$, so this implies that E^a and $\varphi_* E^a$ have the same curl. Moreover, since E^a is a Central Field with respect to the regions R_i, so is $\varphi_* E^a$ with

[6]Since isometries preserves the metric, $(\varphi_* \epsilon^{bca})(\varphi_* \epsilon_{bca}) = \epsilon^{bca} \epsilon_{bca} = \pm n!$. Thus, $\varphi_* \epsilon^{bca}$ is a volume element too. But ϵ^{bca} and $-\epsilon^{bca}$ are the unique volume elements, so $\varphi_* \epsilon^{bca} = \pm \epsilon^{bca}$.

respect to the regions $\varphi(R_i)$. Therefore, the premises of Proposition 2 are satisfied, and it follows that $E^a = \varphi_* E^a$. □

What I would like to emphasize about this result is that *even for Curie's own example*, the truth of Curie's hazard depends on a significant amount of background structure. It is not an a priori fact about causes and effects. Indeed, the argument does not go through in the more general context of pseudo-Riemannian manifolds, as are adopted in general relativity, except when restricted to a spacelike hypersurface where the metric g_{ab} becomes positive definite. In particular, the proof of Proposition 2 makes crucial use of the non-degenerate metric available in Riemannian manifolds. I do not know if this theorem can be generalized to the pseudo-Riemannian case, but if it can, then it would likely be established by a rather different argument.

3.3. **General electromagnetic fields.** Curie's hazard fares worse when applied to general electromagnetic fields in spacetime. The natural analogue of Curie's statement for electrostatics simply fails when translated into this language.

Electromagnetism is naturally formulated on a smooth manifold M with with a metric g_{ab} that is symmetric and invertible, but not necessarily non-degenerate. Such a pair (M, g_{ab}) is called a *pseudo-Riemannian manifold*. To do electromagnetism, we assume the existence of a vector field J^a representing charge-current density, and an anti-symmetric tensor field F_{ab} representing the electromagnetic field, which satisfy Maxwell's equations[7]:

$$\begin{aligned} \nabla_{[a} F_{bc]} &= \mathbf{0} \\ \nabla_a F^{ab} &= J^b. \end{aligned} \tag{1}$$

These general equations reduce in certain contexts to the usual Maxwell equations (for an overview see Malament 2012, §2.6).

What is Curie's hazard in this context? If we take the cause to be the charge-current J^a (instead of just the charge ρ) and take the effect to be the electromagnetic field F_{ab} (instead of just the electric field E^a), then Curie's statement would be that a symmetry of the charge-current J^a is a symmetry of the electromagnetic field F_{ab}. That statement is false.

The problem is that the charge-current J^a does not uniquely determine an electromagnetic field F_{ab} up to isometry without further specification. This makes it possible to find explicit counter-examples to Curie's hazard, such as the following.

[7]In fact, an even more general formulation is available in terms of the Hodge star operator (for an overview see Baez and Muniain 1994, §1.5), although this will not concern us here.

Counterexample. Let F_{ab}, J^a be a solution to Maxwell's equations, and let $\varphi : M \to M$ be an isometry that does not preserve F_{ab},

$$\varphi_* F_{ab} - F_{ab} = H_{ab} \neq \mathbf{0},$$

but such that H_{ab} is divergence-free, $\nabla_a H^{ab} = \mathbf{0}$. For example, this occurs when F_{ab} is the field for a plane wave (as in Figure 2) and φ is a rotation; then $\nabla_a F^{ab} = \mathbf{0}$ and so $\nabla_a H^{ab} = \mathbf{0}$, but $H_{ab} \neq \mathbf{0}$. Since diffeomorphisms preserve the zero vector, this implies,

$$\varphi_* J^b = \mathbf{0} = J^b.$$

Thus, if J^b is the "cause" and F_{ab} the "effect," then a symmetry of the cause fails to be a symmetry of the effect. Without specifying some initial and boundary conditions such as those considered in the previous subsection, a symmetry of J^b need not be a symmetry of F_{ab}.

A persistent believer in Curie might still draw a more optimistic conclusion. It is easy to see that the converse expression of Curie's claim is true. Suppose we consider F_{ab} to be the "cause" and J^b the "effect", and let $\varphi_* F_{ab} = F_{ab}$. Then applying Proposition 1 we have,

$$\varphi_* J^b = \varphi_*(\nabla_a F^{ab}) = \nabla_a \varphi_* F^{ab} = \nabla_a F^{ab} s = J^b.$$

So, every symmetry of the electromagnetic field F_{ab} is a symmetry of the charge-current field J^a. One could conclude from this that Curie simply mistook cause for effect: the appropriate cause in this example is the electromagnetic field F_{ab}, and the appropriate effect the charge-density current J^a. I do not know what would justify this kind of conclusion; on the contrary, the argument of Norton (2014) suggests it would be little more than dubious causal metaphysics.

Another possible route is to adopt initial and boundary conditions that guarantee J^a will determine a unique electromagnetic field F_{ab}. This is not so easy to do. Wald (1984, Chapter 10, Problem 2) points out some (fairly restrictive) circumstances under which F_{ab} is unique, by demanding that $J^a = \mathbf{0}$, and also that the values of the electric and magnetic fields be on a Cauchy surface. Under these circumstances, one has for *any* isometry $\varphi : M \to M$ that $\varphi_* J^a = J^a = \mathbf{0}$, and so,

$$\nabla_a \varphi_* F^{ab} = \varphi_* \nabla_a F^{ab} = \varphi_* J^b = \mathbf{0} = J^b = \nabla_a F^{ab}.$$

Thus, since F_{ab} is the unique field satisfying Maxwell's equations under these circumstances, it follows that $\varphi_* F_{ab} = F_{ab}$ for all isometries. In other words, Curie's hazard is made true, in that a symmetry of J^a is a symmetry of F_{ab}, in the restrictive circumstances of $J^a = \mathbf{0}$ when there is a complete absence of charge-current in spacetime.

But this is only possible in the presence of this or some similar initial and boundary conditions that render F_{ab} unique.

The point I would like to make about all this is not that Curie's hazard is totally misguided, but rather that very specific structures must be in place for it to be true. Without a number of particular background facts, Curie's hazard can fail, even in his own example of electromagnetism.

4. From electromagnetism to symmetry violation

In this section, I will identify a sense in which the failure of Curie's hazard can provide evidence of symmetry violation. In fact, its failure provides an indicator of symmetry violation in some of the most famous historical examples of symmetry violation: the parity violation detected by Chien-Shung Wu in 1956, and the CP-violation detected by Val Cronin and James Fitch in 1964. I will also now turn to the statement that philosophers and physicists have more commonly come to call "Curie's principle."

4.1. **Curie's principle in skeletal form.** The statement known as "Curie's principle" can be cast in an incredibly general form. Here is how to get there from the example of electromagnetism. Under very particular circumstances, a symmetry of the charge-current distribution is also a symmetry of the electromagnetic field. For Curie's two-plate capacitor, we have seen that sufficient circumstances are the electrostatics of central fields: (1) Gauss' law, (2) Electrostatics, and (3) Central Field. Let me now summarise these properties as the statement that the relation between cause and effect is "symmetry preserving," formulated as follows.

Proposition 3 (Curie Principle)**.** *Let C and E be two sets, and let $\sigma_c : C \to C$ and $\sigma_e : E \to E$ be two bijections. If $D : C \to E$ is a mapping such that,*

(symmetry preservation) $D\sigma_c^{-1}x = \sigma_e^{-1}Dx$ *for all* $x \in C$,

then $\sigma_c x = x$ (symmetric cause) only if $\sigma_e Dx = Dx$ (symmetric effect). If D is a bijection, then $\sigma_c x = x$ if and only if $\sigma_e Dx = Dx$.

Proof. If $\sigma_c x = x$, then $\sigma_e Dx = \sigma_e D(\sigma_c^{-1}x) = \sigma_e(\sigma_e^{-1}D)x = Dx$. If D is a bijection then it has an inverse, so $\sigma_e Dx = Dx$ only if $x = (D^{-1}\sigma_e^{-1}D)x = D^{-1}(D\sigma_c^{-1})x = \sigma_c^{-1}x$ and hence $\sigma_c x = x$. □

The 'σ's are to be interpreted as "the same" symmetry[8], such as a fixed rotation (or whatever), applied to each of the sets C and E. The 'D' mapping captures a sense

[8]One may wish to cash this out as Norton (2014) does in terms of an isomorphism that carries σ_c to σ_e. Or (as is now fashionable) one may interpret this as meaning that C and E are two categories related by a functor $\mathfrak{F}$ with σ_c a morphism of C and σ_e a morphism of E satisfying $\mathfrak{F}(\sigma_c) = \sigma_e$. The

in which "causes determine effects." Note that this formulation explicitly excludes "time reversing" symmetries like T and CPT, since they are typically expressed as mappings between causes and effects[9].

Proposition 3 is similar to some existing formulations of Curie's principle[10]. One sense in which it differs slightly is that it does not presume causes *uniquely* determine effects. When they do not, then the converse statement need not be true, that a symmetry of an effect is necessarily a symmetry of the cause. Elena Castellani[11] has emphasized out that Curie himself viewed his principle as asymmetric:

> In practice, the converse... [of Curie's hazards] are not true, i.e., the effects can be more symmetric than their causes. Certain causes of asymmetry might have no effect on certain phenomena (Curie 1894)[12]

Proposition 3 captures this asymmetry: if causes do not bijectively determine effects, then the converse of Curie's hazard is not guaranteed. However, if a cause does (bijectively) determine a unique effect that satisfies symmetry preservation, then Curie's hazard is true in both directions.

The bare-bones construal of Curie's principle of Proposition 3 can be applied in all kinds of ways. Here are a few, limited only by the imagination.

Example 1 (Electromagnetism)**.** We have already seen the example in which C be is the set charge distributions, and E the set of electric fields on a Riemannian manifold. Here is another way to look at it (which is essentially just the proof of the theorem). Given conditions (1)-(3), Proposition 2 provides a mapping $D : \rho \mapsto E^a$ that determines a unique E^a for each ρ. Proposition 1 then implies[13] that $D\varphi^* = \varphi^* D$ for any isometry φ. Therefore, given (1)-(3), a symmetry of the charge distribution $\varphi_* \rho = \rho$ is also a symmetry of the electric field $\varphi_* E^a = E^a$.

Example 2 (General Relativity)**.** Let C and E both refer to the set of symmetric 2-place tensor fields on a relativistic spacetime (M, g_{ab}), thinking of an element $T_{ab} \in C$ as energy-momentum and an element $G_{ab} \in E$ is the Einstein tensor. Let $D : T_{ab} \mapsto$

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difficulty is that "spurious" identifications of symmetries may still occur, as identified by Norton (2014, fn.4).

[9]This is required in order to get a true principle; for time-reversing symmetries, Curie's principle badly fails (Roberts 2013a).

[10]C.f. Ismael (1997), Belot (2003), Earman (2004, pg.175-176), Mittelstaedt and Weingartner (2005, pg.231), Ashtekar (2014) and Norton (2014).

[11]Personal communication.

[12]Translation from Brading and Castellani (2003, pg.312).

[13]Namely, Proposition 1 implies that if $\rho = \nabla_a E^a$ (and hence that $D\rho = E^a$), then $\varphi^* \rho = \varphi^* \nabla_a E^a = \nabla_a \varphi^* E^a$, and thus $D\varphi^* \rho = \varphi^* E^a = \varphi^* D\rho$.

$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$ be the map determined by Einstein's equation. The symmetries of C and E are determined by an underlying diffeomorphism $\varphi : M \to M$, and identifying $\sigma_c = \sigma_e = \varphi_*$ we trivially find that $D\sigma_c = \sigma_e^{-1} D$.

Example 3 (Particle Physics)**.** Let $C = \mathcal{H}_{in}$ and $E = H_{out}$ be identical copies of a Hilbert space $\mathcal{H}$ representing the in-states and out-states of a scattering experiment. Let $D = S : \psi_{in} \mapsto \psi_{out}$ be the scattering matrix. Then the condition that a symmetry σ (a unitary operator) satisfy $D\sigma_{in} = \sigma_{out}^{-1} D$ just amounts to the condition that it commute with the scattering matrix. When this condition obtains, there is a sense in which Curie's hazard is true, that a symmetry of causes (viewed as an "in" state) gives rise to a symmetry of an effect (viewed as an "out" state).

It is this last example that is significant for the history of symmetry violation, to which we will now turn.

4.2. **Curie's failure implies symmetry violation.** When S is a scattering matrix, Proposition 3 says that Curie's hazard holds for a symmetry σ if and only if the S-matrix is "invariant" under that symmetry. For a long time it was presumed that the laws of nature *must* be invariant under symmetries like parity (P) and the combination of charge conjugation and parity (CP). However, in the mid-20th century this presumption was dramatically disproven, when first parity invariance and then CP invariance were both found to be violated in weak interactions.

The significance of Curie's principle for those discoveries can be seen by casting Proposition 3 in the equivalent "contrapositive" form: if Curie's hazard fails, in that either $\sigma_c(x) = x$ and $\sigma_e(Dx) \neq Dx$ or else $\sigma_e(Dx) = Dx$ and $\sigma_c(x) \neq x$, then we have a case of symmetry violation: $D\sigma_c^{-1} \neq \sigma_e^{-1} D$. Applying this principle to particle decays is a little subtle, but not much. That application is stated and proved in Roberts (2013b, Fact 2) as follows.

Proposition 4 (Scattering Curie)**.** *Let S be a scattering matrix, and $R : \mathcal{H} \to \mathcal{H}$ be a unitary bijection. If there exists a decay channel $\psi^{in} \to \psi^{out}$, i.e. a non-zero amplitude $\langle \psi_{out}, S\psi_{in} \rangle$, such that either,*

(1) *(in but not out) $R\psi^{in} = \psi^{in}$ but $R\psi^{out} = -\psi^{out}$, or*
(2) *(out but not in) $R\psi^{out} = \psi^{out}$ but $R\psi^{in} = -\psi^{in}$,*

then,

(3) *$RS \neq SR$.*

This principle is precisely what was used in the very first revelations that the laws of nature are symmetry violating. For example, parity — the "mirror" transformation that reverses total orientation (or "handedness") of a system — has been long known

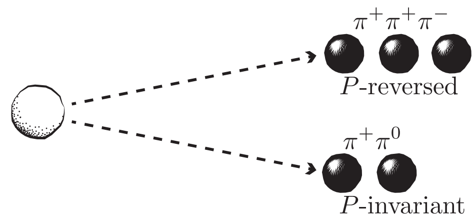

FIGURE 3. The P-violating interaction suggested by Lee and Yang.

to preserve the two-pion state $\pi^+\pi^0$, but reverse the phase of the three-pion state $\pi^+\pi^+\pi^-$:

$$P\pi^+\pi^0 = \pi^+\pi^0$$
$$P\pi^+\pi^+\pi^- = -\pi^+\pi^+\pi^-.$$

The originating particle state for the first was originally called τ and the second θ. Both appeared in the interactions of charged strange mesons, and both were soon found to have a very similar lifetime and rest mass. The famous question thus arose: might these be the very same particle? This was known as the $\theta - \tau$ puzzle. Here is where Curie's principle appears: if θ and τ are the same, then parity symmetry is violated by Proposition 4. For whether or not parity preserves the originating particle state, it would still sometimes decay into a state with different parity, as in Figure 3. This led Lee and Yang to suggest:

> One might even say that the present $\theta - \tau$ puzzle may be taken as indication that parity conservation is violated in weak interactions. This argument is, however, not to be taken seriously because of the paucity of our present knowledge concerning the nature of the strange particles. (Lee and Yang 1956, pg.254).

Their hesitant suggestion was famously vindicated experimentally by Chien-Shiung Wu and her collaborators that same year, in an elegant experiment that was quickly repeated[14].

Curie's principle was even more directly applied in the discovery of CP-violation a few years later. A number of simple theoretical models had arisen in which the observed parity-violation was explained in a way that required CP-invariance. This requirement thus was tested by James Cronin and Val Fitch at Brookhaven, by observing a beam of neutral K-mesons or kaons. They began with a "long-lived" neutral kaon state K_L, which was known to have its phase reversed by the CP transformation;

[14]Confirming results were reported by Wu et al. (1957) and by Garwin et al. (1957).

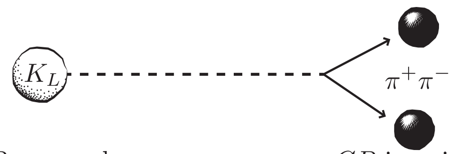

FIGURE 4. The CP-violating interaction discovered by Cronin and Fitch.

the two-pion state $\pi^+\pi^-$, on the other hand, was preserved by parity:

$$CPK_L = -K_L$$
$$CP\pi^+\pi^- = \pi^+\pi^-.$$

In a small but unmistakable number of decays, Cronin and Fitch found[15] the K_L state to decay into $\pi^+\pi^-$, as in Figure as in Figure 4. Again Curie's principle appears in the form of Proposition 4, which implies that, contrary to what the early models suggested, CP symmetry is violated.

These were two of the most important experimental discoveries of 20th century physics. Nobel prizes were awarded for each. And both were crucially underpinned by Curie's principle. In this sense, Curie was not mistaken when he suggested that "there is interest in introducing into the study of physical phenomena the symmetry arguments familiar to crystallographers" (Curie 1894)[16].

4.3. **Norton on Curie's Truism.** Norton (2014) has convincingly argued that the only true formulation of Curie's principle that does not invoke dubious causal metaphysics is a near-tautology. Namely, suppose one presumes that,

> *Determination respects symmetries:* Causes admitting symmetries are mapped to effects that admit those same symmetries.

Then Curie's claim that symmetries of the causes are symmetries of the effects is obviously true. Norton refers to this as "Curie's Lemma," pointing out:

> "there is little substance to it. It is a tautology implementing as an easy modus ponens 'A, if A then B; therefore B.' That simplicity does make precise the sense that the principle somehow has to be true." (Norton 2014, pg.6)

Let me add two comments about the little bit of substance that the truism retains, in light of what we have discussed so far.

First, establishing the truth of the premise that "Determination respects symmetries" may by itself amount to a deep result. It is analogous to the "symmetry

[15] The discovery was reported in Christenson et al. (1964).

[16] Translation from Brading and Castellani (2003, pg.311).

preservation" premise in my formulation of Curie's principle in Proposition 3, which says that given a mapping $D : C \to E$ between sets and a symmetry represented by,

$$\text{(symmetry preservation)}\ D\sigma_c^{-1}x = \sigma_e^{-1}Dx \text{ for all } x \in C.$$

This may be far from obvious for a given choice of causes C, effects E, and a determination relation D. The theorem formulated in section on electromagnetism establishes it for central fields in electrostatics, which is established by premises (1)-(3). But although the proof itself is straightforward, it does rely on some facts such as Stokes' theorem and the uniqueness of a compatible derivative operator that are not exactly trivial (details can be found in the Appendix).

Second, statements of the truism may have empirical significance that is non-trivial. We have seen that the discoveries of parity violation and of CP violation both involved the existence of decay modes in scattering experiments that have different symmetries from the originating states. Curie's principle establishes that such an observation is enough to tell us something interest that the laws of nature, in that there exist possible trajectories whose symmetry-transformed counterparts are not possible. In particular, the unitary evolutions corresponding to the S matrix for a weak interaction are symmetry-violating.

Curie's principle is of course still a pretty insubstantial statement in this context, in that it is still a simple fact about mappings between sets as in Proposition 3. However, this lack of substance is also a strength: the piddling amount of mathematical structure in Curie's principle assures that it is very robust. Thus, using Curie's principle to establish that the laws of nature are symmetry-violating provides evidence that is extremely resilient to theory change, even as new mathematical structures come and go[17].

5. Conclusion

Curie managed to hazard a conjecture that is of interest both when it is true and when it is false. The original hazard requires very special circumstances in order to be true. We have verified mathematically that one such circumstance is that of Curie's example, when one restricts attention to the electrostatics of central fields. However, its formulation as a general statement about electromagnetic currents and fields is false.

When we draw out the special circumstances under which Curie's hazard holds, we find a skeletal but true proposition about sets. This proposition captures what many philosophers of science have in mind when referring to "Curie's principle." Although so bare as to be nearly a triviality, formulating Curie's principle in this

[17]See Ashtekar (2014) for a more elaborate argument on this point.

way allows one to identify it among the arguments for the great symmetry-violating experiments of the mid-20th century. Viewed from this perspective, Curie's principle is indeed a simple and true statement, which managed to become one of the very fruitful symmetry principles of modern physics.

Appendix

Definitions. Let M and $\tilde{M}$ be smooth manifolds, each with a metric g_{ab} and $\tilde{g}_{ab}$, assumed here to be smooth symmetric invertible tensor fields, which are non-degenerate but not necessarily positive-definite. I adopt Penrose's abstract index notation for this discussion.

A *derivative operator* ∇ (or a 'covariant derivative' or a 'connection') maps an index a and an arbitrary tensor like λ^{bc}_{d} to another tensor, written $\nabla : (a, \lambda^{bc}_{d}) \mapsto \nabla_a \lambda^{bc}_{d}$. It is defined by the following properties, which we adopt following Malament (2012, §1.7).

(1) ∇ commutes with addition, index substitution and contraction on tensor fields.
(2) ∇ satisfies the Leibniz rule with respect to tensor multiplication.
(3) If ξ^a is a vector field and α a scalar field, then $\xi^a \nabla_a \alpha = \xi(\alpha)$. That is, $\xi^a \nabla_a \alpha$ is the "directional derivative" that ξ assigns to α.
(4) ∇ is torsion-free, in that if α is a scalar field, then $\nabla_a \nabla_b \alpha = \nabla_b \nabla_a \alpha$.

Let ∇ be a derivative operator on M, and suppose that it is *compatible* with the metric g_{ab} in that $\nabla_a g_{bc} = \mathbf{0}$. Let $\tilde{\nabla}$ similarly be a derivative operator on $\tilde{M}$ satisfying $\tilde{\nabla}_a \tilde{g}_{bc} = \mathbf{0}$. I will write $\varphi : M \to \tilde{M}$ to indicate a diffeomorphism, with pushforward φ_* and pullback φ^*.

Preserving derivatives. As a simple example, consider first the case of the derivative of a scalar field, $\nabla_a \alpha$. Every diffeomorphism φ "preserves" covariant derivatives of a scalar field, in that,

$$\varphi_*(\nabla_a \alpha) = \tilde{\nabla}_a \varphi_* \alpha. \tag{2}$$

This statement can be quickly verified: if α is any scalar field at $p \in M$ and $\tilde{\xi}^a$ is any vector at $\varphi(p) \in \tilde{M}$, then,

$$\tilde{\xi}^a \varphi_*(\nabla_a \alpha) = (\varphi^* \tilde{\xi}^a)(\nabla_a \alpha) = (\varphi^* \tilde{\xi})(\alpha) = \tilde{\xi}(\alpha \circ \varphi^{-1}) = \tilde{\xi}^a \tilde{\nabla}_a \varphi_* \alpha.$$

Equation 2 does not always hold when α is replaced with an arbitrary tensor. However, it does when we further restrict φ to be an isometry — and in fact for a slightly weaker condition. It is established by the following.

Lemma 1. *Let $\varphi : M \to \tilde{M}$ be a diffeomorphism. Then the equality,*

$$\varphi_*(\nabla_a \lambda^{bc}_d) = \tilde{\nabla}_a \varphi_* \lambda^{bc}_d$$

holds for an arbitrary tensor field like λ^{bc}_d if and only if $\tilde{\nabla}_a \varphi_* g_{ab} = \mathbf{0}$, where g_{ab} is the metric compatible with ∇. In particular the equality holds if φ is an isometry.

Proof. The 'only if' direction is trivial, since if the above equality holds for all tensors, then in particular,

$$\tilde{\nabla}_a \varphi_* g_{bc} = \varphi_*(\nabla_a g_{bc}) = \varphi_* \mathbf{0} = \mathbf{0},$$

where the penultimate equality applies compatibility, and the final equality the fact that $\varphi_* \mathbf{0} = \varphi_*(\mathbf{0} + \mathbf{0}) = \varphi_* \mathbf{0} + \varphi_* \mathbf{0}$.

For the 'if' direction, consider the mapping $\hat{\nabla}$ defined by,

$$\hat{\nabla} : (a, \lambda^{bc}_d) \mapsto \varphi^*(\tilde{\nabla}_a \varphi_* \lambda^{bc}_d).$$

where a is an index. The first step is to show that this mapping is a derivative operator. It obviously commutes with addition, index substitution and contraction because all three maps do (φ_*, φ^* and $\tilde{\nabla}_a$). It is also easy to check that it satisfies the Leibniz rule and the torsion-freeness condition. Moreover, for all vectors ξ^n and all scalar fields α, $\hat{\nabla}$ satisfies the condition that,

$$\xi^a \hat{\nabla}_a \alpha = \xi^a \varphi^*(\nabla_a \varphi_* \alpha) = \xi^a \nabla_a \varphi^* \varphi_* \alpha = \xi^a \nabla_a \alpha = \xi(\alpha),$$

where the second equality is an application of Equation 2. Therefore $\hat{\nabla}$ is a derivative operator. Note that this argument required only that φ be a diffeomorphism.

The second step is to observe that $\hat{\nabla}$ is compatible with the metric:

$$\hat{\nabla}_a g_{bc} = \varphi^* \tilde{\nabla}_a(\varphi_* g_{bc}) = \varphi^* \mathbf{0} = \mathbf{0},$$

where the second equality applies our assumption. Compatibility holds in particular when φ is an isometry, since then $\tilde{\nabla}_a(\varphi_* g_{bc}) = \tilde{\nabla}_a(\tilde{g}_{bc}) = \mathbf{0}$.

Finally, we use the fact that there is a unique derivative operator compatible with a given metric (Malament 2012, Prop. 1.9.2). So, $\hat{\nabla}$ and ∇ are the same. Therefore,

$$\nabla_a \lambda^{bc}_d = \hat{\nabla}_a \lambda^{bc}_d = \varphi^*(\tilde{\nabla}_a \varphi_* \lambda^{bc}_d)$$

Pushing-forward the left and right sides with φ_*, we thus have that,

$$\varphi_*(\nabla_a \lambda^{bc}_d) = \tilde{\nabla}_a \varphi_* \lambda^{bc}_d.$$

□

As a special case of this lemma we have Proposition 1 from page 5.

Proposition 1. *If $\varphi : M \to M$ is an isometry and λ^{bc}_d an arbitrary tensor field, then $\varphi_*(\nabla_a \lambda^{bc}_d) = \nabla_a \varphi_* \lambda^{bc}_d$.*

Non-isometries. An example of a non-isometry that preserves covariant derivatives is any 'constant' conformal transformation, i.e. a conformal transformation $\varphi_* g_{ab} = \Omega^2 \tilde{g}_{ab}$ for which the conformal factor Ω is a constant scalar field, $\nabla_a \Omega = 0$. Then,

$$\tilde{\nabla}_a \varphi_* g_{ab} = \tilde{\nabla}_a (\Omega^2 \tilde{g}_{ab}) = \Omega^2 \tilde{\nabla}_a g_{ab} = \mathbf{0}.$$

Since the premises of the proposition are satisfied, this transformation φ preserves covariant derivatives.

However, these are the *only* conformal transformations that preserve covariant derivatives. If Ω is any conformal factor with non-zero covariant derivative, then applying the chain rule we have,

$$\tilde{\nabla}_a \varphi_* g_{ab} = \tilde{\nabla}_a \Omega^2 g_{ab} = g_{ab} \tilde{\nabla}_a (\Omega^2) + \Omega^2 \underbrace{\tilde{\nabla} \tilde{g}_{ab}}_{=\mathbf{0}} = 2\Omega \tilde{g}_{ab} \tilde{\nabla}_a \Omega \neq \mathbf{0}.$$

So, conformal transformations do not in general preserve covariant derivatives.

Proposition 2. *Let (M, g_{ab}) be an oriented simply-connected 3-dimensional Riemannian manifold, and let ξ^a and χ^b be two vector fields that each satisfy the Central Field assumption with respect to some (possibly different) region. If* $\mathrm{div}(\xi) = \mathrm{div}(\chi)$ *and* $\mathrm{curl}(\xi) = \mathrm{curl}(\chi)$*, then $\xi^a = \chi^a$.*

Proof. Let $\lambda^a = \xi^a - \chi^a$. We will show that $\lambda^a = \mathbf{0}$. By the linearity of the divergence and curl we have,

$$\mathrm{div}(\lambda) = \mathrm{div}(\xi) - \mathrm{div}(\chi) = 0,$$
$$\mathrm{curl}(\lambda) = \mathrm{curl}(\xi) - \mathrm{curl}(\chi) = \mathbf{0}.$$

A vanishing curl $\mathrm{curl}(\lambda) = \epsilon^{abc} \nabla_a \lambda_b$ is only possible if $\nabla_{[a} \lambda_{b]} = \mathbf{0}$, i.e. if λ_b is closed[18]. But a closed covector on a simply connected manifold is exact, meaning that it may be expressed as a gradient,

$$\lambda_a = \nabla_a \phi$$

for some scalar field ϕ (Malament 2012, Prop. 1.8.3).

Now, we have assumed that ξ^a vanishes on the boundary and outside of some region R_1, and χ^a similarly for some region R_2. Both ξ^a and χ^a thus vanish on the boundary and outside of the combined region $R = R_1 \cup R_2$, and therefore so does $\lambda^a = \xi^a - \chi^a$. That is, λ^a is a central field with respect to the region R. We thus have,

$$(3) \qquad \int_R \lambda_a \lambda^a = \int_R (\nabla_a \phi)(\nabla^a \phi) = \int_R \nabla_a (\phi \nabla^a \phi) = \int_{\partial R} \eta_a \phi \nabla^a \phi = \int_{\partial R} \eta_a \phi \lambda^a = 0,$$

[18] An antisymmetric tensor $\xi_{[a} \nabla_b \lambda_{c]}$ can always be written in terms of the volume element as $\xi_{[a} \nabla_b \lambda_{c]} = k \epsilon_{abc} \epsilon^{def} \xi_{[d} \nabla_e \lambda_{f]}$ for some constant k (Malament 2012, §1.11). And a vanishing curl implies $k \epsilon_{abc} \epsilon^{def} \xi_d \nabla_e \lambda_f = \mathbf{0}$ for any arbitrary vector ξ^d. But then the total antisymmetry of ϵ_{abc} implies that $\mathbf{0} = k \epsilon_{abc} \epsilon^{def} \xi_{[d} \nabla_e \lambda_{f]} = \xi_{[d} \nabla_e \lambda_{f]}$. Since ξ^d was arbitrary, this requires $\nabla_{[e} \lambda_{f]} = \mathbf{0}$.

where second equality follows from the chain rule and the fact that $\nabla_a \nabla^a \phi = \nabla_a \lambda^a = 0$; the third equality applies Stokes' theorem (Wald 1984, Appendix B, B.2.26); and the last equality applies the assumption that $\nabla^a \phi = \lambda^a = \mathbf{0}$ on the boundary ∂R.

Finally, g_{ab} is assumed to be positive definite. Thus, $\lambda^a \lambda_a$ is strictly non-negative, so Equation 3 is only possible if $\lambda^a = \mathbf{0}$. □

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