Local Reduction in Physics

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Abstract

A conventional wisdom about the progress of physics holds that successive theories wholly encompass the domains of their predecessors through a process that is often called "reduction." While certain influential accounts of inter-theory reduction in physics take reduction to require a single "global" derivation of one theory's laws from those of another, I show that global reductions are not available in all cases where the conventional wisdom requires reduction to hold. However, I argue that a weaker "local" form of reduction, which defines reduction between theories in terms of a more fundamental notion of reduction between *models* of a single fixed system, is available in such cases and moreover suffices to uphold the conventional wisdom. To illustrate the sort of fixed-system, inter-model reduction that grounds inter-theoretic reduction on this picture, I specialize to a particular class of cases in which both models are dynamical systems. I show that reduction in these cases is underwritten by a mathematical relationship that follows the broad prescriptions of Nagel/Schaffner reduction, and support this claim with several examples. Moreover, I show that this broadly Nagelian analysis of inter-model reduction encompasses several cases that are sometimes cited as instances of the "physicist's" limit-based notion of reduction.

1 Introduction

According to the most commonly told story about the progress of physics, successive theories in physics come ever closer to revealing the true, fundamental nature of reality. This convergence rests on the supposition that later theories bear a special relationship to their predecessors often called "reduction," which minimally requires one theory to encompass the domain of application of another. More specifically, the conventional wisdom tells us that Newtonian mechanics "reduces to" special relativity, [1] special relativity to general relativity, classical mechanics to quantum mechanics, quantum mechanics to relativistic quantum mechanics, relativistic quantum mechanics to quantum field theory, thermodynamics to statistical mechanics, and more. In order to assess the truth of the conventional wisdom, however, it is necessary to gain a

[1]As Nickles noted several decades ago, two opposing conventions have arisen in the literature on inter-theoretic reduction, one (employed most commonly in the philosophical literature) that takes a less encompassing theory to "reduce to" a more encompassing one, and the other (employed most commonly in the physics literature) that takes the more encompassing theory to "reduce to" the less encompassing one. Here, I will adopt the first of these conventions. It should also be noted that the two concepts of reduction that Nickles discusses in his paper differ on more substantive points than this choice of convention, which I discuss further below [25].

more precise sense of what is needed in a given case to show that one theory reduces to another.

In his widely cited 1973 paper, Nickles distinguished two types of approach to reduction in physics: first, the approach commonly employed by philosophers, which originates in Ernest Nagel's well-known account of reduction, and second, the approach commonly employed by physicists that requires one theory to be a "limit" or "limiting case" of another [25]. Since Nickles' paper, these two accounts have tended to dominate philosophical discussion concerning issues of the general methodology of reduction in physics. As commonly presented, both strongly suggest - and in some cases, state explicitly - that reduction between theories in physics should rest on a single "global" derivation of a high-level theory's laws from those of a low-level theory. Here, I argue by means of a particular example that global reduction is not always available in cases where the conventional wisdom requires reduction to hold. However, I argue that it is possible to a define a weaker "local" notion of reduction in physics that suffices to uphold the conventional wisdom by ensuring the subsumption of one theory's domain by another. This notion of reduction is "local" in the sense that it permits the reducing theory to account for the reduced theory's success through numerous context-specific derivations that are relativized to different systems in the high-level theory's domain. These derivations concern the specific *models* that the theories use to describe a single fixed system, rather than the theories as a whole.

This paper has two main goals, which are mutually supporting. The first is to motivate and develop a local account of inter-theoretic reduction in physics. Inter-theoretic reduction in physics, understood minimally as the requirement that one theory subsume the domain of another, does not require anything as strong as global reduction directly between theories; local reduction suffices for this purpose, and moreover avoids difficulties that afflict global approaches. I then argue that local reduction between theories should be understood in terms of the more basic notion of reduction between models of a single fixed system. The second goal is to illustrate what is meant by fixed-system, inter-model reduction by giving an account of this concept in a special class of cases where both models of the system in question are dynamical systems, and to show that such cases can be analyzed in terms of a certain model-based adaptation of the Nagel/Schaffner approach to reduction. I further show that this broadly Nagelian analysis of inter-model reduction encompasses many cases that have been cited as instances of "physicists'" limit-based notion of reduction, as well as providing a more precise characterization of these cases than do existing formulations of the limit-based approach. Finally, I suggest how this model-based adaptation of the Nagel/Schaffner approach might be extended to fixed-system, inter-model reduction involving models that are not dynamical systems.

The present analysis of reduction is given in two parts, corresponding respectively to the two goals just described. Part I, which consists of Sections

2 and 3, is largely non-technical and concerns issues of general methodology. As suggested, its purpose is to motivate and present a certain local, model-based approach to inter-theoretic reduction and to explain how this strategy avoids certain difficulties that afflict more global approaches. In Section 2, I briefly review two approaches to reduction - global Nagelian and global limit-based - that are often taken as the focus of philosophical discussions of inter-theoretic reduction in physics, and highlight some of their limitations. In Section 3, I sketch a local approach to inter-theoretic reduction in physics that relies on the more basic notion of fixed-system reduction between models and respond to one major objection that such an approach is likely to elicit.

Part II, which consists of Sections 4, 5 and 6, provides a detailed technical analysis of fixed-system, inter-model reduction in a particular set of cases where both models of the system in question are dynamical systems, as well as briefly discussing possible expansions of this analysis to fixed-system, inter-model reduction involving other types of model. Section 4 describes a general mathematical relationship between dynamical systems models that serves to underwrite many real instances of fixed-system, inter-model reduction in physics. In a certain strong sense, this mathematical relationship constitutes an application of the criteria for Nagel/Schaffner reduction to the context of fixed-system, inter-model reduction between dynamical systems models. Section 5 shows how this general relationship serves to characterize reduction across a wide range of particular cases, and to subsume a number of cases that are commonly cited as examples of the physicist's limit-based notion of reduction. Section 6 briefly discusses possibilities for extending and generalizing this strategy for inter-model reduction beyond the set of cases discussed here: first, to an analysis of the relationship between symmetries of the two models involved in a reduction, and second, to an analysis of cases where one or both of the models involved in the reduction is not a dynamical system but some other kind of model (e.g., stochastic, non-dynamical, etc.).

The distinct portions of the analysis given in Parts I and II complement each other in a number of important ways. Part I serves to frame the analysis of reduction between dynamical systems given in Part II within a more general picture of inter-theoretic reduction and in particular to situate this analysis relative to the two accounts of inter-theoretic reduction in physics first distinguished by Nickles. By the same token, Part II provides a concrete illustration of the sort of fixed-system, inter-model reduction that is taken as the basis for the local approach to inter-theoretic reduction described in Part I.

1.1 A Few Points of Terminology

Before proceeding, it is worth taking a moment to clarify several points of terminology.

Because debates about reduction are often frought with ambiguity as to what, precisely, is meant by reduction, I should clarify my use of the term here.

I do not attach my usage to any specific account of reduction - e.g., Nagelian, limit-based, New Wave, functionalist. Rather, I use it to designate a certain general concept that, I take it, all, or most, of the many specific accounts aim to make more precise. "Reduction," then, is taken to designate the general requirement that two descriptions of the world "dovetail" in such a manner that one description entirely encompasses the range of successful applications of the other. That is, reduction on this usage requires *subsumption* of one description's domain of applicability by the other, while the specific sense in which the two descriptions "dovetail" in order to achieve this is deliberately left vague, so as not to bias its usage toward any particular account.

As Nickles has noted, the usage of the term "reduction" most common among philosophers takes the less accurate and encompassing description in a reduction to "reduce to" the more accurate and encompassing description, whereas the usage most common among physicists takes the more accurate, encompassing description to "reduce to" the less accurate and encompassing description. In what follows, I will always adopt the philosopher's convention, even when discussing the physicist's limit-based notion of reduction, so that if theory T_2 is a "limiting case" of T_1, we will say that T_2 "reduces to" T_1.

I will also reserve the term "high-level" to refer to the description that is purportedly reduced and "low-level" to refer to the description that purportedly does the reducing. This usage generalizes another use of the "high-level/low-level" distinction, which presupposes that the high-level description is in some sense a coarse-graining of the low-level description, or that the high-level description is in some sense "macro" and the low-level description in some sense "micro." Here, no such assumption is made. For example, where the relation between Kepler's and Newton's theories of planetary motion is concerned, Kepler's theory would count on our usage as the "high-level" and Newton's as the "low-level" theory even though Kepler's theory is not in any normal sense a coarse-graining of Newton's. While some authors have emphasized the distinction between "inter-level" reductions (e.g. thermodynamics to statistical mechanics) and "intra-level" reduction (e.g. Newtonian mechanics to special relativity, or Kepler's to Newton's theory of planetary motion), the picture of reduction presented here does not rely on this distinction and treats both kinds of reduction on a par. [2]

Henceforth, when I speak of "Nagelian" reduction, the reader should take this to refer specifically to the Nagel/Schaffner account of reduction, which allows for approximative derivations rather than requiring exact derivations. While Nagel/Schaffner reduction is widely framed within a syntactic view of theories - as opposed to the semantic, model-theoretic view adopted here - and is often taken to require global rather than local derivations, my use of the label "Nagelian" here does not presuppose these characteristics. Rather,

[2] As with the term "reduce," it is worth noting that one occasionally finds the high-/low-distinction inverted, so that the "high-level" description is the more encompassing and the "low-level" description the less encompassing of the two.

what is taken to be constitutive of "Nagelian" reduction on my usage is the general requirement that it be possible to derive, on the basis of the low-level description and through the use of bridge principles, approximate versions of the laws or constraints or equations that serve to characterize the high-level description. My usage does not presuppose any view as to whether theories are understood syntactically or semantically - although the specific local approach to reduction advocated here fits much more naturally with a semantic view. Moreover, my usage does not assume any commitments as to the particular nature of these bridge principles - such as whether they are to be understood as empirically established laws or definitions - apart from their role in enabling a translation or comparison between the frameworks of the two descriptions in question. Rather than taking the term "Nagelian" to designate a specific set of precisely defined formal requirements for reduction, I use it here to designate a certain broad *strategy* for reduction.

Since the local approach to inter-theoretic reduction that I describe here is grounded in a certain account of inter-model reduction, I should say something about what I take to be the relationship between theories and models. For the purposes of this discussion, it will suffice to note that the manner in which any theory serves to describe a physical system in its domain is through some particular model of that theory. Moreover, the specification of any such model entails much narrower commitments than those that serve to characterize the theory itself. For example, specification of a particular quantum or classical model of a material object requires commitments to a particular form for the Hamiltonian (or force law or Lagrangian), including particular values for quantities like mass and charge, while the *theories* of quantum and classical mechanics themselves are compatible with many functional forms of the Hamiltonian and many values for these parameters. One of the central points of my discussion here is that it is sometimes not just the more general specifications that serve to characterize the high- and low-level theories in question that are relevant to underwriting the success of the high-level theory in a given case, but also the narrower, more context-dependent specifications that characterize the particular models of the two theories.

Part I: Local Reduction in Physics

In the literature on reduction across the sciences and in philosophy of mind, concerns about multiple realization have lead many philosophers to espouse a more "local" form of reduction that allows a low-level description to account for a high-level description's successes through many context-specific derivations that employ context-specific bridge principles. As a whole, the literature on reduction specifically within physics - which merits focused attention in part because of the special mathematical issues that it raises - has been slow to absorb this development, in that discussions of the general methodology of inter-theoretic reduction in physics very often seem to presuppose a global understanding of reduction. The goal of Part I of this article is to argue that insights about local reduction that have arisen in the general philosophy of science and philosophy of mind literatures ought to be imported into the analysis of reduction specifically within physics, and to suggest how this should be done.

In emphasizing a local approach to inter-theoretic reduction in physics, I do not mean to suggest that it is not possible to effect global reduction, or something approximating it, in some cases. Local reduction is weaker than global reduction and so includes global reduction as a special case. In fact, local reduction accommodates a whole spectrum of cases varying in the extent of their "global-ness": for inter-theoretic reductions at the least global extreme of this spectrum, a separate derivation is required for each separate system in the domain of the high-level theory in order to effect the necessary subsumption of domains; at the most global extreme, it is possible to effect a fixed-system, inter-model reduction for every system in the high-level theory's domain on the basis of a single derivation that applies uniformly across all such systems (and so does not depend essentially on details that characterize some systems but not others). In cases of inter-theoretic reduction that lie between these extremes, the derivations that underwrite fixed-system, inter-model reduction will rely on general results and mechanisms that apply across a wide range of systems, while also requiring reference to system-specific details at certain points in the derivation. An example of an inter-theoretic reduction at the global extreme of the spectrum is the reduction of Kepler's theory of planetary motion (which consists of models that obey Keplter's three laws) to Newton's theory of gravitation. The domain of Kepler's theory consists the motions of the various planets around the Sun (as well as other similar solar systems in the cosmos). Because the same general derivation connects models of Kepler's theory to models of Newton's theory irrespective of the particular planet in this domain that is being represented, the reduction of Kepler's theory to Newton's theory lies at the global end of the spectrum. On the other hand, not all inter-theoretic reductions required by the conventional "imperialist" view of the progress of physics admit of such global derivations, as I show explicitly in Section 3.1.

2 Nickles' Two Senses of Reduction

Since Nickles' 1973 paper, most philosophical discussion about the general methodology of reduction in physics has revolved around the Nagelian and limit-based approaches. Conventional presentations of both approaches treat inter-theory reduction as a matter of deriving one theory from another (whether through Nagelian deduction or a limiting process). This manner of framing the issue strongly suggests that inter-theoretic reduction is being taken in these presentations to be a matter of effecting a single derivation of the high-level theory's laws from the low-level theory, rather than many separate derivations specialized to different contexts. This in turn suggests that it is the general assumptions characterizing the two theories, rather than the more specific details characterizing the theories' models of particular systems, that are relevant to showing how the low-level theory serves to encompass the high-level theory's domain of success. Implicitly if not explicitly, these approaches seem to demand a global rather than a local form of reduction. While it is not impossible in some cases to interpret conventional formulations of both approaches as allowing for many local derivations, what can be said with relative definiteness is that these formulations do not make the need for local derivations, or the fact these local derivations concern specific models of the theories rather than the theories themselves, at all explicit; and both of these points, I take it, do bear emphasizing explicitly. Moreover, as I discuss below, where the Nagel/Schaffner approach is concerned, the extensive philosophical literature that takes multiple realizability to preclude application of this approach does explicitly construe it as a global form of reduction.

2.1 Nagel/Schaffner Reduction

Reduction on the Nagel/Schaffner approach can be understood as a three-step process, starting with the basic ingredients of a low-level theory T_l, a high-level theory T_h, and a set of bridge principles (see, for example, [2] and [12]):

1. Derive an "image" theory T_h^* for some restricted boundary or initial conditions within the low level theory T_l. The "laws" of the image theory T_h^* take the same form as the laws of T_h, but relate quantities that are defined using the terms and concepts of T_l.

2. Use bridge principles to replace terms in T_h^*, which belong to the vocabulary of the low-level theory, with corresponding terms belonging to the high-level theory. This yields the "analogue" [3] theory T_h'. Like the "laws" of T_h^*, those of T_h' have the same form as the laws of T_h, but employ the terms and concepts of T_h rather than of T_l.

3. If the analogue theory T_h' is 'strongly analogous' to the high level theory T_h, the high level theory has been reduced to T_l. The 'strong analogy' relation is sometimes also characterised as approximate equality, close agreement, or good approximation and can be understood in any of these senses.

Note that reduction on the Nagel/Schaffner approach does not require derivation of the high-level theory's laws themselves, but rather of some suitable approximations to them. This is the key point that distinguishes Nagel/Schaffner

[3]The labels "image" and "analogue" are not common to all presentations of the Nagel/Schaffner approach. However, the substantive points of procedure described here are.

reduction from Nagel's original approach. A further point worth noting is that there remains some dispute as to whether the bridge principles of Nagel/Schaffner reduction are empirically discovered laws or mere definitions, and whether they are biconditional or merely conditional relations. The popular term "bridge law," which is often used to designate bridge principles, suggests a clear bias toward interpreting them as empirical laws; here, I use the term "bridge principle" so as to avoid the suggestion of bias toward any particular view on this matter.

2.2 Difficulties with Nagelian Reduction: Multiple Realization

For a comprehensive review of, and response to, the many critiques that have been levelled against Nagelian approaches to reduction, Dizadji-Bahmani *et al*'s 2011 article "Who's Afraid of Nagelian Reduction?" is excellent [12]. In the present context, I will focus exclusively on critiques grounded in multiple realization (MR) because such critiques have served as the major impetus for considering more local forms of reduction, particularly in the philosophy of mind and general philosophy of science literatures.

As its name suggests, multiple realization occurs when a single element of a high-level description (e.g., model, theory or whole science) is realized by more than one element of some lower-level description. A classic example from the philosophy of mind literature is the high-level psychological property of pain, which can be multiply realized in human brains, dog brains, badger brains and so on, all of which have different low-level biological (and physical) descriptions [27]. In the context of Nagelian reduction, realization of an element in the high-level theory by some element of the low-level theory is signified by means of a bridge principle. On many formulations of the Nagel/Schaffner approach, bridge principles are required to be bi-conditional identity statements identifying natural kinds in the high-level theory with natural kinds in the low-level theory. But, as Fodor has argued, the best one can do in cases where multiple realization occurs is to identify the relevant concept in the high-level description with a *disjunction* of associated elements in the low-level description, and it is simply false in most cases that such a disjunction will be a natural kind of the low-level description (for example, in the sense that it occurs naturally in the laws or equations of the low-level theory)[16], [5]. In this manner, multiple realization typically precludes the existence of bridge principles of the sort that are required by these formulations of Nagelian reduction.

Implicitly, this sort of MR-based critique presupposes a global interpretation of the Nagel/Schaffner approach, since its argument assumes that bridge principles are required to identify an element of the high-level description with the *same* "natural kind" element of the low-level description across all contexts where the high-level description applies. In cases of multiple realization, there simply do not exist such global bridge principles. The unavailability of these bridge principles, in turn, precludes the possibility of the sort of global derivation required by global formulations of Nagelian reduction.

2.3 Limit-Based Reduction

On the limit-based approach to reduction, a high-level theory T_h reduces to a low-level theory T_l if T_h is a "limit" or "limiting case" of T_l. Somewhat more precisely, T_h reduces to T_l if there exists some set of parameters $\{\epsilon_i\}$ defined within T_l such that

$$\lim_{\{\epsilon_i \to 0\}} T_l = T_h \tag{1}$$

[4] [25],[2]. Unlike Nagelian reduction, the notion of a limit-based approach to reduction, as first explicitly identified by Nickles, seems to arise not from any clear-cut statement of the general requirements for this kind of reduction, but rather from an assortment of suggestive mathematical results, all of which involve or somehow gesture at the use of mathematical limits. It also seems to arise in part from a manner of speaking that is often employed in discussions of inter-theory relations in physics, as exemplified by references to the "nonrelativistic limit" of special relativity, the "classical limit" of quantum mechanics, the "thermodynamic limit" of statistical mechanics, the "geometric optics limit" of wave optics, and so on. Various facets of this approach to reduction have been explored by many authors, including Batterman, Butterfield, Rohrlich, Berry, Ehlers, and Scheibe, among others [1], [7], [9], [28], [4], [14], [30], [31]. Recently, Norton has highlighted a role for limiting procedures in physics outside the context of reduction - specifically, in distinguishing between the activities of approximation and idealization and in fending off potential confusions caused by conflation of the two [26]. While it is clear that limits have a central role to play in our understanding of inter-theory relations generally, where reduction specifically is concerned, the vague and schematic relation $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$ appears to be as close to a statement of general criteria for limit-based reduction as has been given in the literature.

2.4 Difficulties with the Limit-Based Approach

Perhaps the most serious concern about the limit-based approach to reduction is that, in spite of its mathematical nature, it is extremely vague in its characterization of the general relationship that it takes to underwrite reduction. The expression $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$ is not mathematically well-defined, as there is no precise or general definition of what it is for one theory to be a limit or limiting case of another. Moreover, it is unclear from this expression whether

[4] Note that if one has $\lim_{\{\epsilon_i \to \infty\}} T_l = T_h$, or $\lim_{\{\epsilon_i \to a\}} T_l = T_h$ where $0 < a < \infty$, one can always redefine the parameters $\{\epsilon_i\}$ so that the limit approaches 0.

the prescription to take the limit is to be understood literally or in some loosened sense. For example, if we are to understand the claim that Newtonian mechanics is the limit as $\frac{v}{c} \to 0$ of special relativity literally, then the claim is patently false, since the limit of special relativity as $\frac{v}{c} \to 0$ is a theory in which nothing moves, not Newtonian mechanics (assuming we take c, the speed of light, to be constant, as it is for all real physical systems). For this paradigmatic case to be regarded as an instance of limit-based reduction, it seems necessary to adopt a more liberal construal of the term "limit" - for example, by making use of first- or higher-order approximations in Taylor expansions in $\frac{v}{c}$ since strictly speaking, it is only the *zero*-th order term of such an expansion gives the actual *limit* of this series as $\frac{v}{c} \to 0$. In other cases, including discussions of the thermodynamic limit of statistical mechanics, where the number of degrees of freedom in a system is taken to infinity, the limiting process is interpreted literally - for example, when it is pointed out that the only way to recover the discontinuities of certain functions in thermodynamics from statistical mechanics is to literally take the limit as the number of degrees of freedom approaches infinity. Beyond the points of vagueness already mentioned, it is not clear on this approach *which parts* of T_h and T_l must be related by these "limits" in order for the relation $\lim_{\{\epsilon_i \to 0\}} T_l = T_h$ to hold; presumably, not just any limiting relation between any two parts of the theories will do. Finally, limit-based approaches tend to differ on what constraints, if any, should be placed on the parameters ϵ_i - for example, whether they are supposed to be dimensionless, or may also be dimensionful constants of nature, for example.

Given that this approach offers very little by way of precise characterizations or clear commitments as to the nature of reductive relations in physics, it seems possible to take many cases that we might wish to characterize as successful reductions and categorize them as instances of limit-based reduction. But if existing formulations of the limit-based approach succeed at accommodating many cases in physics, it is largely for the reason that they tell us so little, and are so vague, about the general requirements for reduction. It seems possible to count any reduction as an instance of limit-based reduction as long as the reduction somehow incorporates a procedure that may liberally be construed as "taking a limit." The worry, then, is that this approach, at least in its existing formulations, may give the false impression of providing an authoritative, general account of reduction in physics in spite of having offered little by way of general insight beyond the assertion that, in cases of reduction, *something like* a limit is *somehow* involved in the explanation of the high-level theory's success on the basis of the low-level theory.

Recalling the focus here on comparing global and local forms of reduction, I should point out that since the limit-based approach is conventionally formulated in terms of limiting relations between whole theories, it seems most natural to read this approach as a global strategy for reduction. However, given the vagueness of existing formulations of this approach, it does not appear that anything precludes interpreting this relation on a more local basis. Even so, such a formulation is still vulnerable to worries about precisely the sort of vagueness that allows for this vast flexibility of interpretation.

3 Local Reduction in Physics

It was largely in response to critiques of global Nagel/Schaffner reduction that a number of authors, mostly in the philosophy of mind literature, were prompted to advocate for a more local approach to reduction in which a lower-level description accounts for the successes of a higher-level description through many context-specific derivations employing context-specific bridge principles, rather than through a single global derivation employing the same set of bridge principles for all systems in the high-level description's domain. In particular, Kim has argued that while multiple realization rules out "structure-independent" reductions of psychology to physical science, it does allow for "structure-specific" local reductions between these levels [22]. Following Kim, Dizadji-Bahmani *et al* also have advocated a local response to anti-reductionist arguments from multiple realizability in the context of their general defence of Nagelian reduction [12]. Similar views can be found in the work of Churchland, Hooker, Schaffner, Enc and Lewis [11], Ch. 7; [20]; [29]; [15]; [23]; [5]. The main goal of the present analysis is show explicitly how this sort of local approach, which has been developed primarily in discussions about reduction in philosophy of mind and general philosophy of science, can be imported into methodological discussions of inter-theoretic reduction specifically within physics.

3.1 The Need for Local Reduction in Physics: An Illustrative Example

An example, concerning the relationship between classical and quantum mechanics, will serve to make my general point that global reductions are not available in all cases where the conventional wisdom requires reduction to occur, and that only a weaker and more local (but still highly non-trivial) form of reduction is available in such cases. As we will see, the source of trouble for global approaches to reduction in physics is that it is often not just the broad generalities characterizing the theories - which are common to all models of each of the theories - but also details that differentiate the various models of a given theory, that play a role in explaining why the high-level theory works in a given case. In short, system- or context-specific details often play an indispensable role in accounting for the high-level theory's success on the basis of the low-level theory, and this precludes the sort of generality demanded by global forms of reduction.

The conventional wisdom about the progress of physics asserts that quantum mechanics has strictly superseded quantum mechanics in the sense that any system whose behavior can be accurately modelled in classical mechanics also can be modelled, and modelled more accurately, in quantum mechanics. [5] Here, I will argue that any demonstration to this effect could not possibly, even in principle, take the form of a global reduciton. Consider two systems in the domain of classical mechanics, both of which are described by the same classical model of a simple harmonic oscillator: the first system is a mass on a spring; the second is an electric charge of the same mass moving along a path bored through the axis of a uniform spherical charge distribution. (One can show that in the second case, there will be a restoring force on the charge that varies linearly with its distance from the center of the sphere.) Assume moreover that frictional/radiation effects can be ignored in both systems, and that the linear restoring force in both cases is characterized by the same effective spring constant. Assuming that macroscopic bodies do belong to the domain of quantum mechanics, it is clear in this case that the quantum mechanical models that serve to describe these two physical systems are going to be radically different from each other, and that the process of accounting for the success of the classical harmonic oscillator model on the basis of these quantum models is going to play out very differently in the two systems. In the second case, the classical potential generated by the electric field in the classical model will be the same potential that appears in the Schrodinger equation of the underlying quantum model of the charge's behavior. In the first case, the fact that one can employ a harmonic oscillator potential in the classical model in order to describe the motion of the block is something that needs to be explained in terms of the complex microscopic constitution of the spring - at the microscopic level, this potential will be wildly fluctuating on the length scale of the atoms and molecules making up the spring. A global reduction of classical to quantum mechanics is thus out of the question, since distinct derivations of the classical model's success on the basis of the quantum model are required for each of the two systems.

While global reduction between classical and quantum mechanics fails in this case, there remains a non-trivial, and more local, sense of inter-theoretic reduction in physics, which I describe in the next section, that has the potential to accommodate this sort of example. One thing that people might - and, I take it, often do - mean by the claim that "classical mechanics reduces to quantum mechanics" is that every physical system whose behavior can be modeled in classical mechanics also can be modeled in quantum mechanics at least as accurately and in at least as much detail. This manner of construing the term "reduction" allows for the possibility that for each system separately, there is a direct relationship between the quantum and the classical model *of that system*

[5] There is a sense in which this particular case might appear to be less than an ideal example, given the various added complexities brought about by the notorious interpretational difficulties that afflict quantum mechanics, which seem as though they should bear quite heavily on the relation between quantum and classical descriptions of real physical systems. However, these difficulties do not affect my central point here, as long as we are safe in assuming that the correct interpretation of quantum mechanics, whatever that happens to be, is able to underwrite the success of classical descriptions on a case-by-case basis.

that permits us to understand why the classical model succeeds at describing the system's behavior given that the quantum model is the more detailed and accurate of the two descriptions. Arguments from multiple realization present no reason for doubting that such system-specific quantum-mechanical accounts of the success of classical mechanics are available in such cases.

3.2 A Local, Model-Based Approach to Inter-Theoretic Reduction

The example just discussed suggests a local picture of inter-theoretic reduction in physics where reduction between theories may consist of many local, fixed-system reductions between models - which in turn may rely on assumptions specific to individual systems - rather than requiring a single global derivation that references only the broad generalities that characterize the theories as a whole. Let us provisionally suppose a notion of reduction between two models of a single fixed system, $reduction_M$, whereby the low-level model accounts for the success of the high-level model at tracking the behavior of the system in question; the nature of this relationship will be further illustrated in the second half of this paper through the detailed analysis of $reduction_M$ in a particular class of cases where both models are dynamical systems. Let us designate our local concept of reduction between theories $reduction_T$, which we define as follows:

> **Criteria for Local Inter-Theoretic Reduction:** Theory T_h $reduces_T$ to theory T_l iff for every system K in the domain of T_h - that is, for every system K whose behavior is accurately represented by some model M_h of T_h - there exists a model M_l of T_l also representing K such that M_h $reduces_M$ to M_l.

There are several crucial points to note about this local, model-based concept of inter-theoretic reduction. First, it presupposes a semantic understanding of theories as families of models, rather than a syntactic view which treats theories as axiomatic sets of propositions. Second, it is not simply a two-place relation between the theory T_h and the theory T_l, but rather a *three*-place relation between T_h, T_l, and that class of real systems in the world that are well-described by models of T_h. Since reduction is understood here to require that one description dovetail (in some appropriate sense) with another specifically in those cases where the reduced description *works*, some specification of the reduced description's domain of success is required in order to assess whether any dovetailing between the theories is sufficient to underwrite the required subsumption; here, the relevant dovetailing between theories generally occurs on a local level, between the different models that they use to describe systems in the high-level theory's domain. Third, $reduction_M$, characterized broadly, requires that M_l furnish at least as accurate a description of K as M_h does in those cases where M_h successfully tracks (to within some margin of approximation) K's behavior. Fourth, the precise nature of relationship between models M_h and M_l that serves to underwrite $reduction_M$ in a given case varies depending on the particular mathematical form of these models. Fifth, given the emphasis here on the distinction between theories and their individual models, it is also important to distinguish between what is meant here by the domain of a theory, on the one hand, and the domain of a model that the theory uses to describe a particular physical system. The domain of a single model of a theory, relative to some physical system K, consists of the set of circumstances under which the model accurately tracks the system's behavior. The domain of a theory, as understood here, consists of the range of physical systems whose behavior is well-described (under some fairly robust range of circumstances) by some particular model of the theory, *as well as* the set of circumstances under which each of these models succeeds in tracking the behavior of the system that it describes. Sixth, the account of $reduction_M$ that I develop in Part II for the particular case of reductions between dynamical systems models follows, in a certain strong but qualified sense, the prescriptions of a localized Nagel/Schaffner approach. As I discuss in Section 6.2, the broadly Nagelian character of $reduction_M$ in the case of dynamical systems likely extends to cases of $reduction_M$ involving other kinds of model as well. For this reason, the strategy for inter-theory reduction in physics described here can be understood as a local, model-based formulation of the Nagel/Schaffner approach.

3.3 A Worry about Local Reduction

The local approach to inter-theoretic reduction described in the last section addresses worries about multiple realization by allowing domain-relative, local derivations that employ context-specific bridge principles. One concern about this sort of move, however, is that in its acceptance of disjointed, local derivations of multiply realized high-level regularities, this strategy foregoes a certain legitimate demand for *explanation* of MR - that is, an explanation that alleviates our sense of mystery as to how systems with such disparate low-level descriptions can all give rise to the same high-level regularity, presumably by identifying some salient commonality among them.

The appropriate response to this sort of worry, I believe, begins by distinguishing two problems concerning inter-theory relations: first, the problem of demonstrating the sort of subsumption that I take to define reduction, or more precisely of showing that any system that can be accurately modelled in the high-level theory can be modelled at least as accurately in the low-level theory; second, the problem of explaining multiple realization. The response then is simply to acknowledge that local approaches to reduction address the first problem but not the second. Put more bluntly, the response is simply that it is not the *job* of reduction, as the notion is understood here, to explain how multiple realization comes about. Given the vast differences in the mathematical and conceptual frameworks of theories between which reduction is purported to hold, the task of understanding precisely how one and the same physical system can consistently be described by models of both theories simultaneously poses a significant enough challenge that it deserves not to be conflated with the task of explaining how multiple realization comes about. Reduction already has enough on its plate, so to speak. Indeed, there are many who doubt, and some, including pluralists like Cartwright and Dupre, who outright deny that it is possible to effect the sort of subsumption required by local approaches to reduction - that is, who deny the possibility of the sort of theoretical "imperialism" that seeks to include macroscopic systems like the moon within the domain of quantum mechanics, or thermodynamic systems within the domain of statistical mechanics [10], [13]. Local reduction facilitates such imperialism, but without providing the sort of explanation of multiple realization that is sometimes called for.

Having made these remarks, it is worth pointing out that a good deal of important philosophical analysis has been done on the question of how to explain multiple realization in physics. In particular, Batterman has noted that the phenomenon of universality in physics just *is* multiple realization, and that in physics one *does* find explanations of universality in the form of renormalization group analyses [3]. However, I emphasize again that this sort of issue is distinct from (though closely related to) the issue of how to effect the sort of theoretical subsumption required by reduction, and that it is only this latter issue that I seek to address here.

Part II: The Case of Dynamical Systems

My purpose in the second half of this paper is to illustrate more precisely, through a specific class of cases, what is meant by "fixed-system, inter-model reduction" ("$reduction_M$") in the local picture of inter-theoretic reduction ("$reduction_T$") spelled out in Section 3.2. In the class of cases that I consider here, both models of the system in question are dynamical systems models and, moreover, are assumed to share a common time parameter. In Section 4, I describe a general mathematical relationship, which I call "dynamical systems reduction," "DS reduction," or "$reduction_{DS}$," that provides a template for reduction between dynamical systems models across a wide range of cases, as I show explicitly with several examples in Section 5. Moreover, I show in Section 4.4 that there is a strong sense in which DS reduction follows the broad prescriptions of Nagel/Schaffner reduction, albeit in a form adapted to the specialized context of fixed-system reduction between dynamical systems models. Part of the significance of $reduction_{DS}$ is that it provides an example of a general mathematical relationship between models whereby one model can improve on the scope, detail and accuracy of another model in its description of a physical system, fully encompassing the latter model's domain of application. It also serves to illustrate how models employing radically different mathematical and conceptual frameworks can both consistently (within some margin of approximation) and accurately describe the behavior of one and the same physical system.

As I argue in Section 6, certain central features of $reduction_{DS}$ can likely be carried over to cases of $reduction_M$ in which the models in question do not meet the preconditions for $reduction_{DS}$ - for example, in cases of reduction between dynamical systems models that do not share a time parameter, or in cases where one or both of the models M_l and M_h is stochastic (e.g., models of Brownian motion, or of GRW quantum mechanics), or non-dynamical (e.g. the Ideal Gas Model), or where there is no global splitting of the space of possible solutions of the model into state spaces at different times. I also suggest how various features of DS reduction might be extended to provide an analysis of the relationship between symmetries of different models of the same system. I leave it to future work to give a more detailed analysis of $reduction_M$ in such contexts. However, I argue in Section 7 that the strategy for reduction in these other cases is also likely to be Nagelian in the broad, liberalized sense described in Section 1.1. As was noted in the Introduction, one important general aspect of the approach to reduction adopted here is to begin narrowly by considering reduction in relatively specialized contexts and then consider what generalities can be drawn across ever wider ranges of cases, rather than demanding a high level of generality at the outset.

4 Fixed-System Reduction Between Dynamical Systems Models

Dynamical systems models occur widely throughout physics: in Hamiltonian models of non-relativistic and relativistic classical mechanics and non-relativistic and relativistic classical field theory, in Schrodinger picture models of non-relativistic quantum mechanics, relativistic quantum mechanics and quantum field theory, and in heat diffusion equations in thermodynamics, to name a few cases.

The notion of a dynamical system has been defined in a number of distinct ways that vary both in generality and in their specific requirements. However, all definitions capture some variation on the intuitive notion of a deterministic rule that governs the time evolution of a point in some mathematical space, which in real-world applications usually represents the state of a physical system. More formally, a *dynamical system* is a triple $M = (S, \mathcal{T}, D)$, where the "state space" S is a set, the "timeset" $\mathcal{T} \subset \mathbb{R}$ is an additive semigroup, and the "evolution operator" $D : S \times \mathcal{T} \to \mathcal{S}$ is a one-parameter group of transformations of S satisfying the properties $D(x, 0) = x$ and $D(D(x, t_1), t_2) =$

$D(x, t_1 + t_2)$ for all $x \in S$ and $t_1, t_2 \in \mathcal{T}$ [6]. For our purposes here, I will further assume that S is a differentiable manifold endowed with a norm, that $\mathcal{T} = \mathbb{R}$, that D is continuous, and that D is an invertible function of x for each time t. In dynamical systems that occur in physics, the evolution operator D is usually determined by some set of first-order differential equations of motion,

$$\frac{dx}{dt} = f(x), \tag{2}$$

where $f(x)$ is a continous function of x. The functions f and D are related by the equation $\frac{\partial}{\partial t} D(x_0, t) = f(D(x_0, t))$. That is, $D(x_0, t)$ for fixed x_0 specifies a solution to the equation (2).

4.1 *Reduction$_M$* for Dynamical Systems

The analysis of reduction between dynamical systems provided in this section draws on the work of several authors. All explore variations on the notion that reduction between dynamical descriptions of a system requires *commutation* between the operation of dynamical evolution and some other operation, such as coarse-graining, that maps elements of the low-level state space into elements in the high-level state space. More precisely, the thought is that application of the low-level dynamics for time t followed by application of the mapping from the low- to the high-level state space should approximately equal the result of first applying this mapping and then applying the high-level dynamics for time t. Wallace explores variations of this idea in the context of an investigation into the quantum-mechanical (specifically, Everettian) arrow of time [33]. [6] Giunti discusses this requirement in the context of reduction between dynamical systems generally, although the specific criteria that he imposes are too strong to apply to very many, if any, realistic cases in physics [18]. Yoshimi discusses a similar approach in the context of a general discussion about supervenience, with a view toward applications in philosophy of mind [35]. Butterfield suggests a formulation of this commutation condition that is quite similar to (but formulated independently of) Yoshimi's, with a view to applications in physics; however, he points out a number of ways in which this condition may need to be weakened in order to be applied to realistic cases in physics [8]. The formulation of the commutation-based requirement for reduction that I describe here differs from those of Wallace, Yoshimi and Butterfield in that it takes the high-level and the low-level models to be specified *independently* of each other, whereas on these other formulations the high-level dynamics are specifically *defined* as the dynamics induced by the low-level dynamics through some coarse-graining procedure. Giunti's formulation, like the one suggested here, does take the high- and low-level models to be independently specified, though requires the operation connecting the state spaces of

[6] It was also Wallace who first introduced me, in conversation, to this way of thinking about reduction.

the models specifically to be an injection and requires the commutation to be exact; both of these requirements preclude application to most realistic cases in physics. As I show in the next section, the formulation of the requirement that I present here is refined in such a way as to allow for broad application specifically to inter-model reduction in physics.

Assume, then, that two distinct dynamical systems models $M_h = (\mathbb{R}, S_h, D_h)$ and $M_l = (\mathbb{R}, S_l, D_l)$, which share a time parameter t, both serve to describe the same physical system K. (This is the case, for example, with non-relativistic and relativistic models of a proton, or between non-relativistic quantum-mechanical and relativistic quantum-field-theoretic models of an electron, or between classical and quantum mechanical models of the center of mass of a golf ball, and in a wide range of other instances.) Assume, moreover, that M_h succeeds at tracking the dynamical behavior of some subset of the degrees of freedom characterizing system K within margin of error δ over timescale τ for initial conditions x_{h0} within some domain of states $d_h \subset S_h$. Then define the relation $reduction_{DS}$ as follows:

> M_h $reduces_{DS}$ to M_l iff there exists a differentiable function $B : S_l \to S_h$ that does not depend explicitly on time and a nonempty subset $d_l \subset S_l$ in the domain of B such that $d_h \subset B(d_l)$, and for any $x_l \in d_l$,
>
> $$|B\big(D_l(x_l, t)\big) - D_h\big(B(x_l), t\big)|_h < 2\delta, \tag{3}$$
>
> or more concisely,
>
> $$B\big(D_l(x_l, t)\big) \approx D_h\big(B(x_l), t\big), \tag{4}$$
>
> for all $0 \leq t \leq \tau$.

The notation $| \circ |_h$ designates the norm on S_h, and the margin of error 2δ is taken to be implicit in the $\approx$ of (4). Note that the expression $B\big(D_l(x_l, t)\big)$ on the left-hand side of (4) represents the trajectory on S_h induced by the low-level dynamics D_l via the function B, while the expression $D_h\big(B(x_l), t\big)$ on the right-hand side of (4) represents the trajectory prescribed by the high-level dynamics for the initial condition $B(x_l) \in S_h$. The mathematical relationship between M_h and M_l specified by $reduction_{DS}$ thus requires that the function B - let us call it a "bridge map" - commute with the operation of time evolution, where the time evolution is specified by the low-level dynamics if it comes before application of the bridge map and by the high-level dynamics if it comes after application of the bridge map. The requirement that $d_h \subset B(d_l)$ is imposed for the purpose of ensuring that all possible trajectories of M_h that would accurately track the behavior of the relevant degrees of freedom of K are well-approximated by trajectories induced via B by the low-level dynam-

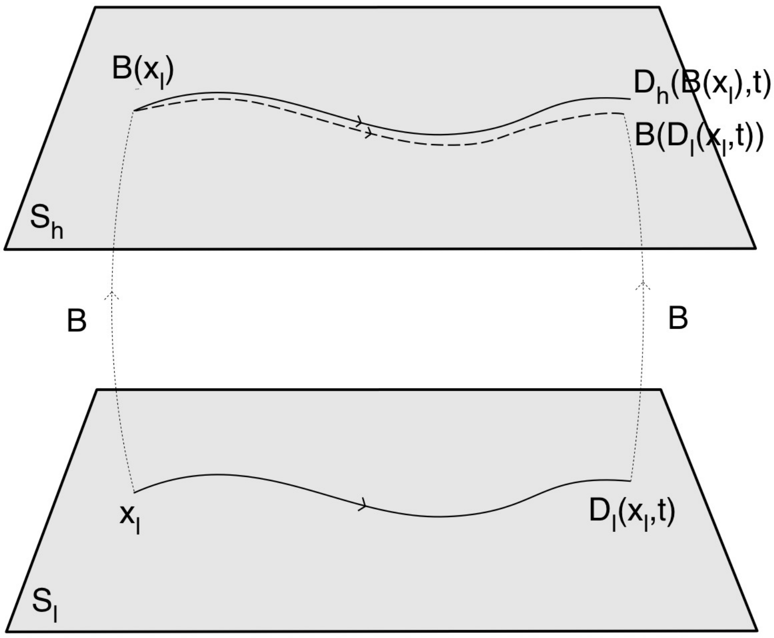

Figure 1: *Reduction$_{DS}$* requires the existence of a time-independent "bridge map" B from the state space of the low-level model, S_l, to that of the high-level model, S_h, that approximately commutes with the operation of dynamical evolution for initial states x_l in a certain domain $d_l \subset S_l$. This amounts to the requirement that the trajectories on S_h induced via B by the low-level dynamics approximate the trajectories prescribed by the corresponding high-level dynamics.

ics. In fact, in cases of reduction one expects the trajectory $B\big(D_l(x_l,t)\big)$ to track the behavior of these degrees of freedom more accurately than the trajectory $D_h(B(x_l),t)$ if the low-level model M_l is the more accurate of the two descriptions. That is, we expect that over timescale τ, $B\big(D_l(x_l,t)\big)$ tracks the behavior of the relevant physical degrees of freedom of K to within a margin of error γ such that $\gamma < \delta$. This in turn requires that the trajectories $D_h\big(B(x_l),t\big)$ and $B\big(D_l(x_l,t)\big)$ agree with each other to within a margin of 2δ. Thus, the physical property of system K that is represented by x_h in the high-level model is represented in the low-level model by $B(x_l)$, so that there is a strong sense in which the high-level state and the function of the low-level state associated with the bridge map *co-refer* in the range of contexts where both successfully track the behavior of the system. [7]

I should clarify here that *reduction$_{DS}$* is not being presented as a nec-

[7]This claim should be interpreted with the understanding that these quantities are not exactly equal and that they only serve to approximate the true behavior of the relevant physical degrees of freedom in K.

essary or sufficient condition for $reduction_M$ in cases where two dynamical systems sharing a time parameter are purported to describe the same physical system (although I suspect that it does serve to characterize most such instances). It is not being put forward as a sufficient condition because, if the quantity $B(x_l)$ in the low-level model is supposed to stand in as the low-level model's surrogate for x_h, one should expect it to mimic x_h's behavior in ways other than through its dynamical evolution - for example, through its symmetry transformation properties. This point is discussed further in Section 6. $Reduction_{DS}$ is also not being put forward as a necessary condition in order to leave open the possibility that there might be other ways in which dynamical systems models that share a time parameter and describe the same physical system may relate to each other so as to facilitate the kind of subsumption required of reduction. [8] Rather, the relation specified by $reduction_{DS}$ is being suggested here as a *template* for reduction that is useful for the treatment of a wide range of examples in this class of cases. Moreover, it suggests a strategy for reduction in some inter-model relations that are not as yet well-understood (e.g., between models of non-relativistic quantum mechanics and interacting quantum field theory, both of which can be formulated as dynamical systems using Schrodinger picture formulations) as well as suggesting extensions to inter-model reduction involving other kinds of model.

4.2 Reduction as a Three-Place Relation

I wish to address a concern about DS reduction that may occur to some readers: specifically, that given any two dynamical systems models in which the low-level state space has at least the same cardinality as the high-level one, the criteria for DS reduction will be satisfied trivially if we allow δ to be sufficiently large, and τ and d_h to be sufficiently small. This concern can be addressed by highlighting the fact that it is not just *any* values of δ, τ and d_h are consistent with the criteria for DS reduction; rather, these parameters are constrained by the nature of the fit between the model M_h and the behavior of the system K itself. Recall that the parameter δ serves to characterize the degree of accuracy with which M_h tracks the behavior of K; τ characterizes the timescales over

[8] Assuming that the Everett interpretation is an empirically viable formulation of quantum mechanics, the reduction of classical Newtonian models to quantum Everettian models in the context of certain classically behaving systems will require reference to the branching structure of the quantum state in Everett's theory. In this case, as in the cases considered here, reduction concerns two dynamical systems models (a classical Hamiltonian model and a model specifying the unitary evolution of the quantum state) sharing a time parameter. However, unlike the cases considered here, the quantum-mechanical quantity that co-refers with the phase space point in the classical model is not a function of the *whole* low-level state, but only of some particular *component* associated with one particular *branch* of that state that emerges dynamically through decoherence (where the branch in turn corresponds to one particular "world" in the Everettian Many Worlds picture). Reductions such as this, in which the low-level model describes a multiverse that emerges from the dynamics, are not included in the present analysis and require separate treatment.

which it does so; d_h characterizes the set of initial states for which M_h succeeds in tracking the system's behavior within this timescale and margin of error. Determining the values of these parameters is therefore an *empirical* matter, to be assessed on the basis of the observed fit between the high-level model and system. [9] Once we have fixed these parameters - or ranges of possible values for them - the task of assessing whether the formal requirements of DS reduction are met for a given pair of models become a question purely of mathematics. Moreover, once these parameters have been set, the criteria for DS reduction represent a highly non-trivial constraint on the mathematical relationship between the two models.

It should be emphasized here that the two models M_h and M_l by themselves do not provide us with sufficient information to assess whether the requirements for DS reduction are met between them. A third element - namely, the system itself, and more specifically a comparison of the system's behavior with the behavior prescribed by M_h - is needed in order to assess whether the criteria for DS reduction are satisfied. This should come as no surprise since $reduction_M$ requires that the low-level model dovetail structurally with the high-level model *only in those circumstances where the high-level model "works" at tracking the behavior of the system K.* And, of course, the question as to what, precisely, these circumstances are is an empirical one requiring reference to the system itself.

The criteria for DS reduction reflect a more general approach to reduction in which reduction is, in a certain sense, regarded not as a two place relation between high- and low-level descriptions, but as a *three*-place relation between high-level description, low-level description and that portion of the physical world that both represent. This follows naturally from the particular construal of "reduction" adopted here, which requires that the low-level description subsume the domain of application of the high-level description; in order to determine what this domain of application is, it is necessary to consider not only the descriptions themselves, but the quality and scope of agreement between the high-level description and the portion of the world it is supposed to represent.

4.3 "Differential" Conditions for DS Reduciton

Assume that the dynamics of the high- and low-level models are specified by first-order differential equations of the form (2), so that $\frac{dx_h}{dt} = f_h(x,t)$ and

[9] Having said this, it is also important to acknowledge that it may not always be a simple matter to assess the precise parameters of agreement between our models and the systems they represent. Nevertheless, it is often possible at the very least to place empirically determined *bounds* on δ, τ and d_h. For example, we know that current models of quantum field theory will likely cease to track the dynamical behavior of systems they represent for initial states of momentum near the Planck scale. We do not expect any successor to QFT to dovetail with the theory's prescriptions for the behavior of states in this domain.

$\frac{dx_l}{dt} = f_l(x,t)$. [10] Then it is possible to state a stronger, "differential" version of (4) that requires the low-level quantity $B(x_l)$ to approximately satisfy the high-level equations of motion. Making the variable substitution $x'_h \equiv B(x_l)$, we can write this requirement compactly as

$$\frac{dx'_h}{dt} \approx f_h(x'_h, t) \tag{5}$$

or more explicitly as,

$$\boxed{\frac{dB\left(x_l\left(t\right)\right)}{dt} \approx f_h\Big(B\big(x_l(t)\big), t\Big)}. \tag{6}$$

If this condition holds over timescale τ and the margin of error characterizing the approximate equality in (6) is η, then condition (4) will be satisfied over timescale τ and within margin of error $2\delta \equiv \eta\tau$. [11] As we will see in the next section, in typical cases, the approximate equality (6) will continue hold only as long as $x_l(t)$ remains in some restricted domain $d'_l \subset S_l$. Also, since the validity of the relation (6) hinges on the behavior of $B\left(x_l\left(t\right)\right)$, which in turn depends on the behavior of $x_l(t)$, which in turn depends on the low-level dynamical equations of motion, the condition (6) should be deduced from the low-level dynamics, together with the state restriction $x_l \in d'_l$.

4.4 DS Reduction as a Special Case of Local Nagel/Schaffner Reduction

Recall that global Nagel/Schaffner reduction, described in Section 2, distinguishes four 'theories': a low-level theory T_l, a high-level theory T_h, an "image" theory T^*_h, and an "analogue" theory T'_h. T^*_h is formulated in terms of the concepts of T_l and deduced directly from T_l, typically for some restricted boundary or initial conditions. T'_h is then obtained from T^*_h by straightforward bridge principle substitution, and is formulated in terms of the concepts of T_h. If the reduction is successful, T'_h will be "strongly analogous" to T_h, where strong analogy signifies approximate agreement of some sort. DS reduction, if effected through proof of the differential condition described in Section 4.3, follows essentially the same series of steps, but with models rather than theories.

By analogy with the four "theories" of global Nagel/Schaffner reduction, in DS reduction one has a low-level model M_l, a high-level model M_h, an "image model" M^*_h and an "analogue model" M'_h. The image model M^*_h is formulated using elements of the model M_l - that is, in terms of mathematical structures defined on M_l's state space - and can be deduced from M_l solely on

[10] Where, recall, $\frac{\partial}{\partial t} D_h(x_{h0}, t) = f_h(D_h(x_{h0}, t))$ and $\frac{\partial}{\partial t} D_l(x_{l0}, t) = f_l(D_l(x_{l0}, t))$

[11] To see this, integrate both sides of (6) from $t = 0$ up to any time t less than τ, employing the substitution $f_h(B(x_l(t))) = \frac{\partial}{\partial t} D_h(B(x_{l0}), t)$, the fact that $B(D_l(x_{l0}, 0)) = B(x_{l0}) = D_h(B(x_{l0}), 0)$, and the fact that $x_l(t) = D_l(x_{l0}, t)$.

the basis of a restriction to a particular domain of states in S_l. Its dynamics are given by the relation,

"Image Model" Dynamics:

$$\frac{d}{dt}B\big(x_l(t)\big) \approx f_h\big(B\big(x_l(t)\big), t\big) \tag{7}$$

which holds for x_l in some domain $d_l' \subset S_l$. Note that this relation approximately takes the same form as the high-level equation of motion $\frac{dx_h}{dt} = f_h(x_h, t)$, but with x_h replaced by its counterpart $B(x^l)$ in the low-level model. Recall from Section 4.3 that satisfaction of the image model dynamics within appropriate margins suffices to ensure satisfaction of the condition (4). The analogue model M_h' is then obtained from the image model through the bridge map substitution,

Bridge Map Substitution:

$$x_h' \equiv B(x^l) \tag{8}$$

and the analogue dynamics are specified by the relation,

"Analogue Model" Dynamics:

$$\frac{dx_h'}{dt} \approx f_h(x_h', t). \tag{9}$$

Note that this dynamical equation is identical to the high-level equation of motion, apart from the approximate nature of the given equality. The domain of applicability of this equation within S_h is the image domain $B(d_l')$. Note that the expression $B(x_l)$, which occurs in the image model, is an expression built from structures within the low level model M_l - in this sense, the image model is formulated in the mathematical 'language' of the low-level model. On the other hand, the more condensed notation of the analogue model conceals the detailed construction of x_h' from quantities in the low-level model M_l, regarding x_h' simply as a point in S_h rather than as a quantity constructed from elements of M_l; in this sense, one may view the analogue model as formulated in the mathematical 'language' of the high-level model. For reduction to take place, the analogue model M_h' must be "strongly analogous" to the high level model M_h. In DS reduction, the relationship of strong analogy is unambiguous, and specifically requires that

'Strong Analogy':

$$\left|x'_h(t) - x_h(t)\right|_h < 2\delta \ \forall\ 0 \leq t \leq \tau, \quad (10)$$

where τ again is the reduction timescale and the measure of approximation between the analogue and high-level dynamics is provided by the norm $|\circ|_h$ on the high-level state space. Note that this "strong analogy" claim is just the condition (3) rewritten using bridge map substitution $x'_h(t) \equiv B\big(D_l(x_{l0}, t)\big)$ and the definition $x_h(t) \equiv D_h\big(B(x_{l0}), t\big)$.

5 Examples of DS Reduction

In this section, I show that the conditions for DS reduction are satisfied by a range of different model pairs. I do so specifically by showing that the relation (6) holds for a certain choice of bridge map and a certain domain of states in the low-level state space, offering qualitative remarks concerning the timescale over which the relation (6) continues to hold. My presentation of each example will follow a common outline: **a.** specification of the state spaces and dynamical equations of the two models; **b.** specification of the bridge map between the models; **c.** specification of the domain d'_l where Eq. (6) holds between the models; **d.** statement of the condition (6) for the particular pair of models being considered; **e.** restatement of this condition, in the form (5), which more directly resembles the high-level dynamical equation; **f.** discussion of factors affecting the timescale on which (6) holds. Where necessary, proof of (6) is deferred to the Appendix.

5.1 CM/QM

Let the high-level model M_h be a model of classical mechanics whose state space is some N-particle, $6N$-dimensional phase space $S_h \equiv \Gamma_N$ and whose dynamics D_h are given by the solutions to the Hamilton equations $(\frac{dX}{dt}, \frac{dP}{dt}) = (\frac{\partial H}{\partial P}, -\frac{\partial H}{\partial X})$, with $H = \frac{P^2}{2M} + V(X)$. In the notation of Section 4.3, we have $x_h \equiv (X, P)$ and $f_h(x_h) \equiv (\frac{\partial H}{\partial P}, -\frac{\partial H}{\partial X}) = (\frac{P}{M}, -\frac{\partial V}{\partial X})$.

Let the low-level model M_l be a model of non-relativistic quantum mechanics whose state space is some N-particle Hilbert space $S_l \equiv \mathcal{H}_N$ and whose dynamics D_l are given by the solutions to the Schrodinger equation $i\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$, with $\hat{H} = \frac{\hat{P}^2}{2M} + V(\hat{X})$. In the notation of Section 4.3, we have $x_l \equiv |\psi\rangle$ and $f_l(x_l) \equiv -i\hat{H}|\psi\rangle$.

Consider the function $B : \mathcal{H}_N \to \Gamma_N$ from the low-level to the high-level state space that maps a quantum state into the phase space point associated with the expectation values of position and momentum in this state,

$$B(x_l) \equiv (\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle) = (\langle\hat{X}\rangle, \langle\hat{P}\rangle). \quad (11)$$

Consider further the domain of narrow wave packet states:

$$d_l' \equiv \{|\psi\rangle \in \mathcal{H}_N \| \psi\rangle = |q, p\rangle \text{ for some } q, p \in \Gamma_N\}, \tag{12}$$

where $|q, p\rangle$ denotes a narrow wave packet with average position q and average momentum p, and the required standard of narrowness in X is defined relative to the scale of spatial variation of the potential V. On the basis of the low-level Schrodinger dynamics, one can show that for $|\psi\rangle$ in d_l', the relation (6) holds relative to this particular case, so that the quantity associated with the bridge map approximately satisfies the high-level Hamilton equations:

$$\frac{d}{dt}(\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle) \approx \left(\frac{\partial H}{\partial P}\bigg|_{\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle}, -\frac{\partial H}{\partial X}\bigg|_{\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle} \right) = \left(\frac{P}{M}\bigg|_{\langle\hat{P}\rangle}, -\frac{\partial V(X)}{\partial X}\bigg|_{\langle\hat{X}\rangle} \right), \tag{13}$$

or, more compactly, employing the variable substitutions $(X', P') \equiv (\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle)$,

$$\frac{d}{dt}(X', P') \approx \left(\frac{\partial H}{\partial P}\bigg|_{X',P'}, -\frac{\partial H}{\partial X}\bigg|_{X',P'} \right) = \left(\frac{P}{M}\bigg|_{P'}, -\frac{\partial V(X)}{\partial X}\bigg|_{X'} \right). \tag{14}$$

This claim follows straightforwardly from Ehrenfest's Theorem, which states that $\frac{d\langle\hat{P}\rangle}{dt} = -\langle\frac{\partial \widehat{V(X)}}{\partial X}\rangle$, from the fact that $\frac{d\langle\hat{X}\rangle}{dt} = \frac{\hat{P}}{M}$, and from the well-known result that for wave packets whose position-space width is narrow by comparison with the characteristic length scales on which $V(X)$ varies, $\frac{d\langle\hat{P}\rangle}{dt} \approx -\frac{\partial V(X)}{\partial X}\big|_{\langle\hat{X}\rangle}$ [24].

On timescales where wave packets remain in d_l', the relation (13) ensures that quantum expectation values of position and momentum approximately satisfy the classical dynamics of the high-level model, and that (4) is satisfied. These timescales in turn will depend on two general factors: 1) the maximum spread in position and momentum of wave packets allowed by the definition of d_l', which in turn will depend on the required accuracy of the approximation in (13) and on the scale of spatial variation of the potential V; 2) the dynamics of the low-level model, and in particular the mass M and the strength of chaotic effects associated with the Hamiltonian H. In general, the larger M is, the slower wave packets in $\mathcal{H}_N$ will tend to spread; moreover, it is typically the case that the smaller the Lyapunov exponent characterizing chaotic divergences of trajectories in the *classical* phase space of the system, the slower the rate of spreading of wave packets in $\mathcal{H}_N$ [36]. [12]

[12]I should note that the low-level model in this case will not suffice to model macroscopic systems like the center of mass of a planet or a baseball, since any realistic quantum model of such a system must take account of the system's interaction with its environment and, in particular, the effects of decoherence (the reduction between quantum and classical models of such systems is a more intricate matter, which I will not consider here). However, in smaller systems like large molecules, effects of decoherence can often be neglected and the

5.2 NRQM/RQM

Let the high-level model M_h be a model of nonrelativistic quantum mechanics of a spin-1/2 particle whose state space is the Hilbert space of 2-spinors $S_h \equiv \mathcal{H}_P$ [13] and whose dynamics D_h are given by the solutions to the Pauli equation, $i\frac{\partial}{\partial t}\phi^{\alpha}(x,t) = \left\{\frac{1}{2m}\left[\sigma \cdot (\hat{p} - qA(x))\right]^2 + qV(x)\right\}^{\alpha\beta} \phi^{\beta}(x,t)$, where $\phi^{\alpha}(x,t)$ are 2-spinor functions on 3-D space, $\alpha, \beta = 1, 2$, σ_i are the Pauli matrices, m is the particle's mass, q is its electromagnetic charge, and $A(x)$ and $V(x)$ are fixed background electromagnetic potentials. [14] In the notation of Section 4.3, we have $x_h \equiv \phi^{\alpha}(x,t)$ and
$f_h(x_h) \equiv -i\left\{\frac{1}{2m}\left[\sigma \cdot (\hat{p} - qA(x))\right]^2 + qV(x)\right\}^{\alpha\beta} \phi^{\beta}(x,t)$. [15]

Let the low-level model M_l be a model of relativistic quantum mechanics for a spin-1/2 particle whose state space is the Dirac Hilbert space of 4-spinors $S_l \equiv \mathcal{H}_D$ and whose dynamics D_l are given by the solutions to the Dirac equation $i\frac{\partial}{\partial t}\psi^a(x,t) = \left[\alpha \cdot (-i\nabla - q\vec{A}(x)) + \beta m + qV(x)\right]^{ac}\psi^c(x,t)$, where $a = 1, 2, 3, 4$ and likewise for b and c, where repeated spinor indices have been summed over, α_i and β are the Dirac matrices, m is the mass of the particle, q is its electromagnetic charge, and $A(x)$ and $V(x)$ are fixed background electromagnetic potentials. In the notation of Section 4.3, we have $x_l \equiv \psi^a(x,t)$ and $f_l(x_l) \equiv -i\left[\alpha \cdot (-i\nabla - q\vec{A}(x)) + \beta m + qV(x)\right]^{ac}\psi^c(x,t)$. [16]

Consider the function $B : \mathcal{H}_D \to \mathcal{H}_P$ from the low-level to the high-level state space given by

$$B(x_l) \equiv e^{imt} P^{\alpha}_{a} \psi^a(x,t), \tag{15}$$

where P^{α}_{a} is the projector onto the subspace of the 4-spinor space corresponding to the upper two components of any spinor. Because of the time-dependent factor e^{imt}, this bridge map may seem to violate the requirement that bridge maps not depend explicitly on time. However, the violation is only apparent

evolution of the molecule's center of mass in some cases can be accurately described by the unitary evolution of a pure state ("bucky ball" interference experiments with large molecules have shown this explicitly, as discussed, for example, in [32], Ch. 6). The relatively large mass of such molecules by comparison with smaller particles like electrons, protons and neutrons slows the rate at which pure state wave packets spread, thereby allowing narrow wave packets to maintain their approximately classical evolutions over longer timescales in these systems. On such timescales, both the classical and quantum models are adequate to describe the motion of the molecule's center of mass to within a certain reasonable margin of error.

[13] Where the "P" is for "Pauli."

[14] Unless explicitly stated otherwise, I employ the Einstein summation convention over repeated indices throughout.

[15] The reader should note that for purposes of concision, I am abusing notation here, since the state x_h is not properly described by the components $\phi^{\alpha}(x,t)$, but rather by the expression $x_h \equiv |\phi\rangle = \sum_{\alpha} \int dx\ \phi^{\alpha}(x,t)\ |x^{\alpha}\rangle$, where $|x^{\alpha}\rangle$ is a state of position x and spin α (say, in the z-direction). Likewise, the expression for $f_h(x_h)$ employs a similar abuse of notation, as does my notation describing the low-level relativistic Dirac model.

[16] As in the high-level model, the reader should note the abuse of notation for purposes of concision.

since the state space is in fact the *projective* Hilbert space, and multiplication of a Hilbert space vector in either the high-or low-level model by an overall phase (whether time dependent or not) does not affect the projective representation of the state. Now consider the domain of low-momentum 4-spinors,

$$d_l' = \left\{ \psi^a(x) \in \mathcal{H}_D \middle| \psi^a(x) = \sum_{i=1,2} \int_0^\mu d^3k \ \tilde{\psi}^i(k) u_i^a(k) e^{-ikx}, \frac{\mu}{m} << 1 \right\}, \quad (16)$$

where $u_i^a(k)$ are positive energy eigenstates of the Dirac Hamiltonian indexed by momentum k and the spin i, and the upper limit μ on the momentum integral imposes the restriction to low-momentum states. On the basis of the low-level Dirac dynamics, one can show that for $\psi^a(x) \in d_l'$, the relation (6) holds for the given bridge map and models, so that $B(x_l)$ in this case approximately satisfies the high-level Pauli equation:

$$\frac{\partial}{\partial t}\left(e^{imt} P_a^\alpha \psi^a(x,t)\right) \approx -i \left\{ \frac{1}{2m} \left[\sigma \cdot (-i\nabla - qA(x))\right]^2 + qV(x) \right\}^{\alpha\beta} \left(e^{imt} P_a^\beta \psi^a(x,t)\right) \quad (17)$$

or, more compactly, employing the variable substitution $\phi'^\alpha(x,t) \equiv e^{imt} P_a^\beta \psi^a(x,t)$,

$$\frac{\partial}{\partial t}\phi'^\alpha(x,t) \approx -i \left\{ \frac{1}{2m} \left[\sigma \cdot (-i\nabla - qA(x))\right]^2 + qV(x) \right\}^{\alpha\beta} \phi'^\beta(x) \quad (18)$$

for all spatial positions x (not to be confused with the high-level state x_h or the low-level state x_l). Proof of this relation can be found in [19], or in most good textbooks on relativistic quantum mechanics, such as [24]. The timescales on which $\psi^a(x,t)$ remains in d_l' will depend primarily on the choice of background fields $A(x)$ and $V(x)$; the domain d_l' will be preserved as long as these background fields do not transfer significant amounts of momentum (that is, on the order of m) to the spinor field.

5.3 NRQM/RQFT

Let the high-level model M_h be model of N free spinless particles in non-relativistic quantum mechanics whose state space is some N-particle Hilbert space $S_h \equiv \mathcal{H}_N$ and whose dynamics D_h are given by the solutions to the non-relativistic Schrodinger equation for N free particles all with mass m: $i\frac{\partial}{\partial t}\psi(x_1, ..., x_N, t) = -\sum_{i=1}^N \frac{1}{2m}\nabla_i^2 \psi(x_1, ..., x_N, t)$. In the notation of Section 4.3, we have $x_h \equiv \psi(x_1, ..., x_N, t)$ and $f_h(x_h) \equiv i\sum_{i=1}^N \frac{1}{2m}\nabla_i^2 \psi(x_1, ..., x_N, t)$. [17]

Let the low-level model M_l be a model of a free massive scalar quantum field in relativistic quantum field theory whose state space is the free-particle

[17] I employ the same abuse of notation here described in the footnotes of the previous section.

Fock space of Klein-Gordon quantum field theory, $S_l \equiv \mathcal{F}_{KG}$, and whose dynamics D_l are given by the solutions to the Schrodinger equation for a free Klein-Gordon quantum field theory, $i\frac{\partial}{\partial t}|\Psi\rangle = \frac{1}{2}\int d^3x \left[\hat{\pi}^2(x) + \left(\nabla\hat{\phi}(x)\right)^2 + m^2\hat{\phi}^2(x)\right]|\Psi\rangle$, where $\hat{\phi}(x)$ is the Hilbert space operator associated with the scalar field, and $\hat{\pi}(x)$ the Hilbert space operator associated with its conjugate momentum. Using the usual expansions of $\hat{\phi}(x)$ and $\hat{\pi}(x)$ in terms of creation and annihilation operators $\hat{a}_k^\dagger$ and $\hat{a}_k$, the Schrodinger equation can be re-written, $i\frac{\partial}{\partial t}|\Psi\rangle = \left(\int \frac{d^3k}{(2\pi)^3} E_k \hat{a}_k^\dagger \hat{a}_k + C\right)|\Psi\rangle$, where $E_k \equiv \sqrt{k^2+m^2}$ and C signifies a constant (associated with the vacuum energy) that diverges with the cutoff of the theory. In the notation of Section 4.3, we have $x_l \equiv |\Psi\rangle$ and $f_l(x_l) \equiv -i\frac{1}{2}\int d^3x \left[\hat{\pi}^2(x) + \left(\nabla\hat{\phi}(x)\right)^2 + m^2\hat{\phi}^2(x)\right]|\Psi\rangle = \left[\int \frac{d^3k}{(2\pi)^3}\left(E_k \hat{a}_k^\dagger \hat{a}_k + C\right)\right]|\Psi\rangle$.

Consider the function $B : \mathcal{F}_{KG} \rightarrow \mathcal{H}_N$ from the low-level to the high-level state space given by

$$B(x_l) \equiv e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle. \tag{19}$$

As in the previous example, the requirement that the bridge map not depend explicitly on time is preserved despite the exponential $e^{i(Nm+C)t}$ since it is projective Hilbert space rather than the Hilbert space itself that represents possible physical states. Now, consider the domain of states in the low-level space consisting of N-particle states of momenta k_1,...,k_N much less than m:

$$d_l' = \left\{|\Psi\rangle \in \mathcal{H}_{KG} \middle| |\Psi\rangle = \int_0^\mu d^3k_1...d^3k_N\ \tilde{\psi}(k_1,...,k_N)\hat{a}_{k_N}^\dagger...\hat{a}_{k_1}^\dagger|0\rangle, \frac{\mu}{m} << 1\right\}. \tag{20}$$

On the basis of the low-level free RQFT dynamics, one can show that for $|\Psi\rangle$ in d_l', the relation (6) holds relative to this particular case, so that,

$$i\frac{\partial}{\partial t}\left(e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle\right) \approx -\sum_{i=1}^{N}\frac{1}{2m}\nabla_i^2\left(e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle\right). \tag{21}$$

or, more compactly, employing the variable substitution $\psi'(x_1,...,x_N) \equiv e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|$

$$i\frac{\partial}{\partial t}\psi'(x_1,...,x_N) \approx -\sum_{i=1}^{N}\frac{1}{2m}\nabla_i^2\psi'(x_1,...,x_N). \tag{22}$$

The proof of this relation is given in the Appendix. If $|\Psi\rangle$ begins in d_l' at $t=0$, the relation (21) will be preserved for all time since there are no interactions that might result in a change of particle number or a transfer of momentum between particles in the low-level state space. [18]

[18]However, it is important to note that scalar quantum field theories such as the one

As an extension of this example, one might consider the reduction of the non-relativistic model considered here to a model of interacting relativistic quantum field theory - for example, massive scalar quantum field theroy with a $\hat{\phi}^3$ or $\hat{\phi}^4$ interaction. In such a case, we would wish to underwrite the non-relativistic quantum description of N low-momentum particles widely separated in space on the basis of the interacting quantum field theory. This is a substantially more difficult task than in the case of free quantum field theory, for the question of precisely how to identify those QFT states of the interacting theory that behave as N-particle states, or of whether one can even do this, remains a matter of some dispute in the foundations of quantum field theory [34], [17]. Nevertheless, one expects that the reduction, if it can be effected, will follow the same basic pattern as in the case discussed here, except that the domain of states in the field theory for which the bridge map function approximately satisfies the non-relativistic equation of the high-level model will likely impose a restriction not only to states of low momentum, but also to states in which the N particles are in some quantifiable sense widely separated in space (so that they do not interact).

5.4 NM/SR

Let the high-level model M_h be a model of Newtonian mechanics whose state space is the non-relativistic phase space $S_h \equiv \Gamma_{NR}$ of a single massive charged particle in a background electromagnetic field and whose dynamics D_h are given by the solutions to the corresponding non-relativistic Hamilton equations, $(\frac{dX}{dt}, \frac{dP}{dt}) = (\frac{\partial H_{NR}}{\partial P}, -\frac{\partial H_{NR}}{\partial X})$, with $H_{NR} = \frac{(P-qA(X))^2}{2M} + q\phi(X)$, where q is the particle's charge, M its mass, and $\phi(X)$ and $A(X)$ external electrostatic and magnetic vector potentials, respectively. In the notation of Section 4.3, $x_h \equiv (X, P)$ and $f_h(x_h) \equiv (\frac{\partial H_{NR}}{\partial P}, -\frac{\partial H_{NR}}{\partial X}) = \left(\frac{P-qA}{M}, q\frac{(P_i-qA_i)}{M}\frac{\partial A_i}{\partial X} - q\frac{\partial \phi}{\partial X}\right)$. Using a number of vector identities and the definitions of the electric and magnetic fields in terms of the potentials ϕ and A, one can show that combined, these two equations yield the non-relativistic Lorentz Force Law: $M\frac{d^2X}{dt^2} = qE + qv \times B$.

Let the low-level model M_l be a model of relativistic classical mechanics whose state space is the relativistic phase space $S_l = \Gamma_{SR}$ of a single relativistic massive particle in a background electromagnetic field and whose dynamics D_l are given by the solutions to the corresponding relativistic Hamilton equations, $\frac{dx}{dt} = \frac{\partial H_{SR}}{\partial p}$, $\frac{dX}{dt} = -\frac{\partial H_{SR}}{\partial x}$, with $H_{SR} = \sqrt{(p - qA)^2 + M^2} + q\phi$, where q is the particle's charge, M its mass, and $\phi(x)$ and $A(x)$ external electrostatic and

considered here are more a pedagogical tool for illustrating the basic principles of quantum field theory than they are a description of any real physical system that would also be well-described by a model of non-relativistic quantum mechanics. Thus, it is unlikely that there is any real system K to which both models apply. However, the example is nevertheless instructive in that it does serve to illustrate the sort of inter-model relation that satisfies the requirements of DS reduction. Moreover, this sort of reduction between quantum-mechanical and quantum-field-theoretic models might be extended to more realistic QFT models.

magnetic vector potentials, respectively. In the notation of Section 4.3, we have $x_h \equiv (x, p)$ and $f_h(x_h) \equiv (\frac{\partial H_{SR}}{\partial p}, -\frac{\partial H_{SR}}{\partial x}) = \left(\frac{p-qA}{\gamma M}, q\frac{(p_i - qA_i)}{\gamma M}\frac{\partial A_i}{\partial x} - q\frac{\partial \phi}{\partial x},\right)$. Using a number of vector identities and the definitions of the electric and magnetic fields in terms of the potentials ϕ and A, one can show that, combined, these two equations yield the relativistic Lorentz Force Law: $\frac{d}{dt}(\gamma M v) = qE + qv \times B$.

Consider the function $B : \Gamma_{SR} \rightarrow \Gamma_{NR}$ from the low-level to the high-level state space that identifies the point (X, P) in Γ_{NR} with with the point (x, p) in Γ_{SR}, so that $(X, P) \equiv (x, p)$:

$$B(x_l) \equiv (x, p). \tag{23}$$

Note that the bridge map in this case is trivial in a sense since it maps the relativistic phase space point into the non-relativistic phase space point with the same position and momentum (though it should be noted that the dynamics of the two models will relate momentum and velocity differently). Now consider the domain,

$$d_l' \equiv \{(x, p) \in \Gamma_{SR} | = \frac{p - qA(x)}{\gamma m} << 1\}, \tag{24}$$

where $v = \frac{p-qA}{\gamma m}$ is the velocity of the particle (this follows from the well-known relationship between relativistic canonical momentum and velocity, $p = \gamma m v + qA(x)$). [19]

On the basis of the low-level relativistic dynamics, one can show that for (x, p) in d_l', the relation (6) holds relative to this particular case, so that the quantity associated with the bridge map (namely, position and momentum in the relativistic phase space) approximately satisfies the nonrelativistic Hamilton equations:

$$\frac{d}{dt}(x, p) \approx \left(\frac{\partial H_{NR}}{\partial P}\bigg|_{x,p}, -\frac{\partial H_{NR}}{\partial X}\bigg|_{x,p} \right) \tag{25}$$

or, employing the variable substitutions $(X', P') \equiv (x, p)$,

$$\frac{d}{dt}(X', P') \approx \left(\frac{\partial H_{NR}}{\partial P}\bigg|_{X',P'}, -\frac{\partial H_{NR}}{\partial X}\bigg|_{X',P'} \right). \tag{26}$$

The proof of this relation is straightforward and simply invokes the fact that for $v << 1$, $\gamma \equiv \frac{1}{\sqrt{1-v^2}} \approx 1$. The relation (25) will continue to hold as long as (x, p) remains in d_l' - that is, as long as $v << 1$. The timescales on which this holds, in turn, are likely to depend on the initial value of (x, p) in d_l' and on the strength and form of the fields $\phi(x)$ and $A(x)$.

[19] Note that I have set $c = 1$, and were we to include factors of c in our analysis, we would see that the domain restriction amounts to the requirement that $\frac{v}{c} << 1$.

5.5 Open-Subsystem NRQM/Closed-System NRQM

Let the high-level model M_h be a quantum-mechanical model of an open system S whose state space $S_h = \mathcal{D}(\mathcal{H}_S)$ is the space of density operators on S's Hilbert space $\mathcal{H}_S$, and whose dynamics D_h is given by solutions to the density matrix master equation $i\frac{d\hat{\rho}_S}{dt} = [\hat{H}_S, \hat{\rho}_S] - i\Lambda[\hat{X}, [\hat{X}, \hat{\rho}_S]]$. In the notation of Section 4.3, we have $x_h \equiv \hat{\rho}_S$ and $f_h(x_h) \equiv -i[\hat{H}_S, \hat{\rho}_S] - \Lambda[\hat{X}, [\hat{X}, \hat{\rho}_S]]$. Let us assume here that the density matrix and its dynamics are defined independently of and without reference to any lower-level pure state description. [20]

Let the low-level model M_l be a model of non-relativistic quantum mechanics in the closed system SE, whose state space $S_l \equiv \mathcal{H}_S \otimes \mathcal{H}_E$ is the tensor product space of S's Hilbert space and the Hilbert space of its environment E; let the dynamics D_l of the model be given by solutions to the Schrodinger equation $i\frac{d}{dt}|\psi\rangle = \left(\hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{H}_I\right)|\psi\rangle$ for the closed system SE; the system Hamiltonian $\hat{H}_S$, the environment Hamiltonian $\hat{H}_E$, and the interaction Hamiltonian $\hat{H}_I$ will not be precisely specified here, as the following remarks apply for a wide range of forms for these operators and a wide range of systems. For examples of detailed models, like the well-known Caldeira-Leggett model, in which a precise form for these Hamiltonians is specified, as well as for a detailed derivation of an approximate high-level density matrix master equation of the form specified in M_h from these models, the reader may wish to consult [32], Ch.'s 4,5 and [21], Ch.3. In the notation of Section 4.3, we have in this case that $x_l \equiv |\psi\rangle$ and $f_l(x_l) \equiv -i\left(\hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{H}_I\right)|\psi\rangle$. [21]

Now, consider the function $B : \mathcal{H}_S \otimes \mathcal{H}_E \rightarrow \mathcal{D}(\mathcal{H}_S)$ from the low-level to the high-level state space given by taking the trace over the environment of the pure-state density operator associated with the state $|\psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$:

$$B(x_l) \equiv Tr_E(|\psi\rangle\langle\psi|). \tag{27}$$

The relevant domain $d_l' \subset \mathcal{H}_S \otimes \mathcal{H}_E$ in this particular case is harder to specify than in the earlier examples. However, it is generally assumed to involve the restriction to states in which the temperature of the environment is high in comparison to certain other relevant energy scales characterizing the system and environment; see [32], Ch.5.

On the basis of the low-level Schrodinger dynamics for SE - assuming any among a range of suitable low-level Hamiltonians - it widely thought that one can show that for $|\psi\rangle$ in some restricted domain d_l', the quantity $Tr_E(|\psi\rangle\langle\psi|)$ associated with the bridge map approximately satisfies the high-level density matrix master equation, thereby also satisfying the condition (6):

[20] That is, let us forget momentarily about the association between the density matrix of S and the trace over the environment of the pure state density matrix of some larger system SE.

[21] Thanks to David Wallace for this example.

$$i\frac{dTr_E\hat{\rho}_{SE}}{dt} \approx [\hat{H}_S, Tr_E\hat{\rho}_{SE}] - i\Lambda[\hat{X}, [\hat{X}, Tr_E\hat{\rho}_{SE}]], \tag{28}$$

or, more compactly, employing the variable substitution $\hat{\rho}'_S \equiv Tr_E(|\psi\rangle\langle\psi|)$,

$$i\frac{d\hat{\rho}'_S}{dt} \approx [\hat{H}_S, \hat{\rho}'_S] - i\Lambda[\hat{X}, [\hat{X}, \hat{\rho}'_S]]. \tag{29}$$

For a derivation of this equation and a discussion of the various assumptions on which it relies, see [32], Ch. 4. As with the matter of specifying the domain d'_l, the matter of specifying precisely the timescale on which this relation holds is more complicated than in the other examples considered; however, existing analyses strongly suggest that the high-level equations should hold to good approximation over macroscopically long timescales, as observed in experimental applications of density matrix master equations like the one considered here.

5.6 "Macro" CM/"Micro" CM

Let the high-level model M_h be a non-relativistic model of classical mechanics prescribing the Newtonian evolution of some N centers of mass (e.g. of planets in the solar system) interacting via some time-independent potential V dependent only on the distance between the centers of mass (e.g. a Newtonian gravitational potential). The state space will be a $6N$-dimensional phase space $S_h \equiv \Gamma_{macro}$ and the dynamics D_h will be given by the solutions to the Hamilton equations $(\frac{dX}{dt}, \frac{dP}{dt}) = (\frac{\partial H_{macro}}{\partial P}, -\frac{\partial H_{macro}}{\partial X})$, with $H_{macro} = \sum_{i=1}^{N} \frac{P_i^2}{2M_i} + 1/2\sum_{i\neq j} V(X_i - X_j)$. In the notation of Section 4.3, we have $x_h \equiv (X, P)$ and $f_h(x_h) \equiv (\frac{\partial H_{macro}}{\partial P}, -\frac{\partial H_{macro}}{\partial X})$
$= \Big(\frac{P_1}{M_1}, ..., \frac{P_N}{M_N}; -\frac{\partial}{\partial X_1}\sum_{j\neq 1} V(X_1 - X_j), ..., -\frac{\partial}{\partial X_N}\sum_{j\neq N} V(X_N - X_j)\Big)$.

Let the low-level model M_l also be a non-relativistic model of classical mechanics whose state space $S_l \equiv \Gamma_{micro}$ is the phase space of some n point particles, where $n > N$, and whose dynamics D_l are given by the solutions to the Hamilton equations $(\frac{dx}{dt}, \frac{dp}{dt}) = (\frac{\partial H_{micro}}{\partial p}, -\frac{\partial H_{micro}}{\partial x})$, with $H_{micro} = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + 1/2\sum_{i\neq j} v(x_i - x_j)$. In the notation of Section4.3, we have $x_h \equiv (X, P)$ and $f_h(x_h) \equiv (\frac{\partial H_{micro}}{\partial p}, -\frac{\partial H_{micro}}{\partial x})$
$= \Big(\frac{p_1}{m_1}, ..., \frac{p_n}{m_N}; -\frac{\partial}{\partial x_1}\sum_{j\neq 1} v(x_1 - x_j), ..., -\frac{\partial}{\partial x_n}\sum_{j\neq n} v(x_N - x_j)\Big)$.

Now, consider a grouping of the n particles of the low-level model into N sets, with each set corresponding to those particles contained in one of the bodies whose center of mass is described by the high-level model. Let n_1 be the number of particles contained in the first body, n_2 the number in the second, and so on up to n_N in the N^{th}, where $n_1 + n_2 + ... + n_N = n$. Given this grouping, consider the function $B : \Gamma_{micro} \to \Gamma_{macro}$ from the low-level to the high-level state space given by,

$$B(x_l) \equiv \left(\frac{\sum_{\alpha_1=1}^{n_i} m_{\alpha_1} x_{\alpha_1}}{\sum_{\alpha_i=1}^{n_1} m_{\alpha_1}}, ..., \frac{\sum_{\alpha_N=1}^{n_N} m_{\alpha_N} x_{\alpha_N}}{\sum_{\alpha_N=1}^{n_N} m_{\alpha_N}}; \sum_{\alpha_1=1}^{n_1} p_{\alpha_1}, ..., \sum_{\alpha_N=1}^{n_N} p_{\alpha_N} \right). \quad (30)$$

For notational convenience, define $X'_i \equiv \frac{\sum_{\alpha_i=1}^{n_i} m_{\alpha_i} x_{\alpha_i}}{\sum_{\alpha_i=1}^{n_i} m_{\alpha_i}}$ and $P'_i \equiv \sum_{\alpha_i=1}^{n_i} p_{\alpha_i}$. Moreover, assume that $M_i = \sum_{\alpha_i} m_{\alpha_i}$ and $V(X_i - X_j) = n_i n_j \; v(X_i - X_j)$, where M_i and V are the masses and potential function of the high-level model. Now consider the domain,

$$d'_l = \left\{ (x,p) \in \Gamma_{micro} \middle| |x_{\alpha_i} - X'_i| << L_v \; \forall \; i \leq i \leq N \text{ and } \forall \; 1 \leq \alpha_i \leq n_i \right\}, \quad (31)$$

where L_v characterizes the typical length scale over which v varies, determined by v's derivatives (see the Appendix for a more precise definition of L_V). On the basis of the low-level "micro" Hamilton dynamics, one can show that for (x,p) in d'_l, the relation (6) holds relative to this particular case, so that the quantity associated with the bridge map approximately satisfies the high-level "macro" Hamilton equations, written out schematically as,

$$\frac{d}{dt}(X', P') \approx \left(\frac{\partial H_{macro}}{\partial P} \bigg|_{X',P'}, -\frac{\partial H_{macro}}{\partial X} \bigg|_{X',P'} \right) = \left(\frac{P'}{M}, -\frac{\partial V(X')}{\partial X'} \right). \quad (32)$$

Proof of this relation is given in the Appendix. [22] The timescale on which this relation holds will depend on timescales on which the low-level initial condition (x_0, p_0) remains in d'_l as it evolves according to the low-level dynamics.

5.7 A Note on Limit-Based Approaches to Reduction

It is worth noting here that a number of the examples discussed in this section - in particular, those discussed in Sections 5.2, 5.3, 5.4 and 5.6 - can in a certain sense be cited as instances of limit-based reduction. All of these cases involve domain restrictions of the form $\epsilon << 1$ for some dimensionless parameter ϵ, which, although they are not strictly speaking mathematical limits, call to mind the sort of thing people often mean when they talk of one theory's being a limit or limiting case of another. I only wish to point out here that the relations (4) and (6) offer a far more precise characterization of the common structure that these various reductions possess, as well as of the structure they possess in common with other cases, than does the claim that one model is a limit or limiting case of another - a claim that, as we have seen, may be interpreted in any number of ways.

[22] Note that variable substitutions have already been incorporated into this statement of condition (6), as writing it out explicitly in terms of the variables of the low-level model would be more cumbersome than illuminating in this case.

6 Extending DS Reduction

In this section, I consider possible extensions of DS reduction, first to include an analysis of the relations between symmetries of different dynamical systems models and then to an analysis of the relation $reduction_M$ outside the class of cases to which $reduction_{DS}$ applies.

6.1 DS Reduction and Symmetry

Given that $B(x_l)$ and x_h serve to represent the same physical degrees of freedom (albeit within the context of different models), it is natural to ask whether these two quantities share other physically relevant transformation properties besides their dynamical evolution - in particular, their transformation properties under certain symmetry operations. More specifically, one can inquire as to whether the sort of approximate commutation condition that holds between the bridge map and the dynamical evolution of the two models extends to the symmetries of the models, so that if $T_h(x_h)$ is a symmetry of the high-level model, there exists some symmetry $T_l(x_l)$ of the low-level model such that for states x_l in some subset $d_l'' \subset d_l' \subset S_l$,

$$T_h(B(x_l)) \approx B(T_l(x_l)). \tag{33}$$

We can perform a cursory initial examination of the examples considered in Section 5 to check whether such a condition holds in these cases. Consider first the case presented in Section 5.1, concerning the relation between quantum and classical dynamical systems models; it is straightforward to see that rotations of the quantum state in Hilbert space will induce rotations in the phase space through the bridge map given there, and likewise for translations, reflections and Galilean boosts. Similarly, regarding the relation between non-relativistic and relativistic models of quantum mechanics considered in Section 5.2, it can be shown that the action of a Lorentz boost on the 4-spinor space of the relativistic model induces a corresponding Galilean boost on the 2-spinors of the non-relativistic model when acting within the domain of low-momentum 4-spinors [19]. In the relation between the N-particle model of NRQM and the model of free scalar quantum field theory discussed in Section 5.3, spatial translations, rotations and reflections acting on the state of the RQFT model induce corresponding transformations in the non-relativistic model; it would be interesting to see whether Lorentz boosts within the range of low-momentum N-particle states in the QFT state space likewise induce approximate Galilean transformations on the state in the NRQM model. In the example of Section 5.4 concerning the relationship between Newtonian and relativistic models of classical mechanics, rotations, translations, and reflections on the relativistic phase space all induce corresponding transformations on the non-relativistic phase space; likewise, Lorentz boosts acting within the domain of low-velocity states of the relativistic phase space induce transformations that approximate corresponding Galilean transformations on the non-relativistic phase space. In the example of Section 5.5, a rotation on the pure state of the total closed system SE should induce a corresponding rotation on the reduced density matrix describing S, and likewise for translations and reflections. In the example of Section 5.6, it is relatively straightforward to see that rotations, translations, reflections and Galilean boosts acting on the micro-level state space all induce corresponding transformations on the macro-level state space.

In all of these cases, it is also natural to ask whether the bridge map respects the composition properties of the various symmetry transformations - that is, whether the transformation induced via the bridge map by the composition of two symmetry transformations in the low-level model approximates the composition of the corresponding transformations in the high-level model. Without checking this explicitly in each case, it seems *prima facie* likely that this condition (with appropriate qualifications) should hold in the model pairs discussed in the above examples. What the preliminary considerations of this subsection all suggest is that the relationship between $B(x_l)$ and x_h is in fact much richer and more comprehensive than is suggested merely by the similarity of dynamical behavior discussed in Sections 4 and 5. Detailed elaboration of these claims is postponed to a future article.

6.2 Extending DS Reduction to Other Types of Model

It is natural to ask which aspects of DS reduction can be extended to fixed-system, inter-model reduction - what I have called *reduction*$_M$ - in cases outside the class of model pairs considered here - for example, in cases where one or both of the models is not a dynamical system, or where the two dynamical systems do not naturally share a time parameter. It seems that at least two aspects of DS reduction are likely to carry over to these cases: the use of bridge maps and domain specificity. It also seems that the following general Nagelian strategy employed in DS reduction may also apply in these other cases: namely, to prove on the basis of the low-level model's constraints (dynamical or otherwise), together with a restriction to a certain subset in the space of the low-level model, that the quantity associated with the bridge map approximately satisfies the constraints of the high-level model; this amounts to deriving a certain "image" of the high-level model's constraints within the framework of the low-level model; one can then employ bridge map substitutions to write down an "analogue" form of these relations, which is "strongly analogous" to the constraints imposed by the high-level model. As in the case of DS reduction, one can interpret "strong analogy" in a precise sense if the high-level model includes specification of a norm on the high-level space.

7 Conclusion

I have argued against a general, if not always explicit, tendency in the literature on inter-theory relations in physics to conceive of inter-theoretic reduction as an exclusively global affair - that is, as primarily a matter of deriving, in a completely general way that is insensitive to the particularities of different systems, the laws of one-theory from those of another. Taking a cue from certain authors primarily in the philosophy of mind literature, I have advocated a more local approach to inter-theory reduction in physics that is sensitive to such particularities. More precisely, I have argued that such an approach should focus not on derivations directly between theories, but on derivations that relate the particular models that the high- and low-level theories use to describe individual systems in the high-level theory's domain. I have shown that in the particular class of cases where both models of the system in question are dynamical systems, this sort of fixed-system reduction between models follows the main strategic prescriptions of Nagel/Schaffner reduction (suitably adapted to the context at hand). I have also suggested in a preliminary way how these prescriptions might be extended to fixed-system inter-model reduction in other kinds of cases. Within the specialized context of reduction between dynamical systems, I have shown that a particular mathematical relationship precisely characterizes the manner in which the low-level model underwrites the success of the high-level model across a wide range of cases. Moreover, I have shown that while a number of these cases also might be cited as instances of limit-based reduction, the criteria for DS reduction offer a much more precise and comprehensive characterization of these cases than does the vague claim that one model is a "limit" or "limiting case" of another. [23] While there is still much work to be done in elaborating the local, model-based approach to inter-theoretic reduction in physics described here - in particular, in examining the relationship between the symmetries of different models, and in extending this approach to fixed-system reduction involving other model-types - I hope that the preceding analysis has served to underscore the value of analyzing reduction first in specific cases and then expanding outward to see what generalities can be drawn, rather than demanding an unrealistically high level of generality at the outset.

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[23] Whether it is possible to give a more precise meaning to such a claim that extends across the range of examples considered here is not something I have the space to comment on here. Suffice it to say that no *existing* formulations of the limit-based approach succeed in avoiding the vagueness that has grounded my critique of this approach here.

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A Proof of Eq. (21)

A general state $|\Psi\rangle \in \mathcal{F}_{KG}$ in the low-level model can be expanded in the form

$$|\Psi\rangle = \psi_0|0\rangle + \sum_{n=1}^{\infty} \int d^3k_1...d^3k_n \ \tilde{\psi}_n(k_1, ..., k_n) \ \hat{a}^\dagger_{k_N}...\hat{a}^\dagger_{k_1}|0\rangle \tag{A-1}$$

where $\psi_0 \equiv \langle 0|\Psi\rangle$ and $\tilde{\psi}_n(k_1, ..., k_n) \equiv \langle 0|\hat{a}_{k_1}...\hat{a}_{k_N}|\Psi\rangle$. The QFT Schrodinger equation for the low-level model straighforwardly entails that,

$$\begin{aligned} &i\frac{\partial}{\partial t}\psi_0 = C\psi_0 \\ &i\frac{\partial}{\partial t}\tilde{\psi}_n(k_1, ..., k_n, t) = \left(C + \sqrt{|k_1|^2 + m^2} + ... + \sqrt{|k_n|^2 + m^2}\right) \tilde{\psi}_n(k_1, ..., k_n, t) \ \ \forall n. \end{aligned} \tag{A-2}$$

If we now restrict the state $|\Psi\rangle$ to lie in the domain d' of low-momentum N-particle states defined by Eq. (20), then we may make the approximation $\sqrt{|k_i|^2 + m^2} \approx m + \frac{1}{2m}k_i^2$. Within this domain and under this approximation, Eq. (A-2) yields,

$$\begin{aligned} &i\frac{\partial}{\partial t}\psi_0 = 0 \\ &i\frac{\partial}{\partial t}\tilde{\psi}_n(k_1, ..., k_N, t) \approx \left(C + Nm + \frac{1}{2m}k_1^2 + ... + \frac{1}{2m}k_N^2\right) \tilde{\psi}_N(k_1, ..., k_N, t). \end{aligned} \tag{A-3}$$

We can return the position representation $\psi_n(x_1, ..., x_N, t)$ of N-particle states using the expansion $\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left(\hat{a}_k e^{ikx} + \hat{a}^\dagger e^{-ikx}\right)$. Doing this, we get $\psi_n(x_1, ..., x_N, t) \equiv \langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle = \int \frac{d^3k_1}{(2\pi)^3}...\frac{d^3k_N}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_1}}}...\frac{1}{\sqrt{2E_{k_N}}} \ \tilde{\psi}_N(k_1, ..., k_N, t)e^{ik_1x}...e^{ik_Nx}$. Together with this expression, (A-3) entails,

$$i\frac{\partial}{\partial t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle \approx \left(C + Nm - \frac{1}{2m}\nabla_1^2 - ... - \frac{1}{2m}\nabla_N^2\right) \langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle \tag{A-4}$$

This in turn entails that,

$$i\frac{\partial}{\partial t}\left(e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle\right) \approx -\sum_{i=1}^{N} \frac{1}{2m}\nabla_i^2 \left(e^{i(Nm+C)t}\langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_N)|\Psi\rangle\right). \tag{A-5}$$

B Proof of Eq. (32)

I prove the X- and P-components of the relation (32) separately. For the X-components,

$$\frac{dX_i'}{dt} = \frac{d}{dt}\left(\frac{\sum_{\alpha_i} m_{\alpha_i} x_{\alpha_i}}{\sum_{\alpha_i} m_{\alpha_i}}\right) = \frac{\sum_{\alpha_i} p_{\alpha_i}}{\sum_{\alpha_i} m_{\alpha_i}} = \frac{P_i'}{M_i'}, \tag{B-1}$$

where in the second equality I have used the low-level Hamilton equation $\frac{dx_{\alpha_i}}{dt} = \frac{p_{\alpha_i}}{m_{\alpha_i}}$. For the P-components,

$$\begin{aligned}\frac{dP_i'}{dt} &= \frac{d}{dt}\left(\sum_{\alpha_i} p_{\alpha_i}\right)\\ &= -\sum_{\alpha_i} \frac{\partial}{\partial x_{\alpha_i}} \sum_{\alpha_j \neq \alpha_i} v(x_{\alpha_i} - x_{\alpha j})\\ &\approx -\sum_{\alpha_i} \sum_{\alpha_j \neq \alpha_i} \frac{\partial}{\partial X_i'} v(X_i' - X_j')\\ &= -\frac{\partial}{\partial X_i'}\left(\sum_{j\neq i} n_i n_j v(X_i' - X_j')\right)\end{aligned} \tag{B-2}$$

$$= -\frac{\partial}{\partial X_i'}\left(\sum_{j\neq i} V(X_i' - X_j')\right). \tag{B-3}$$

In going from the first to the second line I have used the low-level Hamilton equation, $\frac{dp_{\alpha_i}}{dt} = -\frac{\partial}{\partial x_{\alpha_i}} \sum_{\alpha_j \neq \alpha_i} v(x_{\alpha_i} - x_{\alpha j})$. In going from the second to third line, I have used the definition $x_{\alpha_i} \equiv X_i' + r_{\alpha_i}$ and the fact that $\frac{\partial v}{\partial x_{\alpha_i}}\big|_{X_i'+r_{\alpha_i}, X_j'+r_{\alpha_j}} \approx \frac{\partial v}{\partial x_{\alpha_i}}\big|_{X_i', X_j'} + \frac{\partial^2 v}{\partial x_{\alpha_i} \partial x_{\alpha_i}}\big|_{X_i', X_j'} r_{\alpha_i} + \frac{\partial^2 v}{\partial x_{\alpha_j} \partial x_{\alpha_i}}\big|_{X_i', X_j'} r_{\alpha_j} + \ldots$, so that if $|r_{\alpha_i}|$ is sufficiently small, all but the lowest order terms in the expansion can be ignored and $\frac{\partial v}{\partial x_{\alpha_i}}\big|_{X'+r} \approx \frac{\partial v}{\partial x_{\alpha_i}}\big|_{X'}$. Define L_v to be the length scale (determined by v's derivatives) such that all higher-order terms in the expansion can be neglected when $|r_{\alpha_i}| << L_v \ \forall \alpha_i$. We can then rewrite $\frac{\partial v(x_{\alpha_i} - x_{\alpha_j})}{\partial x_{\alpha_i}}\big|_{X'}$ as $\frac{\partial v(X_i' - X_j)}{\partial X_i'}$. In going from the third line to the fourth, I have used the fact that under the approximation made in the previous step, $v(X_i' - X_j')$ does not vary with changes in the index α_i for fixed i, or in α_j for fixed j, so that $\sum_{\alpha_i=1}^{n_i} \frac{\partial}{\partial X_i'} v(X_i' - X_j') = n_i v(X_i' - X_j')$ and $\sum_{\alpha_j \neq \alpha_j} \sum_{\alpha_i=1}^{n_i} \frac{\partial}{\partial X_i'} v(X_i' - X_j') = \sum_{j\neq i} n_j n_i v(X_i' - X_j')$. In the final step, I have used the assumption $V(X_i - X_j) = n_i n_j \ v(X_i - X_j)$.

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