

# Defining determinism

Thomas Müller and Tomasz Placek\*

## Abstract

The paper puts forward a branching-style framework for the analysis of determinism and indeterminism of scientific theories, starting from the core idea that an indeterministic system is one whose present allows for more than one alternative possible future. We describe how a definition of determinism stated in terms of branching models supplements and improves current treatments of determinism of theories of physics. In these treatments we identify three main approaches: One based on the study of (differential) equations, one based on mappings between temporal realizations, and one based on branching models. We first give an overview of these approaches and show that current orthodoxy advocates a combination of the mapping- and the equations-based approaches. After giving a detailed formal explication of a branching-based definition of determinism, we consider three concrete applications, and end with a formal comparison of the branching- and the mapping-based approach. We conclude that the branching-based definition of determinism most usefully combines formal clarity, connection with an underlying philosophical notion of determinism, and relevance for the practical assessment of theories.

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# 1 Introduction

In this paper we describe how a definition of determinism based on branching models supplements and improves current treatments of determinism of scientific theories in physics. Our focus is on a definition of determinism that takes a scientific theory as input, and delivers a verdict as to the theory's determinism as output, providing one bit of information. This may seem to be a simple matter, but in practice, a number of subtle issues are involved: In which form is a theory fed into the definition? When does a (semi-)formal definition have a claim to providing an explication of the philosophical notion of determinism, rather than something else? And what is the use of the definition: Does failure of determinism signal a defect in the theory (this may be a practitioner's sentiment), or rather a useful insight about the theory or perhaps even about metaphysical issues (which may be a philosopher's view)? It turns out that in the actual assessment of a physical theory as deterministic or indeterministic, all these matters play an important role, so that deciding about a theory's determinism is a delicate practice rather than a simple call upon a definition. Still, a general definition provides a useful overarching perspective on the determinism of scientific theories, functioning both as a guideline for practical assessment and as an interface to discussions in other areas of philosophy. Our contribution is aimed at this general level.

In current philosophy of science there are three subtly different approaches to defining determinism for physical theories, which we label DEQN, DMAP, and DBRN. Figure 1 gives a schematic overview. According to DEQN, a theory is assessed via a study of the behavior of its defining (differential) *equations*. Definition DMAP is in terms of *mappings* between the linear temporal developments of systems allowed for by the theory. Finally, DBRN is in terms of partially ordered, *branching* models of such systems' behavior. Our main claim is that DBRN gives the most useful general definition of determinism for physical theories, despite the fact that DMAP enjoys the status of current orthodoxy in philosophy of science, and DEQN is typically invoked by practitioners. We believe that this insight into the usefulness of a DBRN-type analysis of determinism in fact carries over to other areas of philosophy as well.

We argue for this main claim in the following way. In order to provide a basis for discussion, in §2, we briefly describe the general background notion of determinism and introduce the three approaches to determinism of scien-

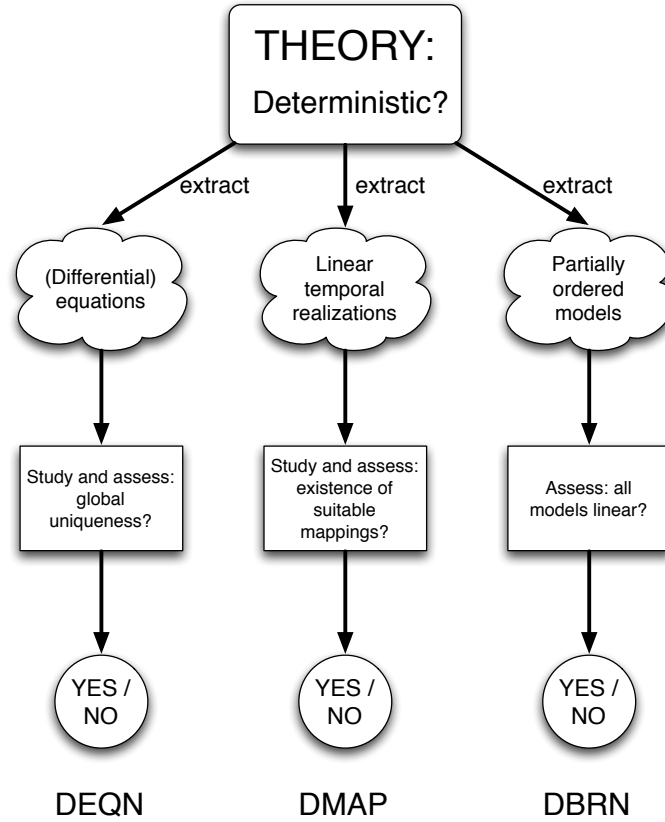


Figure 1: Three definitions of determinism for a theory: DEQN, based on equations; DMAP, based on mappings between linear temporal realizations, and DBRN, based on branching models.

tific theories in physics, DMAP, DEQN, and DBRN. §3 offers an overview of currently dominant definitions of determinism, thereby showing in which way DMAP enjoys the status of orthodoxy, and which role DEQN plays in the actual assessment of theories. §4 provides the DBRN definition of determinism in formal detail, including questions of topology. In §5, which comprises the bulk of the paper, we compare the three mentioned approaches to defining determinism, with a view to making good our claim of the usefulness of DBRN. We proceed in two steps: First (§5.1), we give three examples of the application of the three approaches to physical theories, referring to Newtonian mechanics, quantum mechanics and general relativity. We show in

which way the DBRN definition comes naturally, including the construction of an explicit mathematical model. A perhaps surprising result from our case studies is that the DMAP definition, despite its status as official orthodoxy, is hardly ever used, and that the DEQN definition is inappropriate when it comes to quantum mechanics. In a second step (§5.2), we offer a formal comparison of the DMAP and the DBRN definitions. We point out the subtle role that a class of isomorphisms plays for the DMAP analysis, and we show that the DBRN and DMAP definitions can give rise to different assessments in such a way that DMAP classifies a theory as indeterministic too easily. Finally, §6 sums up the paper and recapitulates our conclusions.

## 2 Determinism in philosophy of science: three approaches

The question whether our world is deterministic or not—whether the future is genuinely open, or whether there is just one real possibility for the future—is one of the fundamental concerns of metaphysics. And the impact of that question is not limited to theoretical metaphysical speculation. Determinism is a topic that cuts across many philosophical sub-disciplines, including ethics, action theory and philosophy of science.

In philosophy of science, the question of determinism is addressed in relation to scientific theories, and provides an important means of assessing theories in various respects. There are many reasons to ask the question whether a given scientific theory is deterministic or not. One is metaphysical: We may be convinced that the theory gives an appropriate picture of what the world is like, and therefore use the theory in order to find out about the determinism or indeterminism of the world as a whole. Another one is epistemological: Finding out whether a theory is deterministic, can tell us something about in-principle limitations on predictions (or retrodictions) that the theory affords. It has also turned out that asking the question about determinism is often a good way to deepen one's understanding of the theory itself, since many subtle technical issues have to be addressed in order to provide a verdict on whether the theory is deterministic or not.

Assessing a given theory as to its determinism or indeterminism is a tricky issue. Earman [1986] has done a great service to the philosophy of science community by tracing out in detail the questions involved for physical

theories, and the decisions that need to be made along the way. His work closely follows the discussions of practitioners, showing how issues such as the behavior of differential equations, the identification of gauge degrees of freedom, the notion of a physical state, and considerations of the physicality of specific set-ups can all play a role. (We will mention some of these issues later on.) Nevertheless, what is being done in assessing the determinism of a theory, must in some important sense be the same in all cases, for otherwise, there would be no unity of the notion of determinism of a theory to begin with. It is therefore reasonable to ask what it is that is the same in all these cases, that is, what the common definition of determinism is that is being employed.

## 2.1 Determinism: the core idea and how to spell it out

The core idea behind the philosophical (metaphysical) notion of determinism is that given the way things are at present, there is only one possibility for the future to turn out.<sup>1</sup> Accordingly, indeterminism (which we equate with the negation of determinism) amounts to there being more than one possibility for the future to turn out, given the way things are at present. This core idea lends itself immediately to a branching representation of indeterminism: graphically, indeterminism can be pictured as a tree-like structure of possible histories overlapping at present and branching into the future. This idea of branching histories has been rigorously developed in tense logic (by Prior [1967] and Thomason [1970]), leading also to fruitful applications in other fields such as the theory of agency (Belnap et al. [2001]; Horty [2001]). In a branching approach, alternative future possibilities are represented by models that are ‘modally thick’, in the sense of containing more than one overlapping history (see Müller [2012] for a comparison with other modal notions). This feature is central to the DBRN definition to be detailed below (see esp. §4.1).

The core idea of determinism appeals to future possibilities, but this notion does not immediately transfer to mathematical physics. After all, the concept of future possibilities does not belong to the repertoire of physics; it thus needs to be spelled out what this concept amounts to in the context of a given physical theory. Laplace’s [1820, p. 4] popular metaphor of a demon computing the future of the universe suggests a characterization of

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<sup>1</sup>It is possible to express this core idea without tensed notions, by saying that each event permits at most one possible subsequent course of events. We will stick to the tensed version in terms of present state and future development in what follows.

determinism in terms of laws of nature. The resulting doctrine is called nomological state-determinism: Given the laws and a state of the world, there is only one way the world can turn out to be. Both the notion of state and the notion of laws require analysis.<sup>2</sup>

Nomological state-determinism with respect to a given theory can be spelled out in at least two different ways. The first requires a philosophical concept of laws of nature and the notion of a possible world satisfying (or being governed by) such laws. Less metaphysically, the concepts of a theory and of a theory's models rather than those of laws of nature and of possible worlds can be taken as starting points. The DMAP analysis of determinism employs exactly these two conceptual ingredients: A theory  $T$  is said to be deterministic just in case whenever models  $w$  and  $v$  of  $T$  agree with respect to their state at one time, then  $w$  and  $v$  agree with respect to their states at all times.

The other way to spell out nomological state-determinism begins with the observation that a theory's mathematical representation typically supplies all the resources needed for assessing the theory's determinism: A theory's defining differential equations assume the role of laws of nature, and solutions to these equations stand in for physically possible worlds. On the assumption that different solutions to a theory's defining equations represent different physically possible worlds, determinism then boils down to the existence of a unique solution for each appropriate initial value—this is the essence of the DEQN approach.

## 2.2 The three approaches in more detail

We now turn to characterizing in more detail the three approaches: DEQN, DMAP, and DBRN. All three presuppose that a theory is given to us as an object to be diagnosed as to its determinism or indeterminism—the general structure is depicted in Figure 1.<sup>3</sup>

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<sup>2</sup>Thanks to Balázs Gyenis for a discussion of this point.

<sup>3</sup>As Wilson [1989] remarks, this picture may be unjustified when it comes to assessing, e.g., the determinism or indeterminism of classical mechanics: Breakdowns of the determinism of the theory will normally lead to the incorporation of additional assumptions or additional bits of theory, rather than a flat-out admission of indeterminism. The point remains, however, that at any given stage of practical assessment, one can consider 'what's currently on the table', and a definition of determinism has to apply at any such stage.

**DEQN.** The equation-based definition of determinism represents the mathematical perspective of present physicists, focusing on a theory’s defining (differential) equations. The leading question is whether for each initial condition there exists a unique solution of these equations. It turns out that the answer depends on what kind of differential equations one considers. For ordinary differential equations (ODEs) there are general methods that allow for a conditional statement of the existence and uniqueness of solutions. In contrast to this tractable landscape of ordinary differential equations, there are no useful general results concerning the existence and uniqueness of solutions to partial differential equations (PDEs).

It is important to distinguish here between global existence and local existence of solutions, where ‘global’ refers to the full range of the time parameter, and ‘local’ indicates a neighborhood (possibly arbitrarily small) of a given moment of time. The question of the existence and uniqueness of solutions thus splits into two problems. First, is there a unique local solution for each moment of time? And if the answer to that question is positive: Are such local solutions uniquely extendible to the full, global range of the time parameter? Now, for an ODE  $\frac{dx}{dt} = f(x, t)$ , the Peano theorem establishes that for every initial condition there is at least one local solution of the equation—provided that the function  $f$  is bounded and continuous. Further, the Picard-Lindelöf theorem states that, provided the function  $f$  satisfies the so-called Lipschitz condition, for every initial condition an ODE has at most one local solution. These results extend to ODEs of arbitrary order and carry over to sets of ODEs as well.<sup>4</sup> As to the extendibility of local solutions to an ODE to a global solution, in general, the answer is in the negative, though for some classes of ODEs, under certain conditions, extendibility holds. This is highly pertinent to the assessment of determinism, as it is global uniqueness that naturally corresponds to determinism, whereas the mentioned theorems conditionally assure the existence and uniqueness of merely local solutions. Non-extendability then points to a possible failure of determinism due to lack of a unique global solution—despite there being unique local solutions everywhere (for some topological details, see §4.3). Note thus that mathematics alone indicates how subtle determinism or its failure, indeterminism, can be.

A disadvantage of DEQN is that not all theories can be described neatly

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<sup>4</sup>For a rigorous statement of the mentioned theorems and some useful discussions, see Arnol’d [1992].

in a form that allows for the application of the DEQN recipe. A notable case in point is quantum mechanics, for which the defining Schrödinger equation is very well behaved, but which is often regarded as a main example of an indeterministic theory: the Born rule, which is an integral part of quantum mechanics, prescribes probabilities for possible measurement outcomes (see §5.1.2 below for discussion).

**DMAP.** The mapping-based approach to defining determinism amounts to current orthodoxy in philosophy of science (see §3). That approach takes determinism to be a matter of the existence of suitable mappings in the whole space of a theory's temporal realizations. The approach is grounded in Montague's pioneering formal investigations of deterministic theories from a logical point of view (Montague [1962]). Speaking abstractly, the diagnosis of determinism according to DMAP is a two-stage affair. In a first step, all of the individual realizations of the linear temporal development of systems falling under the theory are put side by side. These are the separate possible ways a world could be that are admitted by the theory. Depending on the theory in question, these could be all the solutions to the theory's defining equations, or a class of temporal realizations that is given in some other, perhaps more complex manner (quantum mechanics in a consistent histories formulation would be a case in point here; see §5.1.2). In a second step, this class of temporal developments is checked for instances of indeterminism, in the following way: If there are two realizations that can be identified at one time,<sup>5</sup> but whose future segments after that time cannot be identified, this signals indeterminism. If the test fails, that is, if all realizations that can be identified at one time, can also be identified at all future times, then the theory is deterministic. The type of mapping that is used to identify different realizations at different times, plays a subtle but crucial role for this definition (see §3).

The DMAP definition corresponds to a divergence analysis of future possibilities, which is popular in current metaphysics (Lewis [1986]). Individual realizations (ways a world could be) are 'modally thin' in that they harbor no possibilities. Possibilities are present only extrinsically, via the existence of suitable mappings between the realizations.

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<sup>5</sup>Instead of identification at a time, for some theories it is necessary to consider identification over an arbitrarily short interval of time, or over initial segments of temporal developments.



**DBRN.** An alternative understanding of future possibilities underlies the branching-based DBRN characterization of the determinism of theories. Doing justice to the philosophical idea of alternative future possibilities, in a branching conception a single model is so construed that it can contain multiple possibilities. The existence of possibilities is intrinsic to a model, and a model can therefore be ‘modally thick’. Rather than opting for linearly ordered temporal realizations as in DMAP, a branching model is generally only partially ordered in a tree-like manner; the individual realizations form linear chains (histories) within that partial ordering. Within one partial ordering, these histories are bound together by overlapping up to a certain time, so that there is no need to look for the identifying mappings needed for DMAP. The diagnosis of indeterminism is very simple: If there is a model that is not linearly ordered (such a model contains more than one history), then the theory is indeterministic. A deterministic theory is one all models of which are linear.

### 2.3 Representing indeterminism

Figure 2 shows what indeterminism, that is, failure of determinism, looks like according to the three mentioned definitions. For DEQN, such a failure comes down to a differential equation admitting globally different solutions for the same initial data. For DMAP, indeterminism is witnessed by the existence of two linear temporal realizations that can be mapped at one time (lower arrow) but not at all future times (upper, crossed arrow). For DBRN, a witness of indeterminism is a model that is branching rather than linear.

## 3 Orthodoxy: DMAP, with invocations of DEQN

In order to have a point of reference for our work on a formal branching-style analysis of determinism, we first characterize the dominant approach to defining determinism for scientific theories. This dominant approach owes much to the work of Jeremy Butterfield and John Earman. Their views are summarized in two influential encyclopedia entries (Butterfield [2005]; Earman [2006]) and an earlier book (Earman [1986]), on which we will focus.<sup>6</sup> Both authors define determinism along the lines of DMAP: any two realizations

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<sup>6</sup>For affirmations of a similar general outlook, see, e.g., Bishop [2006] and Hofer [2010].

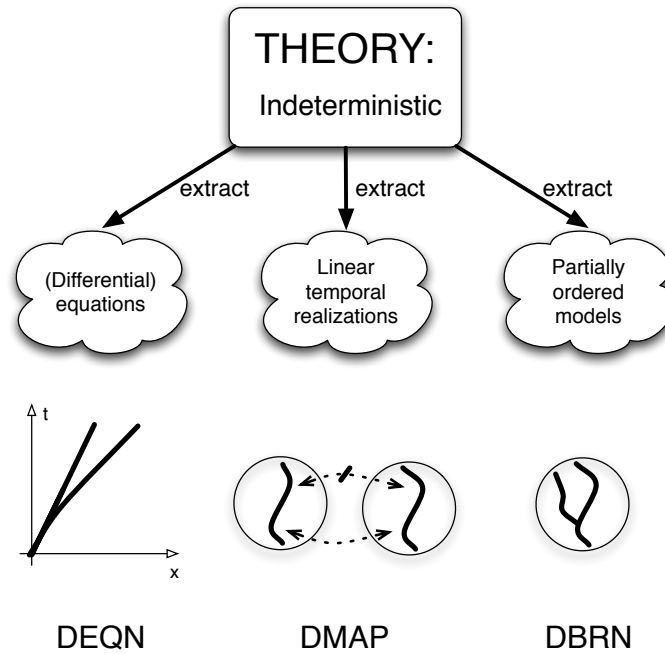


Figure 2: Failure of determinism according to the three definitions of determinism: DEQN, DMAP, and DBRN.

(separate mathematical structures) that agree (can be suitably identified by a mapping) at one time, have to agree at all later times.

Butterfield starts by stressing the need to study determinism in terms of isolated systems. A model<sup>7</sup> of a theory is ‘a sequence of states for such a single system, that conforms to the laws of the theory’ (Butterfield [2005]). The concept of state, as Butterfield notes, is philosophically loaded: states should be maximal and intrinsic, and in many actual theories, different mathematical states correspond to the same physical state (see below). This precludes a direct use of DMAP in terms of identity. Butterfield accordingly appeals to the notion of isomorphism:

Determinism is [...] a matter of isomorphic instantaneous slices  
implying that the corresponding final segments are isomorphic

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<sup>7</sup>In this section we stick to Butterfield’s use of ‘model’ for what we generally call ‘realization’ or ‘history’, i.e., for a single, linear temporal time-course of development of a system.

(where ‘corresponding’ means ‘starting at the time of the instantaneous slice’). That is: we say that a theory is deterministic if, and only if: for any two of its models, if they have instantaneous slices that are isomorphic, then the corresponding final segments are also isomorphic. (Butterfield [2005])

The notion of isomorphism appealed to above needs clarification. Butterfield says that while he uses ‘model’ in the broad sense of philosophy of science, he uses ‘isomorphism’ in the ‘usual sense used by logicians’ (Butterfield [2005]). A theory’s model is thus not a model in the logical sense, but a realization. For Newtonian, special relativistic, and general relativistic theories, a theory’s realization takes the form  $\langle \mathcal{M}, O_1, \dots, O_n \rangle$ , where  $\mathcal{M}$  is a differentiable manifold, and  $O_i$  are geometrical object fields on  $\mathcal{M}$ . Realizations of that form are used quite generally in discussions of a theory’s determinism in philosophy of science.<sup>8</sup>

An isomorphism, in the logical sense, is a structure-preserving bijection between the domains of two models, where the relevant structure depends on the characteristics of a language: its class of constants, its relation symbols, and its function symbols (Hodges [1993], 5ff.). There is no intuitively adequate notion of isomorphism that is language-independent.<sup>9</sup> Since Butterfield does not say which language and which symbols need to be considered, the admissible class of mappings (isomorphisms) remains underspecified. Charitably one can read the reference to isomorphisms as a promissory note: For each given theory, a linguistic presentation shall be specified, from which the sought-for notion of isomorphism would follow.<sup>10</sup>

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<sup>8</sup>Note that a defining feature of manifolds, local Euclidicity, together with the Hausdorff property (typically assumed for manifolds in physics applications), implies that there is no branching in  $\mathcal{M}$ . As a consequence, there is no way for  $\mathcal{M}$  to represent alternative possible events;  $\mathcal{M}$  is only interpretable as a totality of spatiotemporal events. One might worry that to account for alternative possible events via branching, either local Euclidicity or the Hausdorff property for spacetimes must be sacrificed. We will return to this worry and dismiss it in section 4.3.

<sup>9</sup>See (Halvorson [2012]) for similar issues that arise for attempts to specify a theory in language-independent terms.

<sup>10</sup>For general relativity the promissory note is repaid (although not in a strictly logical sense, as no language is specified) in (Butterfield [1989]). ‘Isomorphism’ is used there in the sense applicable to manifolds. A *diffeomorphism* is a smooth bijection between two manifolds  $\mathfrak{M}$  and  $\mathfrak{M}'$ . Two models  $\langle \mathfrak{M}, O_i \rangle$  and  $\langle \mathfrak{M}', O'_i \rangle$  are called *isomorphic* iff there is a diffeomorphism  $d$  between the manifolds  $\mathfrak{M}$  and  $\mathfrak{M}'$ , and for the objects  $O_i$ , we have  $d^*(O_i) = O'_i$  (where  $d^*(O_i)$  is the object  $O_i$  dragged along by the diffeomorphism  $d$ ). The

In view of the difficulties involved in specifying the correct notion of isomorphism, it is tempting to phrase the concept of determinism in terms of *identical* instantaneous states, but this calls for carefully distinguishing between states as represented within a given theory and physical states. This issue is stressed by Earman, whose point of departure is the explication of determinism for pre-relativistic, pre-quantum theories. Such theories specify a set of candidates for genuine physical magnitudes, call it  $\mathcal{O}$ . It is assumed that each such magnitude takes a definite value at every moment of time  $t \in \mathbb{R}$ . The explication then is as follows:<sup>11</sup>

A *history*  $H$  is a map from  $\mathbb{R}$  to tuples of values of the basic magnitudes, where for any  $t \in \mathbb{R}$  the state  $H(t)$  gives a snapshot of behavior of the basic magnitudes at time  $t$ . The world is *Laplacean deterministic* with respect to  $\mathcal{O}$  just in case for any pair of histories  $H_1, H_2$  satisfying the laws of physics, if  $H_1(t) = H_2(t)$  for some  $t$ , then  $H_1(t) = H_2(t)$  for all  $t$ . (Earman [2006], 1370)

Here, the correspondence between the set  $\mathcal{O}$  and the set of genuine physical states is crucial. Since elements of  $\mathcal{O}$  may have mathematical surplus structure, the failure of the requirement that ‘if  $H_1(t) = H_2(t)$  for some  $t$ , then  $H_1(t) = H_2(t)$  for all  $t$ ’ need not signal indeterminism. It is a typical situation in physics, and not some mere philosophical possibility of theoretical underdetermination, that a theory’s mathematical descriptions correspond many-one to physical states, so that the identity in the above quotation needs to be replaced by a broader notion of agreement. Consider classical electromagnetism, where the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  are derived from a scalar potential  $\phi$  and a vector potential  $\mathbf{A}$ . The relation between  $\mathbf{A}, \phi$  and  $\mathbf{E}, \mathbf{B}$  is many-one: for any smooth function  $\psi$ , the potentials

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definition of determinism is then as follows:

A theory with models  $\langle \mathfrak{M}, O_i \rangle$  is  $\mathbf{S}$ -deterministic, where  $\mathbf{S}$  is a kind of region that occurs in manifolds of the kind occurring in the models, iff: given any two models  $\langle \mathfrak{M}, O_i \rangle$  and  $\langle \mathfrak{M}', O'_i \rangle$  containing regions  $S, S'$  of kind  $\mathbf{S}$  respectively, and any diffeomorphism  $\alpha$  from  $S$  onto  $S'$ :  
if  $\alpha^*(O_i) = O'_i$  on  $\alpha(S) = S'$ , then:  
there is an isomorphism  $\beta$  from  $\mathfrak{M}$  onto  $\mathfrak{M}'$  that sends  $S$  to  $S'$ , *i.e.*  
 $\beta^*(O_i) = O'_i$  throughout  $\mathfrak{M}'$  and  $\beta(S) = S'$ . (Butterfield [1989, 9])

<sup>11</sup>Although Earman focuses on future- *and* past-oriented determinism, whereas Butterfield analyses future-oriented determinism, each explication can be readily extended to accommodate both versions of determinism.

$\mathbf{A}$ ,  $\varphi$ , and the potentials

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \quad \varphi' = \varphi - \frac{\partial\psi}{\partial t}$$

represent the same physical situation, and the same  $\mathbf{E}$  and  $\mathbf{B}$  fields. The transformation  $\mathbf{A} \mapsto \mathbf{A}'$ ,  $\varphi \mapsto \varphi'$  is called a gauge transformation. The arbitrariness of the choice of  $\psi$  means that the theory has surplus mathematical structure ('gauge freedom'). For a theory with gauge freedom, the fact that two realizations have the same mathematical state at a certain  $t$ , but different states at some later  $t'$ , is not of itself indicative of indeterminism: it could also be that the states at  $t'$  represent the same physical state by different mathematical means. It will not do to demand that only theories without gauge degrees of freedom are considered, since there are good scientific reasons for allowing that kind of freedom in our physical theories.<sup>12</sup> Thus, to decide the question of determinism of a theory requires to decide whether the divergence of realizations results from gauge freedom or not. This is conceptually difficult, as one main motivation for believing gauge freedom to be operative in a theory is to maintain determinism. We will return to this issue in a more formal setting in §5.2, where we compare the DMAP and DBRN approaches to determinism.

Our final observation is that although both Butterfield and Earman explicate determinism in terms of mappings (DMAP), they both add a gloss to the effect that for theories given via differential equations, their definitions correspond to the existence of unique solutions to such equations,<sup>13</sup> which amounts to DEQN. The nature of this correspondence is, however, left open.

**Four marks of orthodoxy** To summarize, the current orthodoxy in treating the question of determinism has the following four marks:

1. A theory is represented by the class of its realizations ('models')—possible total time courses of evolution of a system to which the theory

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<sup>12</sup>As (Earman [2006], 1381) points out, attempts at treating gauge degrees of freedom as physical quantities subject to dynamical laws are generally not fruitful.

<sup>13</sup>Cf. (Butterfield [2005], 98): the given 'definition [...] corresponds to such a set of equations having a unique solution for future times, given the values at the initial time'. Similarly, (Earman [2006], 1371f.): '[...] [T]he laws of physics typically take the form of differential equations, in which case the issue of Laplacean determinism translates into the question of whether the equations admit of an *initial value formulation*, i.e. whether for arbitrary initial data there exists a unique solution agreeing with the given initial data'.

applies. The realizations are ‘modally thin’: a single realization does not contain different possibilities. Such possibilities (whose existence implies indeterminism) thus have to be represented via relations between realizations.

2. Accordingly, the question of whether a theory is deterministic or not, is understood as a question about the structure of the class of its realizations, to be spelled out in terms of suitable mappings: determinism means that agreement of two realizations at one time implies agreement at all later times.
3. The notion of ‘agreeing at a time’ is crucial, and gives rise to complications. The agreement is meant to be with respect to *physical states* of the system, but the class of realizations contains mathematical objects that may have surplus structure, for instance due to gauge freedom.
4. It is assumed that the practical assessment of determinism or indeterminism of a given theory mostly depends on the behavior of that theory’s defining differential equations: a strong link is claimed between determinism in the mapping sense (DMAP) and the well-posedness of the initial value problem, if the laws of a theory in question are formulated by differential equations.

Points (1)–(3) show that the orthodox definition of determinism amounts to DMAP, with point (3) constraining the suitable mappings. Point (4), however, makes a strong link to a DEQN-style definition of determinism.

## 4 Branching-style determinism (DBRN)

We turn now to a branching characterization of determinism. As we said, that characterization is an attempt to capture directly the core idea of determinism: the present has exactly one possible future. The crucial concept of alternative future possibilities is analyzed by means of branching histories; a system is assessed as indeterministic iff<sup>14</sup> some initial segment of its evolution can be continued in more than one alternative way. Although a branching analysis of future possibilities is rarely used in philosophy of science, it has been rigorously developed in logic and its applications, for instance in theories

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<sup>14</sup>We follow the custom of using ‘iff’ as an abbreviation for ‘if and only if’.

of agency. The branching concept is also sometimes used in natural sciences (see below). The general definition of determinism according to DBRN is as follows:

**Definition 1 (Determinism of a theory)** *A theory is indeterministic iff it has at least one faithful indeterministic model. The theory is deterministic iff all of its faithful models are deterministic.*

So far, this is of course just passing the buck. To see in which way branching is involved, we need to specify a sense of ‘model’, ‘faithful’ and ‘indeterministic’ such that a model of a theory can be both faithful and indeterministic.

## 4.1 Models and realizations

In the orthodox DMAP approach described in §3, ‘model’ is often used as a synonym for ‘realization’, or ‘history’. Our proposed usage here is broader: A model of a theory is whatever fulfills the requirements of that theory. Thus, a model can be modally thick (containing structures representing alternative possibilities), or modally thin (containing no such structures). ‘Realization’, on the other hand, is tied to a linear temporal development, and thus, realizations are always modally thin. More formally, a *realization*  $\langle T, <, S, f \rangle$  of a theory specifies a function  $f$  from a linearly ordered set of times  $\langle T, < \rangle$  to mathematical states  $S$ .<sup>15</sup> The function  $f$  must be admitted by the dynamics of the theory; typically, it must be a solution to the theory’s dynamical equations given some initial data.

With a view to Def. 1, we are looking for a broader formal notion of a *model* of a theory that allows such a model to be intrinsically indeterministic. We can take a lead from logic (see for instance (Thomason [1970])), where indeterministic models are discussed in the context of so-called ‘branching time’,<sup>16</sup> and from the stochastic processes literature (for example, (van Kampen [2007], Ch. III)), in which a model of a stochastic process includes not just a single realization, but many incompatible realizations. The consistent histories approach to quantum mechanics (see (Griffiths [2002])) and

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<sup>15</sup>We will mostly take  $\langle T, < \rangle$  to be  $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$ , but we also allow for discrete time or other temporal structures.

<sup>16</sup>See also (Belnap et al. [2001], Ch. 7A) and, for the related framework of branching space-times, (Belnap [1992, 2012]).

§5.1.2 below) also employs a ‘branching time’ representation. In all these approaches, a model is allowed to be ‘modally thick’ by representing different, incompatible future possibilities in one mathematical structure.

We will thus take a model  $\langle M, <, S, f \rangle$  to specify a function  $f$  from a possibly branching, tree-like partial ordering  $\langle M, < \rangle$  of *moments* to the allowed states  $S$ , and such that the restriction of  $f$  to each realization accords with the theory’s dynamics. Formally, we require of  $\langle M, < \rangle$ , for  $x, y, z \in M$ :

- asymmetry: if  $x < y$ , then not  $y < x$ ,
- transitivity: if  $x < y$  and  $y < z$ , then  $x < z$ ,
- backwards linearity: if  $x < z$  and  $y < z$ , then either  $x = y$  or  $x < y$  or  $y < x$ , and
- connectedness: for any  $x$  and  $y$  there is some  $z$  such that  $z \leq x$  and  $z \leq y$ .

A *chain* in  $\langle M, < \rangle$  is a linear subset; by virtue of Zorn’s lemma, there are maximal chains in  $M$ . A tree-like order  $\langle M, < \rangle$  can contain more than one maximal chain, as the lower right of Figure 2 (see also Figure 3).<sup>17</sup> It is usually sensible to require that all maximal chains in  $M$  be order-isomorphic, for example, all isomorphic to  $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$ .

Based on this notion of a model, the following definition of determinism is adequate:

**Definition 2** *A model  $\langle M, <, S, f \rangle$  is indeterministic iff  $\langle M, < \rangle$  contains more than one maximal chain. The model is deterministic iff  $\langle M, < \rangle$  contains just one maximal chain.*

The existence of more than one maximal chain in  $\langle M, < \rangle$  means that there is more than one realization in the model  $\langle M, <, S, f \rangle$ , since each maximal chain  $h \subseteq M$  in the model specifies a realization  $\langle h, <|_h, S, f|_h \rangle$ , simply by restricting  $f$  to  $h$ , as  $\langle h, <|_h \rangle$  is a linear order. Accordingly, a deterministic model of a theory contains just one realization, whereas an indeterministic model bundles together a number of realizations in one branching tree. These realizations branch in the following sense: for any pair  $h, h'$  of distinct

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<sup>17</sup>Provably,  $M$  contains more than one maximal chain iff it contains at least one upward fork, i.e., three moments  $x, y, z$  for which  $x < y$ ,  $x < z$  but there is no common upper bound for  $y$  and  $z$ , which holds iff neither  $y \leq z$  nor  $z < y$ .



maximal chains,  $h \cap h' \neq \emptyset$  (by connectedness), and for every  $x \in h \cap h'$ , we have  $f|_h(x) = f(x) = f|_{h'}(x)$ . Viewed from such an  $x$ , the realizations  $\langle h, <|_h, S, f|_h \rangle$  and  $\langle h', <|_{h'}, S, f|_{h'} \rangle$  thus describe (alternative) possible future developments of the system in question. We may therefore interpret realizations as (alternative) possibilities, and call them possible histories. As a single model can contain different histories, a model in our sense can capture modality intrinsically.

## 4.2 Faithfulness

There is one loose end left to tie: So far, it would be possible to produce an indeterministic model from a linearly ordered, deterministic model  $\langle M, <, S, f \rangle$ , simply by adding a disjoint copy of a final segment of  $M$  to create a forward-branching structure, and extending  $f$  on the new branch by copying. This would be indeterminism on the cheap. We will require a *faithful* branching model to contain no difference in the ordering without a corresponding difference in states.<sup>18</sup> On a strict reading, faithfulness requires a difference in states at, or immediately after, the splitting of any two histories. In the case of what we will call case-(b) branching (such that branching histories have a first moment of disagreement, see §4.3), this boils down to the requirement that the first moments of difference in two histories have different states assigned.<sup>19</sup> In case-(a) branching, where there is a maximal moment in the intersection of two histories, faithfulness means that states should be different immediately after such a maximal moment. This can be made more precise by adding a further structure that identifies moments occurring at the same time in different histories.<sup>20</sup>

In §5.2, where we construct a DBRN representation of a system out of a DMAP representation of a system, we will use a weaker concept of faithfulness which simply requires that two branching histories must be different

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<sup>18</sup>In some cases we may want to drop the requirement of faithfulness. E.g., if we *know* that a process is indeterministic and thus, has to be modeled by a branching model, but the assigned states agree on two different maximal chains, this may signal that the theory is incomplete. In the present context, however, our aim is to *diagnose* (in)determinism, and so we are methodologically required to assume completeness.

<sup>19</sup>In the case of Figure 3(b), we thus want  $f(0_1) \neq f(0_2)$ .

<sup>20</sup>See (Belnap et al. [2001], 194–196) on instants as partitions of the set of moments. In the case of Figure 3(a), if these instants have the plausible form  $\langle t_1, t_2 \rangle$  for  $t \in \mathbb{R}^+$ , so that we can identify the time of a moment  $t_1$  on the upper track with the time of  $t_2$  on the lower track, we thus want that for any  $t > 0$  there is some  $t' < t, t' > 0$ , s.t.  $f(t'_1) \neq f(t'_2)$ .

state-wise. In that section we will also discuss how to formally represent sameness of physical states, given that physical states might have non-unique mathematical representations, resulting from gauge freedom or from the symmetries of a theory.

### 4.3 Two types of branching topologies

Consider a branching order  $\langle M, < \rangle$  in which there are two maximal chains  $h_1$  and  $h_2$ , both order-isomorphic to  $\mathbb{R}$ . What does such a model look like? Mathematically, this is equivalent to asking how we can glue together two copies of the real line,  $\mathbb{R}_i = \{ \langle t, i \rangle \mid t \in \mathbb{R} \}$ ,  $i = 1, 2$ , via an equivalence relation  $\sim$  on  $\mathbb{R}_1 \cup \mathbb{R}_2$  such that  $M = \mathbb{R}_1 \cup \mathbb{R}_2 / \sim$ , with the obvious ordering. Let us agree that the branching should happen at 0, so that  $\langle t, 1 \rangle \sim \langle t, 2 \rangle$  for  $t < 0$  but not for  $t > 0$ . There are two possibilities, pointing to two different sensible options for branching topologies: Either (case (a)), we add the condition  $\langle 0, 1 \rangle \sim \langle 0, 2 \rangle$ —in this case, the histories  $h_1$  and  $h_2$  have a maximum  $0 := \{ \langle 0, 1 \rangle, \langle 0, 2 \rangle \}$  in their intersection. Or (case (b)) we demand  $\langle 0, 1 \rangle \not\sim \langle 0, 2 \rangle$ —then, there is an upper bounded chain in  $M$  that has two different minimal upper bounds  $0_i := \{ \langle 0, i \rangle \}$ ,  $i = 1, 2$ . See Figure 3 (a) and (b), respectively. Both of these cases are related to a failure of determinism in the following sense (assuming that the histories after the branching represent physically distinct developments of the system; the mentioned times are arbitrary): Both in case (a) and in case (b), at  $t = -1$ , there is a unique state of the system, but at  $t = 1$ , two different states are possible, corresponding to different moments  $1_1$  and  $1_2$ . So, the system at  $t = -1$  has two alternative possible future developments. The cases are different, however, in that at  $t = 0$ , there is a unique state in case (a), but not in case (b). In case (a), the branching of the system’s development happens immediately after  $t = 0$ , whereas in case (b), the branching has happened already at  $t = 0$ , but at no time prior to  $t = 0$ . Considered as topological spaces,<sup>21</sup> if one imposes the plausible restriction that open subsets of  $\mathbb{R}$  in the copies away from the branching should be open and the resulting space should be connected, one sees that case (a) fails to be locally Euclidean (any open set containing 0 has to contain some initial segment of both tracks), whereas case (b) is a generalized manifold (a locally Euclidean space, in which however the Hausdorff

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<sup>21</sup>For topological notions, see, e.g., (Munkres [2000]).

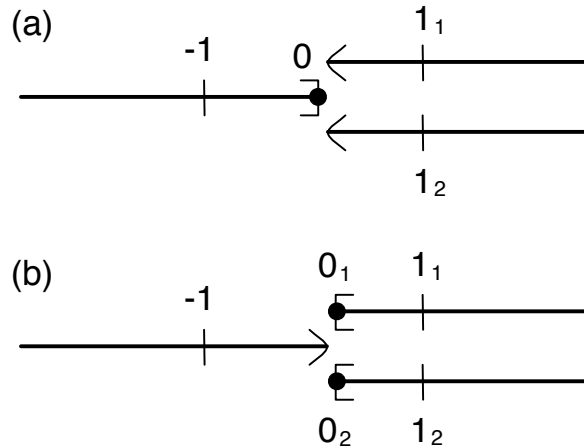


Figure 3: Two types of branching. Both (a) and (b) depict two continuous histories branching at point 0. In (a), point 0 is the shared maximum in the intersection. In (b), the intersection of the histories has no maximum, and points  $0_1$  and  $0_2$  are different minimal upper bounds of the intersection.

condition fails for  $0_1$  and  $0_2$ ).

In terms of the behavior of differential equations, that is, taking the function  $f$  to be provided by solutions to such equations, case (a) represents a situation in which unique initial data can be given for some time  $t$ , but there is no unique local solution immediately after  $t$ . Case (b) represents a different and somewhat trickier affair, since a system of differential equations giving rise to this case need not exhibit any failure of local uniqueness. In Figure 3(b), at any time  $t^- < 0$ , there is a unique solution for at least a little while (e.g., until  $t^-/2$ ), and for the moments  $t_i^+$ , where  $t^+ \geq 0$  and  $i = 1, 2$ , the development along  $h_i$  is unique for all times. Indeterminism reigns, however, in the sense that, like in case (a), the state at any  $t < 0$  does not determine a unique state for all  $t > 0$ . In a discussion of such cases from the point of view of DEQN, it is not local well-posedness (this is satisfied), but the existence of solutions for some longer time, or the extendability of locally unique solutions past some crucial time, that is decisive. A branching representation helpfully allows one to understand indeterminism as a local affair even in case (b): In a topologically obvious sense, the points  $0_1$  and  $0_2$ , which witness the first instance of non-uniqueness, are actually closer than

any two distinct points in case (a); they are non-Hausdorff related, meaning that they cannot be separated by disjoint open sets.

A worry may arise at this point that the DBRN approach permits branching within a system’s history, or that time (or space-time, in a spatiotemporal extension of the framework—see (Belnap [1992])) is not Hausdorff. A physical spacetime without the Hausdorff property has dire consequences, described in (Earman [2008]).<sup>22</sup> The worry, however, is groundless, since in each possible realization, time has the topological structure of  $\mathbb{R}$  (or, respectively, space-time is a Hausdorff manifold). It is just that an indeterministic model represents the different possibilities for a system’s future temporal development in one single structure. A mapping-based representation of the system has to represent exactly the same modal facts; it just brushes under the carpet the topological aspect of the development of the system’s states by using a non-overlapping representation.—For a discussion of topological facts about (in)determinism in space-time theories, as well as for a proof that each spacetime in Belnap’s (1992) theory of branching space-times is Hausdorff, see (Placek and Belnap [2012]), (Müller [2013]), and (Placek, Belnap, and Kishida [2014]).

## 5 Comparing the approaches

In comparing the three approaches, DEQN, DMAP, and DBRN, we will focus on two aspects. First, we would like to learn how the three approaches work in actual cases, that is, how a particular theory (or a system falling under the theory) is assessed through the lens of each approach. To this end, we explore three examples: Norton’s dome (§5.1.1), quantum mechanics (§5.1.2), and general relativity (§5.1.3). We draw some general conclusions from these examples in §5.1.4.

Second, since both DMAP and DBRN offer rigorous definitions of determinism, we will investigate how the respective representations of determinism and indeterminism are formally related. Do these two approaches always deliver the same verdicts with respect to a theory’s determinism or indeterminism? Answers to these questions might cast some light on which of the approaches provides the more adequate analysis of determinism. We turn to a formal comparison of DMAP and DBRN in §5.2.

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<sup>22</sup>The same paper voices the worry that branching implies a failure of the Hausdorff property.

## 5.1 Case studies

### 5.1.1 Norton's dome

Norton's (2008) dome is a system of Newtonian physics whose indeterminism results from a failure of the Lipschitz condition mentioned above in §2. At time  $t = 0$ , a point particle of unit mass is at the apex  $r = 0$  of a dome, whose surface satisfies the constraint  $h = (2/3g)r^{3/2}$ , where  $g$  is the strength of the homogeneous vertical gravitational field,  $r$  is the radial distance from the apex, and  $h$  is the vertical distance from the apex. Since the dome is discussed in the context of point particle mechanics and the mass point is restricted to one dimension, the set of system states is the set of possible radial distances,  $S = \mathbb{R}$ .<sup>23</sup>

Newton's second law yields

$$d^2r/dt^2 = r^{1/2}. \quad (1)$$

Given the initial data  $r(0) = 0, dr/dt(0) = 0$ , eq. (1) has a stationary solution  $r_\infty(t) = 0$  as well as a family of solutions  $r_b$ , parametrized by the real-valued parameter  $b \geq 0$ :

$$r_b(t) = \begin{cases} 0 & \text{if } t \leq b \\ (t - b)^4/144 & \text{if } t > b. \end{cases} \quad (2)$$

Let us look at the three approaches to defining determinism in turn. DEQN offers the most straightforward analysis, delivering the verdict of indeterminism: There is no unique solution to eq. (1) for the initial data. On its own, DEQN does not say how massive this indeterminism is, that is, how many different possibilities there are. There are continuum many functions  $r_b(t)$ ,  $b \in \mathbb{R}_0^+$ , which suggests continuum many possibilities. But on the assumption that the point particle has been located at the apex of the dome forever before  $t = 0$ , all  $r_b$ -type solutions are related by a time-translation symmetry, which suggests that all these solutions represent a single possibility.<sup>24</sup> In that case, there would be exactly two possibilities,  $r_\infty$  and  $r_b$ .

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<sup>23</sup>We stick to this simple choice of  $S$  here in order to keep the exposition simple. Nothing of substance is changed if we take the system's state at a moment to specify, not just the particle's radial distance from the apex ( $S = \mathbb{R}$ ), but that radial distance together with the particle's instantaneous momentum ( $S = \mathbb{R}^2$ ).

<sup>24</sup>It is important that the particle has been at the apex for all  $t < 0$ . If there is some first time  $t_0 < 0$  at which the particle is placed on the apex, the solutions have to satisfy  $r_b(t) = 0$  only for  $t_0 \leq t \leq b$ , and will not be time-translation symmetric. This would speak against counting them as just one possibility.

To apply the DMAP analysis, one needs first to have a grip on the set of realizations. Recall that the canonical form of a realization is based on manifolds (see §3), and observe that the real line is a differentiable manifold. Accordingly, each  $\langle \mathbb{R}, <_{\mathbb{R}}, r_b \rangle$  as well as  $\langle \mathbb{R}, <_{\mathbb{R}}, r_{\infty} \rangle$  is a realization. To decide on determinism, we then ask if realizations with isomorphic initial segments are globally isomorphic. As above, any two  $r_b$ -type realizations are plausibly related by a time-translation  $tr_t : r_b \rightarrow r_{b+t}$ , a clear case of isomorphism. Thus, these realizations are not witnesses for indeterminism. Nevertheless, DMAP delivers the verdict of indeterminism, which is secured by the existence of the  $r_{\infty}$  solution: This solution cannot be derived from any of the  $r_b$  by a time-translation.

DBRN calls for a construction of a branching-style model for Norton's dome. We begin with an auxiliary set  $\tilde{M} := \{\langle t, b \rangle \mid t, b \in \mathbb{R}, b \geq 0\}$ , define the relation  $\approx$  on  $\tilde{M}$  by putting  $\langle t, b \rangle \approx \langle t', b' \rangle$  iff  $t = t'$  and  $(b = b'$  or  $(t \leq b$  and  $t \leq b')$ ) (which is provably reflexive, symmetric, and transitive) and define our base set  $M$  as the quotient structure, (i)  $M := \tilde{M}/\approx$ . We write elements of  $M$  as  $[t, b] := \{\langle t', b' \rangle \in \tilde{M} \mid \langle t', b' \rangle \approx \langle t, b \rangle\}$  and define the relation  $\leq$  on  $M$  as (ii)  $[t, b] \leq [t', b']$  iff  $t < t'$  and  $[t, b] = [t', b']$ . It can be proved that  $\langle M, \leq \rangle$  satisfies the postulates for a tree-like partial ordering.  $\langle M, \leq \rangle$  has a family  $\{h_b\}_{b \in \mathbb{R}_{0, \infty}^+}$  of maximal chains, where  $\mathbb{R}_{0, \infty}^+ = \mathbb{R}^+ \cup \{0\} \cup \{\infty\}$ . We will associate  $h_{\infty}$  with the stationary solution, and  $h_b$ , for  $0 \leq b < \infty$ , with a solution in which the mass point begins to move immediately after time  $t = b$ .

We now construct the actual branching model. The function  $f$  for the model is (iii)

$$f([t, b]) = \begin{cases} 0 & \text{if } t < b \\ (t - b)^4/144 & \text{if } t \geq b \end{cases} \quad (3)$$

Now we can define:

**Definition 3** *The branching model for Norton's dome is the quadruple  $\langle M, \leq, \mathbb{R}, f \rangle$ , where  $M, \leq$ , and  $f$  are defined by conditions (i), (ii), and (iii), respectively.*

Note that the model has a 'stationary' history,  $\langle h_{\infty}, \leq, \mathbb{R}, f|_{h_{\infty}} \rangle$ , representing the mass point remaining stationary on the dome's apex, as well as a family of 'dynamic' histories of the form  $\langle h_b, \leq, \mathbb{R}, f|_{h_b} \rangle$  ( $0 \leq b < \infty$ ), representing the mass point remaining on the apex till time  $b$ , and then moving in accordance

with eq. 2.<sup>25</sup> The model exhibits (a)-type branching, since every intersection  $h_b \cap h_{b'}$  (with  $b \neq b'$ ) has a maximum  $[c, c]$  (where  $c = \min\{b, b'\}$ ). Finally, the model constructed above is faithful in our sense: If two maximal chains  $h_b$  and  $h_{b'}$  branch at a moment, then there is a difference in states assigned to elements of  $h_b$  and of  $h_{b'}$  immediately after that moment. The verdict thus is: Norton's dome is indeterministic.

We draw some morals from the application of DEQN, DMAP, and DBRN to Norton's dome in §5.1.4—after discussing two more cases.

### 5.1.2 Quantum mechanics

With respect to the determinism question, standard quantum mechanics is the odd one out: That theory is based on a very well behaved differential equation (suggesting determinism), but its essential ingredient is a probabilistic algorithm that answers what, and how probable, are the possible results of a measurement (which suggests indeterminism). In order to pass a final verdict about the determinism of quantum mechanics one would thus need to resolve the conflict between these two aspects of the theory (known as the measurement problem).

It is our contention, however, that controversies surrounding determinism of quantum mechanics partially derive from a failure to distinguish between various senses of determinism, as captured in the three approaches, DEQN, DMAP, and DBRN. Without thus proposing a solution to the measurement problem, or any other grand thing, we will sketch how determinism of quantum mechanics is to be analyzed through the lens of each of the three approaches, focusing in particular on a branching-style representation of that theory.

For the DEQN approach, the main fact is the form of the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

which governs the temporal evolution of isolated quantum-mechanical systems. The question then is whether for different Hamiltonians  $\hat{H}$  (which characterize different quantum-mechanical systems) the Schrödinger equation yields a set of differential equations that has a unique solution for each appropriate initial value. There is a wealth of information on this subject

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<sup>25</sup>Note that the model involves continuous branching, since if the mass point is at rest at some  $t \geq 0$ , it can start moving at any later time.

(and related subjects) in the mathematical literature.<sup>26</sup> One class of results points to Newtonian systems for which uniqueness of solutions does not obtain, but the quantum counterparts of which are deterministic: in contrast to the former systems, for the latter systems there exist unique solutions to the ensuing set of equations. Such results suggest that quantum mechanics is more deterministic than classical mechanics. On the other hand, certain Hamiltonians<sup>27</sup> allow for multiple temporal evolutions. It is then a matter of controversy whether those Hamiltonians are physically meaningful. Although this question is pertinent to the issue of determinism of quantum mechanics, it is clear that considerations of the evolution equation alone fail to provide an adequate picture of quantum mechanics—DEQN simply ignores the quantum probabilistic algorithm.

One might hope that DMAP may be more helpful, as it is not restricted to comparing solutions to the Schrödinger equation: Its data for comparison are the theory’s realizations, and there seems to be no obstacle to include in the latter other entities referred to by quantum mechanics, in particular, measurement results. There is, however, virtually no DMAP-style analysis of quantum mechanics, although a Montague-style language-based notion of realization seems to be readily available.

In contrast to the above mentioned modally-flat notion of realization, DBRN calls for producing branching models for quantum mechanics. There is a formalism for standard QM, the so-called consistent histories approach ((Gell-Mann and Hartle [1993]), (Griffiths [2002])), that explicitly employs a branching time representation. A *family of histories* is a set  $\{Y^h \mid h \in H\}$  of mutually exclusive histories (chains of projectors) of the form<sup>28</sup>

$$Y^h = P_h^1 \odot P_h^2 \odot \dots \odot P_h^m,$$

where at each chosen time  $t_i$ ,  $i = 1, \dots, m$ , the family contains projectors  $P_h^i$  from an exhaustive set of  $n_i$  mutually exclusive projectors  $P_1^i, \dots, P_{n_i}^i$ ,

$$P_j^i \cdot P_k^i = \delta_{jk} P_j^i; \quad \sum_{j=1}^{n_i} P_j^i = I.$$

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<sup>26</sup>A good introduction to this field is (Earman [1986]). For a more mathematically advanced treatment, see (Earman [2006]).

<sup>27</sup>Those that admit multiple self-adjoint extensions; cf. (Earman [2006], 1401).

<sup>28</sup>A more general setup in terms of POVMs is possible; see, e.g., (Peres [2000]). See also (Müller [2007]) for more details about the relation of consistent histories and branching time representations.



A consistency condition constrains the set of admissible families of histories. Such a family of histories directly specifies a model in the sense of Definition 2, and when such a family has more than one member, that model is indeterministic. Topologically, the indeterminism is of the case-(b) variety, as *at the times* chosen, different projectors are assigned. This assignment of different projectors also ensures that the branching model is faithful in our sense.

The formalism can be, and usually is, extended to allow for a branch-dependent selection of the set of projectors at any given moment. This also leads to a faithful branching model in our sense, and the same criterion for (in)determinism applies: once a family contains more than one history, it is indeterministic. Note that the formalism allows one to assign probabilities for the members of a branching family in agreement with experimental results.

### 5.1.3 General relativity

In philosophical discussions of the determinism of general relativity (GR), the DMAP approach is the most prominent one. As the theory has realizations of the required form (differentiable manifolds), and a notion of isomorphism for segments of such models can be rigorously defined (via diffeomorphisms), GR is amenable to the DMAP analysis. Since the late 1980's the DMAP analysis has been applied to GR in a particular way in the philosophy of physics, having been driven by Einstein's hole argument.<sup>29</sup> This argument is an appeal to a special transformation on manifolds, the so-called hole diffeomorphism, which is used to produce a different manifold out of a given one, with the resulting pair of manifolds (seemingly) witnessing indeterminism of GR. Importantly, all parties in the debate were unanimous that this phenomenon does not show indeterminism in GR. The consensus is rather that the argument raises the question whether two differential manifolds related by a diffeomorphism represent one physical spacetime or two.

In contrast to philosophers' uses of the DMAP analysis in the hole argument, physicists concerned with the (in)determinism of GR appeal to the DEQN analysis. They investigate whether or not GR admits a globally well-posed initial value problem, that is, whether the data on an appropriate space-like slice of a spacetime can be uniquely extended to that spacetime. This question is typical of the DEQN approach.<sup>30</sup> In this section we investi-

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<sup>29</sup>See (Earman and Norton [1987]) and (Butterfield [1989]).

<sup>30</sup>Since physicists see the well-posedness of the initial value problem as a criterion any

gate whether the practitioners’ approach to the initial value problem exhibits similarities to DBRN characterization of determinism, and whether these can be used to construct branching models for GR.

The basic context of the practitioners’ debate on the initial value problem in GR is provided by so-called globally hyperbolic spacetimes.<sup>31</sup> Such spacetimes admit a global time function; surfaces of constant value of such a function are Cauchy surfaces. Thus, for a globally hyperbolic spacetime we are in the familiar setting of a global time with momentary states (identified with Cauchy surfaces with appropriate data on them).<sup>32</sup> In this context the starting point of both the determinism issue and the initial value problem is the question of whether a 3-dimensional Riemannian manifold with appropriate data on it fixes a unique 4-dimensional Lorentzian manifold of a specified kind. ‘A specified kind’ refers here to a particular form of Einstein’s field equations, depending on whether a spacetime includes matter, then what the model for matter is, and whether the equations include a non-zero cosmic constant  $\Lambda$ , or not. These decisions are also highly relevant to what the appropriate initial data (3-dimensional Riemannian manifolds with some objects on them) are.

We consider here only globally hyperbolic spacetimes that are vacuum solutions of the Einstein equations, that is, for which the Ricci curvature tensor  $R_{\alpha\beta} = 0$ . In this case the initial data are triples  $\langle \mathcal{M}, g, k \rangle$ , where  $\mathcal{M}$  is a 3-dimensional manifold,  $g$  is a Riemannian metric, and  $k$  a symmetric covariant tensor (coding incremental changes of  $g$  in the direction normal to the manifold). Further, for  $\langle \mathcal{M}, g, k \rangle$  to be embeddable in a globally hyperbolic spacetime satisfying Einstein’s equations, it should satisfy certain equations, called initial value constraints. Triples  $\langle \mathcal{M}, g, k \rangle$  satisfying these constraints are called ‘vacuum data sets’.

The following theorem (Choquet-Bruhat and Geroch [1969]) is relevant to whether or not GR is deterministic in the vacuum case:

Let  $\langle \mathcal{M}, g, k \rangle$  be a vacuum data set. Then there is a unique, up to isometry, maximal vacuum Cauchy development (MVCD) of  $\langle \mathcal{M}, g, k \rangle$ .

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good theory should satisfy, their interest is biased toward determinism. It is thus striking to see them concede that in some cases GR is indeterministic.

<sup>31</sup>A spacetime is globally hyperbolic if it admits a Cauchy surface—for further pertinent definitions see (Wald [1984], 200ff.).

<sup>32</sup>Another determinism-friendly feature of a globally hyperbolic spacetime is that the domain of dependence of a Cauchy surface is, by definition, identical to the whole spacetime, which, roughly speaking, excludes influences coming from nowhere.

To explain, a vacuum Cauchy development of  $\langle \mathfrak{M}, g, k \rangle$  is a globally hyperbolic 4-dimensional Lorentzian spacetime in which  $\langle \mathfrak{M}, g, k \rangle$  is embeddable. Two spacetimes  $\langle \mathfrak{M}, g_{\alpha\beta} \rangle$  and  $\langle \mathfrak{M}', g'_{\alpha\beta} \rangle$  are isometric if there is a diffeomorphism  $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}'$  such that  $\varphi^*(g_{\alpha\beta}) = g'_{\alpha\beta}$  (it is not required that other objects be dragged along by  $\varphi$ ).

Theorems similar to the one quoted above hold for the Einstein equations with other data.<sup>33</sup> From our perspective, the interesting point is that these theorems do not prohibit a maximal Cauchy development of an initial data set from having more than one non-isometric extension—the theorems only prohibit these extensions from being globally hyperbolic. A case in point is provided by the so-called polarized Gowdy spacetime. This is a globally hyperbolic spacetime, defined for a restricted set of values of one coordinate, a vacuum solution to the Einstein equations, and an MVCD of an appropriate initial data set. When this spacetime is extended for the full range of the coordinate, some of its maximal extensions turn out not to be isometric (Chruściel and Isenberg [1993], 1623ff). These non-isometric extensions might be viewed as possible histories of a faithful indeterministic model, yielding the verdict that GR is indeterministic.<sup>34</sup>

The issue is, however, complicated. The mentioned non-isometric maximal extensions of the polarized Gowdy spacetime do not admit global time functions and contain closed timelike curves (CTCs). This spells trouble for any definition of determinism based on partial orderings, since there is no natural antisymmetric ordering on a CTC. Accordingly, the notion of alternatives for the future, which is basic to the core idea of indeterminism, makes sense only locally, but not globally any more.<sup>35</sup>

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<sup>33</sup>Notably, (Ringström [2009], Thm. 16.6) proves the existence of a maximal globally hyperbolic development of the data for a specific model with matter, the so-called non-linear scalar field model (*ibid.*, 147). In this case the initial data sets are different from their counterparts in the vacuum solution case, and embeddability applies to the matter field as well.

<sup>34</sup>Non-isometric extensions of a maximal Cauchy development are deemed non-generic by the Strong Cosmic Censorship Conjecture. In this spirit, (Chruściel and Isenberg [1993]) prove that non-isometric extensions of a polarized Gowdy spacetime are rare, in a measure-theoretical sense, in the set of all extensions of that spacetime. As there is little ground to equate ‘rare’ with ‘non-physical’, the example cannot be discounted easily. For a discussion of the Strong Cosmic Censorship Conjecture in the context of polarized Gowdy spacetimes, see (Chruściel et al. [1990]).

<sup>35</sup>This calls for spelling out our Definition 2 of indeterminism in terms of modal forks, that is: ‘A model is indeterministic iff it contains at least one modal fork’; for the definition

Certainly more work is needed to fully develop the core idea of indeterminism with respect to spacetimes admitting CTCs. With respect to our approach laid out in §4, we note the following. The main definition of the DBRN approach, Def. 1, is still adequate due to its abstract nature. It will, however, be necessary to extend the definition of an indeterministic model (Def. 2) such that spacetimes admitting CTCs are covered as well, by taking local alternatives into account.<sup>36</sup> In parallel to this development, it will be necessary to scrutinize the arguments of the practitioners. For example, invoking the DEQN approach, it is claimed that ‘[t]he fact that there are inequivalent maximal extensions means that the initial data do not uniquely determine a maximal development. In this sense, the general theory of relativity is not deterministic’ (Ringström [2009], 18). Depending on details of the definition, DMAP and DBRN may here come to opposing verdicts; see our discussion of the interrelation of the two approaches in §5.2 below.

#### 5.1.4 Morals from the applications

We have illustrated above how the three approaches to determinism fare in analyzing the theories of Newtonian mechanics, non-relativistic quantum mechanics, and general relativity. Our first observation is that it is DEQN that is mostly used in the cases considered (and also, in actual discussions among practitioners). The existence of non-unique solutions to Newton’s equations in Norton’s dome indicates indeterminism; in a similar vein, the fact that Einstein’s field equations allow for non-isometric extensions to a maximal vacuum Cauchy development counts against determinism of general relativity. The behavior of the Schrödinger equation, however, does not account for the general sentiment that quantum mechanics is an indeterministic theory: In this case, the DEQN approach is severely limited, as it does not accommodate the quantum measurement algorithm.

Second, the DMAP characterization of determinism does not seem to be really used. That is, in cases like that of Norton’s dome, the construction of realizations needed by the DMAP analysis is straightforward and completely relies on solutions to the theory’s defining equations. The construction does

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of modal forks, see (Placek et al. [2014], 423). The two formulations coincide in the context of branching time, as any two histories in BT form such a fork. The formulation in terms of modal forks is, however, also applicable in the context of more complex branching theories in which histories are not linearly ordered.

<sup>36</sup>A theory of this sort is developed in (Placek [2014]).

not add any new value to what is achieved by the DEQN analysis. In the case of quantum mechanics, DMAP looks promising, as it offers a chance to account for measurement results, apart from the Schrödinger evolution. However, that promise of a DMAP analysis of quantum mechanics has never been fulfilled, as far as we know.

Third, approaching the question of determinism of GR from the perspective of the initial value problem refers one to the DEQN approach. Non-isometric spacetimes that witness indeterminism in GR might then be viewed as diverging realizations according to the DMAP definition. Since these spacetimes contain CTCs, a full DBRN analysis will need to be based on an extended notion of an indeterministic spacetime model.

Finally, we have seen that the branching analysis of determinism comes naturally—a DBRN-style representation is often quite literally out there. For quantum mechanics, DBRN-style models are immediately available, in the formalism of quantum histories. For other cases, such models need to be constructed. We showed such an explicit construction for Norton’s dome. These constructions are quite natural, and we conclude that the the notion of an indeterministic DBRN model supplies a useful representation of a theory’s indeterminism.

## 5.2 Formal comparison of the the DMAP and DBRN frameworks

We now turn to a formal comparison of the DMAP and DBRN frameworks, in order to find out about their interrelation. The comparison will be at the level of a single system falling under the theory in question, which means that we are treating the set of the system’s states,  $S$ , as fixed and given.<sup>37</sup> We will compare formal mapping and branching representations of the system’s dynamics, using the following data format:

A mapping representation of the system’s dynamics is a pair

$$\mathfrak{M} = \langle (\mathfrak{M}_j)_{j \in J}, A \rangle; \quad \mathfrak{M}_j = \langle T_j, \langle_j, f_j \rangle, \rangle,$$

with  $J$  some index set. Here, the  $\mathfrak{M}_j$  are the realizations characterizing the system: any  $\langle T_j, \langle_j \rangle$  is a linear ordering of times (typically,  $\langle \mathbb{R}, \langle_{\mathbb{R}} \rangle$ ), and

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<sup>37</sup>Given an assessment of single systems as deterministic or indeterministic, the verdict transfers immediately to the theory itself: A theory is indeterministic iff there is at least one indeterministic system falling under it.

$f_j : T_j \mapsto S$  is a specification of system states for times  $t \in T_j$ . Furthermore,  $A$  is a class of isomorphisms between realizations, allowing for the fact that different mathematical structures may represent the same physics (see our discussion of gauge transformations in §3 above.). Technically, each  $\alpha \in A$  is a mapping between realizations that preserves their structure, which means that it specifies an order-preserving bijection identifying the times across different realizations, and it maps corresponding system states onto physically equivalent system states. In line with typical considerations in physics, we will assume that the set of isomorphisms  $A$  has the structure of a group, i.e., elements of  $A$  can be combined such that (i) there is a neutral element (the identity mapping,  $id$ ), (ii) each element  $\alpha \in A$  has an inverse  $\alpha^{-1} \in A$  for which  $\alpha\alpha^{-1} = \alpha^{-1}\alpha = id$ , and (iii) composition of elements is associative, that is,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ . As we will see, for the DMAP approach it is crucial to identify the right set  $A$  of isomorphisms; the verdict as to a theory's determinism depends sensitively upon the choice of  $A$ .

A branching representation of the system's possible developments has the form

$$\mathfrak{B} = \langle (\mathfrak{B}_i)_{i \in I}, A \rangle; \quad \mathfrak{B}_i = \langle B_i, <_i, f_i \rangle,$$

with  $I$  some index set. The  $\mathfrak{B}_i$  are the faithful branching models of the system's development, in which  $\langle B_i, <_i \rangle$  is a branching (tree-like) partial ordering (see §4.1) and  $f_i : B_i \mapsto S$  is a specification of system states for moments  $m \in B_i$ . The set  $A$  is a group of isomorphisms between branching models: if such models  $\mathfrak{B}_k$  and  $\mathfrak{B}_l$  are connected by some  $\alpha \in A$ , this means that they represent the same physics. Similarly as in the DMAP case, such  $\alpha \in A$  thus specifies both an order-preserving bijection between the sets of moments  $B_k$  and  $B_l$ , and a mapping of physically equivalent states. Observe however that in contrast to the DMAP case,  $\alpha \in A$  relates (typically) non-linear models. Derivatively  $\alpha$  specifies isomorphisms between linear realizations as well, which can be used to check a model's faithfulness (see below). In this sense, two realizations  $\langle h_1, <_{|h_1}, S, f_{|h_1} \rangle$  and  $\langle h_2, <_{|h_2}, S, f_{|h_2} \rangle$  that belong to branching models  $\mathfrak{B}_1 = \langle B_1, <_1, f_1 \rangle$  and  $\mathfrak{B}_2 = \langle B_2, <_2, f_2 \rangle$ , respectively, are isomorphic if for some  $\alpha \in A$ :

$$\alpha_{|h_1}(\langle h_1, <_{|h_1}, S, f_{|h_1} \rangle) = \langle h_2, <_{|h_2}, S, f_{|h_2} \rangle,$$

where  $h_i$  are maximal chains in  $\langle B_i, <_i \rangle$ . We will say that an isomorphism  $\alpha_h$  restricted to a maximal chain  $h$  of some branching model is a *linearization* of  $\alpha$ . There is a certain subtlety concerning the issue where the linearizations

come from. Clearly, the set  $A$  contains automorphisms among branching models (minimally, the identity), and these give rise to linearizations (one takes  $\mathfrak{B}_1 = \mathfrak{B}_2$  in the formula above.) But the linearizations can also be derived from isomorphisms reaching across different branching models (take  $\mathfrak{B}_1 \neq \mathfrak{B}_2$  above). (These linear mappings are partial automorphisms within a branching model, but they are not derived from an automorphism, but from an isomorphism between different models.) Faithfulness of a branching model is assessed by both sorts of isomorphisms between linear realizations. A branching model  $\mathfrak{B}_i$  will be declared unfaithful if it has two maximal chains  $h_1$  and  $h_2$  such that  $\alpha_{|h_1}(\langle h_1, <_{|h_1}, S, f_{|h_1} \rangle) = \langle h_2, <_{|h_2}, S, f_{|h_2} \rangle$  for some  $\alpha \in A$ . Observe that this isomorphism-based assessment uses a weaker notion of faithfulness, which does not require a difference immediately after branching—it only requires absence of total isomorphism of linear realizations (compare §4.2 above). This weaker constraint will make it easier to meet the demand of deriving a branching representation from a mapping representation (see below).

Given the above data structures, our definitions of indeterminism (and thereby, of determinism as indeterminism’s negation) take the following form:

**DBRN.** A system with faithful branching representation  $\mathfrak{B}$  is indeterministic iff there is some  $\mathfrak{B}_i$ ,  $i \in I$ , for which there are  $m, m' \in M_i$ ,  $m \neq m'$ , such that neither  $m <_i m'$  nor  $m' <_i m$ . (I.e., indeterminism corresponds to there being a non-linear, branching structure among the  $\mathfrak{B}_i$ .)

Note that the set of isomorphisms  $A$  plays no direct role in the assessment of a theory’s determinism according to this recipe. We did, however, have to assume that the branching representation was faithful, and as discussed above, linearizations of the isomorphisms in  $A$  provide a security check for faithfulness.

**DMAP.** In terms of the DMAP approach, the core idea of determinism translates into the thought that agreement of two realizations up to some time implies their total agreement.<sup>38</sup> Thus, indeterminism means that there are two realizations that agree up to some time, but disagree later on. Here, ‘agreement’—both with respect to states and with respect to times—has to

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<sup>38</sup>This idea is somewhat more general than the idea that agreement *at some time*, or *in some small region around some time*, should imply global agreement. Our choice makes the DMAP/DBRN comparison somewhat more transparent.

be spelled out in terms of isomorphisms. In line with (Butterfield [2005]), the definition is as follows: A system with mapping representation  $\mathfrak{M}$  is indeterministic iff there are realizations  $\mathfrak{M}_k = \langle T_k, <_k, f_k \rangle$ ,  $\mathfrak{M}_l = \langle T_l, <_l, f_l \rangle$ ,  $k, l \in J$ , for which there is some  $t_0 \in T_k$  and some  $\alpha \in A$  such that

- for all  $t \leq t_0$ ,  $f_k(t) = (\alpha f_l)(t)$ , i.e., the states on an initial segment can be identified, but
- there is no  $\beta \in A$  mapping  $\mathfrak{M}_k$  wholly onto  $\mathfrak{M}_l$ , i.e., no isomorphism  $\beta \in A$  for which  $f_k = \beta f_l$ .<sup>39</sup>

It should be clear from the form of the definition that the choice of  $A$  matters greatly. Minimally, the set of isomorphisms has to contain the identity, but it is a difficult matter to decide which other mappings are to be included for a given system. The verdict about determinism can depend on that choice.

**Comparing DMAP and DBRN.** We now move to our main task, which is to establish whether the verdicts as to a system's determinism delivered by DMAP and by DBRN agree or not. Technically, we will tackle these questions by describing how to derive a mapping representation from a branching representation and conversely, and then checking the verdicts as to determinism.

**DBRN  $\rightarrow$  DMAP.** Starting with a given branching model  $\mathfrak{B} = \langle (\mathfrak{B}_i)_{i \in I}, A \rangle$ , we derive the corresponding mapping representation in three steps: For each  $\mathfrak{B}_i, i \in I$ , that is, for each individual branching model, we extract the linear realizations, lump these together, and derive the appropriate set of isomorphisms between realizations from the given  $A$ . In more detail: For  $i \in I$ , the branching model  $\mathfrak{B}_i = \langle B_i, <_i, f_i \rangle$  has histories  $h_i^k, k \in J_i$ , where  $J_i$  is an index set enumerating the histories in branching model  $\mathfrak{B}_i$ . We individuate these histories as a set of realizations by restriction:

$$\mathfrak{C}_i = \{ \langle h_i^k, <_{i|h_i^k}, f_{i|h_i^k} \rangle \mid k \in J_i \}.$$

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<sup>39</sup>Our notation,  $\alpha f_l$  or  $\beta f_l$ , indicates that an isomorphism  $\alpha \in A$  maps a function  $f_l$  to a function  $\alpha f_l$ , thereby taking care of two things in accordance with our discussion above: mapping of times and mapping of states.



We construct the set  $A_{\mathfrak{M}}$  of isomorphisms by restricting the isomorphisms  $\alpha \in A$  to linear realizations:

$$A_{\mathfrak{M}} = \{\alpha' \mid \alpha' \text{ a linearization of some } \alpha \in A\}.$$

Finally, we collect all the linear realizations in one set  $\cup_{i \in I} \mathfrak{C}_i$ , arriving at the mapping structure

$$\mathfrak{M} = \langle \cup_{i \in I} \mathfrak{C}_i, A_{\mathfrak{M}} \rangle.$$

Let us now consider how the verdict of determinism or indeterminism for the branching structure  $\mathfrak{B}$  fares with respect to the derived mapping structure  $\mathfrak{M}$ .

If  $\mathfrak{B}$  is indeterministic, then  $\mathfrak{M}$  will be diagnosed as indeterministic as well. This can be seen as follows: Indeterminism of  $\mathfrak{B}$  means that there is some  $\mathfrak{B}_i$  containing histories  $h_i^k, h_i^l$  that are not globally isomorphic (by faithfulness). These two histories reappear, by construction, as realizations  $\mathfrak{M}_k = \langle T_k, <_k, f_k \rangle$  and  $\mathfrak{M}_l = \langle T_l, <_l, f_l \rangle$  in  $\mathfrak{M}$ . We can show that these realizations provide a witness of indeterminism in the mapping sense. As (by the definition of a branching model)  $h_i^k \cap h_i^l \neq \emptyset$ , there is some  $t_0 \in T_k$  such that  $f_k(t') = f_l(t')$  for all  $t' \leq t_0$ . So, the realizations  $\mathfrak{M}_k$  and  $\mathfrak{M}_l$  have an isomorphic initial segment (using the identity as isomorphism, which belongs to  $A_{\mathfrak{M}}$  by construction). But these realizations are not globally isomorphic, since that would contradict the faithfulness of the original branching representation  $\mathfrak{B}$ .

If  $\mathfrak{B}$  is deterministic, the verdict as to the derived  $\mathfrak{M}$  need not coincide, depending on how much information about isomorphisms is given through  $A$ . As stated above, for branching structures the set  $A$  plays a double role. On the one hand,  $A$  can provide information about the global fact that two different branching models picture the same physical situation, because (for instance) they correspond to a different choice of gauge. On the other hand  $A$  gives rise to a set of linearized isomorphisms; this latter set provides a local criterion for faithfulness of an individual branching model  $\mathfrak{B}_i$  from  $\mathfrak{B}$ : no two histories within one such branching model may be globally isomorphic. (Note that any two of them are, by overlap of histories, isomorphic on an initial segment, with the isomorphism provided by the identity.)

A deterministic  $\mathfrak{B}$  means that all branching models  $\mathfrak{B}_i$  are in fact linear, i.e., contain just a single history. Thus, each  $\mathfrak{B}_i$  already has the mathematical structure of a realization  $\mathfrak{M}_i$ . Following exactly the same procedure as in the

indeterministic case described above, we arrive at the mapping structure

$$\mathfrak{M} = \langle (\mathfrak{B}_i)_{i \in I}, A_{\mathfrak{M}} \rangle.$$

Whether this  $\mathfrak{M}$  is judged to be deterministic or indeterministic, now depends on whether  $A_{\mathfrak{M}}$  gives rise to partial but not global isomorphisms. To illustrate how the verdicts could diverge, consider the case of electromagnetism described in §3 above. A branching representation for a system falling under that theory will contain only linear models. The different models correspond to different initial conditions, or to different choices of gauge, or both. The verdict on this representation will be determinism—after all, all branching models are linear, there is no case of branching. This holds even if the gauge transformations are excluded from the set of isomorphisms (for instance, if  $A$  contains just the identity). In that case, the derived mapping representation will, however, be judged to be indeterministic: Some realizations will agree initially, but diverge later, due to a difference in gauge. The set  $A_{\mathfrak{M}}$  will be too small to capture the fact that these realizations picture the same physics by different mathematical means. This is exactly the dialectics of diagnosing gauge freedom via spurious indeterminism described in §3 above. So we see that in order to have a reliable verdict of determinism in the mapping representation, care needs to be taken to correctly identify all the physical isomorphisms for the system in question. There is no formal procedure for that step.

**DMAP**  $\rightarrow$  **DBRN**. Let us now consider how the verdict of determinism or indeterminism for a mapping structure  $\mathfrak{M}$  fares with respect to a derived branching structure  $\mathfrak{B}$ . It turns out that the construction is somewhat involved and not unique, but the verdicts agree.

We will derive a branching representation from a mapping representation by successively constructing branching models from appropriate sets of realizations. In such a set, any two realizations must be partially isomorphic, but not globally isomorphic. More formally, let us call a subset  $\{\mathfrak{M}_k \mid k \in K\}$  of the set of realizations *good* iff for any two  $k, l \in K$ ,  $k \neq l$ , there is some  $t_0 \in T_k$  and some  $\alpha \in A$  such that

- for all  $t' \leq t_0$ ,  $f_k(t') = (\alpha f_l)(t')$ , i.e., the states on an initial segment can be identified, but
- there is no  $\beta \in A$  mapping  $\mathfrak{M}_k$  wholly onto  $\mathfrak{M}_l$ , i.e., no isomorphism  $\beta \in A$  for which  $f_k = \beta f_l$ .

By Zorn's lemma, there are maximal good sets (note that a singleton set is good), and the whole set of realizations of the given mapping representation can be partitioned into good sets: Successively take out a maximal good set and identify the next maximal good set in what remains. Note, however, that this partitioning is not unique.<sup>40</sup>

For any resulting good set  $\{\mathfrak{M}_k \mid k \in K\}$  we can derive a branching model in the following way.<sup>41</sup> We pick one realization  $\mathfrak{M}_0 = \langle T_0, <_0, f_0 \rangle$ , which will form a reference history in the resulting branching model. By goodness, for every other realization  $\mathfrak{M}_k$  we can pick an isomorphism  $\alpha_k \in A$  that identifies  $\mathfrak{M}_0$  and  $\mathfrak{M}_k$  up to some  $t_k \in T_0$  (but not for all times). Call the mapped realization

$$\mathfrak{M}'_k = \langle T'_k, <'_k, f'_k \rangle := \langle \alpha_k(T_k), \alpha_k(<_k), \alpha_k(f_k) \rangle.$$

So, for all  $k \in K - \{0\}$  we have

$$f_0(t') = f'_k(t') \quad \text{for all } t' \leq t_k.$$

Exactly as in the construction for Norton's dome in §5.1.1 above, we now define a base set and derive the branching model by dividing out an equivalence relation. To save some ink, we set  $T'_0 := T_0$ ,  $<'_0 := <_0$ ,  $f'_0 := f_0$ . We set

$$\tilde{B} := \bigcup_{k \in K} (T'_k \times \{k\}).$$

For the equivalence relation  $\approx$ , we set

$$\langle t, n \rangle \approx \langle s, m \rangle \quad \text{iff} \quad t = s \quad \text{and for all } t' \leq t \text{ we have } f'_n(t') = f'_m(t').$$

Next, our set of moments (the base set for the partial branching order) is

$$B := \tilde{B} / \approx,$$

and we define the state-assignment function  $f$  on  $B$  to be

$$f([\langle t, n \rangle]) = f'_n(t).$$

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<sup>40</sup>This is connected to the fact that the relation of being partially but not globally isomorphic is not transitive, i.e., it is not an equivalence relation.

<sup>41</sup>The following construction is not unique, and it is not guaranteed to deliver branching models that are intuitively satisfying. The construction does, however, fulfill all formal requirements.

Note that the definition of  $\approx$  guarantees well-definedness, meaning that for  $\langle s, m \rangle \in [\langle t, n \rangle]$  we have  $f'_n(t) = f'_m(s)$ . It remains to define the partial ordering  $<$  on the set of moments  $B$ :

$$[\langle t, n \rangle] < [\langle s, m \rangle] \text{ iff } t <'_n s \text{ and for all } t' \leq'_n t \text{ we have } f'_n(t') = f'_m(t').$$

Pulling things together, adding all these branching models for all the good sets into which the given set of realizations was partitioned, will give a full branching representation for the system in question. It remains to specify the set  $A_{\mathfrak{B}}$  of isomorphisms. For the verdict of determinism or indeterminism, this subtle issue is, however, not important, so that we can set  $A_{\mathfrak{B}}$  to contain just the identity.

It is easy to see that by the given construction, a verdict as to determinism or indeterminism of the mapping structure is retained in the branching structure. In the case of determinism, all good sets are singletons, giving rise to only linear, deterministic branching models. In the case of indeterminism, there will be at least one non-trivial good set, giving rise to a faithful branching model with at least two histories. Such a non-linear structure triggers the verdict of indeterminism.

**Summing up.** As we saw, the relations between two representations of determinism, as offered by DMAP and DBRN, are somewhat intricate. Their verdicts with respect to a system's determinism usually agree, but not always. That is, if DBRN diagnoses a system as indeterministic, DMAP will concur. However, if DBRN's verdict is 'determinism', a DMAP analysis might disagree. In the opposite direction verdicts agree, that is, if DMAP deems a system as deterministic / indeterministic, DBRN will come with the same diagnosis. Although the two approaches agree on the verdict in such cases, they might view the underlying details differently, as the DMAP representation of determinism (or indeterminism) does not translate uniquely into a DBRN representation of determinism (or indeterminism). This last subtlety and a possible divergence of verdicts about determinism derives from the different roles the set of isomorphisms plays in the two approaches, and from the status of the faithfulness assumption.

The dialectics here is as follows. Given a physical system in whose determinism or indeterminism we are interested, we need to construct a mathematical representation with respect to which we can study the question of determinism in a formally precise way—that is the overarching framing of

the determinism issue in philosophy of science. The three approaches under discussion here differ with respect to their mathematical representation of a system. We have pointed out that DEQN makes good sense in most, but not all applications, quantum mechanics being a notable exception. Here we have considered the interrelation between the two remaining approaches, DMAP and DBRN. In our discussion of case studies in §5.1 we have seen that the actual construction of DBRN models typically leads to models whose faithfulness is guaranteed. In the case of Norton's dome, the behavior of the differential equation secured the necessary difference in physical state (here, position of the particle); in quantum mechanics, the assignment of different projectors in different histories did the same. A set of isomorphisms between branching models, which we have included in our formal description in this section, can be helpful to provide a more adequate picture, showing that two different branching models may depict the same physical situation. This, however, does not affect the verdict as to determinism or indeterminism, which is based on the ordering structure of the individual branching models whose faithfulness is assured beforehand. Our discussion of the DMAP approach, on the other hand, shows that the choice of the set of isomorphisms is crucial for the assessment as to determinism or indeterminism. Trouble can arise if too few isomorphisms are identified, since then, a spurious assessment of indeterminism threatens. There is nothing in the construction of a DMAP representation of a system that secures the identification of a physically adequate group of isomorphisms.

Our diagnosis of this state of things is as follows. Branching is the natural representation of indeterminism—we directly understand a non-linear branching structure as representing an indeterministic scenario. Typically, an investigated problem contains information about grouping of realizations into sets of alternative mutually possible developments: this information comes in a statement of an initial value problem, or a system's symmetries, or even similarities of processes considered. This information, and the partition of realizations it affords, is lost, or is not being used, in the mapping-based account. Instead, a lot of mathematical surplus structure needs to be considered. The worry is that in the end, getting this surplus structure right will only be possible in case we have a different underlying representation (for example, a branching-based representation) that anchors our assessment in the first place.

## 6 Conclusions

Our aim in this paper was to elaborate formally the core idea of determinism according to which a deterministic system has no alternative future developments, and to apply the resulting framework to theories of physics. That framework is based on branching theories that are well known in tense logic; a novelty of this paper consists in showing how to construct branching models for theories of physics. A salient feature of branching models is their modal thickness: a single model of that sort has resources to represent alternative possible evolutions of a system. Technically speaking, a branching model may contain more than one history (maximal chain), in which case we call it ‘indeterministic’; otherwise we call it ‘deterministic’. We say that a theory is indeterministic if it has at least one faithful indeterministic model; otherwise, the theory is deterministic. This topic of a branching-style analysis of determinism of theories was discussed in §4.

To locate our analysis with respect to extant debates on determinism, in §2 we singled out three styles of thinking about determinism: DEQN characterizes determinism in terms of solutions to a theory’s defining equations, DMAP proceeds in terms of mappings between linear temporal realizations admitted by the theory, and DBRN uses the concept of alternative possible future continuations. In §3 we focused on the dominant approach to determinism in current philosophy of science, as exemplified in writings of Earman and Butterfield. We pointed out that both these authors advocate a position that combines the DMAP and DEQN approaches; this combination seems, however, to call for further elaboration, as no proof of the equivalence of (or of other logical relations between) DMAP and DEQN is known. Another critical issue is the orthodoxy’s appeal to the notion of isomorphism, which (we claim) is used rather loosely, since a theory’s models are not required to be models in the logical sense. Two further troubling issues, which arise in a similar way for other approaches, are the identification of times across different realizations, and the possible difference between mathematical and physical states.

Having provided a detailed exposition of the DBRN framework in §4, in section §5.1 we compared the three approaches with respect to how they apply to particular cases. Focusing on Norton’s dome and quantum mechanics, we found (perhaps surprisingly) that the DMAP analysis is not used in the literature. In contrast, the formalism of consistent histories for quantum mechanics is immediately translatable into branching models. Branching

models also provide a natural representation of Norton’s dome. By adding some extra structure these branching models can be transformed into DMAP models, but no extra value appears to be provided by such a move. For General Relativity, the core idea of indeterminism becomes problematic in the absence of a global time function. All three approaches need to be based on the notion of a local alternative for the future. This is natural for the DEQN approach; DMAP and DBRN may come to diverging conclusions. This highlights the fact that these approaches are closely related, but not equivalent.

Accordingly, in §5.2 we investigated the formal interrelations between the DMAP and DBRN definitions of determinism. The comparison highlights that DMAP models need some extra structure as compared to DBRN models. This extra structure (coded in a set of isomorphisms) reflects two decisions: which times across realizations should be identified and which mathematical states represent the same physical state. We described the representations of determinism offered by DMAP and DBRN, respectively, and showed how one representation can be derived from the other, noting that there is no uniqueness in the construction of a DBRN representation out of a DMAP representation. We also showed that although in most cases the verdicts of the two representations of determinism agree, a divergence is possible: a DBRN verdict of indeterminism might be rejected in the DMAP approach.

These discrepancies are a consequence of the different role isomorphisms play in the two representations of determinism, and we believe these different roles cut to the bone of the controversy between the two approaches. Informally speaking, whether a system is deterministic or not, depends on whether it has a possible development to which there is a true alternative. For instance, multiple developments from shared initial conditions are true alternatives, but multiple developments produced by exercising gauge freedom, or freedom of coordinate choice, are not true alternatives. Accordingly, any good analysis of determinism requires a way of partitioning a set of realizations into subsets of ‘truly alternative’ realizations. In DBRN this is achieved by lumping subsets of realizations into tree-like branching models, whereas in DMAP a similar effect is simulated via a set of isomorphisms. Now, a typical question of determinism contains information about partitioning a set of realizations into subsets of truly alternative realizations. This kind of information is directly used in the construction of a DBRN model; in contrast, it is not directly used in the mapping-based account. Instead, a lot of mathematical surplus structure is postulated to derive a partition of

realizations mimicking that of the DBRN approach. Whether that surplus structure is of the right sort, appears to be decided by consulting the information utilized in the construction of branching models. This, we believe, tells strongly in favor of the greater simplicity and conceptual primacy of the DBRN approach relative to the DMAP approach.

The final message of this paper is that branching is a natural representation of a theory's indeterminism, which moreover is rendered mathematically rigorous by the definitions we proposed. It is naturally used in the particular cases we considered. Branching represents exactly the kind of structure that is needed to assess a theory's determinism or indeterminism.

## Acknowledgments

Müller's research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement nr 263227, and from the Dutch Organization for Scientific Research, grant nr NWO VIDI 276-20-013. Placek's research was funded by a research grant 'Mistrz 2011' of the Foundation for Polish Science, contract nr 5/2011.

Thomas Müller  
FB Philosophie, Universität Konstanz  
Konstanz, Germany.  
email: Thomas.Mueller@uni-konstanz.de

Tomasz Placek  
Department of Philosophy, Jagiellonian University  
Krakow, Poland.  
email: Tomasz.Placek@uj.edu.pl



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