The Future of the Concept of Infinite Number

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Abstract: In 'The Train Paradox', I argued that sequential random selections from the natural numbers would grow through time. I used this claim to present a paradox. In response to this proposed paradox, Jon Pérez Laraudogoitia has argued that random selections from the natural numbers do not grow through time. In this paper, I defend and expand on the argument that random selections from the natural numbers grow through time. I also situate this growth of random selections in the context of my overall work on infinite number, which involves two main claims: 1) infinite numbers, properly understood, are the infinite natural numbers in a nonstandard model of the reals, and 2) ω is potentially infinite (not actually infinite).

§1: Random Selections from the Natural Numbers

Imagine a black box that randomly selects a natural number. Each number from the natural numbers, {1, 2, 3…}, has the same probability of selection.[1] How do sequential random selections behave through time? In 'The Train Paradox', I argued that sequential random selections from the natural numbers would grow through time.[2] That is, a later selection would be larger than an earlier one. I went on to present a paradox by considering two random selections, two observers, and the relativity of simultaneity. By bringing in the relativity of simultaneity, under certain assumptions two random selections occur in different orders for two observers. For observer M, integer m is randomly selected before integer n. For observer N, integer n is randomly selected before m. If random selections grow through time, observer M will find that m is less than n; observer N will find that n is less than m. But two numbers cannot both be less than each other.[3] What is the way out of this paradox?

In response to this proposed paradox, Jon Pérez Laraudogoitia has argued that random selections from the natural numbers do not grow through time.[4] If true, this would dispel the paradox. In this paper, I demonstrate that there is at least a *prima facie* argument to the conclusion that random selections from the natural numbers grow through time. For other structures there is no such argument. This distinction, I suggest, is important for our conception of infinite number. After discussing the debate about the growth of random selections in section 1, I outline my position on infinite number in section 2. Part of this position is an argument that the apparent growth of random selections provides evidence that the natural numbers are not a completed, actual infinity. In section 3 of the paper, I discuss the importance of arriving at the correct conception of infinite number.

Let us begin by considering the argument that random selections from the natural numbers grow through time. Note than any such random selection seems preposterously low, (almost) certain to be bested by subsequent selections. For example, if 1,037 is randomly selected by the black box, then a sensible reaction would be ‘that is an absurdly low number, future random selections will be greater’. The same holds no matter what number is selected.[5] Why is any number ‘absurdly low’? The natural numbers begin at 1, and so have a smallest element, but they do not have a largest element:

1, 2, 3, 4…

Random selections are thus ‘skewed left’. That is, any random selection is too close to 1. Put in terms of two sequential random selections, once we fix a first random selection, it is then almost certain that a subsequent selection will be larger, because there are only finitely many numbers less than or equal to the first selection, but infinitely many numbers greater. Using the example of 1,037, there are only 1,036 natural numbers less than 1,037, but infinitely many greater.

Those, like Laraudogoitia, who argue that random selections do not grow through time, stress the mathematical difficulties and take a timeless view. See Laraudogoitia (2013) and Bartha (2011) for a discussion of the mathematical problems. From a mathematical-timeless perspective we find that strongest argument to the conclusion that random selections do not grow through time. And indeed, these issues are real and difficult.[6] I suggest, however, that both Laraudogoitia and Bartha beg the question against the order of selection mattering. That is, they both assume that the order of selection does not matter. Laraudogoitia assumes an ‘evident argument of symmetry’ that implies that the order of selections does not matter.[7] Bartha does the same when he writes, ‘The legitimate Symmetry assumption here is that the order … should not affect the probabilities’.[8]

When we shift from this mathematical-timeless view and place an observer within time, as I did in ‘The Train Paradox’, the argument that random selections from the natural numbers grow through time becomes strongest. In a week, Monday happens before Wednesday, which happens before Friday. Imagine that on Monday a random positive integer, n, is selected and observed. Perhaps it is 1,037. On Wednesday, you are able to bet on whether Friday’s selection will be over or under 1,037. I suggest that the over is the only sensible bet, and that the over is the only sensible bet Wednesday no matter what number was selected Monday. Again, there are infinitely many numbers over n, only finitely many under n. From an in-time perspective, random selections seem to grow through time. This was the argument I presented in ‘The Train

Paradox'.[9]

Laraudogoitia's position is that over repeated bets (where a random selection is made Monday and Friday, and the bet is made Wednesday on which number is larger) the following will occur. Monday's number will be larger than Friday's about half of the time, and vice versa. The bet of over, made on Wednesday, is certain to win. Yet this bet will lose about half of the time, which is not problematic (according to Laraudogoitia), as the number selected on Monday varies. So for example, if 1,037 is selected on Monday, the over is the sensible bet on Wednesday. But all this bet means is that over the long run, almost always, when 1,037 is selected on Monday, Friday's number will be larger.

Does Laraudogoitia's position make sense? At the least, the person betting on Wednesday is in a most unfortunate position if Laraudogoitia is correct; the Wednesday bettor is repeatedly placing certain bets, and then losing half of the time. This strikes me as problematic in and of itself. And there are more specific worries. First, the bettor is subject to a Dutch book every week. That is, prior to Monday's selection, following Laraudogoitia and reasoning that Monday's number has a 50% chance to be larger, our bettor accepts a bet to win 4 units if Monday's number is larger versus losing 2 units if Friday's number is larger (or equal). Then on Wednesday, after Monday's number is selected and observed, the person, reasoning that Friday's number is certain to be larger, accepts a bet to win 1 unit if Friday's number is larger (or equal) versus losing 5 units if Monday's number is larger. No matter what numbers are selected, the person happily accepts these bets based on their positive expected values. And the person loses 1 unit, on bets that the person happily accepted on Laraudogoitia's reasoning.[10]

Furthermore, imagine that the person has bet that Friday's number will be larger than Monday's number. He is given the option, on Wednesday, of having Friday's selection be not from all finite natural numbers, but rather from the numbers 1 to 100*m, where m is Monday's selection. This gives the person a 99% chance of having Friday's number be larger. Would swapping make sense? On one hand, the person is swapping from a certain bet to a 99% bet, and the person would be tossing out infinitely many winning numbers. Yet on Laraudogoitia's reasoning, it seems that a 99% bet would be preferable, as the person, over the long run, finds that Friday's number is larger than Monday's only half of the time.[11] Laraudogoitia owes us some guidance is this case. To repeat the question: should a person swap from the certain bet to a 99% bet on Wednesday?

An additional problem is as follows. Let us add a random selection, and consider random selections Monday, Tuesday, and Friday. Now it seems that Laraudogoitia must claim that Friday's number will be largest only 1/3 of the time. And yet the person who is, on Wednesday, presented with the maximum of Monday and Tuesday's numbers, and asked if he would like to bet that Friday's number will be larger, is in much the same circumstance as the person who is presented with only Monday's number. But now this even more unfortunate person finds that he wins only 1/3 of the time, where yet again he thinks that he is constantly betting certain bets. Due to these reasons I have concerns with Laraudogoitia's position.

Do random selections from the positive integers grow through time? Intuitions are split on the answer to this question. I claim that observed selections would grow uniformly through time. Laraudogoitia claims that numbers grow through time (for any selection, going forward 0% of numbers will be smaller than that number), but in a manner such that pairwise comparisons are 50-50 as to which is larger. At the least, I note that there is a *prima facie* argument to the conclusion that numbers grow uniformly through time (any specific number selected seems too low, and the next selection seems certain to be larger) and, I have claimed, problems arise from denying this claim. Contrast this with random selections from, e.g., {1, 2, 3, 4, 5, 6}, that is, a standard die roll. In this case, there is no argument whatsoever to the conclusion that random selections from this set grow through time. Instead, we may say that random selections from this set ‘bounce around’. In the remainder of the paper, I use the *prima facie* argument (to the conclusion that random selections from the natural numbers grow uniformly through time) as part of an argument that certain infinite structures are merely potentially infinite. I think that it is important to see that argument in the context of my general claims regarding infinite number. Therefore I now turn to summarizing and synthesizing my position on infinite number. We return to the growth of random selections in due course.

§2: Infinite Number: Two Main Claims

My recent research has focused on infinite number. The purpose of this section is to summarize and outline my position on infinite number, which involves two main claims:

1) Infinite numbers, properly understood, are the infinite natural numbers in a nonstandard model of the reals, and

2) ω is potentially infinite (not actually infinite).

Let us take these points in turn. Claim 1) flows out of a consideration of the question: Which objects are the infinite numbers? That is, consider the finite whole (natural) numbers: numbers like 3, 103, and 1,037. How should such numbers be extended into the infinite? I suggest that the infinite numbers, properly understood, are the infinite natural numbers in a nonstandard model of the reals. The argument, at its core, is simply that such infinite natural numbers look and behave very much like the finite natural numbers. I argue to this conclusion in Gwiazda (2011, 2012b, and 2014). As I wrote in Gwiazda (2014):

> …let us consider some examples of this similarity of structure and behavior. Any finite number, when you consider order, has a first element with a successor, a last element with a predecessor, and middle elements with both of these. Infinite numbers share this structure ... There is a first element that has a successor, a last element with a predecessor, and middle elements that have both predecessors and successors. Any infinite number (in a nonstandard model of the reals, which I sometimes drop for brevity) is even or odd. Any infinite number is prime or composite. If you add one to any infinite number, you get a larger number; it is larger by one. So I suggest that when we ask this overarching question -- which objects are the infinite numbers? – or even – what does an infinite number look like? -- the infinite numbers are infinite numbers in a nonstandard model of the reals. They simply behave and look very much like finite numbers.

Infinite numbers, properly understood, are infinite natural numbers in a nonstandard model. The important point is that we are not starting with a structure, the natural numbers, and asking: how many?[12] Instead we ask: which objects are the infinite (natural) numbers? Then, I claim it becomes clear that the infinite natural numbers are the infinite natural numbers in a nonstandard model of the reals. Once this is accepted, it quickly follows that there is no number of natural numbers. Simply put, any finite number counts too few natural numbers; any infinite number counts too many natural numbers. Additionally, correctly identifying the infinite (natural) numbers dissolves many paradoxes of the infinite. I return to these points, below.

The second key point that I have argued for is that ω is merely potential. Tying this claim into the first claim, a person could agree that the infinite natural numbers in a nonstandard model of the reals are the correct extension of the finite natural numbers into the infinite, but still ask (for example): what happens when a lamp button is pressed a first time, a second time, a third time, and so on (for brevity, let us call this 'ω times')? Or, what happens when you shine a light on a Zeno sphere?[13] I suggest that a lamp button cannot be pressed ω times, just as an object cannot have ω layers. Generally, a supertask cannot be completed, nor can ω be physically instantiated, because ω is merely potentially infinite. But what does it mean to say that ω is merely potential? ω, being merely potential, is not fixed, determined, or actual. It is not present all at once. This position is most commonly associated with Aristotle. It has fallen out of favor since Cantor.[14] Let us turn to four arguments to the conclusion that ω is merely potential.

First, consider a basic, straightforward supertask: tennis balls numbered 1, 2, 3... are to be burned at times 1/2, 3/4, 7/8,... Can all of the tennis balls be burned? If we assume that ω is a completed, actual infinity, with all of the tennis balls existing simultaneously, then it seems that they can all be burned. Burn ball 1 at time 1/2, ball 2 at time 3/4, ball 3 at time 7/8, and so on. Then it is not possible to name (give the number of) a tennis ball that was not burned. And so by *assuming* that ω is an actual, determined infinity, we can argue that supertasks are completable. However, I suggest that this argument begs the question by assuming that ω is an actual infinity.

Turning to the argument that supertasks cannot be completed, note that tennis balls are being burned 1-by-1. Given this, we may ask: From how many tennis balls remaining is it possible, when burning tennis balls 1-by-1, to arrive at 0 tennis balls remaining? The obvious answer, I suggest, is 1. If and only if 1 tennis ball remains, and we burn 1 tennis ball, do we arrive at 0 tennis balls. For example, if 7 tennis balls remain, and we burn 1, then 6 remain (not 0). More problematically, if countably many balls remain and we burn 1, then countably many still remain. And so the first argument that ω is merely potential is that even the simplest of supertasks is not completable. When performing tasks 1-by-1, the only way to get to 0 tasks is from 1, and yet at no time does 1 task remain. This argument is presented in Gwiazda (2012a). Also, in ‘The Lawn Mowing Puzzle’ I asked if it was possible to have 7 lawns to mow, and get to 0 lawns to mow without going through 6. A constraint is that lawns must be mowed 1-by-1. On the standard Cantorian view, a person can accept lawns to mow numbered 8, 9, 10,… and then perform a supertask, that is, mow lawn 1 by time 1/2, lawn 2 by time 3/4,…. This would accomplish the task of getting from 7 to 0 without going through 6; however, I suggest that it is ridiculous to think that one can get to 0 tasks remaining without going through 1 (or from 7 tasks remaining to 0 without going through 6).

A second argument may be called the 'no progress argument'. Imagine that tennis balls numbered 1, 2, 3... are to be burned. Ball 1 is burned at time 1/2. Has any progress been made? Notice that 1, 2, 3... is isomorphic to 2, 3, 4... and so it seems that the answer is: none.[15] That is, on the standard Cantorian picture, no progress whatsoever has been made towards completing the overall task of burning all of the tennis balls. How can infinitely many examples of no progress amount to progress? Without a clear answer to this question, it is not clear how any progress is ever made towards the completion of a supertask.[16] To put this point another way, if we were trying to devise a task that could not be completed, a good way to do so would be to have a task such that when any part of the task was completed, the exact same overall task remained. This is the situation that we seem to be in when attempting to complete a task of structure ω, that is, a supertask, by performing tasks 1-by-1.[17]

A third argument is as follows. If a structure is complete, actual, and determined (existing fully formed at one time), then random selections from such a structure would bounce around. That is, being fully formed, determined, and static, some random selections would be larger than others, some smaller. To get a sense of such 'bouncing around', one only need roll a die repeatedly. Again, if a structure is actually infinite (or finite), then random selections bounce around. We then find by the contrapositive, that *if random selections from a structure do not bounce around (for instance if they seem to grow through time), then that structure is not actually infinite (it is merely potentially infinite)*. Because {1, 2, 3, 4, 5, 6} is actual and determined, random selections from it bounce around. By contrast, random selections from the natural numbers seem to grow through time, and so this structure is merely potentially infinite. This is a third argument that ω is potentially infinite, and is the importance of the *prima facie* argument that random selections from the natural numbers seem to grow through time, discussed above. This argument is discussed in Gwiazda (2013a).[18]

A fourth piece of evidence that ω is merely potential is that there are many puzzles and paradoxes surrounding ω. Below, I suggest that we should be wary of drawing conclusions that flow from considering ω a completed, actual infinity. Here I note that the fact that so many puzzles and paradoxes surround ω provides evidence that it is not an actual, infinite entity. As a specific example of the types of puzzles that arise surrounding the infinite, consider the following case, which is a version of what is often called Ross's Paradox. I modify a version of Barrett and Arntzenius. Imagine a pit that begins empty and imagine dollar bills with "serial numbers" numbered by the natural numbers: 1, 2, 3... At each step of a supertask, the lowest numbered 10 bills are thrown into the pit, and then the lowest number bill is taken out. And so at time 1/2, bills 1 through 10 go into the pit and bill 1 is taken out. At time 3/4, bills 11 through 20 go into the pit and bill 2 is taken out, etc. The standard story is that after the supertask, no bills remain in the pit, because every bill was removed from the pit at some time. But at the very least this is odd and puzzling. What is odd is that at each step, 9 bills are added to the pit, *and this is the only sort of action that ever occurs*. How adding 9 bills to a pit, even infinitely many times, can leave 0 bills remaining at the conclusion, is not at all clear. We could imagine bouncer Bob, who oversees the pit. At each stage Bouncer Bob makes sure that 9 bills are added to the pit. How, we may ask, does the pit wind up with no bills remaining?

In this section, I have argued for two main claims: 1) Infinite numbers, properly understood, are infinite numbers in a nonstandard model of the reals, and 2) ω is merely potentially infinite. Claim 1) was admittedly presented briefly, but I provided references to three papers where the claim is defended in more detail. With claim 2), I recognize that it is likely that many people will be unfazed and unmoved by any of the arguments. Repeat enough times that 'odd things happen with the infinite' and one can become immune to odd results.[19] I do suggest that some coherent explanation to the problems posed above would be helpful for those wishing to defend the view that ω is actual and determined. Despite these reservations, is the next section I forge ahead and consider the potential importance of the two claims.

§3: The Importance of the Claims

I have made two main claims. Is there any importance to these claims? In the remainder of the paper, I suggest that there is.

The first claim, that the infinite numbers are the infinite natural numbers in a nonstandard model of the reals, dissolves many paradoxes of the infinite. Taking this point seriously means that when one hears ‘infinitely many’ or discusses ‘an infinite number’, these terms refer to an infinite natural number, just as when one hears ‘finitely many’ or discusses ‘a finite number’, these terms refer to a finite natural number. Then for example, if a lamp button is pressed an infinite number of times, no paradox or puzzle remains as to the lamp’s state after these infinitely many presses, as any infinite number is either even or odd. If the button was pressed an even infinite number of times, then the lamp is in its starting state; if the button is pressed an odd infinite number of times, then the lamp’s state is opposite to its starting state. See Gwiazda (2013b) for a further discussion of this point. Similarly, consider the example above of throwing money into a pit. If money is thrown into (and the lowest bill removed from) a pit a certain number of times, then that number is either finite or infinite. If finitely many steps have been completed, n, then there are 9*n dollars in the pit. If infinitely many steps have been completed, N, then there are 9*N dollars in the pit. In both cases there is no puzzle or paradox.

The second claim, that ω is merely potentially infinite, has a negative or cautionary bearing on research. In particular, I suggest that treating ω as an actual, completed infinity and trying to arrive at meaningful results is not wise, as it is not possible to complete ω many tasks as argued above (also see Gwiazda 2012a). As one example, Barrett and Arntzenius (1999) treat ω as an actual infinity and draw conclusions about rationality. ω is merely potentially infinite, and so claims flowing out of treating ω as an actual infinity are worrisome.[20]

Perhaps most importantly, correctly identifying the infinite natural numbers (I have argued in (Gwiazda 2011 and 2012b)) shows that there is no number of natural numbers. Simply put, any finite number counts too few natural numbers; any infinite number counts too many natural numbers. A picture, in increasing order of size, is as follows:[21]

A Finite number, m: | | | |

ω: | | | | | | |....

An Infinite number, M: | | | | | | | | |... … …| | | | |.... … …| | | | |

We might write: $m < \omega < M$. The meaning of $m < \omega$ is that any finite number, m, is an initial segment of ω. The meaning of $\omega < M$ is that ω is an initial segment of the set of natural numbers (finite or infinite) less than M, considered in order. Put in cardinal terms, any finite number is a subset of the natural numbers ($m < \omega$), and the natural numbers are a subset of the set of natural numbers (finite or infinite) less than M ($\omega < M$). I suggest a similarity with the liar paradox. In one form, the liar paradox asks us to consider the sentence 'this sentence is false'. Is this sentence true or false? Neither 'true' nor 'false' is a good answer. Additionally, the liar paradox has repercussions in the foundations of mathematics. When it comes to natural numbers, we can consider the question: How many natural numbers are there? Neither 'a finite number' nor 'an infinite number' is a good answer. Additionally, this puzzle regarding the number of natural numbers has repercussions in mathematics. What are they? For one, if there is not a number of natural numbers, then it becomes less surprising that the Continuum Hypothesis does not seem to have a clear answer. There is not even any number of natural numbers, and so it is not surprising that it is not clear if there is a 'size' between the countable and the continuum.

Let me tell a story as to how the conception of infinite number outlined above led to a recent result. In a fair lottery on the natural numbers, what probability should be assigned to each number's probability of selection? In particular, can an infinitesimal succeed? An infinitesimal is the reciprocal of an infinite number.[22] Taking seriously the idea that the infinite numbers are the infinite numbers in a nonstandard model of the reals, we found that $m < \omega < M$. But then, just as the reciprocal of any finite number is too large to be the probability of the selection of a natural number (because there are more natural numbers than any finite number), so too the reciprocal of any infinite number is too small to be the probability of the selection of a natural number (because there are fewer natural numbers than any infinite number). Restated, starting from $m < \omega < M$ and taking reciprocals, we find that $1/m > 1/\omega > 1/M$. Here I am using $1/\omega$ loosely, to refer to the proper probability of selecting a natural number in a fair lottery, if any such number exists. Finite numbers of the form $1/m$ are the correct probability of a number's chance of selection in a lottery of m items, not countably many. Similarly, infinitesimals of the form $1/M$ are the correct probability of a number's selection in a lottery of M items, not countably many. This argument, put in terms of time and not probability, appears in Gwiazda (2012c). It recently appeared in Pruss (2014), where the argument was put in the context of probability.[23]

To return to the point that began the paper, we imagined that a black box randomly selected natural numbers. In 'The Train Paradox', I argue there is a paradox if such selections grow through time. I also argue that there are problems if random selections do not grow through time; this is my reply to Laraudogoitia in section 1 of this paper. Indeed, I believe that there are fundamental problems with a lottery on a countable set. And in as sense we already knew this. For example, countable additivity fails. There isn't a good probability to assign to such a lottery (at least, there is not a probability that sums to 1 over all possible outcomes). By contrast, random lotteries from finite or infinite numbers are straightforward. A random lottery on the set of natural numbers is not simple or straightforward, precisely because the set of natural numbers is too small to be infinite in number but too large to be finite in number. The latter point is recognized; the former point is not and is a main theme of this paper. What then, are we to make of a lottery on the natural numbers? I believe that such a thing cannot exist -- it is ruled out by the laws of logic. There is no ‘good’ probability measure on the natural numbers. But the reason for this is more fundamental, namely, there is no number of natural numbers. Recognizing this fact, I suggest, should inform our research projects going forward. We must wrestle with the uncomfortable issue that there is no number of natural numbers: “ω is [a] strange middling beast that is too large to be finite in number, but too small to be infinite in number.”[24] Correctly identifying the infinite natural numbers and then wrestling with this fact that no number, finite or infinite, properly counts the set of natural numbers lies in store for the future of the concept of infinite number.

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Notes

[1] There are a number of issues with even this story. How would you become convinced that the integers were randomly selected? Also, mathematically there is no uniform probability distribution on the positive integers that satisfies countable additivity. That is, it is impossible to assign the same probability to the chance of the selection of each positive integer such that the infinite sum of the probabilities adds to 1. Let us put these worries to the side at the outset and accept provisionally that the black box is selecting random positive integers.

[2] Gwiazda 2006.

[3] Gwiazda (2013a) extended this argument to the uncountable case.

[4] Laraudogoitia 2013.

[5] There is a continuous analog. Consider the positive real numbers, and imagine telling someone to go a random distance away from you (from 0). That is, they select a random positive real. No matter where they wind up, a sensible response is ‘why are you so close to me?’. (And indeed, there is no measure on the positive reals that assigns the same probability of selection to, e.g., each unit interval.)

[6] I return to this issue of the mathematical issues. Ultimately I argue that such mathematical issues support my position that the natural numbers are not, themselves, a (natural) number.

[7] Laraudogoitia (2014, p.220).

[8] Bartha (2011, p. 641).

[9] To feel the force of this argument, imagine that it is Wednesday and you are betting the over/under relative to a selection that was made on Monday (and that you know). Are you really indifferent between the over and the under? Furthermore, if you are certain that Friday's number will be larger, wouldn't it be odd to find that Friday's number is only larger 50% of the time?

[10] In Gwiazda 2012d I presented a case where repeatedly opting for a larger expected value leads to a worse outcome. And so I have no fundamental problem with cases where betting based on expected value leads to worse outcomes. However, this Dutch book is problematic in ways that are not analogous to the example I presented in Gwiazda 2012d.

[11] Note that if the 99% probability is considered problematic, the same point can be made by asking if the person should switch to a straightforward 99% bet (perhaps one number is selected from 100, and the person wins if the number is 1).

[12] Nor are we assuming that the correct way to judge the relative sizes of infinite sets is by considering bijections. See the final footnote for a discussion of this point.

[13] A Zeno sphere is a series of concentric shells of radius 1/2, 3/4, 7/8, etc. There is no outermost shell.

[14] Though the position that the infinity is merely potential is not common today, there are defenders. For example, Edward Nelson writes 'The notion of the actual infinity of all numbers is a product of human imagination; the story is simply made up. The tale of ω even has the structure of a traditional fairy tale: "Once upon a time there was a number called 0. It had a successor, which in turn had a successor, and all the successors had successors happily ever after"' (Nelson, p.6). And also, 'The celebration of infinity is the celebration of life, of newness, of becoming, of the wonder of possibilities that cannot be listed in a finished static rubric. The

very etymology of the word infinite is "unfinished." As Aristotle observed, infinity is always potential and never actual or completed' (Nelson, p. 1).

[15] Or speaking of cardinals, we have gone from a countable infinity to a countable infinity.

[16] This example is highlighted by the example of Lazy Lara, discussed in Gwiazda (2012a, p. 3).

[17] At least on the standard Cantorian picture that isomorphic structures (e.g. 1, 2, 3… and 2, 3, 4…) are the same.

[18] Note that random selections from a hyperinteger, M, bounce around, and so this proposed test does not show that M is potentially infinite. Indeed, I believe that infinite integers are examples of the actual infinite. Note also that I argue: 1) if a structure is actual, then selections bounce around and 2) if selections don't bounce around, then the structure is potential. I *do not* suggest 3) if a structure is potential, then selections bounce around, nor 4) if selections bounce around, then the structure is actual.

[19] Ultimately I think that a strange thing is happening with ω, namely, it is obviously not actual and determined. Rather it is merely potential. This is the sort of claim that should be argued from, not to. Yet oddly much of the world seems to be in the grip of some sort of mass Cantorian delusion.

[20] For other examples, see Bacon, Bartha, and Peijnenburg. Bacon makes claims in decision theory. Bartha considers an apple cut into ω many slices. Peijnenburg wrestles with what occurs when Zeno-spheres collide.

[21] It is necessary to consider the natural numbers less than the infinite natural number M to arrive at the comparison between ω and M. Though, on the Cantorian picture, ω is countable and the hyperintegers less than M are uncountable, so the set of hyperintegers less than M is obviously bigger than the cardinality of ω. These points are made in Pruss (2014).

[22] It occurred to me that an infinitesimal probability could be assigned to each number's selection using a specific construction of a hyperreal number. I described this infinitesimal in Gwiazda (2008). The same measure later appeared in more detail in Wenmackers and Horsten (2010). It later occurred to me that an infinitesimal cannot succeed, for the reasons outlined.

[23] In particular, I suggest that this argument undercuts the alpha-theory of Benci and di Nasso. See, for example, Benci and di Nasso (2000).

[24] Gwiazda 2014. I face some fundamental difficulties in trying to gain acceptance of the ideas presented in this paper. Though initially I am responding to a specific point regarding the behavior of a sequence of random selections, I am also trying to place this discussion in a larger context of my overall work on infinite number. This work is spread across many papers and is not Cantorian in nature, which is to say that many people will be initially inclined to disagree with my arguments. Thus in this final footnote, at the risk of some repetition, I restate the overall argument of the paper and reply to one common objections.

The main question that my work on infinite number has addressed is the question: Which objects are the infinite numbers? That is, if you want the infinite analogue of numbers like 3, 7, and 103, what should you point to? The answer, I suggest, is the infinite natural numbers in a non-standard model of the reals. This argument is spelled out in detail in Gwiazda (2011, 2012b, and 2014), but the basis of the argument is simply that these infinite natural numbers look and behave very much like the finite natural numbers. (Also note that they are already called the 'infinite natural numbers'.) Cantor was trying to moving the finite natural numbers into the infinite, but he identified the wrong things. Very often I have found that people do not understand what I mean by the question: Which objects are the infinite natural numbers? For a physical analogue of the type of confusion I claim is occurring, see Gwiazda (2014).

I now address one potential objection, which is that I am confusing, conflating, or at least not clear about whether I am talking about cardinal or ordinal numbers. Generally I am taking order into account, and so it is best to think of the argument as occurring in an ordinal framework. But I argue that none of Cantor's infinite numbers, neither cardinal nor ordinal, are the infinite natural numbers. The infinite natural numbers, which are the infinite natural numbers in a nonstandard model of the reals, have cardinal and ordinal properties, but ones that, I suggest, would surprise the Cantorian. Bijection is not the only possible way to judge the relative sizes of infinite sets. That is, the standard view is that if two infinite sets can be mapped 1-to-1, then they are the same size. But it is possible to judge the relative sizes of infinite sets based on subset, where if one set is a subset of the other, the former is smaller. I believe that subset is the correct way to judge the relative sizes of infinite sets. See Gwiazda (2011) for the argument. See Parker (2009) and Mayberry (2000) for discussions of subset versus bijection, and the legitimacy of subset. Though my two main claims may be unpopular, I hope that there is value in presenting an outline of the claims and their potential importance, while providing further references. My hope is that this paper may serve as that outline.