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# INDISTINGUISHABILITY

SIMON SAUNDERS

By the end of the 19th century the concept of particle indistinguishability had entered physics in two apparently quite independent ways: in statistical mechanics, where, according to Gibbs, it was needed in order to define an extensive entropy function; and in the theory of black-body radiation, where, according to Planck, it was needed to interpolate between the high frequency (Wien law) limit of thermal radiative equilibrium, and the low frequency (Rayleigh-Jeans) limit. The latter, of course, also required the quantization of energy, and the introduction of Planck's constant: the birth of quantum mechanics.

It was not only quantum mechanics. Planck's work, and later that of Einstein and Debye, foreshadowed the first quantum field theory as written down by Dirac in 1927. Indistinguishability is essential to the interpretation of quantum fields in terms of particles (Fock space representations), and thereby to the entire framework of high-energy particle physics as a theory of local interacting fields.

In this essay, however, we confine ourselves to particle indistinguishability in low energy theories, in quantum and classical statistical mechanics describing ordinary matter. We are also interested in indistinguishability as a *symmetry*, to be treated in a uniform way with other symmetries of physical theories, especially with space-time symmetries. That adds to the need to study permutation symmetry in classical theory – and returns us to Gibbs and the derivation of the entropy function.

The concept of particle indistinguishability thus construed faces some obvious challenges. It remains controversial, now for more than a century, whether classical particles *can* be treated as indistinguishable; or if they can, whether the puzzles raised by Gibbs are thereby solved or alleviated; and if so, how the differences between quantum and classical statistics are to be explained. The bulk of this essay is on these questions. In part they are philosophical. As Quine remarked:

Those results [in quantum statistics] seem to show that there is no difference even in principle between saying of two elementary particles of a given kind that they are in the respective places  $a$  and  $b$  and that they are oppositely placed, in  $b$  and  $a$ . It would seem then merely that elementary particles are unlike bodies; it would seem that there are no such denizens of space-time at all, and that we should speak of places  $a$  and  $b$  merely as being in certain states, indeed the same state, rather than as being occupied by two things. (Quine 1990, 35).

He was speaking of indistinguishable particles in quantum mechanics, but if particles in classical theory are treated the same way, the same questions arise.

This essay is organized in three sections. The first is on the Gibbs paradox and is largely expository. The second is on particle indistinguishability, and the explanation of quantum statistics granted that classical particles just like quantum particles can be treated as permutable. The third is on the more philosophical questions raised by sections 1 and 2, and on the question posed by Quine. There is a special difficulty in matters of ontology in quantum mechanics, if only because of the measurement problem.<sup>1</sup> I shall, so far as is possible, be neutral on this this. My conclusions apply to most realist solutions of the measurement problem, and even some non-realist ones.

## 1. THE GIBBS PARADOX

### 1.1 Indistinguishability and the quantum

Quantum theory began with a puzzle over the statistical equilibrium of radiation with matter. Specifically, Planck was led to a certain combinatorial problem: for each frequency  $\nu_s$ , what is the number of ways of distributing an integral number  $N_s$  of ‘energy elements’ over a system of  $C_s$  states (or ‘resonators’)?

The distribution of energy over each type of resonator must now be considered, first, the distribution of the energy  $E_s$  over the  $C_s$  resonators with frequency  $\nu_s$ . If  $E_s$  is regarded as infinitely divisible, an infinite number of different distributions is possible. We, however, consider - and this is the essential point -  $E_s$  to be composed of a determinate number of equal finite parts and employ in their determination the natural constant  $h = 6.55 \times 10^{-27}$  erg sec. This constant, multiplied by the frequency,  $\nu_s$ , of the resonator yields the energy element  $\Delta\epsilon_s$  in ergs, and dividing  $E_s$  by  $h\nu_s$ , we obtain the number  $N_s$ , of energy elements to be distributed over the  $C_s$  resonators. (Planck 1900, 239).<sup>2</sup>

Thus was made what is quite possibly the most successful single conjecture in the entire history of physics: the existence of Planck’s constant  $h$ , postulated in 1900 in the role of energy quantization.

The number of distributions  $Z_s$ , or *microstates* as we shall call them, as a function of frequency, was sought by Planck in an effort to apply Boltzmann’s statistical method to calculate the energy-density  $\bar{E}_s$  of radiative equilibrium as

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<sup>1</sup>See Wallace (2013), Bacciagaluppi (2013).

<sup>2</sup>I have used a different notation from Planck’s for consistency with the notation in the sequel.

a function of temperature  $T$  and of  $Z_s$ . To obtain agreement with experiment he found

$$Z_s = \frac{(N_s + C_s - 1)!}{N_s!(C_s - 1)!}. \quad (1)$$

The expression has a ready interpretation: it is *the number of ways of distributing  $N_s$  indistinguishable elements over  $C_s$  distinguishable cells* – of noting only *how many* elements are in which cell, not *which* element is in which cell.<sup>3</sup> Equivalently, the microstates are *distributions invariant under permutations*. When this condition is met, we call the elements *permutable*.<sup>4</sup> Following standard physics terminology, they are *identical* if these elements, independent of their microstates, have exactly the same properties (like charge, mass, and spin).

Planck’s ‘energy elements’ at a given frequency were certainly identical; but whether or not it followed that they *should* be considered permutable was hotly disputed. Once interpreted as particles (‘light quanta’), as Einstein proposed, there was a natural alternative: why not count microstates as distinct if they differ in which particle is located in which cell, as had Boltzmann in the case of material particles? On that count the number of distinct microstates should be:

$$Z_s = C_s^{N_s}. \quad (2)$$

Considered in probabilistic terms, again as Einstein proposed, if each of the  $N_s$  elements is assigned one of the  $C_s$  cells at random, independent of each other, the number of such assignments will be given by (2), each of them equiprobable.

But whilst (2) gave the correct behaviour for  $\bar{E}_s$  in the high-frequency limit (Wien’s law), it departed sharply from the Planck distribution at low frequencies. Eq.(1) was empirically correct, not (2). The implication was that if light was made of particles labelled by frequency, they were particles that could not be considered as independent of each other at low frequencies.<sup>5</sup>

Eq.(1) is true of bosons; bosons are represented by totally symmetrized states in quantum mechanics and quantum field theory; totally symmetrized states are entangled states. There is no doubt that Einstein, and later Schrödinger, were puzzled by the lack of independence of light-quanta at low frequencies. They were also puzzled by quantum non-locality and entanglement. It is tempting to view all these puzzles as related.<sup>6</sup> Others concluded that light could not after all be made of particles, or that it is made up of both particles and waves, or it

<sup>3</sup>A microstate as just defined can be specified as a string of  $N_s$  symbols ‘ $p$ ’ and  $C_s - 1$  symbols ‘ $|$ ’ (thus, for e.g.  $N_s = 3$ ,  $C_s = 4$ , the string  $p|pp|$  corresponds to one particle in the first cell, none in the second, two in the third, and none in the fourth). The number of distinct strings is  $(N_s + C_s - 1)!$  divided by  $(C_s - 1)!N_s!$ , because permutations of the symbol ‘ $|$ ’ among themselves or the symbol ‘ $p$ ’ among themselves give the same string. (This derivation of (1) was given by Ehrenfest in 1912.)

<sup>4</sup>I take ‘indistinguishable’ and ‘permutable’ to mean the same. But others take ‘indistinguishable’ to have a broader meaning, so I will give up that word and use ‘permutable’ instead.

<sup>5</sup>The *locus classicus* for this story is Jammer [1966], but see also Darrigol [1991].

<sup>6</sup>Or as at bottome the same, as argued most prominently by Howard [1990].

is made up of a special category of entities that are not really objects at all.<sup>7</sup> We shall come back to these questions separately.

For Planck's own views on the matter, they were perhaps closest to Gibbs'.<sup>8</sup> Gibbs had arrived at the concept of particle indistinguishability quite independent of quantum theory. To understand this development, however, considerably more stage-setting is needed, in both classical statistical mechanics and thermodynamics, the business of sections 1.2-1.4. (Those familiar with the Gibbs paradox may skip directly to 1.5.)

## 1.2 The Gibbs paradox in thermodynamics

Consider the entropy of a volume  $V$  of gas composed of  $N_A$  molecules of kind  $A$  and  $N_B$  molecules of kind  $B$ .<sup>9</sup> It differs from the entropy of a gas at the same temperature and pressure when  $A$  and  $B$  are identical. The difference is:

$$-kN_A \log N_A - kN_B \log N_B + k(N_A + N_B) \log(N_A + N_B) \quad (3)$$

where  $k$  is Boltzmann's constant,  $k = 1.38 \times 10^{-16}$  erg  $K^{-1}$ . The expression (3) is unchanged no matter how similar  $A$  and  $B$  are, even when in practise the two gases cannot be distinguished; but it must vanish when  $A$  and  $B$  are the same. This is the Gibbs paradox in thermodynamics.

It is not clear that the puzzle as stated is really paradoxical, but it certainly bears on the notion of identity – and on whether identity admits of degrees. Thus Denbigh and Redhead argue:

The entropy of mixing has the same value...however alike are the two substances, but suddenly collapses to zero when they are the same. It is the absence of any 'warning' of the impending catastrophe, as the substances are made more and more similar, which is the truly paradoxical feature (Denbigh and Redhead [1989, 284].)

The difficulty is more severe for those who see thermodynamics as founded on operational concepts. Identity, as distinct from similarity under all practical measurements, seems to outstrip any possible experimental determination.

To see how experiment does bear on the matter, recall that the classical thermodynamic entropy is an *extensive* function of the mass (or particle number) and volume. That is to say, for real numbers  $\lambda$ , the thermodynamic entropy  $S$  as a function of  $N$  and  $V$  scales linearly:

$$S(\lambda N, \lambda V, T) = \lambda S(N, V, T), \quad \lambda \in \mathbb{R}.$$

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<sup>7</sup>As suggested by Quine. See French and Kraus [2006] for a comprehensive survey of debates of this kind.

<sup>8</sup>See Planck [1912], [1921] and, for commentary, Rosenfeld [1959].

<sup>9</sup>This section largely follows van Kampen [1984].

By contrast the pressure and temperature are *intensive* variables that do not scale with mass and volume. The thermodynamic entropy function for an ideal gas is:

$$S(P, T, N) = \frac{5}{2}Nk \log T - Nk \log P + cN \quad (4)$$

where  $c$  is an arbitrary constant. It is extensive by inspection.

The extensivity of the entropy allows one to define the analogue of a density – entropy per unit mass or unit volume – important to non-equilibrium thermodynamics, but the concept clearly has its limits: for example, it is hardly expected to apply to gravitating systems, and more generally ignores surface effects and other sources of inhomogeneity. It is to be sharply distinguished from *additivity* of the entropy, needed to define a total entropy for a collection of equilibrium systems each separately described – typically, as (at least initially) physically isolated systems. The assumption of additivity is that a total entropy can be defined as their sum:

$$S_{A+B} = S_A(N_A, V_A, T_A) + S_B(N_B, V_B, T_B).$$

It is doubtful that any *general* statement of the second law would be possible without additivity. Thus, collect together a dozen equilibrium systems, some samples of gas, others homogeneous fluids or material bodies, initially isolated, and determine the entropy of each as a function of its temperature, volume, and mass. Energetically isolate them from external influences, but allow them to interact with each other in any way you like (mechanical, thermal, chemical, nuclear), so long as the result is a new collection of equilibrium systems. Then the second law can be expressed as: the sum of the entropies of the latter systems is equal to or greater than the sum of the entropies of the former systems.<sup>10</sup>

Now for the connection with the Gibbs paradox. The thermodynamic entropy difference between states 1 and 2 is defined as the integral, over any *reversible* process<sup>11</sup> that links the two states, of  $dQ/T$ , that is as the quantity:

$$\Delta S = S_2 - S_1 = \int_1^2 \frac{dQ}{T}$$

where  $dQ$  is the heat transfer. If the insertion or removal of a partition between  $A$  and  $B$  is to count as a reversible process, then from additivity and given that negligible work is done on the partition it follows there will be no change in entropy, so no entropy of mixing. This implies the entropy must be extensive. Conversely extensivity, under the same presupposition, and again given additivity, implies there is no entropy of mixing.

Whether or not the removal of a partition between  $A$  and  $B$  *should* count as a reversible process is another matter: surely not if means are available to tell the two gases apart. Thus if a membrane is opaque to  $A$ , transparent to  $B$ ,

<sup>10</sup>See Lieb and Yngvason [1999] for a statement of the second law at this level of generality.

<sup>11</sup>Meaning a process which at any point in its progress can be reversed, to as good an approximation as is required. Necessary conditions are that temperature gradients are small and effects due to friction and turbulence are small (but it is doubtful these are sufficient).

under compression work  $P_A dV$  must be done against the partial pressure  $P_A$  in voiding one part of the cylinder of gas  $A$  (and similarly for  $B$ ), where:

$$P_A = \frac{N_A}{N_A + N_B} P, \quad P_B = \frac{N_B}{N_A + N_B} P.$$

The work  $dW$  required to separate the two gases isothermally at temperature  $T$  is related to the entropy change and the heat transfer by:

$$dQ = dW + TdS.$$

Using the equation of state for the ideal gas to determine  $dW = PdV$

$$PV = kNT$$

where  $N = N_A + N_B$ , the result is the entropy of mixing, Eq.(3). However, there can be no such semi-permeable membrane when the two gases are identical,<sup>12</sup>

Would it matter to the latter conclusion if the differences between the two gases were sufficiently small (were ignored or remained undiscovered)? But as van Kampen argues, it is hard to see how the chemist will be led into any *practical* error in ignoring an entropy of mixing, if he cannot take mechanical advantage of it. Most thermodynamic substances, in practise, are composites of two or more substances (typically, different isotopes), but such mixtures are usually treated as homogeneous. In thermodynamics, as a science based on operational concepts, the meaning of the entropy function does not extend beyond the competencies of the experimenter:

Thus, whether such a process is reversible or not depends on how discriminating the observer is. The expression for the entropy depends on whether or not he is able and willing to distinguish between the molecules  $A$  and  $B$ . This is a paradox only for those who attach more physical reality to the entropy than is implied by its definition. (Van Kampen [1984, 307].)

A similar resolution of the Gibbs paradox was given by Jaynes [1992]. It appears, on this reading, that the entropy is not a real physical property of a thermodynamic system, independent of our knowledge of it. According to Van Kampen, it is attributed to a system on the basis of a system of *conventions* – on whether the removal of a partition is to be counted as a reversible process, and on whether or not the entropy function for the two samples of gas is counted as extensive. That explains why the entropy of mixing is an all-or-nothing affair.

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<sup>12</sup>At least in the absence of Maxwell demons: see section 3.1.

### 1.3 The Gibbs paradox in statistical mechanics

Thermodynamics is the one fundamental theory of physics that might lay claim to being based on operational concepts and definitions. The situation is different in statistical mechanics, where the concept of entropy is not limited to equilibrium states, nor bound to the concept of reversibility.

There is an immediate difficulty, however; for the classical derivations of the entropy in statistical mechanics yield a function that is *not* extensive, even as an idealization. That is, classically, there is *always* an entropy of mixing, even for samples of the same gas. If the original Gibbs paradox was that there was no entropy of mixing in the limit of identity, the new paradox is that there is.<sup>13</sup>

To see the nature of the problem, it will suffice to consider the ideal gas, using the Boltzmann definition of entropy, so-called<sup>14</sup>. The state of a system of  $N$  particles is represented by a set of  $N$  points in the 6-dimensional 1-particle phase space (or  $\mu$ -space), or equivalently, by a single point in the total  $6N$ -dimensional phase space  $\Gamma^N$ . A *fine-graining* of  $\Gamma^N$  is a division of this space into cells of equal volume  $\tau^N$  (corresponding to a division of  $\mu$ -space into cells of volume  $\tau$ , where  $\tau$  has dimensions of [momentum]<sup>3</sup>[length]<sup>3</sup>). A *coarse-graining* is a division of  $\Gamma^N$  into regions with a given range of energy. For weakly interacting particles these regions can be parametrized by the one-particle energies  $\epsilon_s$ , with  $N_s$  the number of particles with energy in the range  $[\epsilon_s, \epsilon_s + \Delta\epsilon_s]$ , and the coarse-graining extended to  $\mu$ -space as well. These numbers must satisfy:

$$\sum_s N_s = N; \quad \sum_s N_s \epsilon_s = E \quad (5)$$

where  $E$  is the total energy. Thus, for any fine-grained description (microstate) of the gas, which specifies how, for each  $s$ ,  $N_s$  particles are distributed over the fine-graining, there is a definite coarse-grained description (macrostate) which only specifies the number in each energy range. Each macrostate corresponds to a definite volume of phase space.

We can now define the Boltzmann entropy of a gas of  $N$  particles in a given microstate: *it is proportional to the logarithm of the volume, in  $\Gamma^N$ , of the corresponding macrostate*. In this the choice of  $\tau$  only effects an additive constant, irrelevant to entropy differences.

This entropy is computed as follows. For each  $s$ , let there be  $C_s$  cells in  $\mu$ -space of volume  $\tau$  bounded by the energies  $\epsilon_s, \epsilon_s + \Delta\epsilon$ , containing  $N_s$  particles. Counting microstates as distinct if they differ in which particles are in which cells, we use (2) for the number of microstates, each with the same phase space volume  $\tau^{N_s}$ , yielding the volume:

$$Z_s \tau^{N_s} = C_s^{N_s} \tau^{N_s}. \quad (6)$$

<sup>13</sup>I owe this turn of phrase to Jos Uffink.

<sup>14</sup>Boltzmann defined the entropy in several different ways; see Bach [1990].

The product of these quantities (over  $s$ ) is the  $N$ -particle phase-space volume of the macrostate  $N_1, N_2, \dots, N_s, \dots$  for just *one* way of partitioning the  $N$  particles among the various 1-particle energies. There are

$$\frac{N!}{N_1! \dots N_s! \dots} \quad (7)$$

partitionings in all. The total phase space volume  $W^B$  (' $B$ ' is for Boltzmann) of the macrostate  $N_1, N_2, \dots, N_s, \dots$  is the product of terms (6) (over  $s$ ) and (7):

$$W^B = \frac{N!}{N_1! \dots N_s! \dots} \prod_s C_s^{N_s} \tau^{N_s} \quad (8)$$

and the entropy is:

$$S^B = k \log W^B = k \log \left[ \frac{N!}{N_1! \dots N_s! \dots} \prod_s C_s^{N_s} \tau^{N_s} \right].$$

From the Stirling approximation for  $x$  large,  $\log x! \approx x \log x - x$ :

$$S^B \approx kN \log N + k \sum_s N_s \log \frac{C_s}{N_s} + kN \log \tau. \quad (9)$$

By inspection, this entropy function is not extensive. When the spatial volume and particle number are doubled, the second and third expressions on the RHS scale properly, but not the first. This picks up a term  $kN \log 2$ , corresponding to the  $2^N$  choices as to which of the two sub-volumes contains which particle.

One way to obtain an extensive entropy function is to simply subtract the term  $kN \log N$ . In the Stirling approximation (up to a constant scaling with  $N$  and  $V$ ) that is equivalent to dividing the volume (8) by  $N!$ . But with what justification? If, after all, permutations of particles did not yield distinct fine-grained distributions, the factor (7) would not be divided by  $N!$ ; it would be set equal to unity. Call this *the  $N!$  problem*. This is itself sometimes called the Gibbs paradox, but is clearly only a fragment of it. It is the main topic of sections 1.5 and 2.1.

## 1.4 The equilibrium entropy

Although not needed in the sequel, for completeness we obtain the equilibrium entropy, thus making the connection with observable quantities.<sup>15</sup>

A system is in equilibrium when the entropy of its coarse-grained distribution is a maximum; that is, when the entropy is stationary under variation of the numbers  $N_s \rightarrow N_s + \delta N_s$ , consistent with (5), i.e. from (9):

$$0 = \delta \bar{S}^B = \sum_s [\delta N_s \log C_s - \delta N_s \log N_s - \delta N_s] \quad (10)$$

<sup>15</sup>For a text-book derivation using our notation, see e.g. Hercus [1950].



where

$$\sum_s \delta N_s = 0; \quad \sum_s \delta N_s \epsilon_s = 0. \quad (11)$$

If the variations  $\delta N_s$  were entirely independent, each term in the summand (10) would have to vanish. Instead introduce Lagrange multipliers  $\alpha$ ,  $\beta$  for the respective constraint equations (11). Conclude for each  $s$ :

$$\log C_s - \log N_s - \alpha - \beta \epsilon_s = 0.$$

Rearranging:

$$N_s = C_s (e^{-\alpha - \beta \epsilon_s}). \quad (12)$$

Substituting in (9) and using (5) gives the equilibrium entropy  $\bar{S}^B$ :

$$\begin{aligned} \bar{S}^B &= kN \log N + k \sum_s N_s (\alpha + \beta \epsilon_s) + kN \log \tau \\ &= kN \log N + kN \alpha + k\beta E + kN \log \tau. \end{aligned} \quad (13)$$

The values of  $\alpha$  and  $\beta$  are fixed by (5) and (12). Replacing the schematic label  $s$  by coordinates on phase space for a monatomic gas  $\vec{x}$ ,  $\vec{p}$ , with  $\epsilon_s$  the kinetic energy  $\frac{1}{2m} \vec{p}^2$ , the sum over  $N_s$  in the first equation of (5) becomes:

$$e^{-\alpha} \int_V \int e^{-\frac{\beta}{2m} \vec{p}^2} d^3x d^3p = N.$$

The spatial integral gives the volume  $V$ ; the momentum integral gives  $(2\pi m/\beta)^{3/2}$ , so

$$e^{-\alpha} = \frac{N}{V} (2\pi m/\beta)^{-3/2}.$$

From the analogous normalization condition on the total energy (the second constraint (5)), substituting (12) and given that for an ideal monatomic gas  $E = \frac{3}{2} NkT$ , deduce that  $\beta = \frac{1}{kT}$ . Substituting in (13), the equilibrium entropy is:

$$\bar{S}^B(N, V, T) = Nk \log V + \frac{3}{2} kN \log 2\pi mkT + \frac{3}{2} Nk + Nk \log \tau.$$

It is clearly not extensive. Compare Eq.(4), which using the equation of state for the ideal gas takes the form (the Sackur-Tetrode equation):

$$S(N, V, T) = Nk \log \frac{V}{N} + \frac{3}{2} kN \log 2\pi mkT + cN$$

where  $c$  is an arbitrary constant. They differ by the term  $Nk \log N$ , as already noted.

## 1.5 The $N!$ puzzle

The  $N!$  puzzle is this: what justifies the subtraction of the term  $Nk \log N$  from the entropy? Or equivalently, what justifies the division of the phase space

volume Eq.(8) by  $N!$ ? In fact it has a fairly obvious answer (see section 2.1): classical particles, if identical, should be treated as permutable, just like identical quantum particles. But this suggestion has rarely been taken seriously.

Much more widely favoured is the view that quantum theory is needed. Classical statistical mechanics is not after all a correct theory; quantum statistical mechanics (Eq.(1)), in the dilute limit  $C_s \gg N_s$ , gives:

$$Z_s = \frac{(N_s + C_s - 1)!}{N_s!(C_s - 1)!} \approx \frac{C_s^{N_s}}{N_s!}$$

yielding the required correction to (6) (setting (7) to unity). Call this the *orthodox* solution to the  $N!$  puzzle.

This reasoning, so far as it goes, is perfectly sound, but it does not go very far. It says nothing about *why* particles in quantum theory but not classical theory are permutable. If rationale is offered, it is that classical particles are localized in space and hence are distinguishable (we shall consider this in more detail in the next section); and along with that, that the quantum state for identical particles is unchanged.<sup>16</sup> But how the two are connected is rarely explained.

Erwin Schrödinger, in his book *Statistical Thermodynamics*, did give an analysis:

It was a famous paradox pointed out for the first time by W. Gibbs, that the same increase of entropy must not be taken into account, when the two molecules are of the same gas, although (according to naive gas-theoretical views) diffusion takes place then too, but unnoticeably to us, because all the particles are alike. The modern view [of quantum mechanics] solves this paradox by declaring that in the second case there is no real diffusion, because exchange between like particles is not a real event - if it were, we should have to take account of it statistically. It has always been believed that Gibbs's paradox embodied profound thought. That it was intimately linked up with something so important and entirely new [as quantum mechanics] could hardly be foreseen. ((Schrödinger [1946, 61].))

Evidently, by 'exchange between like particles' Schrödinger meant the sort of thing that happens when gases of classical molecules diffuse – the trajectories of individual molecules are twisted around one another – in contrast to the behaviour of quantum particles, which do not have trajectories, and so do not diffuse in this way. But as for *why* the exchange of quantum particles 'is not a real event' (whereas it is classically) is lost in the even more obscure question of what quantum particles really are. Schrödinger elsewhere said something more, He wrote of indistinguishable particles as 'losing their identity', as 'non-individuals', in the way of units of money in the bank (they are 'fungible'). That fitted with Planck's original idea of indistinguishable quanta as elements of energy, rather than material things – so, again, quite unlike classical particles.

<sup>16</sup>Statements like this can be found in almost any textbook on statistical mechanics.

On this point there seems to have been wide agreement. Schrödinger's claims about the Gibbs paradox came under plenty of criticism, for example, by Otto Stern, but Stern remarked at the end:

In conclusion, it should be emphasized that in the foregoing remarks classical statistics is considered in principle as a part of classical mechanics which deals with individuals (Boltzmann). The conception of atoms as particles losing their identity cannot be introduced into the classical theory without contradiction. (Stern [1949, 534].)

This comment or similar can be found scattered throughout the literature on the foundations of quantum statistics.

There is a second solution to the  $N!$  puzzle that goes in the diametrically-opposite direction: it appeals only to classical theory, precisely *assuming* particle distinguishability. Call this the 'classical' solution to the puzzle.

Its origins lie in a treatment by Ehrenfest and Trkal [1920] of the equilibrium conditions for molecules subject to disassociation into a total of  $N^*$  atoms. This number is conserved, but the number of molecules  $N_A, N_B, \dots$  formed of these atoms, of various types  $A, B, \dots$  may vary. The dependence of the entropy function on  $N^*$  is not needed since this number never changes: it is the dependence on  $N_A, N_B, \dots$  that is relevant to the extensivity of the entropy for molecules of type  $A, B, \dots$ , which can be measured. By similar considerations as in section 1.3, the number of ways the  $N^*$  atoms can be partitioned among  $N_A$  molecules of type  $A, N_B$  molecules of type  $B, \dots$  is the factor  $N^*/N_A!N_B!\dots$ . This multiplies the product of all the phase space volumes for each type of molecule, delivering the required division by  $N_A!$  for molecules of type  $A$ , by  $N_B!$  for molecules of type  $B$ , and so on (with the dependence on  $N^*$  absorbed into an overall constant).

A similar argument was given by van Kampen [1984], but using Gibbs' methods. The canonical ensemble for a gas of  $N^*$  particles has the probability distribution:

$$W(N^*, q, p) = f(N^*)e^{-\beta H(q,p)}.$$

Here  $(q, p)$  are coordinates on the  $6N^*$  dimensional phase space for the  $N^*$  particles, which we suppose are confined to a volume  $V^*$ ,  $H$  is the Hamiltonian, and  $f$  is a normalization constant. Let us determine the probability of finding  $N$  particles with total energy  $E$  in the sub-volume  $V$  (so  $N' = N^* - N$  are in volume  $V' = V^* - V$ ). If the interaction energy between particles in  $V'$  and  $V$  is small, the Hamiltonian  $H_{N^*}$  of the total system can be approximately written as the sum  $H_N + H_{N'}$  of the Hamiltonians for the two subsystems. The probability density  $W(N, q, p)$  for  $N$  particles as a function of  $\langle N, q, p \rangle = \langle \vec{q}_1, \vec{p}_1; \vec{q}_2, \vec{p}_2; \dots; \vec{q}_N, \vec{p}_N \rangle$  where  $\vec{q}_i \subset V$  is then the marginal on integrating out the remaining  $N'$  particles in  $V'$ , multiplied by the number of ways of selecting  $N$  particles from  $N^*$  particles. The latter is given by the binomial function:

$$\binom{N^*}{N} = \frac{N^*!}{(N^* - N)!N!}.$$

The result is:

$$W(N, q, p) = f(N^*) \binom{N^*}{N} e^{-\beta H_N(N, q, p)} \int_{V'} e^{-\beta H_{N'}(N', q', p')} dq' dp'.$$

In the limit  $N^* \gg N$ , the binomial is to a good approximation:

$$\frac{N^*!}{(N^* - N)!N!} \approx \frac{N^{*N}}{N!}.$$

The volume integral yields  $V'^{N^* - N}$ . For non-interacting particles, for constant density  $\rho = V'/N'$  in the large volume limit  $V' \gg V$  we obtain:

$$W(N, q, p) \approx f(N^*, V^*) \frac{z^N}{N!} e^{-\beta H_N(q, p)}$$

where  $z$  is a function of  $\rho$  and  $\beta$ . It has the required division by  $N!$ .

Evidently this solution to the  $N!$  puzzle is the same as in Ehrenfest and Trkal's derivation: extensivity of the entropy can only be obtained for an open system, that is, for a proper subsystem of a closed system, never for a closed one – and it follows precisely *because* the particles are non-permutable. The tables are thus neatly turned.<sup>17</sup>

Which of the two, the orthodox or the classical, is the 'correct' solution to the  $N!$  puzzle? It is tempting to say that *both* are correct, but as answers to different questions: the orthodox solution is about the thermodynamics of real gases, governed by quantum mechanics, and the classical solution is about the consistency of a hypothetical classical system of thermodynamics that in reality does not exist. But on either line of reasoning, identical quantum particles are treated as radically unlike identical classical particles (only the former are permutable).<sup>18</sup> This fits with the standard account of the departures of quantum from classical statistics: they are explained by permutability. But it is a false dichotomy.

## 2. INDISTINGUISHABILITY AS A UNIFORM SYMMETRY

### 2.1 Gibbs' solution

There is another answer as to which of the two solutions to the  $N!$  puzzle is correct: *neither*. The  $N!$  puzzle arises in both classical and quantum theories

<sup>17</sup>For another variant of the Ehrenfest-Trkall approach, see Swendsen [2002, 2006, Nagle [2004].)

<sup>18</sup>Note added Sep 2016. An exception is Shigeji Fujita, who in a much-neglected article (neglected by me) argued that indistinguishability is inherited from quantum statistical mechanics in the classical limit and therefore that classical particles are permutable just as are quantum particles. See Fujita [1990].

and is solved in exactly the same way: by passing to the quotient space (of phase space and Hilbert space respectively). This is not to deny that atoms really are quantum mechanical, or that measurements of the dependence of the entropy on particle number are made in the way that Ehrenfest et al envisaged; it is to deny that the combinatorics factors thus introduced are, except in special cases, either justified or needed.

Gibbs, in his *Elementary Principles in Statistical Mechanics*, put the matter as follows:

If two phases differ only in that certain entirely similar particles have changed places with one another, are they to be regarded as identical or different phases? If the particles are regarded as indistinguishable, it seems in accordance with the spirit of the statistical method to regard the phases as identical. (Gibbs [1902,187].)

He proposed that the phase of an  $N$ -particle system be unaltered ‘by the exchange of places between similar particles’. Phases (points in phase space) like this he called ‘generic’ (and those that are altered, ‘specific’). The state space of generic phases is the *reduced phase space*  $\Gamma^N/\Pi_N$ , the quotient space under the permutation group  $\Pi_N$  of  $N$  elements. In this space points of  $\Gamma^N$  related by permutations are identified.

The suggestion is that even classically, the expressions (6) and (7) are wrong. (7) is replaced by unity (as already noted): there is just one way of partitioning  $N$  permutable particles among the various states so as to give  $N_s$  particles to each state. But (6) is wrong too: it should be replaced by the volume of *reduced* phase space corresponding to the macrostate (for  $s$ ), the volume

$$\frac{(C_s \tau)^{N_s}}{N_s!}.$$

For the macrostate  $N_1, N_2, \dots, N_s, \dots$  the total reduced volume, denote  $W^{red}$  is:

$$W^{red} = \prod_s \frac{C_s^{N_s} \tau^{N_s}}{N_s!} = \frac{W^B}{N!}. \quad (14)$$

The derivation does not depend on the limiting behaviour of Eq.(1), or on the assumption of equiprobability or equality of volume of each fine-grained distribution (and is in fact in contradiction with that assumption, as we shall see).

Given (14), there is no entropy of mixing. Consider a system of particles all with the same energy  $\epsilon_s$ . The total entropy before mixing is, from additivity:

$$S_A + S_B = k \log \left( \frac{C_A^{N_A} \tau^{N_A}}{N_A!} \frac{C_B^{N_B} \tau^{N_B}}{N_B!} \right). \quad (15)$$

After mixing, if  $A$  and  $B$  are identical:

$$S_{A+B} = k \log \frac{(C_A + C_B)^{N_A+N_B} \tau^{N_A+N_B}}{(N_A + N_B)!}. \quad (16)$$

If the pressure of the two samples is initially the same (so  $C_A/N_A = C_B/N_B$ ), the quantities (15), (16) should be approximately equal<sup>19</sup> – as can easily be verified in the Stirling approximation. But if  $A$  and  $B$  are not identical, and permutations of  $A$  particles with  $B$  particles isn't a symmetry, we pass to the quotient spaces under  $\Pi_{N_A}$  and  $\Pi_{N_B}$  separately and take their product, and the denominator in (16) should be  $N_A!N_B!$ . With that  $S_A + S_B$  and  $S_{A+B}$  are no longer even approximately the same.

Gibbs concluded his discussion of whether to use generic or specific phases with the words, "The question is one to be decided in accordance with the requirements of practical convenience in the discussion of the problems with which we are engaged" (Gibbs [1902, 188].) Practically speaking, if we are interested in defining an extensive classical entropy function (even for closed systems), use of the generic phase (permutability) is clearly desirable. On the other hand, integral and differential calculus is simple on manifolds homeomorphic to  $\mathbb{R}^{6N}$ , like  $\Gamma^N$ ; the reduced phase space  $\Gamma^N/\Pi_N$  has by contrast a much more complex topology (a point made by Gibbs). If the needed correction, division by  $N!$ , can be simply made at the end of a calculation, the second consideration will surely trump the first.

## 2.2. Arguments against classical indistinguishability

Are there principled arguments against permutability thus treated uniformly, the same in the classical as in the quantum case? The concept of permutability can certainly be misrepresented. Thus, classically, *of course* it makes sense to move atoms about so as to interchange one with another, for particles have definite trajectories; in that sense an 'exchange of places' must make for a real physical difference, and in that sense 'indistinguishability' cannot apply to classical particles.

But that is not what is meant by 'interchange' – Schrödinger was just misleading on this point. It is interchange of points in phase space whose significance is denied, not in configuration space over time. Points in phase-space are in 1 : 1 correspondence with the dynamically allowed trajectories. A system of  $N$  particles whose trajectories in  $\mu$ -space swirl about one another, leading to an exchange of two or more of them in their places in space at two different times, is described by *each* of  $N!$  points in the  $6N$ -dimensional phase space  $\Gamma^N$ , each faithfully representing the same swirl of trajectories in  $\mu$ -space (but assigning different labels to each trajectory). In passing to points of the quotient space  $\Gamma^N/\Pi_N$  there is therefore no risk of descriptive inadequacy in representing particle interchange in Schrödinger's sense.

Another and more obscure muddle is to suppose that points of phase space can only be identified insofar as they are all traversed by one and the same trajectory. That appears to be the principle underlying van Kampen's argument:

<sup>19</sup>Should they be exactly equal? No, because it is an *additional* constraint to insist, given that  $N_A + N_B$  particles are in volume  $V_A + V_B$ , that exactly  $N_A$  are in  $V_A$  and  $N_B$  in  $V_B$ .

One could add, as an aside, that the energy surface can be partitioned in  $N!$  equivalent parts, which differ from one another only by a permutation of the molecules. The trajectory, however, does not recognize this equivalence because it cannot jump from one point to an equivalent one. There can be no good reason for identifying the  $Z$ -star [the region of phase space picked out by given macroscopic conditions] with only one of these equivalent parts. (Van Kampen 1984, 307).

But if the whole reason to consider the phase-space volumes of macrostates in deriving thermodynamic behaviour is because (say by ergodicity) they are proportional to the amount of time the system spends in the associated macrostates, then, *just because* the trajectory cannot jump from one point to an equivalent one, it should enough to consider only one of the equivalent parts of the  $Z$ -star. We should draw precisely the opposite conclusion to van Kampen.

However van Kampen put the matter somewhat differently – in terms, only, of probability:

Gibbs argued that, since the observer cannot distinguish between different molecules, "it seems in accordance with the spirit of the statistical method" to count all microscopic states that differ only by a permutation as a single one. Actually it is exactly opposite to the basic idea of statistical mechanics, namely that the probability of a macrostate is given by the measure of the  $Z$ -star, i.e. *the number of corresponding, macroscopically indistinguishable microstates*. As mentioned...it is impossible to justify the  $N!$  as long as one restricts oneself to a single closed system. (van Kampen 1984, 309, emphasis added).

Moreover, he speaks of probabilities of macroscopically indistinguishable microstates, whereas the contentious question concerns *microscopically* indistinguishable microstates. The contentious question is whether microstates that differ only by particle permutations, with all physical properties unchanged – which are in this sense indistinguishable – should be identified.

Alexander Bach in his book *Classical Particle Indistinguishability* defended the concept of permutability of states in classical statistical mechanics, understood as the requirement that probability distributions over microstates be invariant under permutations. But what he meant by this is the invariance of functions on  $\Gamma^N$ . As such, as probability measures, they could never provide *complete* descriptions of the particles (unless all their coordinates coincide) – they could not be concentrated on individual trajectories. He called this the 'deterministic setting'. In his own words:

**Indistinguishable Classical Particles Have No Trajectories.**

The unconventional role of indistinguishable classical particles is best expressed by the fact that in a deterministic setting no indistinguishable particles exist, or - equivalently - that indistinguishable classical

particles have no trajectories. Before I give a formal proof I argue as follows. Suppose they have trajectories, then the particles can be identified by them and are, therefore, not indistinguishable. (Bach 1997, 7).

His formal argument was as follows. Consider the coordinates of two particles at a given time. in 1-dimension, as an extremal of the set of probability measures  $M_+^1(\mathbb{R}^2)$  on  $\mathbb{R}^2$  (a 2-dimensional configuration space), from which, assuming the two particles are impenetrable, the diagonal  $D = \{ \langle x, x \rangle \in \mathbb{R}^2, x \in \mathbb{R} \}$  has been removed. Since indistinguishable, the state of the two particles must be unchanged under permutations (permutability), so it must be in  $M_{+,sym}^1(\mathbb{R}^2)$ , the space of symmetrized measures. It consists of sums of delta functions of the form:

$$\mu_{x,y} = \frac{1}{2} (\delta_{\langle x,y \rangle} + \delta_{\langle y,x \rangle}), \quad \langle x, y \rangle \in \mathbb{R}^2 \setminus D$$

But no such state is an extremal of  $M_+^1(\mathbb{R}^2)$ .

As already remarked, the argument presupposes that the coordinates of the two particles defines a point in  $M_+^1(\mathbb{R}^2)$ , the unreduced space, rather than in  $M_+^1(\mathbb{R}^2/\Pi_2)$ , the space of probability measures over the reduced space  $\mathbb{R}^2/\Pi_2$ . In the latter case, since  $M_+^1(\mathbb{R}^2/\Pi_2)$  is isomorphic to  $M_{+,sym}^1(\mathbb{R}^2)$ , there is no difficulty.<sup>20</sup>

Bach's informal argument above is more instructive. Why not use the trajectory of a particle to identify it, by the way it twists and turns in space? Why not indeed: it that is all there is to being a particle, you have already passed to a trajectory in the quotient space  $\Gamma^N/\Pi_N$ , for those related by permutations twist and turn in exactly the same way. The concept of particle distinguishability is not about the trajectory or the one-particle state: it is about the *label* of the trajectory or the one-particle state, or equivalently, the question of *which* particle has that trajectory, that state.

## 2.3 Haecceitism

Gibbs' suggestion was called 'fundamentally idealistic' by Rosenfeld, 'mystical' by van Kampen, 'inconsistent' by Bach; they were none of them prepared to see in indistinguishability the rejection of what is on first sight a purely metaphysical doctrine – that after every describable characteristic of a thing has been accounted for, there still remains the question of *which* thing has those characteristics.

The key word is 'every'; describe a thing only partly, and the question of which it is of several *more* precisely described things is obviously physically meaningful. But microstates, we take it, are maximal, complete descriptions. If there is a more complete level of description it is the microstate as given by

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<sup>20</sup>Bach's proof, if sound, would imply that corpuscles in the de Broglie-Bohm pilot-wave theory are distinguishable (for discussion of particle indistinguishability in pilot-wave theory, see e.g. Brown et al (1999)).



another theory, or at a deeper level of description by the same theory, and to the latter our considerations apply.

The doctrine, now that we have understood it correctly, has a suitably technical name in philosophy. It is called *haecceitism*. Its origins are medieval if not ancient, and it was in play, one way or another, in a connected line of argument from Newton and Clarke to Leibniz and Kant. That centered on the need, given symmetries, including permutations, not just for symmetry-breaking in the choice of initial conditions,<sup>21</sup> but for a choice among haecceistic differences – in the case of continuous symmetries, among values of absolute positions, absolute directions, and absolute velocities. All parties to this debate agreed on haecceitism. These choices were acts of God, with their consequences visible only to God (Newton, Clarke); or they were humanly visible too, but in ways that couldn't be put into words – that could only be grasped by 'intuition' (Kant); or they involved choices *not* even available to God, who can only choose on the basis of reason; so there could be no created things such as indistinguishable atoms or points of a featureless space (Leibniz).<sup>22</sup>

So much philosophical baggage raises a worry in its own right. If it is the truth or falsity of haecceitism that is at issue, it seems unlikely that it can be settled by any empirical finding. If that is what the extensivity of the entropy is about, perhaps extensivity has no real physical meaning after all. It is, perhaps, itself metaphysical – or conventional. This was the view advocated by Nick Huggett when he first drew the comparison between Boltzmann's combinatorics and haecceitism.<sup>23</sup>

But this point of view is only remotely tenable if haecceitism is similarly irrelevant to empirical questions in quantum statistics. And on the face of it that *cannot* be correct. Planck was, after all, led by *experiment* to Eq.(1). Use of the unreduced state space in quantum mechanics rather than the reduced (symmetrized) space surely has direct empirical consequences.

Against this two objections can be made. The first, following Reichenbach, is that the important difference between quantum and classical systems is the absence in quantum theory of a criterion for the re-identification of identical particles over time. They are, for this reason, 'non-individuals' (this links with Schrödinger's writings<sup>24</sup>). This, rather than any failure of haecceitism, is what is responsible for the departures from classical statistics.<sup>25</sup> The second, following Post [1970] and French and Redhead [1989] is that haecceitism must be consistent with quantum statistics (including Planck's formula) because particles,

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<sup>21</sup>As in e.g. a cigar-shaped mass distribution, rather than a sphere. Of course, this is not really a breaking of rotational symmetry, in that each is described by relative angles and distances between masses, invariant under rotations.

<sup>22</sup>For more on Leibniz see Saunders [2003]. For a compilation of original sources and commentary, see Huggett [1999b].

<sup>23</sup>Huggett [1999a], also endorsed in Albert [2000, 47-8].).

<sup>24</sup>See Schrödinger [1984, 207-210]. The word 'individual' has also been used to mean an object answering to a unique description at a single time (as 'absolutely discernible' in the terminology of Saunders [2003], [2006b]). Note added Sep 2016: and as re-identifiable over time (see my [2016] for further discussion

<sup>25</sup>As recently endorsed by Pooley [2006 section 8].

even given the symmetrization of the state, may nevertheless possess ‘transcendental’ individuality, and symmetrization of the state can itself be understood as a dynamical constraint on the state, rather than in terms of permutability.

Of these the second need not detain us. Perhaps metaphysical claims can be isolated from any possible impact on physics, but better, surely, is to link them with physics where such links are possible. Or perhaps we were just wrong to think that haecceitism is a metaphysical doctrine: it just means non-permutability, it is the rejection of a symmetry.

As for the first, it is simply not true that indistinguishable quantum particles can never be re-identified over time. Such identifications are only exact in the kinematic limit, to be sure, and even then only for a certain class of states; but the ideal gas is commonly treated in just such a kinematic limit, and the restriction in states applies just as much to the reidentification of identical quantum particles that are *not* indistinguishable – that are *not* permutable – but which are otherwise entangled.

This point needs some defence. Consider first the case of non-permutable identical particles. The  $N$  particle state space is then  $\mathcal{H}^N = \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$ , the  $N$ -fold tensor product of the 1-particle state space  $\mathcal{H}$ . Consider states of the form:

$$|\Phi\rangle = \underbrace{|\phi\rangle_a \otimes |\phi\rangle_b \otimes \dots \otimes |\phi\rangle_c \otimes \dots \otimes |\phi\rangle_d}_{N\text{-factors}} \quad (17)$$

where the one-particle states are members of some orthonormal basis (we allow for repetitions). The  $k^{th}$ - particle is then in the one-particle state  $|\phi_c\rangle$ . If the particles are only weakly interacting, and the state remains a product state, the  $k^{th}$ - particle can also be assigned a one-particle state at later times, namely the unitary evolute of  $|\phi_c\rangle$ . Even if more than one particle has the initial state  $|\phi_c\rangle$ , still it will be the case that each particle in that state has a definite orbit under the unitary evolution. It is true that in those circumstances it would seem impossible to tell the two orbits apart, but the same will be true of two classical particles with exactly the same representative points in  $\mu$ -space.<sup>26</sup>

Now notice the limitation of this way of speaking of particles as one-particle states that are (at least conceptually) identifiable over time: it does not in general apply to superpositions of states of the form (17) – as will naturally arise if the particles are interacting, even starting from (17). In general, given superpositions of product states, there is no single collection of  $N$  one-particle states, or orbits of one-particle states, sufficient for the description of the  $N$  particles over time. In these circumstances no definite histories, no orbits of one-particle states, can be attributed to identical but distinguishable particles either.

Now consider identical permutable quantum particles (indistinguishable quantum particles). The state must now be invariant under permutations, so (for

<sup>26</sup>One might in classical mechanics add the condition that the particles are impenetrable; but one can also, in quantum mechanics, require that no two particles occupy the same one-particle state (the Pauli exclusion principle). See sections 2.5, 3.3.

vector states):

$$U_\pi|\Phi\rangle = |\Phi\rangle \quad (18)$$

for every  $\pi \in \Pi^N$ , where  $U : \pi \rightarrow U_\pi$  is a unitary representation of the permutation group  $\Pi^N$ . Given (18),  $|\Phi\rangle$  must be of the form:

$$|\Phi\rangle = c \sum_{\pi \in \Pi_N} |\phi_{\pi(a)}\rangle \otimes |\phi_{\pi(b)}\rangle \otimes \dots \otimes |\phi_{\pi(c)}\rangle \otimes \dots \otimes |\phi_{\pi(d)}\rangle \quad (19)$$

and superpositions thereof. Here  $c$  is a normalization constant,  $\pi \in \Pi_N$  is a permutation of the  $N$  symbols  $\{a, b, \dots, c, \dots, d\}$  (which again, may have repetitions), and as before, the one-particle states are drawn from some orthonormal set in  $\mathcal{H}$ . If non-interacting, and initially in the state (19), the particle in the state  $|\phi_c\rangle$  can *still* be reidentified over time – as the particle in the state which is the unitary evolute of  $|\phi_c\rangle$ .<sup>27</sup> That is to say, *for entanglements like this*, one-particle states can still be tracked over time. It is true that we can no longer refer to the state as that of the  $k^{\text{th}}$  particle, in contrast to states of the form (17), but that labelling – unless shorthand for something else, say a lattice position – never had any physical meaning. As for more entangled states – for superpositions of states of the form (19) – there is of course a difficulty; but it is the same difficulty as we encountered for identical but distinguishable particles.

Reichenbach was therefore right to say that quantum theory poses special problems for the reidentification of identical particles over time, and that these problems derive from entanglement; but not from the ‘mild’<sup>28</sup> form of entanglement required by symmetrization itself (as involved in states of the form (19)), of the sort that explains quantum statistics. On the other hand, this much is also true: permutability does rule out appeal to the reduced density matrix to distinguish each particle in time (defined, for the  $k^{\text{th}}$  particle, by taking the partial trace of the state over the Hilbert space of all the particles save the  $k^{\text{th}}$ ). Given (anti)symmetrization, the reduced density matrices will all be the same. But it is hard to see how the reduced density matrix can provide an operational as opposed to a conceptual criterion for the reidentification of an individual system over time.

What would an operational criterion look like? here is a simple example: a helium atom in the canister of gas by the laboratory door is thereby distinguished from one in the high-vacuum chamber in the corner, a criterion that is preserved over time. This means: the one-particle state localized in the canister is distinguished from the one in the vacuum chamber.

We shall encounter this idea of reference and reidentification by location (or more generally by properties) again, so let us give them a name: call it ‘individuating reference’, and the properties concerned ‘individuating properties’. In quantum mechanics the latter can be represented in the usual way by projection operators. Thus if  $P_{can}$  is the projector onto the region of space  $\Delta_{can}$

<sup>27</sup>As we shall see, there is a complication in the case of fermions (section 3.3), although it does not effect the point about identity over time.

<sup>28</sup>The terminology is due to Penrose [2004, 598]. See Ghirardi et al [2002], 2004] for the claim that entanglement due to (anti)-symmetrization isn’t really entanglement at all

occupied by the canister, and  $P_{cham}$  onto the region  $\Delta_{cham}$  occupied by the vacuum chamber, and if  $|\chi_1\rangle, |\chi_2\rangle$  are localized in  $\Delta_{can}$  (and similarly  $|\psi_1\rangle, |\psi_2\rangle$  in  $\Delta_{cham}$ ), then even in the superposition (where  $|c_1|^2 + |c_2|^2 = 1$ )

$$|\Phi\rangle = c_1 \frac{1}{\sqrt{2}} (|\chi_1\rangle \otimes |\psi_1\rangle + |\psi_1\rangle \otimes |\chi_1\rangle) + c_2 \frac{1}{\sqrt{2}} (|\chi_2\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\chi_2\rangle)$$

one can still say there is a state in which one particle is in region  $\Delta_{can}$  and one in  $\Delta_{cham}$  (but we cannot say which); still we have:

$$(P_{can} \otimes P_{cham} + P_{cham} \otimes P_{can})|\Phi\rangle = |\Phi\rangle. \quad (20)$$

If the canister and vacuum chamber are well-sealed, this condition will be preserved over time. Individuating properties can be defined in this way just as well for permutable as for non-permutable identical particles.

It is time to take stock. We asked whether the notion of permutability can be applied to classical statistical mechanics. We found that it can, in a way that yields the desired properties of the statistical mechanical entropy function, bringing it in line with the classical thermodynamic entropy. We saw that arguments for the unintelligibility of classical permutability in the literature are invalid or unsound, amounting, at best, to appeal to the philosophical doctrine of haecceitism. We knew from the beginning that state-descriptions in the quantum case should be invariant under permutations, and that this has empirical consequences, so on the most straight-forward reading of haecceitism the doctrine is false in that context. Unless it is emasculated from all relevance to physics, haecceitism cannot be true a priori. We wondered if it was required or implied if particles are to be reidentified over time, and found the answer was no to both, in the quantum as in the classical case. We conclude: permutation symmetry holds of identical classical particles just as it does of identical quantum particles, and may be treated in the same way, by passing to the quotient space.

Yet an important lacuna remains, for among the desirable consequences of permutation symmetry in the case of quantum particles are the departures from classical statistics – statistics that are unchanged in the case of classical particles. Why is there this difference?

## 2.4 The explanation of quantum statistics

Consider again the classical reduced phase-space volume for the macrostate  $N_1, N_2, \dots, N_s, \dots$ , as given by Eq.(14):

$$W^{red} = \prod_s \frac{C_s^{N_s} \tau^{N_s}}{N_s!}. \quad (21)$$

In effect, Planck replaced the one-particle phase-space volume element  $\tau$ , hitherto arbitrary, by  $h^3$ , and changed the factor  $Z_s$  by which it was multiplied to obtain:

$$W^{BE} = \prod_s \frac{(N_s + C_s - 1)! h^{3N_s}}{N_s! (C_s - 1)!}. \quad (22)$$

Continuing from this point, using the method of sections 1.3 and 1.4 one is led to the equilibrium entropy function and equation of state for the ideal Bose-Einstein gas. The entire difference between this and the classical ideal gas is that for each  $s$ , the integer  $C_s^N$  is replaced by  $(N_s + C_s - 1)! / (C_s - 1)!$ . What is the rationale for this? It does not come from particle indistinguishability (permutability); that has already been taken into account in (21).

Let us focus on just one value of  $s$ , that is, on  $N_s$  particles distributed over  $C_s$  cells, all of the same energy (and hereafter drop the subscript  $s$ ). At the level of the fine-grained description, in terms of how many (indistinguishable) particles are in each (distinguishable) cell, a microstate is specified by a sequence of fine-grained occupation numbers  $\langle n_1, n_2, \dots, n_C \rangle$ , where  $\sum_{j=1}^C n_j = N$ ; there are many such corresponding to the coarse-grained description  $(N, C)$  (for a single value of  $s$ ). Their sum is

$$\sum_{\substack{\text{all sequences } \langle n_1, \dots, n_C \rangle \\ \text{s.t. } \sum_{k=1}^C n_k = N}} 1 = \frac{(N + C - 1)!}{N! (C - 1)!} \quad (23)$$

as before. But here is another mathematical identity:<sup>29</sup>

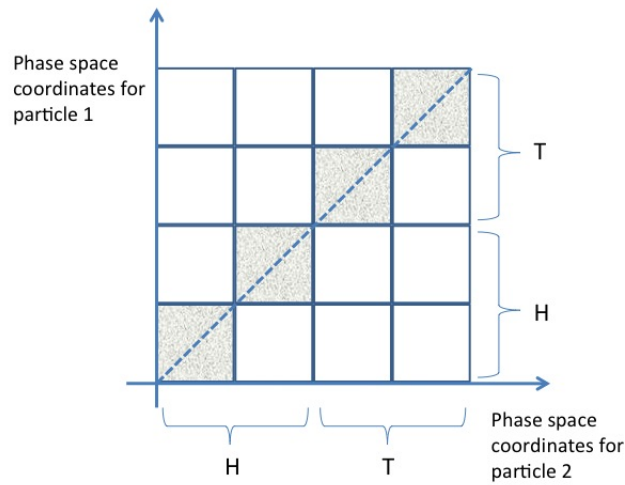
$$\sum_{\substack{\text{all sequences } \langle n_1, \dots, n_C \rangle \\ \text{s.t. } \sum_{k=1}^C n_k = N}} \frac{1}{n_1! \dots n_C!} = \frac{C^N}{N!}. \quad (24)$$

In other words, the difference between the two expressions (21) and (22), apart from the replacement of the unit  $\tau$  by  $h^3$ , is that *in quantum mechanics every microstate*  $\langle n_1, n_2, \dots, n_C \rangle$  *has equal weight, whereas in classical mechanics each is weighted by the factor*  $(n_1! \dots n_C!)^{-1}$ .

Because of this weighting, a classical fine-grained distribution where the  $N$  particles are evenly distributed over the  $C$  cells has a much greater weight than one where most of the particles are concentrated in a small handful. In contrast, in quantum mechanics, the weights are always the same. Given that ‘weight’, one way or another, translates into statistics, particles weighted classically thus tend to repel, in comparison to their quantum mechanical counterparts; or put the other way, quantum particles tend to bunch together, in comparison to their classical counterparts.

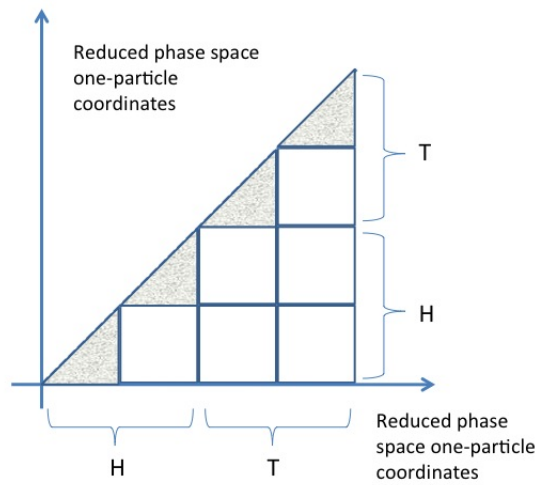
That is what the weighting does, but why is it there? Consider fig.1a, a representation of phase space for  $N = 2$ ,  $C = 4$ . Suppose, for concreteness, we are modelling two classical, non-permutable identical coins, such that the first

<sup>29</sup>A special case of the multinomial theorem (see e.g. Rapp [1972, 49-50]).



**Fig 1a**

Shaded cells have no partners on particle interchange



**Fig 1b**

Shaded cells have half the volume of cells off the diagonal

two cells correspond to one of the coins landing heads ( $H$ ), and the remainder to that coin landing tails ( $T$ ) (and similarly for the other coin).<sup>30</sup> The cells along the diagonal correspond not just to both coins landing heads or both landing tails – they are cells in which the two coins have *all* their fine-grained properties the same. For any cell away from the diagonal, there is a corresponding cell that

<sup>30</sup>Of course for macroscopic coins, the assumption of degeneracy of the energy is wildly unrealistic, but let that pass.

differs only in which coin has which fine-grained property (its reflection in the diagonal). Their combined volume in phase space is therefore twice that of any cell on the diagonal.

The same is true in the reduced phase space, fig.1b. For  $N = 3$  there are three such diagonals; cells along these have one half the volume of the others. And there is an additional boundary, where all three particles have the same fine-grained properties, each with one sixth their volume. The weights in Eq.(24) follow from the structure of reduced phase space, as faithfully preserving ratios of volumes of microstates in the unreduced space. As explained by Huggett [1999], two classical identical coins, if permutable, still yield a weight for  $\{H, T\}$  twice that of the weight for  $\{H, H\}$  or  $\{T, T\}$ , just as for non-permutable coins, that is with probabilities one-half, one-quarter, and one-quarter respectively.

Contrast quantum mechanics, where subspaces of Hilbert space replace regions of phase space, and subspace dimension replaces volume measure. Phase space structure, insofar as it can be defined in quantum theory, is derivative and emergent. Since the only measure available is subspace dimension, each of a set of orthogonal directions in each subspace is weighted precisely the same – yielding, for the symmetrized Hilbert space, Eq.(23) instead.<sup>31</sup>

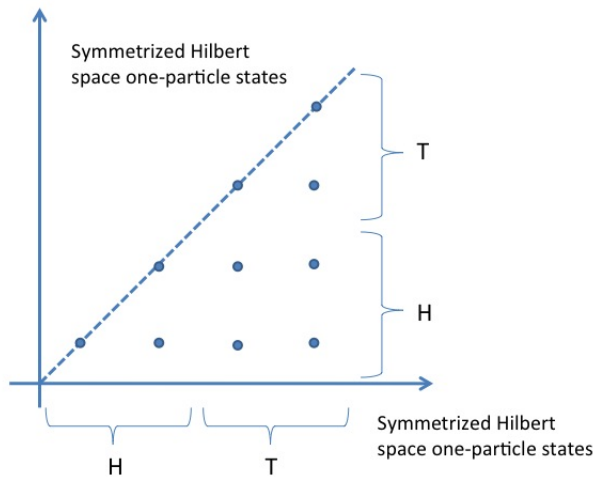
But there are two cases when subspace dimension and volume measure are proportional to one another – or rather, for we take quantum theory as fundamental, for when phase-space structure, complete with volume measure, emerges from quantum theory<sup>32</sup>. One is in the limit  $C \gg N$ , when the contribution from the states along the diagonals is negligible in comparison to the total (fig. 2b), and the other is when the full Hilbert space for non-permutable particles is used. That is why permutability makes a difference to statistics in the quantum case but not the classical: for  $N \approx C$ , as in fig 2a, the dimensionality measure departs significantly from the volume measure (in fig 2a, as five-eighths to one-half). For  $N = 2, C = 2$  there are just three orthogonal microstates, each of equal weight. Take two two-state quantum particles (qubits) as quantum coins, and the probabilities  $\{H, H\}, \{T, T\}, \{H, T\}$  are all one-third.

Is there a remaining puzzle about quantum statistics – say, the non-independence of permutable quantum particles, as noted by Einstein? Statistical independence fails, in that the state cannot be specified for  $N - 1$  particles, independent of the state of the  $N^{th}$ , but that is true of classical states on reduced phase space too (or, indeed, for permutation-invariant states on the unreduced phase space – see Bach [1997]). Find a way to impose a discrete measure on a classical permutable system, and one can hope to reproduce quantum statistics as well (Gottesmann [2005]). Quantum holism has some role to play in the explanation of quantum statistics, but like entanglement and identity over time, less than meets the eye.

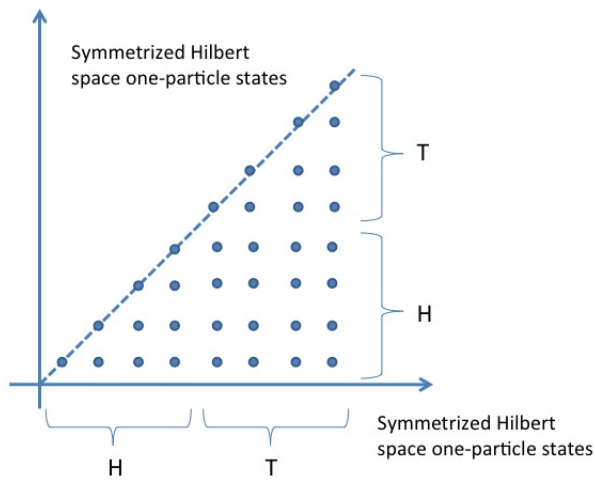
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<sup>31</sup>One way of putting this is that in the quantum case, the measure on phase space must be discrete, concentrated on points representing each unit cell of ‘volume’  $h^3$ . For early arguments to this effect see Planck [1912], Poincaré [1911, 1912].

<sup>32</sup>See Wallace (2013)



**Fig 2a**  
Discrete measure for symmetrized Hilbert space



**Fig 2b**  
As C increases, the discrete measure becomes approximately proportional to dimension of unsymmetrized subspace

## 2.5 Fermions

We have made almost no mention so far of fermions. In fact most of our discussion applies to fermions too, but there are some differences.

Why are there fermions at all? The reason is that microstates in quantum theory are actually rays, not vector states  $|\phi_c\rangle$ , that is, they are 1-dimensional subspaces of Hilbert space. As such they are invariant under multiplication



by complex numbers of unit norm. If only the ray need be invariant under permutations, there is an alternative to Eq.(18), namely:

$$U_\pi|\Psi^{FD}\rangle = e^{i\theta}|\Psi^{FD}\rangle \quad (25)$$

where  $\theta \in [0, 2\pi]$ . Since any permutation can be decomposed as a product of permutations  $\pi_{ij}$  (that interchange  $i$  and  $j$ ), even or odd in number, and since  $\pi_{ij}\pi_{ij} = \mathbb{I}$ , it follows that (18) need not be obeyed after all: there is the new possibility that  $\theta = 0$  or  $\pi$  for even and odd permutations respectively. Such states are *antisymmetrized*, i.e. of the form:

$$|\Psi^{FD}\rangle = \frac{1}{\sqrt{N!}} \sum_{\pi \in \Pi_N} \text{sgn}(\pi) |\phi_{\pi(a)}\rangle \otimes |\phi_{\pi(b)}\rangle \otimes \dots \otimes |\phi_{\pi(c)}\rangle \otimes \dots \otimes |\phi_{\pi(d)}\rangle \quad (26)$$

where  $\text{sgn}(\pi) = 1$  ( $-1$ ) for even (odd) permutations, and superpositions thereof.

An immediate consequence is that, unlike in (19), every one-particle state in (26) must now be orthogonal to every other: repetitions would automatically cancel, leaving no contribution to  $|\Psi^{FD}\rangle$ . Since superpositions of states (19) with (26) satisfy neither (18) or (25), permutable particles in quantum mechanics must be of one kind or the other.<sup>33</sup>

The connection between phase space structure and antisymmetrization of the state is made by the Pauli exclusion principle – the principle that no two fermions can share the same complete set of quantum numbers, or equivalently, have the same one-particle state. In view of the effective identification of elementary phase space cells of volume  $h^3$  with rays in Hilbert space, fermions will be constrained so that no two occupy the same elementary volume. In other words, in terms of microstates in phase space, the  $n_k$ 's are all zeros or ones. In place of Eq.(23), we obtain for the number of microstates for the coarse-grained distribution  $\langle C, N \rangle$  (as before, for a single energy level  $s$ ):

$$\sum_{\substack{\text{fine grainings } n_k \in [0,1] \\ \text{s.t. } \sum_{k=1}^C n_k = N}} 1 = \frac{C!}{(C-N)!N!}. \quad (27)$$

Use of (27) in place of (1) yields the entropy and equation of state for the Fermi-Dirac ideal gas. It is, of course, extensive. A classical phase space structure emerges from this theory in the same limit  $C \gg N$  (for each  $s$ ) as for the Bose-Einstein gas, when the classical weights for cells along the diagonals are small in comparison to the total. Away from this limit, whereas for bosons their weight is too small (as suppressed by the factor  $(n_1! \dots n_C!)^{-1}$ ), for fermions their weight is too large (as not suppressed enough; they should be set equal to zero). Thus fermions tend to repel, in comparison to non-permutable particles.<sup>34</sup>

<sup>33</sup>This is to rule out parastatistics – representations of the permutation group which are not one-dimensional (see e.g. Greiner and Müller [1994]). This would be desirable (since parastatistics have not been observed, except in 2-dimensions, where special considerations apply), but I doubt that it has really been explained.

<sup>34</sup>The situation is a little more complicated, as antisymmetry in the spin part of the overall state forces symmetry in the spatial part – which can lead to spatial bunching (this is the origin of the homopolar bond in quantum chemistry).

### 3 ONTOLOGY

The explanation of quantum statistics completes the main argument of this essay: permutation symmetry falls in place as with any other exact symmetry in physics, and applies just as much to classical systems of equations that display it as to quantum systems.<sup>35</sup> In both cases *only quantities invariant under permutations are physically real*. This is the sense in which ‘exchange between like particles is not a real event’; it has nothing to do with the swirling of particles around each other, it has only to do with haecceistic redundancies in the mathematical description of such particles, swirling or otherwise. Similar comments apply to other symmetries in physics, where instead of haecceistic differences one usually speaks of coordinate-dependent distinctions.

In both classical and quantum theory state-spaces can be defined in terms only of invariant quantities. In quantum mechanics particle labels need never be introduced at all (the so-called ‘occupation number formalism’) – a formulation recommended by Teller [1995]. Why introduce quantities (particle labels) only to deprive them of physical significance? What is their point if they are permutable? We come back to Quine’s question and to eliminativism.

There are two sides to this question. One is whether, or how, permutable particles can be adequate as ontology (section 3.1), and link in a reasonable way with philosophical theories of ontology (sections 3.2 and 3.3). The other question is whether some other way of talking might not be preferable, in which permutability as a symmetry does not even arise (section 3.4).

#### 3.1 The Gibbs paradox, again

A first pass at the question of whether permutable entities are really objects is to ask how they may give rise to non-permutable objects. That returns us to the Gibbs paradox in the sense of section 1.2: How similar do objects have to be to count as identical?

On this problem (as opposed to the  $N!$  problem) section 2 may seem a disappointment. It focused on indistinguishability as a symmetry, but the existence of a symmetry (or otherwise) seems just as much an all-or-nothing affair as identity. But section 2 did more than that: it offered a microscopic dynamical analysis of the process of mixing of two gasses.

In fact, not even the  $N!$  problem is entirely solved, for we would still like to have an extensive entropy function even where particles are obviously non-identical, say in the statistical behaviour of large objects (like stars), and of small but complex objects like fatty molecules in colloids.<sup>36</sup> In these cases we can appeal to the Ehrenfest-Trkaal-van Kampen approach, but only given that

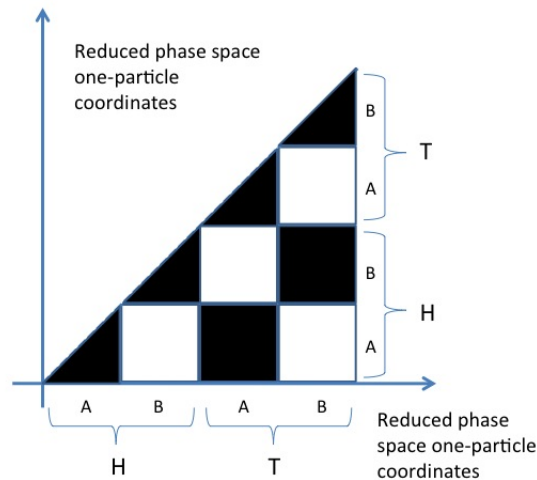
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<sup>35</sup>But see Belot (2013) for pitfalls in defining such symmetries.

<sup>36</sup>This problem afflicts the orthodox solution to the Gibbs paradox, too (and was raised as such by Swendsen [2006]).

we can arrive at a description of such objects as distinguishable: how do we do that, exactly?

The two problems are related, and an answer to both lies in the idea of individuating properties, already introduced, and the idea of phase-space structure as emergent, already mentioned. For if particles (or bound states of particles) acquire some dynamically stable properties, there is no reason that they should not play much the same role, in the definition of effective phase-space structure, as do intrinsic ones. Thus two or more non-identical gases may arise, even though their elementary constituents are identical and permutable, if all the molecules of one gas have some characteristic arrangement, different from those of the other. The two gases will be non-identical only at an effective, emergent level of description to be sure, and permutation symmetries will still apply at the level of the full phase-space. The effective theory will have only approximate validity, in regimes where those individuating properties are stable in time. Similar comments apply to Hilbert-space structures.<sup>37</sup> In illustration,



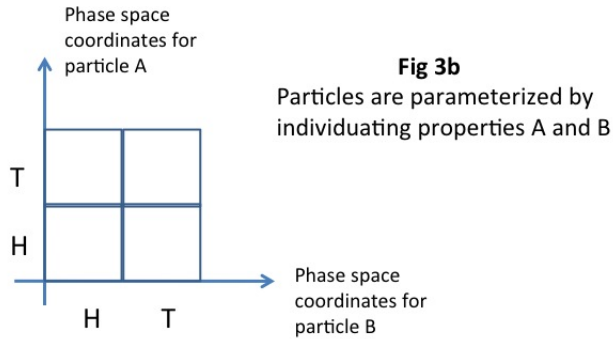
**Fig 3a**

Shaded areas represent dynamically inaccessible regions of phase space

consider again figure1b for two classical permutable coins. Suppose that the dynamics is such that one of the coins always lands on top of the other. Their gravitational potential energy is therefore different.<sup>38</sup> This fact is recorded in the microstate: each coin not only lands either heads ( $H$ ) or tails ( $T$ ), but lands either above ( $A$ ) or below ( $B$ ). It follows that certain regions of the reduced

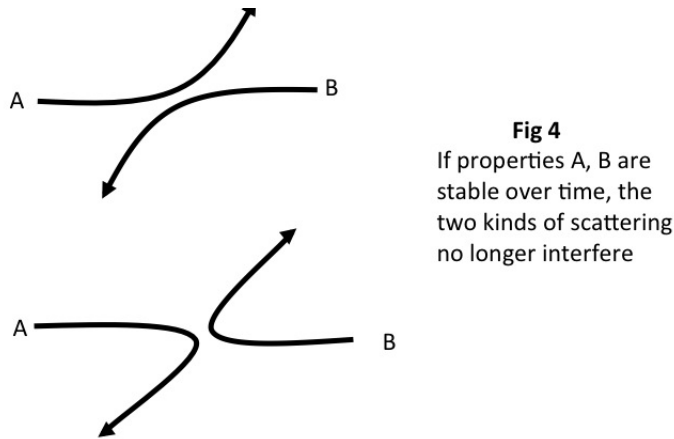
<sup>37</sup>That is, the familiar intrinsic properties of particles (like charge, spin and mass) may be state-dependent: String theory and supersymmetric theories provide obvious examples. See Goldstein et al [2005a,b] for the argument that all particles may be treated as permutable, identical or otherwise.

<sup>38</sup>That makes it harder to maintain the fiction of degeneracy of the energy, but let that pass too.



phase space are no longer accessible, among them the cells on the diagonal for which all the properties of the two coins are the same (shaded, figure 3a). By inspection, the available phase space has the effective structure of an unreduced phase space for *distinguishable* coins, the *A* coin and the *B* coin (figure 3b). It is tempting to add ‘even if there is no fact of the matter as to which of the coins is the *A* coin, and which is the *B* coin’, but there is another way of putting it: the coin which is the *A* coin is the one rotating one way, the *B* coin is the one rotating the other way.<sup>39</sup>

The elimination of the diagonals makes no difference to particle statistics (since this is classical theory), but analogous reasoning applies to the quantum case, where it does. Two quantum coins (qubits), thus dynamically distinguished, will land one head and one tail with probability one half, not one third.



The argument carries over unchanged in the language of Feynman diagrams.

<sup>39</sup>For further discussion, see section 3.3. Whether the *A* coin after one toss is the same as the *A* coin on another toss (and likewise the *B* coin) will make a difference to the effective dynamics.

Thus the two scattering processes depicted in figure 4 cannot (normally) be dynamically distinguished if the particles are permutable. Correspondingly, there is an interference effect that leads to a difference in the probability distributions for scattering processes involving permutable particles from those for distinguishable particles. But if dynamical distinctions  $A$  and  $B$  can be made between the two particles, stable over time (in our terms, if  $A$  and  $B$  are individuating properties), the interference terms vanish, and the scattering amplitudes will be the same as for distinguishable particles.

The same procedure can be applied to  $N = N_A + N_B$  coins,  $N_A$  of which land above and  $N_B$  land below. The result for large  $N_A, N_B$  is an effective phase space representation for two non-identical gases  $A$  and  $B$ , each separately permutable, each with an extensive entropy function, with an entropy of mixing as given by (3). And it is clear this representation admits of degrees: it is an effective representation, more or less accurate, more or less adequate to practical purposes.

But by these means we are a long way from arriving at an effective phase space theory of  $N$  distinguishable particles. That would require, at a minimum,  $N$  distinct individuating properties of the kind we have described – at which point, if used in an effective phase space representation, the original permutation symmetry will have completely disappeared. But it is hardly plausible (for microscopic systems), when  $N$  is large, that a representation like this can be dynamically defined. Even where there are such individuating properties, as with stars and (perhaps) with colloids, it is hard to see what purposes their introduction would serve – their dynamical definition – unless it is to model explicitly a Maxwell demon.<sup>40</sup> On this point we are in agreement with van Kampen. But it must be added: we do better to recognize that the use of unreduced phase space, and the structure  $\mathbb{R}^{6N}$  underlying it, is in general, and at best, a mathematical simplification, introducing distinctions in thought that are not instantiated in the dynamics.

That seems to be exactly what Gibbs thought on the matter. He had, recall, an epistemological argument for passing to reduced phase space – that nothing but similarity in qualities could be used to identify particles across members of an ensemble of gasses – but he immediately went on to say:

And this would be true, if the ensemble of systems had a simultaneous objective existence. *But it hardly applies to the creations of the imagination.* In the cases which we have been considering...it is not only possible to conceive of the motion of an ensemble of similar systems simply as possible cases of the motion of a single system, but it is actually in large measure for the sake of representing more clearly the possible cases of the motion of a single system that we use the conception of an ensemble of systems. The perfect similarity of several particles of a system will not in the least interfere with the

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<sup>40</sup>The memory records of such a demon in effect provide a system of individuating properties for the  $N$  particles.

identification of a particular particle in one case with a particular particle in another. (Gibbs [1902, 188], emphasis added.)

If pressed, it may be added that a mathematician can always construct a domain of objects in set theory, or in one-one correspondence with the real numbers, each number uniquely represented.<sup>41</sup> Likewise for reference to elements of non-rigid structures, which admit non-trivial symmetries – for example, to a particular one of the two roots of  $-1$  in the complex number field, or to a particular orientation on 3-dimensional Euclidean space, the left-handed orientation rather than the right-handed one.<sup>42</sup> But it is another matter entirely as to whether reference like this, in the absence of individuating properties, can carry over to physical objects. The whole of this essay can be seen as an investigation of whether it can in the case of the concept of particle; our conclusion is negative.

The lesson may well be more general. It may be objects in mathematics are always objects of singular thought, involving, perhaps, an irreducible indexical element. If, as structuralists like Russell and Ramsey argued, the most one can hope for in representation of physical objects is structural isomorphisms with objects of direct acquaintance, these indexical elements can be of no use to physics. It is the opposite conclusion to Kant's.

### 3.2 Philosophical logic

A second pass at our question of whether permutable entities can be considered as objects is to ask whether they can be quantified over in standard logical terms. Posed in this way, the question takes us to language and objects as values of bound variables. Arguably, the notion of object has no other home; physical theories are not directly about objects, properties, and identity in the logical sense (namely equality).

But if are to introduce a formal language, we should be clear on its limits. We are not trying to reproduce the mathematical workings of a physical theory in its terms. That would hardly be an ambitious, but hardly novel undertaking; it is the one proposed by Hilbert and Russell, that so inspired Carnap and others in the early days of logical empiricism. Our proposal is more modest. The suggestion is that by formalization we gain clarity on the *ontology* of a physical theory, not rigour or clarity of deduction – or even of explanation. But it is ontology subject to symmetries: in our case, permutability. We earlier saw how invariant descriptions and invariant states (under the permutation group) suffice for statistical mechanics, suffice even for the description of individual trajectories; we should now see how this invariance is to be cashed out in formal, logical terms.

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<sup>41</sup>For further discussion, see Muller and Saunders [2008]. (Set-theory of course yields rigid structures *par excellence*.)

<sup>42</sup>This was also, of course, a key problem for Kant. For further discussion, and an analysis of the status of mirror symmetry given parity violation in weak-interaction physics, see Saunders [2007].

Permutability of objects, as a symmetry, has a simple formal expression: predicates should be invariant (have the same truth value) under permutations of values of variables. Call such a predicate ‘totally symmetric’.

Restriction to predicates like these certainly *seems* onerous. Thus take the simple case where there are only two things, whereupon it is enough for a predicate to be totally symmetric that it be symmetric in the usual sense. When we say:

(i) Buckbeak the hippogriff can fly higher than Pegasus the winged horse

the sentence is clearly informative, at least for readers of literature on mythical beasts; but ‘flies higher’ is not a symmetric predicate. How can we convey (i) without this asymmetry?

Like this: by omitting use of proper names. Let us suppose our language has the resources to replace them with Russellean descriptions, say with ‘Buckbeak-shaped’ and ‘Pegasus-shaped’ as predicates (‘individuating predicates’). We can then say in place of (i)

(ii)  $x$  is Buckbeak-shaped and  $y$  is Pegasus-shaped and  $x$  can fly higher than  $y$

But now (ii) gives over to the equally informative totally symmetric predicate:

(iii)  $x$  is Buckbeak-shaped and  $y$  is Pegasus shaped and  $x$  can fly higher than  $y$ , or  $y$  is Buckbeak-shaped and  $x$  is Pegasus-shaped and  $y$  can fly higher than  $x$ .

The latter is invariant under permutation of  $x$  and  $y$ . Prefacing by existential quantifiers, it says what (i) says (modulo uniqueness), leaving open only the question of which of the two objects is the one that is Buckbeak-shaped, rather than Pegasus-shaped, and vice versa. But continuing in this way – adding further definition to the individuating predicate – the question that is left open is increasingly empty. If no further specification is available, one loses nothing in referring to that which is Buckbeak-shaped, that which is Pegasus-shaped (given that there are just the two); or to using ‘Buckbeak’ and ‘Pegasus’ as mass terms, like ‘butter’ or ‘soil’. We then have from (iii):

(iv) There is Buckbeak and there is Pegasus and Buckbeak can fly higher than Pegasus, or there is Buckbeak and there is Pegasus and Buckbeak can fly higher than Pegasus

With ‘Pegasus’ and ‘Buckbeak’ in object position, (iv) is not permutable. We have recovered (i).

How does this work when there are several other objects? Consider the treatment of properties as projectors in quantum mechanics. For a one-particle projector  $P$  there corresponds the  $N$ -fold symmetrized projector:

$$P \otimes (I-P) \otimes \dots \otimes (I-P) + (I-P) \otimes P \otimes (I-P) \dots \otimes (I-P) + \dots + (I-P) \otimes \dots \otimes (I-P) \otimes P$$

where there are  $N$  factors in each term of the summation, of which there are  $\binom{N}{1} = N$ . For a two-particle projector of the form  $P \otimes Q$ , the symmetrized operator is likewise a sum over products of projections and their complements ( $N$  factors in each), but now there will be  $\binom{N}{2} = N(N-1)$  summands. And so on. The obvious way to mimic these constructions in the predicate calculus, for the case of  $N$  objects, is to define, for each one-place predicate  $A$ , the totally symmetric  $N$ -ary predicate:

$$(v) (Ax_1 \wedge \neg Ax_2 \wedge \dots \wedge \neg Ax_N) \vee (\neg Ax_1 \wedge Ax_2 \wedge \neg Ax_3 \wedge \dots \wedge \neg Ax_N) \vee \dots \vee (\neg Ax_1 \wedge \dots \wedge \neg Ax_{N-1} \wedge Ax_N).$$

The truth of (v) (if it is true) will not be affected by permutations of values of the  $N$  variables. It says only that exactly one particle, or object, satisfies  $A$ , not which particle or object does so. The construction starting with a two-place predicate follows similar lines; and so on for any  $n$ -ary predicate for  $n \leq N$ . Disjuncts of these can be formed as well.

Do these constructions tell us all that we need to know? Indeed they must, given our assumption that the  $N$  objects are adequately described in the predicate calculus without use of proper names, for we have:

**Theorem 1** Let  $\mathcal{L}$  be a first-order language with equality, without any proper names. Let  $S$  be any  $\mathcal{L}$ -sentence true only in models of cardinality  $N$ . Then there is a totally symmetric  $N$ -ary predicate  $G \in \mathcal{L}$  such that  $\exists x_1 \dots \exists x_N Gx_1 \dots x_N$  is logically equivalent to  $S$ .

(For the proof see Saunders [2006a].) Given that there is some finite number of objects  $N$ , anything that can be said of them without using proper names (with no restriction on predicates) can be said of them using a totally symmetric  $N$ -ary predicate.<sup>43</sup>

On the strength of this, it follows we can handle uniqueness of reference as well, in the sense of the ‘that which’ construction, ‘the unique  $x$  which is  $Ax$ ’. In Peano’s notation it is the object  $\iota x Ax$ . Following Russell, it is contextually defined by sentences of the form

(vi) the  $x$  that is an  $A$  is a  $B$

or  $B(\iota x)Ax$ , which is cashed out as:

(vii)  $\exists x(Ax \wedge \forall y(Ay \rightarrow y = x) \wedge Bx)$ .

From Theorem 1 it follows that (vii), supplemented by information on just how many objects there are, is logically equivalent to a sentence that existentially quantifies over a totally symmetric predicate (like (v)). It says that a thing

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<sup>43</sup>This construction was overlooked by Dieks and Lubberdink [2010] in their criticisms of the concept of classical indistinguishable particles. They go further, rejecting indistinguishability even in the quantum case (they consider that particles only emerge in quantum mechanics in the limit where Maxwell-Boltzmann statistics hold sway – where individuating properties in our sense can be defined).



which is  $A$  is a  $B$ , that something is an  $A$ , and that there are no two distinct things that are both  $A$ , without ever saying which of  $N$  things is the thing which is  $A$ .

How much of this will apply to quantum particles? All of it. Of course definite descriptions of objects of definite number is rarely needed in talk of atoms, and rarely available. Individuating properties at the macroscopic level normally provide indefinite descriptions of an indeterminate number of particles. So it was earlier; I was talking of any old helium atom in the canister by the door, any old helium atom in the vacuum chamber, out of an indeterminate number in each case. But sometimes numbers matter: a handful of atoms of plutonium in the wrong part of the human body might be very bad news indeed. Even one might be too many.

Nor need we stop with Russellean descriptions, definite or otherwise. There are plenty of other referential devices in ordinary language that may be significant. It is a virtue of passing from the object level, from objects themselves (the ‘material mode’, to use Carnap’s term), to *talk* of objects (the ‘formal mode’), that the door is open to linguistic investigations of quite broad scope. Still, in agreement with Carnap and with Quine, our litmus test is compatibility with elementary logic and quantification theory.

To conclude: in the light of Theorem 1, and the use of individuating properties to replace proper names, nothing is lost in passing from non-permutable objects to permutable ones. There is no loss of expressive content in talking of  $N$  permutable things, over and above what is lost in restricting oneself to the predicate calculus and abjuring the use of names. That should dissipate most philosophical worries about permutability.

There remains one possible bugbear, however, namely identity in the logical sense (what we are calling equality). Quantum objects have long been thought problematic on the grounds that they pose insuperable difficulties to any reasonable account of logical equality – for example, in terms of the principle of identity of indiscernibles (see below). To this one can reply, too bad for an account of equality; the equality sign can be taken as primitive, as is usual in formal logic.<sup>44</sup> (That is to say, in any model of  $\mathcal{L}$ , if a language with equality, the equals sign goes over to equality in the set-theoretic sense.) But here too one might do better.

### 3.3 Identity conditions

If physical theories were (among other things) directly *about* identity in the logical sense, an account of it would be available from them. It is just because physical theories are *not* like this (although that could change) that I am suggesting the notion of object should be formalized in linguistic terms. It is not spelt out for us directly in any physical theory.

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<sup>44</sup>See Pniower [2004] for arguments to this effect.

But by an ‘account of equality’ I do not mean a theory of logical equality in full generality. I mean a theory of equality only of physical objects, and specific to a scientific language. It may better be called an account of identity conditions, contextualized to a physical theory.

Given our linguistic methods, there is an obvious candidate: exhaustion of predicates. That is, if  $F...s$ .if and only if  $F...t...$ , for every predicate in  $\mathcal{L}$  and for every predicate position of  $F$ , then  $s$  and  $t$  are equal. Call this  $\mathcal{L}$ -equality, denote ‘ $s =_{\mathcal{L}} t$ ’. It is clearly a version of Leibniz’s famous ‘principle of identity of indiscernibles’. This is often paraphrased as the principle that objects which share the same properties, or even the same relational properties, are the same, but this parsing is unsatisfactory in an important respect. It suggests that conditions of the form

$$\forall y(Fsy) \leftrightarrow \forall y(Fty) \wedge \forall y(Fys) \leftrightarrow \forall y(Fyt) \quad (28)$$

are sufficient to imply that  $s$  and  $t$  are equal, but more than this is required for exhaustion of predicates. The latter also requires the truth of sentences of the form:

$$\forall y(Fsy \leftrightarrow Fty) \wedge \forall y(Fys \leftrightarrow Fyt). \quad (29)$$

These are the key to demonstrating the non-identity of many supposed counterexamples to Leibniz’s principle (of distinct objects that appear qualitatively the same; see Saunders [2003]).

$\mathcal{L}$ -equality is the only defined notion of equality (in first-order languages) that has been taken seriously by logicians.<sup>45</sup> It satisfies Gödel’s axioms for the sign ‘=’, used in his celebrated completeness proof for the predicate calculus with equality, namely the axiom scheme:

$$\text{Leibniz’s law } s = t \rightarrow \bigwedge_{F \in \mathcal{L}} (F..s.. \leftrightarrow F..t..)$$

together with the scheme  $s = s$ . Since one has completeness, anything true in  $\mathcal{L}$  equipped with the sign ‘=’ remains true in  $\mathcal{L}$  equipped with the sign ‘ $s =_{\mathcal{L}} t$ ’. The difference between  $\mathcal{L}$ -equality and primitive equality cannot be stated in  $\mathcal{L}$ .<sup>46</sup>

But the notion that we are interested in is not  $\mathcal{L}$ -equality, sameness with respect to every predicate in  $\mathcal{L}$ , but *sameness with respect to invariant predicates constructible in  $\mathcal{L}$* , denote  $\mathcal{L}^*$ . Call equality defined in this way ‘physical equality’, denote ‘ $=_{\mathcal{L}^*}$ ’. With that completeness is no longer guaranteed, but our concern is with ontology, not with deduction.

In summary, we have: $\mathcal{L}$

$$\text{physical equality } s =_{\mathcal{L}^*} t \stackrel{\text{def}}{=} \bigwedge_{F \in \mathcal{L}^*} (Fs \leftrightarrow Ft)$$

<sup>45</sup>It was first proposed by Hilbert and Bernays [1932]; it was subsequently championed by Quine [1960], [1970].

<sup>46</sup>For further discussion, see Quine [1970, 61-64], and, for criticism, Wiggins [2004, 184-88].

and, as a necessary condition for physical objects ('the identity of physical indiscernibles'):<sup>47</sup>

**IPI**  $s =_{\mathcal{L}^*} t \rightarrow s = t$ .

If  $s \neq_{\mathcal{L}^*} t$ , we shall say  $s$  and  $t$  are '(physically) discernible'; otherwise 'indiscernible'.

There are certain logical distinctions (first pointed out by Quine) for equality in our defined sense that will prove useful. Call  $s$  and  $t$  'absolutely discernible' if for an open sentence  $F$  in one free variable,  $Fs$  and not  $Ft$ ; call  $s$  and  $t$  'weakly discernible' (respectively 'relatively discernible') if for an open sentence  $F$  in two free variables  $Fst$  but not  $Fss$  (respectively, but not  $Fts$ ). Objects that are only weakly or relatively discernible are discerned by failure of conditions of the form (29), not (28).

Of these, as already mentioned, weak discernibility is of greater interest from both a logical and physical point of view. Satisfaction of any symmetric but irreflexive relation is enough for weak discernibility:  $\neq$  and  $\neq_{\mathcal{L}}$  are prime examples. And many simple invariant physical relations are symmetric and irreflexive: for example, having non-zero relative distance in a Euclidean space (a relation invariant under translations and rotations). Thus take Max Black's famous example of identical iron spheres  $s$ ,  $t$ , one mile apart, in an otherwise empty Euclidean space. The spheres are weakly discerned by the relation  $D$  of being one mile apart, for if  $Dst$  is true, it is not the case that  $Dxs \leftrightarrow Dxt$  for any  $x$ , since  $Dst$  but not  $Dss$  (or  $Dtt$ ), so  $s \neq_{\mathcal{L}^*} t$ . And, fairly obviously, if  $\mathcal{L}^*$  contains only totally symmetric predicates, physical objects will be at most weakly discernible.

Here as before 's' and 't' are terms, that is variables, functions of variables, or proper names. What difference do the latter make? Names are important to discernibility under  $\mathcal{L}$ -equality. Thus if it is established that  $s$  and  $t$  are weakly  $\mathcal{L}$ -discernible, then, if 's' or 't' are proper names, they are absolutely  $\mathcal{L}$ -discernible. In the example just given, if  $Dst$  and 's' is a proper name, then  $Dsx$  is true of  $t$  but not  $s$ . But the presence of names in  $\mathcal{L}$  makes no difference to  $\mathcal{L}^*$ -discernibility (discernibility by totally symmetric predicates). Thus, even if  $Dxy \in \mathcal{L}^*$ , on entering a proper name in variable position one does not obtain a one-place predicate in  $\mathcal{L}^*$ . Permutable objects are only weakly discernible, if discernible at all.

It remains to determine whether permutable particles *are* discernible at all. In the classical case, assuming particles are impenetrable, they are always some non-zero distance apart, so the answer is positive. Impenetrability also ensures that giving up permutability, and passing to things which are particle states or trajectories, they will be at least weakly discernible. Typically they will be strongly discernible, but as Black's two spheres illustrate (supposing they just sit there), not always.

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<sup>47</sup>For further discussion of this form of the principle of identity of indiscernibles, see Muller and Saunders [2008, 522-23].

It is the quantum case that presents the greater challenge; indistinguishable quantum particles have long been thought to violate any interesting formulation of Leibniz's principle of indiscernibles.<sup>48</sup> But in fact the same options as in the classical case are there available. One can speak of that which has such-and-such a state, or orbit, and pass to states and orbits of states as things, giving up permutability. One-particle states or their orbits, like classical trajectories, will in general be absolutely discernible, but sometimes only weakly discernible – or (failing impenetrability) not even that. Or retaining permutability, one can speak of particles as being in one or other states, and of  $N$  particles as being in an  $N$ -particle state, using only totally symmetric predicates. One then looks for a symmetric and irreflexive relation that they satisfy.

On both strategies there is a real difficulty in the case of bosons, at least for elementary bosons. On the first approach, there may be two bosonic one-particle states, each exactly the same; on the second, there seems to be no general symmetric and irreflexive relation that is always satisfied. But the situation is different when it comes to fermions. On the first approach, given only the mild entanglement required by antisymmetrization, one is guaranteed that of the  $N$  one-particle states, each is orthogonal to every other, so objects as one-particle states are always absolutely discernible; and on the second approach, again following from antisymmetrization, an irreflexive symmetric relation can always be defined (whatever the degree of entanglement). I shall consider them in turn.

The first strategy is not without its difficulties. To begin with, even restricting to mildly-entangled states, *which* one-particle states are to be the objects replacing particles is ambiguous. The problem is familiar from the case of the singlet state of spin: neglecting spatial degrees of freedom the antisymmetrized state is

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|\psi_+^z\rangle \otimes |\psi_-^z\rangle - |\psi_-^z\rangle \otimes |\psi_+^z\rangle) \quad (30)$$

where  $|\psi_\pm^z\rangle$  are eigenstates of spin in the  $z$  direction. But this state can equally be expanded in terms of eigenstates of spin in the  $y$  direction, or of the  $x$  direction: which pair of absolutely discernible one-particle states are present, exactly?

The problem generalizes. Thus, for arbitrary orthogonal one-particle states  $|\phi_a\rangle, |\phi_b\rangle$ , and a two-fermion state of the form:

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|\phi_a\rangle \otimes |\phi_b\rangle - |\phi_b\rangle \otimes |\phi_a\rangle) \quad (31)$$

define the states (the first is just a change of notation):

$$\begin{aligned} |\phi_+^1\rangle &= |\phi_a\rangle, \quad |\phi_-^1\rangle = |\phi_b\rangle \\ |\phi_+^2\rangle &= \frac{1}{\sqrt{2}}(|\phi_a\rangle + |\phi_b\rangle), \quad |\phi_-^2\rangle = \frac{1}{\sqrt{2}}(|\phi_a\rangle - |\phi_b\rangle) \\ |\phi_+^3\rangle &= \frac{1}{\sqrt{2}}(|\phi_a\rangle + i|\phi_b\rangle), \quad |\phi_-^3\rangle = \frac{1}{\sqrt{2}}(i|\phi_a\rangle + |\phi_b\rangle). \end{aligned} \quad (32)$$

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<sup>48</sup>See French and Kraus [2006] for this history.

They yield a representation of the rotation group. One then has, just as for components of spin:

$$\begin{aligned} |\Phi\rangle &= \frac{1}{\sqrt{2}}(|\phi_+^1\rangle \otimes |\phi_-^1\rangle - |\phi_-^1\rangle \otimes |\phi_+^1\rangle) \\ &= \frac{1}{\sqrt{2}}(|\phi_+^2\rangle \otimes |\phi_-^2\rangle - |\phi_-^2\rangle \otimes |\phi_+^2\rangle) = \frac{1}{\sqrt{2}}(|\phi_+^3\rangle \otimes |\phi_-^3\rangle - |\phi_-^3\rangle \otimes |\phi_+^3\rangle) \end{aligned}$$

and an ambiguity in attributing one-particle states to the two particles arises with (31) as with (30). I shall come back to this in section 3.4.

This difficulty can be sidestepped at the level of permutable particles, however. In the case of (30), we may weakly discern the particles by the relation ‘opposite spin’, with respect to any direction in space (Saunders [2003], [2006b], Muller and Saunders [2008]). Thus if  $\sigma^x$ ,  $\sigma^y$ ,  $\sigma^z$  are the Pauli spin matrices, the self-adjoint operator

$$\sigma^x \otimes \sigma^x = \sigma^y \otimes \sigma^y = \sigma^z \otimes \sigma^z \quad (33)$$

has eigenvalue  $-1$  in the singlet state  $|\Psi_0\rangle$ , with the clear interpretation that the spins are anticorrelated (with respect to any direction in space). Asserting this relation does not pick out any direction in space, no more than saying Black’s spheres are one-mile apart picks out any position in space.

For the construction in the generalized sense (32), define projection operators onto the states  $|\phi_\pm^k\rangle$

$$P_\pm^k = P_{|\phi_\pm^k\rangle}, \quad k = 1, 2, 3$$

and define the self-adjoint operators:

$$(P_+^k - P_-^k) \otimes (P_+^k - P_-^k), \quad k = 1, 2, 3.$$

Each has eigenvalue  $-1$  for  $|\Phi\rangle$ , and likewise picks out no ‘direction’ in space (i.e the analogue of (33) is satisfied). Moreover, one can define sums of such in the case of finite superpositions of states of the form (31), by means of which fermions can be weakly discerned.

On the strength of this, one can hope to weakly discern bosons that are composites of fermions, like helium atoms. And even in the case of elementary bosons, self-adjoint operators representing irreflexive, symmetric relations required of any pair of bosons have been proposed.<sup>49</sup> The difficulty of reconciling particle indistinguishability in quantum mechanics with the IPI looks well on its way to being solved.

### 3.4 Eliminativism.

We are finally in a position to assemble the arguments for and against eliminativism – that is, for and against renouncing talk of permutable objects in

<sup>49</sup>See Muller and Seevink [2009]. Their idea was to use certain commutator relations that could not be satisfied were there only a single particle.

favour of non-permutable objects defined in terms of individuating properties, whether points in  $\mu$ -space, trajectories, one-particle states, or orbits of one-particle states. The gain, *usually*, is absolute discernibility. On the other hand we have found that quantification over permutable objects satisfies every conservative guideline we have been able to extract from elementary logic (with the possible exception of identity conditions for elementary bosons). And there remains another conservative guideline: we should maintain standard linguistic usage where possible.

That stacks the odds against eliminativism, for talk of particles, and not just of one-particle states, is everywhere in physics. But even putting this to one side eliminativism would seem to fare poorly, for (anti)symmetrized states are generically entangled, whereupon no set of  $N$  one-particle states will suffice for the description of  $N$  particles. And where such a set is available, given sufficiently mild entanglements, it may be non-unique.

Against this there are two objections. The first is that we anyway know the particle concept is stretched to breaking point in strongly-interacting regimes. There the best we can say is that there are quantum fields, and, perhaps, superpositions of states of different particle number. Where the latter can be defined, one can talk of modes of quantum fields instead. In the free-field limit, or as defined by a second-quantization of a particle theory,<sup>50</sup> such modes are in one-to-one correspondence with one-particle states (or, in terms of Fourier expansions of the fields, in correspondence with ‘generalized’ momentum eigenstates). The elimination of particles in favour of fields and modes of fields is thus independently motivated.

The second objection is that we cannot lightly accept indeterminateness in attributing a definite set of  $N$  one-particle states to an  $N$ -particle system, for it applies equally to particles identified by individuating properties. That is, not even the property of being a bound electron in a helium atom in the canister by the corner, and being one in the vacuum chamber by the door, hold unambiguously. The construction (32) applies just as much to (the antisymmetric version of) (20).<sup>51</sup>

But *this* difficulty we recognize as a fragment of the measurement problem. Specifically, it is the ‘preferred basis problem’: into what states does a macroscopic superposition collapse (if there is any collapse)? – or, if macroscopic superpositions exist: what singles out the basis in which they are written? Whatever settles this question (decoherence, say) will dictate the choice of basis used to express the state in terms of macroscopic individuating properties.<sup>footnote</sup>

Whether such a choice of basis – or such a solution to the preferred basis problem – can extend to a preferred basis at the microscopic level is moot. It depends, to some extent, on the nature of the solution (decoherence only goes down so far). Of course it is standard practise in quantum theory to express

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<sup>50</sup>For a discussion of the relation between second quantized and free-field theories (fermionic and bosonic respectively), see Saunders [1991], [1992]

<sup>51</sup>Note added Sep 2016. This difficulty was pointed out in Ghirardi et al [2002, 84-86], but it was dismissed on the grounds that measurements involving the ‘wrong’ choice of states ‘are extremely difficult to perform and of no practical interest’ (p.86).

microscopic states in terms of a basis associated with physically-interpreted operators (typically generators of one-parameter spacetime symmetry groups, or in terms of the dynamical quantities that are measured). The use of quantum numbers for bound states of electrons in the atom, for energy, orbital angular momentum, and components of angular momentum and spin – in conventional notation, quadruples of numbers  $\langle n, l, m_l, m_s \rangle$  – is a case in point. When energy degeneracies are completely removed (introducing an orientation in space) one can assign these numbers uniquely. The Pauli exclusion principle then dictates that every electron has a unique set of quantum numbers. Use such quadruples as names and talk of permutable particles can be eliminated.

It is now clearer that the first objection adds support to the second. Quadruples of quantum numbers provide a natural replacement for particles in atoms; modes of quantum fields (and their excitation numbers) provide a natural replacement for particles involved in scattering. And in strongly-interacting regimes, even modes of quantum fields give out (or they have only a shadow existence, as with virtual particles). All this is as it should be. Our inquiry was never about fundamental ontology (a question we can leave to a final theory, if there ever is one), but with good-enough ontology, in a definite regime.

In the regime we are concerned with, stable particles of ordinary matter whose number is conserved in time, there is the equivalence between one-particle states and modes of quantum field already mentioned. Let us settle on a preferred decomposition of the field (or preferred basis) in a given context. But suppose that context involves non-trivial entanglement: can entanglements of particles be understood as entanglements of modes of fields?

Surely they can – but on pain of introducing many more modes of the field than there were particles, and a variable number to boot. As with one-particle states so modes of the field: in a general entanglement, arbitrarily many such modes are involved, even given a preferred decomposition of the field, whereas the number of particles is determinate. Just where the particle concept is the most stable, in the regime in which particle number is conserved, eliminativism in favour of fields and modes of fields introduces those very features of the particle concept that we found unsatisfactory in strongly-interacting regimes. That speaks against eliminativism.

This does not, of course, militate against the reality of quantum fields. We recognize that permutable particles are emergent from quantum fields, just as non-permutable particles are emergent from permutable ones. Understood in this way, we can explain a remaining fragment of the Gibbs paradox – the fact that particle identity, and with it permutation symmetry, can ever be exact. How is it that intrinsic quantities, like charge and mass, are identically the same? (their values are real numbers, note). The answer is that for a given particles species, the particles are one and all excitations of a single quantum field – whereupon these numerical identities are forced, and permutation symmetry has to obtain. The existence of *exact* permutation symmetry, in regimes in which particle equations are *approximately* valid, is therefore explained, and with it particle indistinguishability.

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