Platonism in classical logic versus formalism in the proposed non-Aristotelian finitary logic

Radhakrishnan Srinivasan*

Abstract

The main thesis of this paper is that Platonism is inherent in classical infinitary reasoning and that strict formalism inevitably leads one to the author's non-Aristotelian finitary logic (NAFL) proposed in the Philsci preprint ID Code 635. This claim is established by an elementary analysis of Peano Arithmetic and its weak fragments (especially Successor Arithmetic). Hence it is argued that Hilbert's program is decisively settled negatively – classical infinitary reasoning stands refuted from the finitary and formalistic standpoints. The philosophical basis for NAFL is discussed thoroughly and motivated via consideration of several examples, including the Schrödinger cat and four-mirror experiments of quantum mechanics. Particular attention is paid to the delicate interplay between syntax and semantics in NAFL, and the differences in this regard between NAFL and classical/intuitionistic/constructive logics. The meaning of 'existence' of mathematical objects and the concept of negation in NAFL are discussed. NAFL also correctly handles time-dependent truth values for propositions involving future contingencies; this is illustrated with examples, such as, Aristotle's 'There will be a sea battle tomorrow'. That NAFL justifies quantum superposition on the one hand, while emphatically rejecting much of classical infinitary reasoning and the continuum-based relativity theories (see also the PhilSci preprint ID Code 666) on the other, means that the incompatibility between quantum mechanics and the theory of relativity is clearly established in NAFL. Another important implication for quantum mechanics is that NAFL requires the concept of 'measurement' to be confined to the metatheory, i.e., it is not formalizable.

1 Platonism in classical first-order predicate logic

We begin by quoting Stephen Simpson[1] on the current status of the philosophy of mathematics:

> We have mentioned three competing 20th century doctrines: formalism, constructivism, set-theoretical Platonism. None of these

*R & D Group – Exports services, IBM Global Services India Pvt. Ltd., 5th floor, Golden Enclave, Airport road, Bangalore 560017, India. E-Mail: sradhakr@in.ibm.com

> doctrines are philosophically satisfactory, and they do not provide much guidance for mathematically oriented scientists and other users of mathematics. As a result, late 20th century mathematicians have developed a split view, a kind of Kantian schizophrenia, which is usually described as "Platonism on weekdays, formalism on weekends". In other words, they accept the existence of infinite sets as a working hypothesis in their mathematical research, but when it comes to philosophical speculation, they retreat to a formalist stance. Thus they have given up hope of an integrated view which accounts for both mathematical knowledge and the applicability of mathematics to physical reality. In this respect, the philosophy of mathematics is in a sorry state.

In this section, we demonstrate that the "retreat to a formalist stance" in the above quote is not really a viable option in classical first-order predicate logic with equality (FOPL); i.e., Platonism is inherent in classical infinitary reasoning.

The existence of nonstandard models of Peano Arithmetic (PA) in FOPL follows from any one of the completeness, compactness and incompleteness theorems due to Gödel [2] in 1930 and 1931; Skolem [3] first demonstrated the existence of these nonstandard models in the 1930's. The proofs of the (in)completeness and compactness theorems may be formalized in a theory T of intermediate strength between PA and Zermelo-Fraenkel set theory with axiom of choice (ZFC). The proof of the existence of nonstandard integers therefore tacitly assumes the consistency of T; for if T were inconsistent, the (in)completeness and compactness theorems would carry no conviction from the formalistic standpoint, since proof is the ultimate criterion for formalists. The consistency of T in turn requires nonstandard models of T to exist, by Gödel's incompleteness theorems, and so tacitly presumes the existence of nonstandard integers. It is clear that from a formalistic standpoint, one does not have a convincing non-circular proof of either the existence of nonstandard integers or of the consistency of PA. In what follows, we demonstrate that a strict interpretation of formalism requires one to *deny* the existence of nonstandard integers, and hence to deny the consistency of PA and the validity of the proofs of the (in)completeness and compactness theorems. This would leave Platonism as the only viable philosophy for classical infinitary reasoning. In Section 2, we demonstrate that the non-Aristotelian finitary logic (NAFL) developed by the author [4] is the appropriate logic which embodies formalism.

1.1 Definition of formalism

Traditionally, formalism has been defined as the formal manipulation of symbols according to certain well-defined rules. Thus mathematics consists solely of syntax; semantics will be acceptable to formalists only when the concept of truth itself is formalized, as in model theory. In other words, the axioms of mathematics do not reflect any higher reality; the Platonic world of mathematical objects and the corresponding Platonic truths about these objects do not exist. This leads to the following definition of formalism.

Definition 1 (Formalism). *According to formalism, the only legitimate 'truths' regarding formal propositions in the language of a theory are the theorems provable in that theory and the truths in models for that theory, when suitably formalized in a stronger theory (i.e., the metatheory). The theorems of a given theory do not reflect any metamathematical (Platonic) 'reality'. Equivalently, no such metamathematical truths ought to be deducible from the theorems of a given theory; for that would mean that such informal/metamathematical/Platonic truths have been used in formulating the axioms of that theory.*

To understand the implications of the above definition, consider first the theory PA in FOPL and let GC stand for Goldbach's conjecture. Let P be a suitable formalization, in ZFC, of the proposition "GC is undecidable in PA". Suppose that ZFC proves P (although we do not know whether this is the case). From this proof, we can certainly deduce that PA is consistent and that GC is true in the standard model of PA. From Definition 1, it follows that these additional facts must also be formalizable and provable in ZFC; in fact the proof of P is also a proof in ZFC: (a) of the existence of the standard model of PA (suitably formalized), (b) of GC in the standard model of PA and (c) of con(PA) (which formalizes "PA is consistent"). If any of (a), (b) and (c) above were to be either not formalizable in the language of ZFC or not provable in ZFC despite the assumed provability of P, then one would have to conclude that ZFC is an illegitimately defined theory according to the formalistic philosophy outlined in Definition 1. Of course, this is not the case in this specific instance. But in what follows, we demonstrate that Definition 1 implies the inconsistency and/or illegitimacy of PA and in fact, PRA (Primitive Recursive Arithmetic), as formulated in FOPL.

1.2 The failure of formalism in FOPL

In this subsection, whenever we mention PA, it is to be understood that our comments apply equally well to PRA. To get to the root of the problem in classical infinitary reasoning, we begin by considering the most elementary theory postulating infinitely many entities, namely, Successor Arithmetic (SA). The two axioms of SA are:

$$\begin{gathered}\forall x\, \neg(S(x)=0), \\ \forall x \forall y (S(x)=S(y) \Rightarrow x=y).\end{gathered} \tag{1}$$

Here S is the successor function. The induction axiom scheme (not included in SA) is defined by

$$P(0)\ \&\ \forall x(P(x) \Rightarrow P(S(x))) \Rightarrow \forall x P(x), \tag{2}$$

where P represents any property of the natural numbers expressible in the language of SA. We will later consider (2) as a proposition in SA and comment on the fact that classical logic requires (2) to be undecidable in SA.

Consider the number-theoretic proposition Con(SA), which formalizes "SA is consistent". It is known that

$$\mathrm{PA} \vdash \mathrm{Con(SA)}, \tag{3}$$

which should be read as "PA proves Con(SA)". Indeed, PA proves the consistency of every finite subset of its axioms and this includes SA. Keeping in mind Definition 1, let us carefully examine the consequences of (3).

Let N denote the infinite class of all natural numbers in classical arithmetic, i.e., $N = \{0, S(0), S(S(0)), ...\}$. Note that N is the standard model for SA. In the terminology of nonstandard arithmetic, every natural number is *standard finite* by definition (as opposed to *nonstandard finite*); we reserve the word 'integer' for numbers that may possibly be nonstandard finite. We state the main result of this subsection in the following metatheorem:

Metatheorem 1. *Let Q be the proposition that formalizes the existence of N as an infinite class. Definition 1 and equation (3) imply that* $\mathrm{PA} \vdash Q$. *Consequently, nonstandard models for* PA *cannot exist and by Gödel's incompleteness theorems,* PA *is inconsistent.*

The question of how Q can be a legitimate proposition of PA is addressed in Remark 1. The simplest proof of metatheorem 1 is as follows.

Proof. The consistency of SA (i.e., Con(SA)) is equivalent in FOPL to the existence of a model for SA, which must either be a standard model (N) or a nonstandard one. To prove the metatheorem, one only needs to demonstrate that the existence of a nonstandard model for SA implies Q (as is done in the ensuing paragraph); this would make

$$\mathrm{Con(SA)} \Leftrightarrow Q. \tag{4}$$

From (4), (3) and Definition 1 it would then follow that

$$\mathrm{PA} \vdash Q. \tag{5}$$

To see why (5) must follow, note the example for GC discussed below Definition 1. For (4) makes the truth of Q essential to formalize the very notion of a model for SA, i.e., to formalize 'SA is consistent' via Con(SA). Definition 1 then forces the conclusion that PA must prove Q; in the absence of such a proof, we would be able to draw the conclusion from (4) that Q is informally/metamathematically/Platonically true, in violation of Definition 1. A proof in PA of Q would amount to a denial in PA of the existence of nonstandard integers and the metatheorem would follow.

The proof that the existence of a nonstandard model for SA implies Q is as follows. The very notion of a nonstandard integer requires the existence of the class N_s of 'all' standard non-negative integers; a nonstandard positive integer, *by definition*, is required to be greater than every member of N_s. Of course, N_s can only be defined by putting it in one-to-one correspondence with

N; such a mapping would be called "external" in nonstandard analysis. So it is impossible to formalize the existence of a nonstandard model for SA without first postulating Q, i.e., the existence of N. It follows that the existence of *any* model for SA, or equivalently, con(SA), is logically equivalent to Q, and (4) follows. □

Remark 1. The reader's first reaction upon looking at (5) will probably be "PA supposedly does not even formalize the notion of 'standard finite' required to formulate the proposition Q and so cannot possibly prove Q". But this claim is true if and only if one already presumes that PA is consistent and that therefore nonstandard models for PA exist. Since metatheorem 1 falsifies this claim, one has to re-examine it in that light. Our contention is that *every* universally quantified proposition, in particular, Con(SA), is logically equivalent to Q. When PA proves that something is true for 'all' integers, Definition 1 demands that the notion of 'all' *must* also be formalized as the existence of an infinite class of integers which *must* be taken as provable in PA. But the proof of metatheorem 1 shows that the existence of *any* such infinite class of integers (i.e., *any* model for SA) is in fact logically equivalent to Q. One example of a universally quantified proposition provable in PA is

$$\forall n \exists m \; (m > n).$$

The truth of this proposition clearly allows us to draw the inference (via a proof by contradiction) that infinitely many integers of the form $\{0, 1, 2, ...\}$ exist and therefore Q must be 'true'. Definition 1 would make PA an illegally defined theory if such an inference were not formalizable and provable in PA; taking Q as 'true-but-unprovable' is unacceptable in formalism.

In fact the following proposition, provable in SA, is also logically equivalent to Q:

$$\forall x \exists y \; (y = S(x)).$$

This means that SA proves its own consistency; this is not surprising, because as we will argue below in Remark 3, the very notion of SA as a theory can only be formalized with the unavoidable assumption that Q is 'metamathematically true'. Definition 1 then demands that Q be provable in SA. Of course, this does not make SA inconsistent because SA is too weak for Gödel's theorems to apply. It follows that nonstandard integers do not exist, period. So in fact even Gödel's incompleteness theorems and the completeness/compactness theorems of FOPL do not go through, because, as noted earlier, they require theories at least as strong or even stronger than PRA to formulate and metatheorem 1 shows that PA (PRA) is inconsistent. There is clearly a problem of circularity here; the inconsistency of PA (PRA) is deduced from Gödel's theorems, which however, are unreliable because of said inconsistency. In Section 2, we observe that a new logic [4] which accepts the requirement of metatheorem 1 that PA $\vdash Q$, but rejects the validity of Gödel's theorems because of their self-referential nature, will be needed to justify the axioms of PA from the point of view of formalism.

Remark 2. Another way of looking at Definition 1 is as follows. If a proposition were to be undecidable or cannot even be formalized in a theory, then the metatheory that formulates such a theory should in principle be able to either affirm or deny that proposition via a proof/refutation, or not even decide/formalize such a proposition. Thus formalism requires that the truth or falsity of such a proposition should not be essential in specifying that theory. In the case of Q, one finds that it is essential to assume it as 'true' in formulating *any* infinitary axiomatic theory of FOPL (see Remark 3 below), in particular, SA. So Q cannot be denied in the metatheory that formalizes SA, i.e., Q must be 'really' true in every model for SA. Definition 1 then requires SA $\vdash Q$. An important consequence of this fact is that the induction axiom scheme (2) must be taken as provable in SA as explained below. It is the assumed undecidability of (2) in SA that leads to the existence of nonstandard models for SA, denied herein. To see why, consider the theory SA*, defined as SA with an additional predicate St and the following additional axiom:

$$St(0)\ \&\ \forall x(St(x) \Rightarrow St(S(x))). \tag{6}$$

Clearly, every natural number n will satisfy $St(n)$. But the assumed undecidability of (2) implies $\forall x St(x)$ is not provable in SA* and so there must exist a nonstandard model for SA* in which $\exists x \neg St(x)$ is true; such an x would be a nonstandard integer. Definition 1 shows that (2) must in fact be provable in SA and hence SA*; this means that nonstandard integers do not exist and St becomes a superfluous predicate that may be thought of as defining 'standard finite'. Observe that the undecidability of (2) in SA is established in FOPL only by circular reasoning; the theory that establishes such undecidability *presumes* the existence of nonstandard integers (via its own consistency, which is a tacit assumption); so what such a theory really establishes is the tautology 'If (2) is undecidable in SA, then (2) is undecidable in SA'.

Remark 3. The proposition Con(SA) literally denotes "There does not exist a proof of '0=S(0)' in SA". Of course, this raises the question of what "There does not exist", "proof" and "SA" denote. It is clear that the theory SA can only be defined, in particular, by identifying its infinite class of well-formed formulae (wff's), each of which is a string of *standard finite* (as opposed to *nonstandard finite*) length, and its infinite class of proofs each of which must contain a standard finite number of wff's. Therefore what Con(SA) 'really' denotes is: "The infinite class of SA-proofs does not contain within it a proof of '0=S(0)', or equivalently, does not contain within it a pair of SA-proofs which come to contradictory conclusions". It is absolutely essential to note that the "infinite class" of SA-proofs (or wff's) referred to above can only be formally defined by mapping the SA-proofs (wff's) to the class N of all natural numbers. If such a mapping is not done, the notion of "all SA-proofs" or "all wff's of SA" cannot be said to have been formalized. The requirement that N contain 'all' and *only* standard finite (natural) numbers is also absolutely essential; it is easy to show that if one permits nonstandardly long proofs/wff's, then SA will be inconsistent. One concludes that to even formulate SA as a theory one has to assume the truth of Q. Even within a nonstandard model for SA, 'all' SA-proofs and wff's must necessarily be identified to be of standard finite length, i.e., mapped to N; it is only with such identification that one can meaningfully assert that 'all' theorems of SA are true within such a nonstandard model. As noted earlier, the nonstandardly long candidates for 'wff' or 'proof' must be rejected in order to prove that a given (nonstandard) structure is in fact a model for SA. This fact augments the assertion in the proof of metatheorem 1 that the very existence of nonstandard integers requires Q to be true. The main point of this remark is that it is impossible to formalize the notion of SA as a theory and its consistency via Con(SA) without first formally postulating the existence of N.

Remark 4. Note that Con(SA) is a purely number-theoretic (universally quantified) proposition. To interpret it as meaning 'SA is consistent' (or equivalently, 'There exists a model for SA') requires formulating this latter proposition in a stronger theory than PA and translating it into the PA-proposition Con(SA) using the techniques developed by Gödel [2]. That such an infinitary proposition can even be translated into PA clearly demonstrates that PA 'really' proves the existence of an infinite class. Metatheorem 1 forces us to conclude (4) and (5). Hence Definition 1 implies that Gödel really translated the proposition Q (i.e., 'There exists the standard model N of SA') into Con(SA). The claim that 'There exists a model for SA' can be translated into Con(SA) without at the same time translating Q is a matter of Platonic belief. But this (Platonic) leap of faith is unacceptable in formalism because any metatheory that formalizes "There exists a model for SA" must necessarily *prove* its equivalence to Q; to translate one into Con(SA) is to translate the other.

Remark 5. As noted in Remark 1, the objection that PA does not even contain the predicate 'standard' and so cannot possibly formulate Q is met by the fact that if nonstandard integers do not even exist, such a predicate is unnecessary. As observed earlier, in nonstandard arithmetic, the class of 'all' standard integers N_s is first defined by putting it into one-to-one correspondence with N. But *by fiat*, such a mapping, required to define a nonstandard integer, is deemed 'external'; only 'internal' propositions are legitimate in nonstandard arithmetic. Hence the class N_a of 'all' nonnegative integers and the class N_s in nonstandard arithmetic do not coincide, even though their definitions are *identical* except for the word 'standard'. The definition of N_a is given by

$$0 \in N_a \ \& \ \forall x \ (x \in N_a \Rightarrow S(x) \in N_a).$$

To define N_s, one merely introduces the predicate 'standard' as follows:

$$0 \in N_s \ \& \ \forall^{st} x \ (x \in N_s \Rightarrow S(x) \in N_s),$$

where '$\forall^{st} x$' should be read as 'for all standard x'. Thus merely by introducing the word 'standard', the meanings of 'all' in 'all integers' and 'all standard integers' do not coincide in these two otherwise identical definitions of N_a and N_s. Defnition 1 essentially tells us that formalism will not permit such obfuscation of the meaning of the world 'all'; Platonism is inherent in such obfuscation (which would make nonstandard integers 'internally nonstandard finite', but 'externally infinite'). In other words, formalism requires 'all' to have a unique meaning in the two definitions given above.

Remark 6. Let $Con(n)$ be the arithmetical proposition equivalent to the consistency of the first n axioms of PA (suitably enumerated), where n is a natural number (i.e., $n \in N$; for convenience, we take $n = 0$ to denote the null set of axioms). By the compactness theorem,

$$\mathrm{Con(PA)} \Leftrightarrow \forall n\, \mathrm{Con}(n). \tag{7}$$

It is also known that for each *given* natural number m (here 'm' is a constant symbol),

$$\mathrm{PA} \vdash \mathrm{Con(m)}, \quad \mathrm{m} \in N. \tag{8}$$

Note that according to conventional wisdom, (8) is *not* to be interpreted as

$$\mathrm{PA} \vdash \mathrm{Con}(n), \tag{9}$$

which is the same as 'PA proves the *open* formula Con(n)', where n is a free variable ranging over all non-negative integers. Such an interpretation is illegal in FOPL because by universal generalization and (7), (9) would amount to PA $\vdash$ Con(PA) and make PA inconsistent by Gödel's second incompleteness theorem. Metatheorem 1 shows, however, that (8) and (9) are in fact equivalent, since nonstandard integers do not exist; from either of these formulae, one can infer the existence of N and by Definition 1, PA $\vdash Q$. Formalism requires that an 'arbitrary constant' like m in (8) is exactly the same as the free variable n, since both range over 'all' nonnegative integers. To give a different meaning to 'all' in these two contexts when it is clear that the axioms of PA do not imply any such difference is illegal according to formalism.

Remark 7. One may also conclude that Turing's halting problem ought not to be undecidable, for the same reason that Gödel's theorems do not apply: nonstandard integers cannot exist, by metatheorem 1. It follows that Cantor's diagonalization procedure, used by Gödel and Turing to deduce their incompleteness results, must be illegal. Similarly, the axioms for addition and multiplication and (2) (as noted earlier) cannot be undecidable in SA since these axioms can only fail in nonstandard models. In Section 2, we demonstrate that the author's proposed non-Aristotelian finitary logic (NAFL) [4] is consistent with and justifies all these restrictions.

Remark 8. Nevertheless, if one insists that PA does not formalize the notion of "standard finite" and hence does not prove Q despite (3), then one concludes that PA is illegitimately defined from the strictly formalistic standpoint of Definition 1. One option would be to abandon formalism as a valid philosophy of classical mathematics and resort to Platonic existence of N. The argument here would be that since N "really" exists (in a Platonic world), notions such as "all" theorems/proofs/wff's of SA do not have to be formalized by PA, eventhough such notions are required to interpret the PA-proof of Con(SA) as meaning "SA is consistent". The proposition Q is therefore being taken as Platonically 'true', eventhough unprovable in PA. Of course ZFC would prove Q, but then one is taking the axioms of ZFC as Platonically true, i.e., one is assuming the Platonic existence of the "universe of ZFC sets". Hence Platonism is inherent in classical infinitary reasoning.

Remark 9. The arguments of this subsection lead to the strong conclusion that Hilbert's Program is decisively settled *negatively* from the finitary and formalistic standpoints; classical PA (PRA), and hence classical infinitary reasoning, would be *inconsistent* if one insists on Definition 1. From the finitary standpoint, one must also deny the legitimacy of ZFC in formalizing the 'truth' of the existence of N. The second option is to accept this conclusion, i.e., to reject Platonism and insist on Definition 1 as valid. Since this would make PA (PRA) inconsistent, a new logic would be required to implement formalism as a philosophy of infinitary reasoning in mathematics. In Section 2, we demonstrate that Definition 1 is consistent with the truth definition given in the Main Postulate of the logic (NAFL) proposed by the author [4]; in other words, strict formalism inevitably leads one away from classical logic and into NAFL.

2 Formalism in non-Aristotelian finitary logic

In Section 1, we saw that a consistent (non-circular) interpretation of formalism is not possible within classical logic; formalism requires one to reject Gödel's and Turing's incompleteness results and hence the Cantor diagonalization argument as self-referential; nonstandard integers also cannot exist. In NAFL [4], these restrictions are justified rigorously by demonstrating that infinite sets cannot exist, and so N (the class of all natural numbers) must be a proper class. Diagonalization will then be seen to fail because of the restriction that one cannot quantify over infinitely many infinite (proper) classes, as explained later. Other important implications for quantum mechanics, the relativity theories and non-Euclidean geometries are also discussed.

2.1 The Main Postulate of NAFL

Definition 1 seemingly provides an intuitively correct definition for formalism; yet it is not sustainable within classical logic. The flaw is in the requirement of classical logic that the law of the excluded middle (LEM) must necessarily apply to an undecidable proposition in a consistent theory. It is this particular requirement that makes Platonism inherent in classical logic. NAFL does away with LEM and also the law of non-contradiction (LNC) via the truth definition given by its Main Postulate (see Sec. 2 of [4]), which provides the appropriate generalization of Definition 1. This makes formalism sustainable in NAFL, but with severe restrictions on classical infinitary reasoning.

The fundamental assumption of NAFL is that the *only* metamathematical objects that exist are axiomatic theories (which have the same rules of inference as classical FOPL theories) and their *models*, which are structures in which all the axioms of these theories and their logical consequences are assigned 'true'; propositions undecidable in an axiomatic theory may be assigned 'true', 'false' or 'neither true nor false'. This last option makes NAFL models different from classical models as will become clear from the explanations below. Formal mathematical objects do not have any existence outside of the theories in which they are postulated. There are no 'truths' outside of these axiomatic theories for formal propositions of NAFL; this is in keeping with Definition 1. Platonic truths do exist in NAFL, but these are regarding propositions *about* axiomatic theories which *cannot* be formalized in consistent NAFL theories, because of their self-referential nature (and also because the theories themselves are not formal objects); examples of such non-formalizable propositions are provability/refutability/undecidability of formal propositions with respect to the theories in which they are formulated, and the consistency of NAFL theories. In NAFL, a theory is consistent if and only if it has a model; this definition also holds in FOPL, but the NAFL notion of consistency is a metamathematical one and is not quite the same as that of classical logic, for reasons that will be explained later. It immediately follows that in NAFL, truths for formal propositions can only exist with respect to axiomatic theories; there cannot exist any absolute truths in just the *language* of a theory.

This raises the question of what it means for a proposition Ψ to be true or false in a consistent NAFL theory T. If Ψ is provable/refutable in T, then it is certainly true/false in T, i.e., a model for T will assign Ψ to be true/false. If Ψ is undecidable in T, i.e., neither Ψ nor its negation $\neg\Psi$ is provable in T, then the Main Postulate of NAFL provides the appropriate truth definition as follows: Ψ is true/false in T if and only if Ψ is provable/refutable in an *interpretation* T* of T. Here 'interpretation' is defined as an axiomatic theory T* that resides in the human mind and acts as the 'truth-maker' for (a model of) T; it immediately follows that T* must prove all the axioms and theorems of T. Thus far it appears that NAFL is not much different from classical logic and that the NAFL truth definition coincides with Definition 1. The crucial difference occurs when Ψ is undecidable in T*; in this case NAFL interprets Ψ as neither true nor false in T, and metatheorems 1 and 2 of [4] show that both LEM and LNC fail for Ψ, in a non-classical model for T in which $\Psi \& \neg\Psi$ is the case. This non-classical model is essentially a superposition of two or more classical models. Here we are using the words 'classical model' and 'non-classical model' only with respect to the truth values for Ψ. Since the NAFL model for T will, in general, contain some proposition undecided in T*, it will always be non-classical with respect to this proposition.

The essence of the Main Postulate is that an undecidable proposition Ψ in a consistent NAFL theory T is true/false in T if and only if it has been *axiomatically asserted* as true/false by virtue of its provability/refutability in the interpretation T* of T. This essentially means that Ψ is true/false in T if and only if $T^* = T + \Psi / T + \neg\Psi$; this notation is used in [4] to emphasize the fact that Ψ ($\neg\Psi$) is an *axiomatic declaration* with respect to T, regardless of its status as an axiom or otherwise in the theory T*. Note in particular that axiomatic assertions are made by human beings, and that T* resides in the human mind. In the absence of any axiomatic assertions regarding Ψ, (i.e., if Ψ is undecidable in T*), consistency of T demands that there must exist a non-classical model for T in which Ψ is neither true nor false, i.e., $\Psi \& \neg\Psi$ is the case. The (non-classical) interpretation of $\Psi \& \neg\Psi$ is explained in the ensuing subsection.

2.2 The philosophical basis for NAFL

The first question that one may ask is *why* the truth for undecidable propositions in a consistent theory T is axiomatic in NAFL. Here there are two important points to be understood. Firstly, in NAFL there is no Platonic world in which formal propositions are either true or false, independent of an axiomatic declaration (in the interpretation T* of T). Secondly, NAFL interprets LEM for a T-undecidable proposition Ψ to mean 'Ψ is either true or false'; since there is no Platonic world in which such truth/falsity occurs independent of the human mind, the axiomatic nature of truth in NAFL follows and LEM holds if and only if an axiomatic declaration of truth/falsity of Ψ is made (via provability) in T*. In the absence of such an axiomatic declaration, the failure of LEM and LNC follows for the reasons explained in metatheorems 1 and 2 of [4]. In particular, the classical refutation of $\Psi \& \neg\Psi$ in T proceeds as follows: "If Ψ ($\neg\Psi$) is true, then $\neg\Psi$ (Ψ) must be false". This classical argument appeals to *Platonic* truths of Ψ and $\neg\Psi$ in "pre-existing" models of T; i.e., these classical models have to be assumed to "exist" prior to and independent of any axiomatic declarations of truth made in T. In NAFL, on the other hand, there are no such Platonic truths and no "pre-existing" models. The NAFL model of T is generated by its interpretation T*, which is also an axiomatic NAFL theory. Hence the NAFL model of T has only a temporary existence in the human mind and the 'truths' in this model are axiomatic declarations generated *after* T has been specified. In summary, the attempted (classical) refutation of $\Psi \& \neg\Psi$ in T fails in NAFL because such a refutation must necessarily appeal to *axiomatic* (as opposed to the classically Platonic) truths of Ψ and $\neg\Psi$ and so the refutation is valid only in the NAFL theories T+Ψ and T+$\neg\Psi$. Note that this failure of LNC makes LNC equivalent to LEM in NAFL, unlike intuitionistic logic (where LEM fails, but LNC holds for undecidable propositions in a theory).

We first list some results explained in detail in Remarks 1–7 of [4]. The failure of LNC in a non-classical model for T is a unique NAFL phenomenon, and has the following explanation: 'Ψ' in $\Psi \& \neg\Psi$ asserts that '$\neg\Psi$ has not been axiomatically asserted as true (provable) in T*', while '$\neg\Psi$' in $\Psi \& \neg\Psi$ asserts that 'Ψ has not been axiomatically asserted as true (provable) in T*'; as in classical logic (and unlike intuitionistic/constructive logics) there is no difference between Ψ and $\neg\neg\Psi$ in NAFL (see below under the heading 'The concept of negation in NAFL'). An important consequence of the failure of LNC is that the classical assertion 'From $\Psi \& \neg\Psi$, any proposition can be deduced' fails in NAFL. LEM and its negation are not legitimate propositions in NAFL theories. A more general result (as noted in Remark 5 of [4]) is that if ψ and ϕ are undecidable propositions in a theory T, then $\psi \Rightarrow \phi$ is a legitimate proposition of T if and only if ϕ ($\neg\psi$) is *not* a theorem of T+ψ (T+$\neg\phi$) (or equivalently, if and only if $\psi \Rightarrow \phi$ is *not* classically deducible in T). At this stage the reader might wonder how $\psi \Rightarrow \phi$ can be classically deducible in T and yet be an illegitimate proposition of an NAFL theory T, which has, after all, the same rules of inference as classical FOPL theories. This is explained in detail under the ensuing heading, where the syntax and semantics of NAFL theories are considered.

As noted earlier, consistency of an NAFL theory T and the provability (or undecidability) of a formal NAFL proposition in a theory T are propositions that cannot be formalized in NAFL theories. All these 'informal' propositions are in fact valid propositions of the metatheory, which is NAFL itself; as will be explained below, LEM does apply to these Platonic (informal) notions, i.e., they are either true or false, independent of the human mind. For this reason, the notion of 'consistency' is not the same in NAFL and FOPL; in NAFL (unlike FOPL), a proposition can be undecidable in an inconsistent theory (which is rendered inconsistent by the fact that the required non-classical model for that theory cannot exist). So in NAFL (unlike classical/intuitionistic/constructive logics), the assertion that 'any proposition can be deduced in an inconsistent theory' fails.

Syntax of an NAFL theory, syntax of its proofs and the metatheory

We now discuss a very important issue of syntax versus semantics not clearly addressed in [4]. In NAFL, the *syntax of a theory*, which we will henceforth refer to as 't-syntax', consists essentially of all its legitimate propositions, including the axioms, theorems and undecidable propositions; in particular, the t-syntax will not admit certain (classically acceptable) propositions deemed 'illegitimate' or 'not formalizable' in NAFL, such as, LEM (LNC) or its negation for propositions undecidable in that theory. More generally, as noted earlier, the proposition $\psi \Rightarrow \phi$ is illegitimate in the t-syntax if ψ and ϕ are undecidable in T and $\psi \Rightarrow \phi$ is classically provable in T. The notion of provability/undecidability is fixed by a second level of syntax in NAFL, namely, the *syntax of proofs in a theory*, which we will henceforth refer to as 'p-syntax'. The p-syntax of a theory T will include all the rules of inference of FOPL, in addition to *all* the (classically legitimate) propositions in the language of the theory. In particular, the p-syntax is purely classical and will admit some propositions, such as LEM (LNC) or its negation, or the more general example noted above, which are deemed illegitimate in the t-syntax. Thus LEM (LNC) is assumed true in the p-syntax and may occur in *proofs* of legitimate propositions in the t-syntax; but LEM (LNC) and its negation by themselves cannot occur in the t-syntax. An example of this situation is a deduction $\psi \Rightarrow \phi \& \neg\phi$ in the p-syntax of T for a proposition ϕ that is undecidable in a theory T. Such a deduction is a legitimate proof by contradiction (in the p-syntax) of $\neg\psi$ in T *despite* the fact that $\phi \& \neg\phi$ and its negation are illegitimate in the t-syntax and also despite the fact that in the *metatheory* there does exist a non-classical model for T in which $\phi\&\neg\phi$ is the case. This metatheory, of course, is NAFL itself, which provides the semantics for the t-syntax. The notions of consistency of T and those of provability/refutability/undecidability of propositions in T are purely metamathematical notions (permitted only in the metatheory) and cannot even be expressed in the language of T in NAFL (note that Gödel's translation of these notions into number-theoretic propositions is not valid in NAFL essentially because of their self-referential nature). Another reason why the notion of provability or undecidability of propositions is not formalizable in NAFL theories is clearly the fact that the p-syntax admits classical inference rules including propositions illegitimate in the t-syntax; this feature of the p-syntax makes it Platonistic in intent, while the t-syntax is formalistic.

The p-syntax is Platonistic (i.e., admits classical inference rules) because this is the only non-circular way in which the concept of undecidability of propositions or that of consistency of a theory can even be defined. As an example, consider the example of the assumption ψ leading to the conclusion $\neg\psi$ in an NAFL theory T; in the p-syntax of T this is a proof by contradiction of $\neg\psi$. Clearly LNC, i.e., $\neg(\psi\&\neg\psi)$, has been assumed in this proof; the objection that LNC need not hold by the Main Postulate if ψ is undecidable in T will not make sense in the p-syntax because the notion of undecidability of ψ is valid in the first place only *after all proofs* of T are available (note that the undecidability of ψ in T means that the class of all proofs in T does not include within it either a proof of ψ or a proof of $\neg\psi$). This argument also explains why a deduction of $\psi \Rightarrow \phi\&\neg\phi$ in the p-syntax of T is a proof of $\neg\psi$ in T despite the possible undecidability of ϕ in T.

The author believes that intuitionism fails to provide a non-circular definition of the concept of undecidability of propositions in theories because intuitionism does not distinguish between the p-syntax and the t-syntax. As an example, let T_0 be the null set of axioms. In NAFL, every legitimate proposition of T_0 is undecidable in T_0; this assertion is made in the metatheory and there is no 'proof' for it because the p-syntax of T_0 does not admit the concept of undecidability. In fact it is self-evident that this this assertion cannot be proved in T_0 for precisely the same reason that it is true – nothing is provable in T_0. Thus the attempt to formalize the notion of undecidability in T_0 in NAFL will violate the Main Postulate. Intuitionism, on the other hand, insists on a proof for this assertion, which is non-existent, and as a result comes to the conclusion that it is not even valid to assert that every proposition is either provable or refutable or undecidable in intuitionistic theories. Classical logic accepts this last assertion but insists that the concept of provability/undecidability is formalizable. This results in self-reference and Platonism for the truth of formal propositions of a theory, not acceptable in NAFL. Intuitionism also fails to get rid of Platonism because it insists on LNC for undecidable propositions of a theory; this forces intuitionists to look for formal 'proofs' of undecidable propositions outside of the theory (e.g., T_0). In summary, NAFL requires the p-syntax to be Platonistic and the t-syntax to be formalistic; intuitionism fails on both counts by rejecting LEM in the p-syntax and by accepting LNC in the t-syntax.

Note that the failure of LNC (LEM) for an undecidable proposition of an

NAFL theory T in a non-classical model occurs in the metatheory, because models are metamathematical objects. This does not contradict the t-syntax, because LNC (LEM) is not even a legitimate proposition of the t-syntax. Thus no inconsistency is implied by the existence of such a non-classical model. A second important point regarding the metatheory is, as noted earlier, that LEM applies to all (informal) propositions occurring in the metatheory. Thus the metatheory requires that a theory is either consistent or inconsistent, and a formal proposition (in the t-syntax) is either provable or refutable or undecidable in a given theory, possibly independent of the human mind. In summary, the metatheory is scrupulously Platonistic in intent, in contrast to the (formal) theory itself. This is so because NAFL insists that the propositions in the metatheory *must* make sense in the real world. Thus, in the real world, one has either axiomatically declared an undecidable proposition Ψ in a consistent NAFL theory T to be true or false or neither (the last disjunct occurs by default; it is not legal to declare a proposition Ψ to be 'neither true nor false' in T, for that would amount to adding $\Psi \& \neg\Psi$ formally as an axiom, which is not permitted by the t-syntax). It is not legal in the real world for the human mind that interprets T to assert that he/she does not know which of these is the case, or to assert that none of these options hold. In this respect, NAFL differs from both classical logic and intuitionism.

Open formulas and the meaning of 'existence' in NAFL

An immediate consequence of metatheorem 2 of [4] (which asserts the failure of LNC and the existence of a non-classical model for a consistent theory in which a given proposition is undecidable) is that open formulas (with free variables) and formulas with 'arbitrary constants' in them in a given theory are in fact universally quantified formulas with respect to the free variables/arbitrary constants. The reason is that the theory in question clearly does not decide the values of the free variables/arbitrary constants which must therefore be in a superposed state of assuming all possible values. It follows that the two formulae in (8) and (9) are in fact equivalent in NAFL (to a universally quantified formula) and this is in conformity with the requirements of formalism as stated in Remark 6. It also follows that an NAFL theory which asserts the existence of a *unique* x such that some property $P(x)$ holds, but does not provide a construction (value) for that x, is an inconsistent theory. This is so because the uniqueness requirement precludes the existence of a non-classical model for that theory with values of x superposed; but such existence is required for the consistency of an NAFL theory in which the supposedly unique value of 'x' in $\exists x P(x)$ is undecidable.

In NAFL, the issue of the meaning of 'existence' of entities within a theory is resolved as follows. Clearly, NAFL rejects Platonic existence of mathematical entities. In constructive/intuitionistic logics, existence of an entity has the meaning that a *construction* must necessarily be available for that entity; i.e., $\exists x P(x)$ can only be asserted by specifying such a construction for x. NAFL is not quite as restrictive; existence of an entity X in an NAFL theory T is to be interpreted as meaning "X is a legal entity of T", i.e., it is legal to speak of X in propositions/theorems of T. A construction for such an X need not necessarily be available within T. But in a proposition of an NAFL theory T such as $\exists x P(x)$, whether a construction for such an x is required or not depends on whether metatheorem 2 of [4] will permit such a proposition to be undecidable within T. As an example, let x range over the natural numbers. It is known that each natural number has a unique construction available (built up from the null set, for example, in set theory); so in this case, $\exists x P(x)$ *cannot* be undecidable in T; the required non-classical model cannot exist because LEM must unavoidably apply to this proposition by virtue of the definition of 'natural number'.

For an example of non-constructive existence in NAFL, let G stand for a suitable formalization of "God exists" in the null set of axioms T_0; further, let GG formalize "God is great". Clearly, T_0 does not specify any construction for the entity 'God', and so a non-classical model for T_0 in which $G\&\neg G$ is the case does exist. Hence undecidability of G in T_0 is not a contradiction and non-constructive existence is permitted here. Given the axiomatic nature of truth in NAFL, the superposition $G\&\neg G$ means that neither G nor $\neg G$ has been asserted axiomatically in the interpretation T* of T_0, which is therefore non-classical. This non-classical model clearly corresponds to agnosticism, for an agonostic refuses to acknowledge that God is either a legitimate or an illegitimate entity. Next consider the proposition $GG \Rightarrow G$. This is clearly a rule of inference; Platonically one would argue that 'If God is great, then surely God does exist'; in NAFL, $GG \Rightarrow G$ should be interpreted as "If 'God is great' is true then it is certainly a legitimate sentence; it follows that God has to be a legitimate entity and so 'God exists' is true". So $GG \Rightarrow G$ is in the p-syntax of T_0 as a rule of inference. But since G ($\neg GG$) is provable in the theory $\mathrm{T}_0 + GG$ ($\mathrm{T}_0 + \neg G$) and both G and GG are undecidable in T_0, it follows from Remark 5 of [4] that $GG \Rightarrow G$ is not in the t-syntax of T_0 i.e., it is not a legitimate proposition of T_0. Therefore there does exist a non-classical model for T_0 (in the metatheory) in which both $G\&\neg G$ and $GG\&\neg GG$ are the case, i.e., no axiomatic assertions have been made in T* regarding the truth or falsity of either G or GG and so both are neither true nor false. As noted in Remark 5 of [4], in this non-classical model $GG \Rightarrow G$ is also neither true nor false; this would not be permitted if $GG \Rightarrow G$ were in the t-syntax of T_0, since it would then immediately become provable by virtue of its presence in the p-syntax (as a rule of inference); but then the said non-classical model required by NAFL cannot exist and T_0 would become inconsistent. Intuitively, the non-classical model simply asserts that for the agnostic, "If God is great then God exists" is as meaningless as "God exists" and "God is great" and so is just as entitled to have a non-classical truth value of 'neither true nor false'. Careful thought will show that this is the correct formalistic position; it is Platonism that forces $GG \Rightarrow G$ to be a theorem of T_0 when both G and GG are undecidable in T_0. To conclude this subsection, note that the believer and atheist have respectively made the *axiomatic* assertions G and $\neg G$ with respect to T_0 from the NAFL point of view, eventhough they may personally believe that God 'really' exists/does not exist. NAFL essentially states that their Platonic beliefs are irrelevant and they have ultimately made axiomatic declarations of truth in an interpretation T^* of T_0.

The concept of negation in NAFL

As noted earlier, for any proposition ψ, $(\neg\neg\psi \Leftrightarrow \psi)$ holds in the p-syntax of NAFL theories (as in FOPL and unlike intuitionistic/constructive logics). But there are a couple of subtleties regarding NAFL negation that need to be noted.

Firstly, note that in a non-classical model of a consistent NAFL theory T in which $\psi\&\neg\psi$ is the case under the interpretation T^*, the (classical) logical equivalence $\neg\neg\psi \Leftrightarrow \psi$ certainly fails. In fact even $\psi \Rightarrow \psi$ fails since its negation is seen to hold. To avoid confusion, we reserve the notation $\psi \Rightarrow \phi$ to *always* mean the *classical* disjunction $\neg\psi \vee \phi$, whose negation is $\psi\&\neg\phi$. The equivalence between ψ and $\neg\neg\psi$ holds in the non-classical model under a non-classical interpretation which we define as $\neg\neg\psi \leftrightarrow \psi$. Here $\psi \rightarrow \phi$ is a non-classical disjunction given by $\neg\psi \vee \phi \vee (\neg\psi\&\phi)$, i.e., all three disjuncts are possible *even when ϕ is replaced by ψ*. In the non-classical model, ψ expresses that $\neg\psi$ has not been *classically* asserted as an axiom (is not provable) in T^*, $\neg\psi$ expresses that ψ has not been *classically* asserted as an axiom (is not provable) in T^* and their conjunction $\psi\&\neg\psi$ also obviously holds under this interpretation. Similarly, the equivalence $\neg\neg\psi \leftrightarrow \psi$ also holds; both $\neg\neg\psi$ and ψ express the same concept in the non-classical model and one can be substituted for the other.

Secondly, let ψ stand for the outcome 'one' in the roll of a dice, and let $\neg\psi$ stand for its negation, i.e., any one of the outcomes 'two', 'three', 'four', 'five' or 'six'. Consider the null set of axioms T_0. Suppose we add the axiom ψ to T_0 to obtain an interpretation T^*. Then $\neg\psi$ is certainly a legitimate proposition of T_0 and T^* that stands refuted in T^*. But suppose we wish to add $\neg\psi$ to T_0 instead to obtain the interpretation T^*; *now* $\neg\psi$ is also a legitimate proposition of T_0, but it can be a legitimate axiom of T^* if and only if T^* proves one of the outcomes 'two', 'three', 'four', 'five' or 'six'. The bottom line is that a disjunction can be asserted as an *axiom* of a consistent NAFL theory if and only if that theory proves at least one of the disjuncts. On the other hand, the disjunction can still be a *legitimate proposition* in the t-syntax of a theory even if none of the disjuncts are proved, provided the restriction given in Remark 5 of [4] is satisfied. In this example, the said restriction is satisfied and the disjunction $\neg\psi$ is a legitimate proposition of T_0 (because $\neg\psi$ is of the form $A \Rightarrow B$, where A and B are undecidable in T_0 and $A \Rightarrow B$ is not classically deducible in T_0). Of course, if *all* disjuncts are refuted by the theory (as in the above example where $\neg\psi$ is such a disjunction in the theory $T_0 + \psi$), then the (refuted) disjunction itself is in the t-syntax of that theory. This example illustrates the difference between NAFL negation and its classical and intuitionistic counterparts.

2.3 Implications for some classical theories

It turns out that metatheorem 2 of [4] is a killer of classical infinitary reasoning in all but its weakest forms. The theory of finite sets F, described in detail in Sec. 3 of [4], is essentially Gödel-Bernays set theory with classes and without the axiom of infinity (see Cohen [5], pp. 73–78). Note that we have used boldface notation for F in [4]. Since F is equivalent to PA, we will only consider F in this paper. In Sec. 3 of [4], it is shown in detail that the undecidability of existence of infinite sets in F is a contradiction because the non-classical model for F required by metatheorem 2 of [4] cannot exist. It is argued in [4] that infinite sets are self-referential objects and must therefore be banned if F is to be consistent; in other words, NAFL requires that the p-syntax of F must include additional inference rules that would ban such self-referential objects. In particular, Zermelo-Fraenkel set theory would be inconsistent in NAFL. It is easy to show that by similar reasoning, the axioms of F themselves cannot be undecidable in weaker theories and so must be declared as tautologously true in the p-syntax of F (i.e., they cannot be denied).

The most important consequence of metatheorem 2 of [4] is that the consistency of F demands that undecidable propositions cannot exist in F, i.e., all propositions of F in the t-syntax must be either provable or refutable in F. This result may be derived from the fact that nonstandard models of F cannot exist, if F is consistent; indeed, this is obvious from the fact that infinite sets cannot exist in *any* form in F. Alternatively, non-existence of nonstandard integers can be established by arguments similar to that in Sec. 1; the Main Postulate of NAFL requires that the existence of class N of all (*standard finite*) natural numbers be provable in F because it cannot be denied in any model for F and NAFL does not permit the existence of N to be 'true-but-undecidable' in F. Note that the existence of N is a proposition (denoted by Q) about *formal objects* of F, i.e., the natural numbers; so by the Main Postulate, it is not legal in NAFL to assert that Q is not in the t-syntax of F, but is 'true' in the metatheory. This violates the axiomatic nature of truth regarding propositions about formal objects required in NAFL. The existence of an infinite class (such as, N) in an NAFL theory must be equated with the existence of all of its objects, since the infinite class by itself is not an object of any NAFL theory. Thus any universally quantified proposition that asserts the existence of infinitely many natural numbers (such as, the example given in Remark 1) proves Q in F. It is extremely important to note that the consistency of F (or equivalently, consistency of PA) is no longer equivalent to Q, which is a *formal proposition* of F in NAFL. The NAFL notion of consistency is different from the classical one.

It follows that Gödel's and Turing's incompleteness theorems must be illegal in NAFL, for they predict the existence of undecidable propositions in F. The culprit is the Cantor diagonalization principle, used by Gödel and Turing; this principle must be banned in NAFL because it is illegal in NAFL to quantify over infinitely many infinite (proper) classes. Indeed, such quantification speaks of infinite classes as infinite sets, since it tacitly presumes that the 'super-class' of these infinitely many infinite classes exists. Note that quantification over finitely many infinite classes can always be eliminated and so is not a problem. So the notions of consistency and undecidability cannot be formalized in NAFL because of their self-referential nature. It is doubtful if there exist any methods acceptable in NAFL that would establish the existence of undecidable propositions in F (and hence the inconsistency of F). An important corollary of this result is that some of the famous propositions of arithmetic, such as Fermat's last theorem, Goldbach's conjecture, the Twin prime conjecture, etc., must all be decidable in F. Further, Turing's halting problem must also be decidable.

It is easy to demonstrate that Euclid's fifth postulate cannot be undecidable in NAFL with respect to the first four. The reason again is that if the first four postulates are consistent, the non-classical model for these postulates (required to exist by NAFL) in which the fifth postulate is neither true nor false cannot exist; such existence would violate the requirement of the first four postulates that a *unique* straight line must pass through any two given distinct points. Hence non-Euclidean geometries should be banned as self-referential in NAFL, and all five Euclid's postulates must be declared as tautologously true in the p-syntax. It follows that general relativity theory is inconsistent in NAFL. The inconsistency of special relativity theory (SR) in NAFL is argued briefly in [4] and in greater detail in [6], where strong grounds are established to suspect inconsistency of SR even within classical FOPL.

2.4 Quantum superposition justified in NAFL

We consider two examples, namely, the four mirrors and the Schrödinger cat experiments of quantum mechanics. These experiments are well known and we will assume that the reader is aware of how they are set up. We will show how quantum superposition is justified in NAFL and the discuss the role of 'measurement' in quantum mechanics.

The four mirrors experiment

The essential, surprising result of this experiment is that in the absence of a 'measurement', the photon seems to take *both* available paths (call these A and B) simultaneously; any attempt to directly detect this phenomenon fails and the photon is observed to take one of the two paths.

Let ψ be the proposition 'The photon took path A', with the negation $\neg\psi$ denoting 'The photon took path B'; let QM represent an axiomatization of quantum mechanics. Clearly, ψ is undecidable in QM (as will be explained shortly, 'measurement' will have to be part of the metatheory in NAFL and not of the formalism). So in NAFL, the Main Postulate takes over and the truth/falsity of ψ is *axiomatic* in nature; i.e., ψ is true/false if and only if so asserted axiomatically in an interpretation QM* of QM. In the absence of any such axiomatic assertions, consistency of QM in NAFL demands that there *must* exist a non-classical model for QM in which ψ is neither true nor false. Whether such a non-classical model can exist in the present available formulations of QM is another matter; we assume that QM can be 'fixed' such that this is possible. Note that the superposition does *not* mean that 'The photon took path A and the photon took path B'; it only means that the photon took *neither* path A nor path B in the sense that neither of these alternatives have been asserted axiomatically (ie., can be proved) in the interpretation QM*. So NAFL confirms the Copenhagen interpretation of quantum mechanics, rather than the many-worlds interpretation.

It is clear from this description that NAFL treats 'measurement' as precisely equivalent to an *axiomatic declaration* of truth, in the interpretation QM* (which resides in the human mind). Since the notions of 'axiomatic declaration', 'interpretation' and 'model' are all part of the metatheory in NAFL, it follows that 'measurement' must also be outside the formalism (QM). In NAFL, when the human being makes a measurement, he/she simultaneously 'sees' the outcome and makes an axiomatic declaration of the observed outcome in QM*. If the observer chooses to keep the axiomatic declarations in tune with the observed outcomes, then there is a perfect correspondence between such axiomatic declarations and measurement in the real world. The assertion that one cannot measure $\psi \& \neg\psi$ is in correspondence with the NAFL requirement that $\psi \& \neg\psi$ is not in the t-syntax of QM and so can never be formally asserted as an axiom (i.e., the notion of 'undecidability' is not formalizable in NAFL).

It is also clear that NAFL does not assign any objective reality to the concept of a photon as a 'particle'. Indeed, if the photon were to be fixed as a particle in the sense that we denote this term, then $\psi \vee \neg\psi$ must unavoidably be an axiom of QM and hence the NAFL version of QM will be inconsistent (since the non-classical model required by metatheorem 2 of [4] cannot exist and also, $\psi \vee \neg\psi$ is not even in the t-syntax of QM). So the photon must be treated as having a non-constructive existence in NAFL. The superposition $\psi \& \neg\psi$ in the non-classical model only means that the human mind that inteprets QM has not declared either ψ or $\neg\psi$ axiomatically as true in QM* (which proves neither of these propositions). This is certainly true in the real world, provided, as observed earlier, QM* is kept in tune with real-world measurements.

Note that NAFL correctly handles the temporal aspect of truth in this experiment. The superposition $\psi \& \neg\psi$ applies *until* the measurement is actually made (in the detectors), at which point the observer switches to either ψ or $\neg\psi$ as axiomatic declarations in QM* in accordance with the measured result. Thus QM* is 'dynamic' and will change in time to suit real-world measurements/observations. Of course, NAFL does not insist in general that real-world obervations should be available or that QM* should be kept in tune with such observations even if available. Suppose the experiment is started at time $t = 0$ and the outcome ψ is observed at time $t = T$. Does this mean that the photon 'really' took path A for $0 < t < T$? Not in NAFL; NAFL can only tell us the real-world truth that the photon was not *observed/measured* to take either path A or path B during $0 < t < T$. *However*, the aposteriori conclusion 'At time $t \geq T$, I conclude that the photon took path A for $0 < t < T$' can certainly be formalized in NAFL. There is no contradiction here because this conclusion is valid only for $t \geq T$.

The Schrödinger cat experiment

Let ψ denote 'The cat is alive' and $\neg\psi$ denote 'The cat is dead'. Again, ψ is undecidable in QM. If the experiment is started at $t = 0$ and at $t = T$, say,

ψ is observed, then $\psi\&\neg\psi$ is the case (in a non-classical model for QM) for $0 < t < T$ and neither ψ nor $\neg\psi$ is provable in QM* for these times. For $t \geq T$, the observer declares the observed result (ψ in this case) axiomatically in QM*. Once again, the aposteriori conclusion that the cat was 'really' alive for $0 < t < T$ can be reached and formalized only for $t \geq T$; during the interval $0 < t < T$, NAFL only tells us the real-world truth that the cat has not been *observed* to be either alive or dead.

2.5 Two further examples of the NAFL concept of truth

We discuss here the coin toss experiment and Aristotle's example of 'There will be a sea-battle tomorrow'. Let the theory T_0 denote the null set of axioms.

The coin toss experiment

Let the observer toss a coin and before the outcome is observed, cover the coin, say, under the palm of his/her hand. Let ψ stand for 'The outcome is heads' and $\neg\psi$ for the 'The outcome is tails'. Again in NAFL, the superposition $\psi\&\neg\psi$ is the case in a non-classical model for T_0 until the outcome is actually observed to be either heads or tails (at which time the observer makes the appropriate axiomatic declaration in the interpretation T* of T_0). Let ϕ stand for 'The coin has landed flat under the palm of the observer's hand'. The important point of this experiment is that the observer *knows* that ϕ is 'true' in the real world. But in NAFL, ϕ is not part of the t-syntax of T_0, for clearly $\phi \Leftrightarrow \psi \vee \neg\psi$. But ϕ is legitimate in the p-syntax and the metatheory of T_0. No matter how 'real' ϕ seems to the observer, it cannot be formalized as a proposition of T_0 in NAFL. Indeed, ϕ represents the conclusion that the observer will observe either ψ or $\neg\psi$ if he/she lifts the palm of his/her hand; but in NAFL, 'measurement' or 'observation' is equivalent to an axiomatic declaration of truth and so is part of the metatheory and not formalizable, as discussed earlier.

Aristotle's sea-battle example

Finally, consider Aristotle's example 'There is/was/will be a sea-battle on a specified date and time between two specified nations at a specified location', where the appropriate tense is to be used depending on the time at which the proposition is considered; let ψ formalize this proposition in the NAFL theory T_0. The superposition will once again apply until the specified date and time when the observation is made by the observer. The details are similar to the previous examples. The interest in this example is for the case when say, a disaster takes place wiping out the two specified nations or the specified location prior to the specified time and date. The proposition will clearly become meaningless in classical logic. But in NAFL, the proposition can still be formalized; it no longer applies to the real world. The superposition will continue to apply because the observer will not have the chance to make any real-world observations. Of course, the observer may make fictitious declarations of truth in the interpretation T^* of T_0 at any given time.

3 Concluding remarks

In this paper, we have critically examined the concept of formalism and concluded that it is only NAFL, and not either classical or intuitionistic logic, that correctly embodies this concept. The result is that much of classical infinitary reasoning becomes invalid in consistent NAFL theories. Anything 'stronger' than the theory of finite sets F (or equivalently, Peano arithmetic) discussed in Sec. 2.3 will become inconsistent in NAFL. Hence Hilbert's program has been settled negatively – much of classical infinitary reasoning stands refuted from the finitary and formalistic standpoints. However, it is important to note that in inconsistent NAFL theories, it is still not possible to deduce *any* proposition; this is in contrast to inconsistent theories in classical/intuitionistic/constructive logics. It is still possible to permit infinite sets, say, in the p-syntax of F, to *prove* results, such as FLT, in the t-syntax. However, such a 'proof' is not rigorous by NAFL standards. So can the proof be trusted, i.e., is it true that we will never ever find a counter-example to FLT? In general, the answer must be in the negative; but nevertheless such inconsistent (by NAFL standards) infinitary reasoning may still be useful and convenient in real life, such as, in the case of the relativity theories. But while many of the predictions of such inconsistent theories are valid in the sense that they have been confirmed by experiment, *all* such predictions cannot be trusted in the absence of experimental data. For example, did the Big Bang occur? Do black holes exist? Will the universe shrink to a point as predicted by General Relativity? Is time travel possible as predicted by General Relativity? Can the concept of quantum computing be trusted? These are all questions that can only be answered empirically by experiments. A useful line of further research will be to see how infinite sets can be admitted into NAFL theories without sacrificing consistency; infinite sets, and quantification over infinitely many such sets are very useful in real analysis. A second line of research would be to abandon the concept of the continuum and look for a (possibly very complicated) finitary, discrete description of nature in a consistent NAFL theory in which *everything*, including space, time and matter, is quantized. This line of research is suggested by the fact that the very process by which quantum superposition is justified in NAFL is also responsible for the rejection of the relativity theories in NAFL via the constraint of consistency. Thus the suspected incompatibility between quantum mechanics and the relativity theories is clearly established in NAFL.

Dedication

This paper is dedicated to my son R. Anand and my wife R. Jayanti.

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