On the logical consistency of special relativity theory and non-Euclidean geometries: Platonism versus formalism

Radhakrishnan Srinivasan*

Abstract

The Lorentz transformations in the theory of special relativity (SR) lead to a little-investigated phenomenon called relativistic determinism. When two relatively moving inertial observers A and B coincide in space at a given instant, it is possible that a particular distant event is in the future of one of the observers (B), but is in the present or even in the past of the other (A); this is a well-known consequence of the relativity of simultaneity. Hence B's future at the instant of coincidence with A is determined by the fact that A had already seen it at that instant. In this paper, it is argued that Platonism is inherent in relativistic determinism and from the point of view of formalism, a logical inconsistency can be deduced in SR, as formalized in classical first-order predicate logic (FOPL). Similarly, it is argued that Platonism is inherent in non-Euclidean geometries (NEG) and that formalism demands that Euclid's fifth postulate (EP) be provable in plane neutral geometry (NG) consisting of Tarski's axioms (as formalized in FOPL). The essential argument here is that models of NEG can only be constructed by assuming that the postulates of Euclidean geometry (EG) are metamathematically or Platonically 'true'. Formalism demands however that such Platonic truths do not exist and so one concludes that formally, the provability of EP follows from its truth in every model of NG. The classical argument for 'interpreting' NEG within EG must be formally rejected as amounting to assuming the Platonic/metamathematical truth of the Euclidean postulates. So from the point of view of formalism, this argument does not really prove the relative consistency of NEG with respect to EG. An argument for provability of EP in NG is presented in the non-Aristotelian finitary logic (NAFL) proposed by the author.

1 Relativistic determinism – the clash with logic

Consider the theory of special relativity (SR) as formalized in classical first-order predicate logic (FOPL). For details of such a formalization, see the work

*R & D Group, IBM India Software Labs, 5th floor, Golden Enclave, Airport Road, Bangalore 560017, India. E-Mail: sradhakr@in.ibm.com

of Andréka *et al.* [1]; on page 1245, Part VI of [1] an explanation is given for why first-order logic is to be preferred to higher-order logics. A particularly simple set of postulates for SR is given in [2] and we will adopt these for the elementary treatment in this paper. The little-studied phenomenon of relativistic determinism, which is a consequence of the Lorentz transformations and the relativity of simultaneity, is clearly explained in [3]. An example of relativistic determinism which we wish to consider in this paper may be formulated as follows.

Let A and B be relatively moving inertial observers who happen to coincide in space at a given instant defined by $t = 0$ in A's frame and $t' = 0$ in B's frame. Let C be an instantaneous event that is localized in space and distant to both A and B. Let $U(IBC)$ define a non-trivial universe of material objects with certain well-posed initial-boundary conditions IBC. Define the proposition P as "From A's point of view, C occurs in $U(IBC)$ when A's local clock reads $t = 0$" and the proposition Q as "From B's point of view, C occurs in $U(IBC)$ when B's local clock reads $t' = T$". Here $T > 0$ is a constant obtained from the Lorentz transformations as applied to the event C in A's and B's inertial frames. Relativistic determinism asserts that if P is true then Q must be true (or $P \Rightarrow Q$); in other words, B's future at time $t' = 0$ is determined by the fact that A has observed C at precisely that instant (when A and B coincided) and so B must necessarily observe C at $t' = T$.

In order to obtain a logical contradiction from the above scenario, let us further stipulate that the proposition "Event C occurs in $U(IBC)$" is undecidable in SR, i.e., in particular, neither A nor B can either prove or refute this proposition. Such undecidability could occur in many ways, for example, as a result of Gödel's incompleteness theorems; alternatively, C could be a probabilistic event, such as, the outcome of a coin toss experiment or some quantum phenomenon; or else, C could be completely unpredictable as a result of being decided by the instantaneous free will of a human being. It immediately follows that P and Q are undecidable in SR; see the ensuing paragraph for the definition of such undecidability. Note however, that SR requires $P \Leftrightarrow Q$ to be a theorem despite the undecidability of P and Q; this fact immediately makes SR inconsistent in the non-Aristotelian finitary logic (NAFL) proposed by the author in [4] and [5] (in particular, see Remark 5 of [4] and Section 2.2 of [5]). This argument for inconsistency of SR in NAFL is simpler than the one given using inertial frames in [6]. It follows that the philosophy of formalism as embodied by NAFL [5] immediately rejects relativistic determinism. The goal of this paper is to show that an inconsistency can be deduced in SR even within FOPL, if one insists on formalism.

Henceforth, whenever we refer to A (B), it is to be understood that our argument may apply equally well to any observer in A's (B's) set of inertial frames. Note that we require the following restrictions regarding propositions involving P and Q. The truth of P (Q) can be *asserted* (via an observation, for example) or *deduced* in SR *only* by A (B). However, B (A) can consider and either accept or refute in SR any assertion/deduction of the truth of P (Q) made by A (B); but B (A) cannot *assert* or *deduce* the truth of P (Q). The undecidability of P (Q) in SR means that A (B) can neither prove nor refute

P (Q) in SR. $P \Rightarrow Q$ is a theorem in B's (and not A's) frame; in other words, only B has the right to deduce Q in SR from an assertion of P made by A (if B happens to agree with A's assertion). Similarly, $Q \Rightarrow P$ is a theorem in A's (and not B's) frame. In fact $P \Rightarrow Q$ and $Q \Rightarrow P$ are illegitimate propositions in A's and B's frames respectively. The idea behind these restrictions is to allow A (B) to consider the truth of Q (P) without undermining the Lorentz transformations.

In particular, Q is undecidable in SR, which means, as noted above, that B can neither prove nor refute Q in SR. The question we wish to consider is as follows. Given that A has asserted the truth of P, and given that $P \Rightarrow Q$ is a theorem of SR in B's frame, can B accept A's assertion and conclude Q? In the metatheorem that follows, we argue that B in fact has a formal refutation of A's assertion; i.e., B has a proof of $\neg P$ in SR and hence B has no way to conclude Q despite A's assertion of P. However, B does not have the right to use $Q \Rightarrow P$ along with the said proof of $\neg P$ to deduce $\neg Q$, because, as noted above, $Q \Rightarrow P$ is a theorem of A's (and not B's) frame. Hence Q continues to remain undecided in SR (in B's frame) despite A's assertion of P. See Remark 6 below for further clarifications.

Before proceeding to the main result in the metatheorem below, we observe that an additional restriction is necessary, as follows. A and B accept each other's observations/theorems as true/valid if and only if there is no disagreement with (or a refutation of) the observations or any step used in the proof of the said theorems, including the theorems themselves. As an example, suppose A asserts $\neg P$ and concludes $\neg Q$ from the theorem $Q \Rightarrow P$ of A's frame. Then B accepts A's assertion $\neg P$ as true and A's inference $\neg P \Rightarrow \neg Q$ as valid despite that fact that such an inference is illegal in B's frame. Thus B accepts A's conclusion $\neg Q$ as true; i.e., B does not insist that because of the illegality of the inference $\neg P \Rightarrow \neg Q$ in B's frame, there must exist a model for SR in which A asserts $\neg P$ and B asserts Q.

Metatheorem. *Suppose* A *claims the truth of* P. B *has a proof of* $\neg P$ *in* SR. *Formally,* B *must accept this proof rather than* A*'s claim. Hence the theoremhood of* $P \Rightarrow Q$ *does not decide* Q *in* SR *from* B*'s point of view. From the completeness theorem of* FOPL, *it follows that from* B*'s point of view, there must exist a model for* SR *in which* Q *is false despite* A*'s claim of the truth of* P. *The existence of such a model would make* SR *inconsistent from* A*'s point of view, because the Lorentz transformations would be violated, and indeed, there could even be a disagreement between* A *and* B *over whether* C *occurred at all. If such a model does not exist, then* SR *is inconsistent from* B*'s point of view.*

Proof. Define the proposition R as "From B's point of view, C occurs in $U(IBC)$ when B's local clock reads $t' = 0$". Clearly, B has a proof of $P \Rightarrow Q$ and hence a proof of $P \Rightarrow \neg R$, from the Lorentz transformations; B can only expect to observe C at $t' = T$ if P is indeed true as claimed by A. It follows that if B accepts A's claim of the truth of P, then B does indeed have a proof of $\neg R$. Now $Q \& \neg R$ expresses the fact that C is in B's future when B's local clock reads $t' = 0$, which means that from B's point of view, C has not yet occurred when

A and B coincide with their local clocks reading $t = 0$ and $t' = 0$ respectively. To say that C has not yet occurred at this well-defined instant (according to B's definition of simultaneity) is also the same as making the global assertion that no one, including any observer in A's frame, has observed C at that instant from B's point of view. It follows that B can conclude $\neg P$. In other words, B is entitled to draw the inference $Q \& \neg R \Rightarrow \neg P$. See Remark 3 below for further justification of this inference. Since B has concluded $\neg P$ starting from the assumption P, it follows that B has a proof by contradiction of $\neg P$ in SR. Consequently, B concludes that Q remains undecided in SR despite A's claim of the truth of P and the metatheorem follows. □

Remark 1. At the well-defined instant when A and B coincided, defined by $(t = 0,\ t' = 0)$, the truth value of P in A's frame is classically determined; in other words, in A's frame, P is either true or false in FOPL at $t = 0$ *irrespective* of whether A happens to *know* the truth value at that instant. If eventually A determines that P is true, it was *already true* at the instant $t = 0$. So from B's point of view, A's claim of P already 'exists' at the instant $t' = 0$ and by the metatheorem there must exist a model for SR in which subsequent events in B's frame (including actions of B or anyone else in B's frame) falsify A's claim.

Remark 2. Note that one could deny that P is a legitimate proposition in B's frame of reference. But then B still has the problem of proving Q, given A's claim of P. If B is not allowed to consider A's claim at all, then it follows that B still has no proof of Q. In particular, for times t' satisfying $0 \le t' < T$, B still has no proof of Q and concludes that there must exist a model for SR in which Q is false, *given* the data up to (and including) the time t'; B cannot consider that such data may include A's claim of P, by assumption. So formally, the conclusions of the metatheorem would still follow. But if B were to abandon formalism and accept the Platonic truth of P (without formally admitting P as a proposition), then the Platonic truth of Q would also follow. In other words, one could argue that SR is about the real world and P and Q are real-world truths. But even here there is a problem – if B were to accept the Platonic (real-world) truth of P, B would have no option but to conclude that the event C "really" occurred when A and B coincided, and that the truth of Q is nothing but an illusion. One must remember that it is only through formalism that such a conclusion was avoided in the first place.

Remark 3. A second option might be to deny B's inference $Q \& \neg R \Rightarrow \neg P$ made in the above proof. But such a denial is not tenable, as explained below. Note that there is a clear definition of global simultaneity in SR (unlike general relativity) within an inertial frame of reference, such as, that of B. So there is a clear past, present and future for B and to say that an event C can only occur in the future of B at a given instant defined by $t' = 0$ is the same as saying (from B's point of view) that C can also only occur in the future of A at this instant, when A and B coincided. This immediately implies an assertion of $\neg P$ by B, for at the instant of coincidence, B concludes that A still does not have any proof on whether C will occur at all in the future; this is especially clear if C is a probabilistic event, such as, the outcome 'heads' in a coin toss experiment, as considered in the following remark. It is very important to note that the converse does not apply from A's point of view. That is, given that B has observed Q to be true, A concludes P from the Lorentz transformations. But A cannot infer $\neg Q$ from P (in a manner similar to B deducing $\neg P$ from Q), despite the temptation to do so. A can only conclude that P is true and that the local observer in B's frame has observed that event C has occurred at the instant $t = 0$ in A's frame, but B has the *illusion* of observing C at a later time; hence, from A's point of view, B accepts the truth of Q due to a wrong definition of simultaneity. But there is no scope for B to argue in this manner in order to deny the inference $Q \& \neg R \Rightarrow \neg P$; there is no way for A to get the illusion of an event C that has not yet happened, and indeed, *need not happen at all*, from B's point of view. This asymmetry clearly highlights the problem with relativistic determinism.

Remark 4. Let C be the probabilistic outcome 'heads' in a coin toss experiment. Suppose A claims P. From B's point of view, the event C had not yet occurred at $t' = 0$ on B's local clock and so B concludes that A's frame still did not have any evidence of the outcome 'heads' at the instant when they coincided. Hence from B's point of view, at the instant of coincidence with A, A's frame has no information available on whether C will occur at all. B deduces that it certainly cannot be A's point of view that C occurred at $t = 0$ on A's local clock; it follows that B has a refutation of A's claim of P and concludes that either outcome ('heads' or 'tails') is still possible for all times in the interval $0 \leq t' < T$. In particular, since B allows for the outcome 'tails', there is no way (from B's point of view) that A got the illusion of 'heads' via a wrong definition of simultaneity. Conversely, from B's claim of Q, A concludes P and that the local observer in B's frame had already seen the outcome 'heads' at $t = 0$, when A and B coincided. So from A's point of view, B's frame already has evidence of 'heads' at this instant, although B may not yet be aware of it. Hence A cannot refute B's claim of Q, as noted earlier.

Remark 5. Consider the following example. Let A be at the front end of a long platform at rest and let B be at the front end of a long train. The train is adjacent to the platform and is travelling at a very high, constant velocity (close to the speed of light) relative to the platform. Define the event C as the (instantaneous, localized) coincidence of the rear end of the train with the rear end of the platform. Let A and B coincide in space at an instant defined by $t = 0$ and $t' = 0$ respectively on their local clocks. Define P and Q as before. Suppose that there is a localized fault in the track just prior to the rear end of the platform, so that it is completely unpredictable as to whether the wheels of the train will instantaneously derail or pass smoothly over this fault. It follows that P and Q are undecidable in SR. Suppose A claims that P is true; i.e., A determines that no derailment has occurred. From the metatheorem, B has no proof of Q in SR and concludes that there must exist a model for SR in which Q is false despite A's claim. In particular, B allows for the possibility of derailment of the train at the fault at any time $0 \leq t' < T$ despite A's claim to the contrary and this would cause Q to be false. As noted in Remark 1, at $(t = 0,\ t' = 0)$, neither B nor A need actually be aware of A's (eventual) claim in order for this argument to apply. Suppose one insists that SR works in the real world and that B would always find Q to be true if A claims the truth of P. Let there be a non-zero probability $p < 1$ such that derailment occurs in the time interval $0 \leq t' < T$, from B's point of view. In a series of experiments in which A claims P, B would, by hypothesis, find Q to be true. Since B has a formal refutation of P in SR (by the metatheorem), it follows that B will conclude that the laws of probability have been violated in the real world. Perhaps this argument would throw some insight into why SR may be incompatible with quantum mechanics.

Remark 6. The restriction that $P \Rightarrow Q$ $(Q \Rightarrow P)$ can only be a theorem in B's (A's) frame may seem somewhat artificial to the reader. Observe that if $Q \Rightarrow P$ were to be a theorem in B's frame, then B's proof of $\neg P$ would immediately yield a proof of $\neg Q$, which implies a complete rejection of the Lorentz transformations and SR. Indeed, B's proof of $\neg P$ simply expresses that B disagrees with A's claim of P, which is within the spirit of SR; this proof should not be allowed to enable B to conclude $\neg Q$, thus contradicting SR. On the other hand, since SR requires B to refute A's claim of P, such a claim cannot be used by B to prove Q either. This is the main point of this section, namely, that Q continues to remain undecided in SR from B's point of view despite A's claim of P. Note that since $Q \Rightarrow P$ is a theorem in A's frame, A is free to conclude $\neg Q$ from an assertion of $\neg P$ in A's frame. Here B would have no problem in accepting A's conclusion of $\neg Q$ as true because B does agree with any claim of $\neg P$ by A, and B accepts the validity of A's inference $\neg P \Rightarrow \neg Q$ despite its illegality in B's frame.

Remark 7. Note that the existence (non-existence) of the model stated in the metatheorem makes SR inconsistent from A's (B's) point of view. Similarly, B's proof of $\neg P$ will not be accepted by A as correct. This raises serious philosophical issues of who decides the consistency of a theory and the validity of a proof within that theory. The author's opinion is that these issues can be settled only by agreement amongst the entire human race, irrespective of the frame of reference any particular individual happens to be in. In this respect, SR seems to be an illegitimately formulated theory. To assert that there is a reality for the consistency of SR independent of (and possibly contrary to) the deductions of human beings made from the inertial reference frames they happen to be in, seems to be highly questionable. It is clear that SR does not tolerate undecidable propositions of the type required in this paper. This seems to clash with Gödel's incompleteness theorems and rules out consideration of any probabilistic, spatially localized events in SR.

2 Non-Euclidean geometries

It was noted in [5] that Platonism is inherent in classical logic and that the author's proposed non-Aristotelian finitary logic (NAFL) is the only logic that correctly embodies formalism. Here we will first argue the inconsistency of non-Euclidean geometries from the point of view of NAFL. We will then critically examine non-Euclidean geometries in classical first-order predicate logic (FOPL) and explain precisely why Platonism is inherent in these geometries, which must therefore be rejected from the point of view of formalism.

In this paper we confine ourselves (unless otherwise indicated) to plane geometry; the extension to the three-dimensional case is straightforward. We assume that the reader is familiar with the axiom scheme for plane Euclidean geometry given in FOPL by Tarski [7]. We use EG, HG and NG to denote Euclidean, hyperbolic and neutral geometries respectively. Let ψ be Euclid's fifth postulate; then EG = NG + ψ, and HG = NG + $\neg\psi$. We also assume that the reader is familiar with Euclid's original formulation in terms of his five postulates – an excellent elementary account is given by Greenberg [8] who also presents Hilbert's (second-order) axiomatization of Euclidean geometry. A good web reference is due to Royster [9]. Our analysis applies equally well to Hilbert's formulation as well, except that in second-order logic there is no completeness theorem (which we will later require to argue that ψ must be provable in NG).

2.1 Inconsistency of non-Euclidean geometries in NAFL

In the logic NAFL proposed by the author [4, 5] the Main Postulate asserts that an undecidable proposition ϕ in a consistent NAFL theory T (which has the same rules of inference as in FOPL) can be true (false) with respect to T if and only if ϕ is provable (refutable) in an interpretation T* of T. Here T* is also an axiomatic NAFL theory which, like T, resides in the human mind. Provability/refutability of ϕ in T* is essentially equivalent in NAFL to an axiomatic declaration of truth/falsity of ϕ with respect to T. There is no Platonic world in which ϕ can be true or false independent of axiomatic theories and independent of an axiomatic declaration of such truth/falsity made in the human mind via T*. Metatheorems 1 and 2 of [4] explain why the laws of the excluded middle and non-contradiction must fail in the absence of such an axiomatic declaration (i.e., when ϕ is undecidable in T*); in particular, metatheorem 2 asserts that ϕ is neither true nor false with respect to T in this case, which corresponds to a non-classical model for T in which $\phi \& \neg\phi$ is the case. Hence consistency of T demands the existence of such a non-classical model. Here T* is the 'truth-maker' for a model of T, wherein only the theorems of T* are assigned 'true'; every other proposition is in a superposed state of 'neither true nor false'. Note that 'ϕ' in this superposed state of $\phi \& \neg\phi$ in the non-classical model is to be interpreted as '$\neg\phi$ is not provable in T*' and '$\neg\phi$' is to be interpreted as 'ϕ is not provable in T*'. This interpretation is obviously true and so there is no contradiction in the superposition required by NAFL. An important consequence of the above truth definition is that the superposition of any two models for an NAFL theory T must also be a (possibly non-classical) model for T – here the model is is to be understood as 'non-classical' with respect to propositions that are in a superposed state, and 'classical' with respect to other propositions.

A second important consequence of the NAFL truth definition which we will require here is as follows. Suppose an NAFL theory T requires a certain object (such as, 'line' in Euclid's postulates) to be uniquely defined in every model for T. Then consistency of T demands that T must necessarily provide a unique *construction* (or *definition*) for that object. In other words, 'non-constructive existence' of such uniquely defined objects is not permitted; NAFL requires uniqueness to be enforced with respect to the *theory* T, and not just in *models* for T. For example, take T to be Euclid's first four postulates. Classically, one can get a Euclidean model E for T by interpreting 'line' to mean Euclidean straight line, and a hyperbolic model H for T by interpreting 'line' in the hyperbolic sense. In NAFL, the consistency of T demands that the superposition of these two models H and E also be a non-classical model for T in which ψ is neither true nor false, as noted in the previous paragraph. But such a superposed state of $\psi\&\neg\psi$ will violate the requirement of the first postulate of T (i.e., Euclid's first postulate) that a 'line' be uniquely defined by any two of its distinct points; it follows that the required non-classical model for T cannot exist. Thus we conclude that the undecidability of ψ in T makes T inconsistent in NAFL.

To sum up, it is inconsistent in NAFL to assert that T permits both Euclidean and hyperbolic definitions of a straight line, given that T requires straight lines to be uniquely defined by any two of its points; such an assertion clearly implies a lack of uniqueness *with respect to* T. It follows that NAFL does not permit entities like 'point', 'line', 'plane', etc. to be left uninterpreted (or non-constructively defined) in T because of the fact that these entities must have a unique construction available in every model for T. For example, T requires that any given object is either a point or not a point; in set-theoretical terms, the class of all points (which is precisely the 'plane') unavoidably exists [5] in the NAFL version of T and the axiom of extensionality for classes will require that a given object either belongs to or does not belong to that class. Therefore consistency of T in NAFL demands that two classical models for T, in which a given entity is a point in one of the models and not a point in the other, cannot both exist; for the superposition of such classical models cannot be a non-classical model for T as required. It follows that the NAFL version of T does not tolerate any ambiguity in the meaning of 'point'. Similarly, all classically 'uninterpreted' entities must necessarily be constructively defined in the NAFL version of T (in fact, to have their Euclidean meanings, as we will argue shortly).

Another simple argument for the decidability of ψ in the NAFL version of NG is as follows. Suppose, to get a contradiction, that ψ is undecidable in NG. The Saccheri-Legendre theorem of NG (see Chapter 4, pg. 101 of [8]) asserts that the sum S of the degree measures of the three angles in any triangle is less than or equal to 180°. Consider the proposition Ψ defined by '$S = 180°$', with the negation $\neg\Psi$ taken as '$S < 180°$'. It is easy to show that $\Psi \Leftrightarrow \psi$ and hence by hypothesis, Ψ is undecidable in NG; Ψ corresponds to EG and $\neg\Psi$, to HG. In NAFL, the consistency of NG and the assumed undecidability of Ψ demands that there exist a non-classical model for NG in which $\Psi\&\neg\Psi$ is the case. But it is also a theorem of NG that S be uniquely defined, and so formally, the superposed state of $\Psi\&\neg\Psi$ violates this uniqueness requirement. It follows that the required non-classical model for NG cannot exist and so the NAFL version of NG would be inconsistent if ψ were to be undecidable as assumed. The conclusion is that in NAFL, the rules of inference of NG must necessarily be such that ψ be either provable or refutable; under the ensuing heading, we will argue the case for provability.

2.1.1 Proof of ψ in the NAFL version of NG

Classically, consistency of NG demands undecidability of ψ in NG; this is diametrically opposite to consistency in NAFL, which demands *decidability* of ψ in NG, as noted above. In the ensuing subsection we will demonstrate that the classical argument is valid if and only if one accepts Platonism (which implies a rejection of NAFL).

In order to argue for the provability of ψ in NG, we first note that 'point', 'line', 'plane' and other classically uninterpreted entities of NG must necessarily have unique, constructive definitions as demanded by NAFL. Our contention is that it is precisely the addition of ψ to NG which provides such unique, constructive definitions; ψ, together with the axioms of NG, are *essential* in order to have a meaningful NAFL theory. Hence the axioms of EG cannot be denied in NAFL and must be declared as tautologously true. At this stage the reader might wonder why ψ is essential; why not $\neg\psi$? Note that $\neg\psi$ still does not provide *unique* meanings to the uninterpreted terms; there are many possible classical interpretations of HG, such as, the Beltrami-Klein model, Poincaré's models, etc.; see Chapter 7 of [8]. This ambiguity is not acceptable in NAFL because the superposition of these classical models of HG cannot be a (non-classical) model for HG, as demanded by NAFL; it is only ψ that removes all ambiguities. In the ensuing subsection, we argue that Platonism is inherent in $\neg\psi$, which must be rejected by NAFL. The problem we are faced with is how the definitions and rules of inference of classical NG must be modified so that a proof of ψ results in the NAFL version (which, as noted earlier, uses the classical rules of inference). If such a modification is deemed impossible, then NG is inconsistent in NAFL. An attempt at a solution follows.

Playfair's postulate, which is equivalent to Euclid's fifth postulate ψ, asserts ([8], Chapter 1, pg. 17) that for every line l and every point P that does not lie on l, there exists a unique line m through P that is parallel to l. Henceforth, we will refer to Playfair's postulate as ψ. Classically, two lines are parallel if and only if they do not intersect. But this definition is not satisfactory in NAFL as it leads to undecidability of ψ in NG and the ambiguities in the uninterpreted terms noted above. The NAFL definition of 'parallel' is stated as follows.

Definition 1. Two distinct coplanar lines are parallel if and only if they are equidistant at all points, where distance between the lines at a point (on either line) is defined as the length of the perpendicular to the other line dropped from that point. Similarly, a line segment AB that does not lie on a line l is parallel to l if and only if AB is equidistant from l at every point of AB.

Definition 1 is in fact first due to Posidonius as the following quote from ([9], "The Origins of Geometry") shows:

> "Many people have tried to prove the Fifth Postulate. The first known attempt to prove Euclid V, as it became known, was by Posidonius (1st century B.C.). He proposed to replace the definition of parallel lines (those that do not intersect) by defining them as coplanar lines that are everywhere equidistant from one another. It turns out that without Euclid V you cannot prove that such lines exist."

A similar definition was also used later by Geminus (10 B.C. $- \sim 60$ A.D.) in a failed attempt to prove Euclid's fifth postulate from the first four; see the quote below from [10]:

> "Geminus tried the following approach giving a definition of parallel lines:-
>
> *Parallel straight lines are straight lines situated in the same plane and such that the distance between them, if they are produced without limit in both directions at the same time is everywhere the same.*
>
> The 'proof' which Geminus then gave of the parallel postulate is ingenious but it is false. He made an error right at the start of his argument for he assumed that the locus of points at a fixed distance from a straight line is itself a straight line and this cannot be proved without a further postulate. It is interesting, however, that Geminus attempts to prove the parallel postulate and, although it is unlikely to be the first such attempt, at least it is the earliest one for which details have survived."

Of course, we do not wish to repeat the mistakes of these attempts.

Proposition 1. *Given a line l and a point* P *at an arbitrary non-zero distance D from l, there exists a unique line segment* M *through* P *parallel to l, such that* P *is at the midpoint of* M *and* M *is of a given arbitrary non-zero length L. Here D and L are (standard) finite lengths. The line segment* M *will remain parallel to l when extended by an arbitrary (standard) finite length such that* P *continues to remain at the midpoint of M. Here 'parallel' is defined in Definition 1.*

Note that Euclid's second postulate, which is provable in NG, permits the extensions noted in Proposition 1. A proof of Proposition 1 is, of course, impossible in the classical version of NG. In the spirit of Euclid, we will permit 'reasoning from diagrams' as a rule of inference added to those of classical NG, in order to overcome the above difficulty; call the resulting theory NG(NAFL), in which the uninterpreted entities, such as, 'point', 'line' and 'plane', are restricted to necessarily have their Euclidean meanings in any diagrammatic proof. In this paper, we are only concerned with how NG(NAFL) handles ψ; the question of how the axioms and rules of inference of NG must be modified in NAFL to handle a continuum of real numbers (if at all it is possible) is reserved for future work.

Proof of Proposition 1 in NG(NAFL). We depict the line l on the diagram as a (sufficiently long, Euclidean) line segment with arrowheads at the end-points pointing outwards (i.e., away from the center of the line segment). Given the point P, the unique (Euclidean) line segment M is then constructed as in Proposition 1, using a protractor and ruler, after appropriately scaling down (or scaling up) the length L of M and the distance D of P from l; the scale factors for these two scalings in mutually perpendicular directions need not be the same. Note that the line segments of l are also scaled by an identical factor to that of M. We claim that this would be a diagrammatic 'proof' of Proposition 1 in NG(NAFL). Two diagrams suffice for this 'proof', with P on either side of l; any change in L or D would merely imply a change in the scale factors of the diagrams. □

Remark 8. Scaling down (scaling up) the length of a line segment amounts to translating a long (short) line segment into a shorter (longer) one; since there is a one-to-one correspondence between the points of any two such line segments, the said translation is a legitimate proof technique that must be 'wired' into the rules of inference. Does such a 'proof' presume ψ and Euclidean concepts? Probably, but note that real-life diagrams cannot in any sense be equated with the ideal continuum concepts embodied in the axioms of NG. One should simply view this real-life construction as a mechanical procedure that establishes the desired result. Secondly, we will only need a finite construction of the line segment M on these diagrams which can be carried out with a ruler and a protractor; but ψ requires an infinite construction of the line m. The fact is that the truth of Proposition 1 can be indisputably depicted in these diagrams and the only way to 'prove' it without explicitly invoking ψ is to incorporate this fact into the classical rules of inference. This is perhaps not a very desirable state of affairs, but in the absence of alternatives, we will have to accept it. We might rationalize that this diagrammatic 'proof' is simply another way of asserting Proposition 1 as a tautology in NAFL, i.e., it cannot be denied. The diagrams express the Euclidean construction we must unavoidably have in mind, but cannot express in the language of NG without ψ (or its equivalents), when we think of 'line' or 'line segment'. See the ensuing subsection for why such a Euclidean construction is unavoidable.

Proof of ψ in NG(NAFL). Since L and D are *arbitrary* constants, we claim that Proposition 1 provides a *direct, constructive* proof of ψ in NG(NAFL). The reader may balk at this assertion; after all, is not M a *line segment* of *finite* length L, rather than the infinite line m demanded by ψ? The answer is surprisingly simple. See Sec. 2.2, pg. 14 of [5], under the heading "Open formulas and the meaning of 'existence' in NAFL", where it is explained that in NAFL, open formulas (with a free variable) or formulas with an 'arbitrary' constant (such as, L and D above) are in fact universally quantified formulas with respect to the said variable or constant. This is so because the values of L and D, being undecidable and unspecified in NG(NAFL), must be in a superposed state of assuming all possible values. In particular, Proposition 1 is automatically quantified over all possible *standard* values of L and D. Since there are no nonstandard models of arithmetic (and hence, of NG) in NAFL [5], it follows that such quantification is universal and immediately implies a proof of ψ in NG(NAFL) as explained below.

The uniqueness of the line m as required by ψ follows from the fact that in NAFL, a 'line' must be considered as a 'potential' rather than an 'actual' infinity. The superposed state of M with all possible (standard) values of the length L *is* the line m in NAFL. Note that this interpretation requires the universal quantification to be of the form

$$\forall D\, \forall L\, \text{Proposition1},$$

with the quantifier for D being outermost. Thus for each D, L is in a superposed state of assuming all possible values. The line m is to be interpreted as an infinite class consisting of the union of all possible line segments in the above formula, with each line segment M in the union identified uniquely by its given standard finite length L (as defined in Proposition 1). Note that m may be represented by the union of any divergent, strictly increasing sequence of standard finite lengths $\{L_1, L_2, L_3, \ldots\}$ of the segment M; every point of m is a point of some segment of length L_j, $j \geq 1$, in this sequence and the converse also holds. The uniqueness of m immediately follows; if m and m_1 are two lines that are obtained from this construction, every point of m is a point of m_1 and vice versa. An infinite class is not a mathematical object in NAFL [4, 5]. The axiom of extensionality for classes states that a class is identified uniquely by its elements; so the existence and uniqueness of each element M (of given length L) of the infinite class m ensures that m itself exists uniquely. □

Remark 9. Suppose one starts with $L = L_0$, i.e., a fixed segment M of length L_0 (a pure number) in the above proof. From Euclid's Postulate II ([8], Chapter 1) one concludes that this segment can be extended (such that P continues to remain at the midpoint of M) to lengths $L = nL_0$, where $n = 2, 3, 4, \ldots$, i.e., for all *standard* positive values of the integer n. In NAFL, nonstandard models of arithmetic do not exist [5], and so the formula asserting the existence of the segment M of length $L = nL_0$ is universally quantified over all positive (standard) integers n. This amounts to a construction of the line m.

Remark 10. The classical objection to the above proof might be that the diagrammatic construction of the line segment M is only possible for *standard* values of the lengths L and D. In FOPL, Proposition 1 is *not* to be treated as universally quantified; each different (standard) value of L and D corresponds to a different formula which requires a different proof. FOPL, unlike NAFL, maintains a distinction between 'arbitrary but fixed' constants like L and D, and a free variable. There is no way to express 'standard finite' in weak FOPL theories in which Tarski's axioms for NG may be formalized; note that Hilbert's axioms are not in FOPL, but in second-order logic (or many-sorted logic) and do not admit nonstandard models. So Proposition 1 is really a proposition *scheme* in FOPL, consisting of infinitely many instances of the values of 'fixed constants' like L and D, which are required to be standard finite. The classical argument presumably is that the diagrammatic construction of M does not take into account the existence of nonstandard models for NG, in which L and/or D have nonstandard values and in which ψ is possibly false. In NAFL, however, 'standard' is a superfluous predicate [5] and Proposition 1 is indeed a legitimate proposition that is universally quantified as noted in the above proof.

Remark 11. Consider a specific line l and a specific point P at a fixed distance D from l. The *direct, constructive* proof of ψ in NG(NAFL) given above fails in FOPL, because such a 'proof' would be infinitely long, as noted in Remark 10. Here we have kept P and D fixed, but nevertheless the 'proof' would have to cover infinitely many instances of the length L in order to establish a construction for the line m, and would therefore be no proof at all in FOPL. This is the same as saying that ψ cannot be established in NG by directly extending a Euclidean line segment M through arbitrarily large standard finite values, because such a construction does not prove the existence of m, which is infinitely long. For a very simple analogy, note that it would be wrong in FOPL to infer the existence of an infinite set or class of natural numbers (real numbers) from the existence of infinitely many natural numbers (real numbers); a separate axiom would be needed to establish the set or class in question. In NAFL, however, the existence of infinitely many natural numbers immediately establishes the existence of the infinite (proper) class N [4, 5]; a line is similarly modeled as the union of a proper class of infinitely many Euclidean line segments as noted earlier.

Remark 12. By the completeness theorem of FOPL, one would expect that there must exist a nonstandard model for NG in which Proposition 1 (with Euclidean concepts) is true for the specific line l and point P of Remark 11, but ψ fails. The failure of parallelism (as defined by Definition 1) in this model should only appear at nonstandardly long distances from the point P, where Proposition 1 does not apply; these distances are formally classified as 'nonstandard finite' but are 'really' infinite. What is extremely surprising (to the author at least) is that such a nonstandard model cannot exist. This is so because *each instance* of Proposition 1 with Euclidean concepts provides an *indirect* proof of ψ in NG ! To see this, take one particular instance in which the (Euclidean) line segment M parallel to l has been constructed for specific, standard values of L and D. Drop perpendiculars from the end-points of M to the line l and consider the rectangle bounded by M, l and these perpendiculars. The very existence of such a rectangle, whose angle sum is four right angles, is equivalent to and proves ψ in NG ([9], "The Origins of Geometry"). It is very odd indeed that on the one hand, infinitely many instances of the truth of Proposition 1 (with Euclidean concepts) do not prove ψ in NG; on the other hand, each such instance of Proposition 1 does, after all, prove ψ in NG ! One can only conclude that in FOPL, it would be inconsistent to insist that only Euclidean meanings must be retained for the 'uninterpreted' terms of NG; consistency demands that non-Euclidean meanings must necessarily be admitted, and so the indirect diagrammatic proof of ψ would be invalid in NG (as would the diagrammatic proof of Proposition 1). This is tantamount to insisting that 'line' must *necessarily* have a non-constructive existence in NG; it would be impossible to prove in NG that parallel lines even exist, with 'parallel' defined as in Definition 1. This is diametrically opposite to consistency in NAFL, which demands that *only* constructive Euclidean concepts must be admitted in NG(NAFL); that 'parallel' must necessarily be defined as in Definition 1; and that ψ must necessarily be provable in NG(NAFL). The author believes that the NAFL position is not only the logically consistent one from the point of view of formalism, but is also more natural; the diagrammatic proof in NG(NAFL) directly confirms our intuition that Proposition 1 is true in the real world.

2.2 Inherent Platonism in non-Euclidean models of NG

The main thesis of this subsection is that non-Euclidean models of NG can only be constructed by assuming the Platonic/metamathematical truth of the postulates of Euclidean geometry (EG). This, of course, is objectionable from the point of view of formalism even in FOPL. In NAFL, an outright contradiction can be deduced as follows. Truth for the postulates of EG must necessarily be *axiomatic* in NAFL; there is no Platonic world in which the Euclidean postulates are 'really' true. So one has *axiomatically* declared EG to be true and then 're-interpreted' terms like 'point', 'line', 'plane', etc. into their non-Euclidean meanings in order to generate the non-Euclidean model. The axiomatic nature of NAFL truth clearly does not permit such 're-interpretation' of Euclidean objects into non-Euclidean ones, for once we have axiomatically declared EG to be true, there is no scope for any change in the Euclidean meanings of terms which are classically deemed 'uninterpreted'. Yet classically, it is precisely such an argument that is used to prove the relative consistency of non-Euclidean geometries with respect to EG; see Chapter 7 of [8].

In particular, let us consider the example of the Beltrami-Klein (BK) model of hyperbolic geometry (HG) discussed in Chapter 7 of [8]. First EG is assumed to be Platonically 'true' and a circle γ of Euclidean radius r is constructed in the Euclidean plane. The hyperbolic plane is then defined as the interior of γ. A chord of γ is a segment AB joining two points A and B on γ. The segment without its end-points A and B is called an open chord. Hyperbolic lines are defined as open chords of γ, with hyperbolic points retaining the same meaning as Euclidean ones. After similar re-interpretation of other 'uninterpreted' terms of NG from their Euclidean meanings, the axioms of HG thus obtained are 'translated' back into their Euclidean counterparts and proved in EG in order to establish the relative consistency of HG with respect to EG (and thereby the undecidability of ψ in NG, assuming EG to be consistent).

From the point of view of NG(NAFL), the construction of the BK model *explicitly* and illegally assumes the Platonic truth of ψ. To see this, note that the radius of γ is of *arbitrary* (Euclidean) length. Consider two parallel chords of equal length l_* in γ, where 'parallel' is in the sense of Definition 1. Clearly, l_* can be assigned an arbitrary (but constant) value, given that the radius of

γ is also arbitrary. In NG(NAFL) this amounts to an explicit, constructive validation of ψ, as noted in Proposition 1 and its proof. It is also clear that the definitions of various terms (such as, 'length') and proofs of propositions of HG in the BK interpretation use constructions of points on γ and even points and line segments outside of γ [8]; these constructions have only Euclidean meanings and so must be assumed to 'really' exist. In other words, γ and its exterior in the Euclidean plane must necessarily exist Platonically in order to define terms and execute proofs of HG in the BK interpretation. Such Platonic existence is illegal in NG(NAFL) and hence the BK model does not exist from the point of view of NAFL. In fact NAFL predicts that the same situation holds of every conceivable candidate for a model of HG because ψ is provable in NG(NAFL), as noted in the previous subsection. The axioms of EG must necessarily be 'really' true in every model of NG even from the point of view of FOPL; the completeness theorem demands that ψ be provable in NG and this makes NG inconsistent in FOPL, if one insists on formalism. However, it has been demonstrated in [5] that formalism is not a valid philosophy of FOPL.

Can the above situation be generalized to hold of *any* non-Euclidean geometry, whether two- or three-dimensional? Again, NAFL predicts that this is the case, i.e., NAFL supports the 'axiom of closed ortho-curvature' stated in [11] as follows:

> That, with the axiom of closed ortho-curvature, there are no true non-Euclidean geometries (and no spatial dimensions beyond three), but only pseudo-geometries consisting of curves in Euclidean space.

Ross[11] also states two other axioms conjecturing the existence of non-Euclidean geometries and then concludes that these are questions in "physics or metaphysics and are logically entirely separate from the status of geometry in logic or mathematics or from our psychological powers of visual imagination". In NAFL, however, the truth of the axiom of closed ortho-curvature is upheld as a matter of logic and the other axioms rejected; the axiomatic nature of NAFL truth means that if we cannot *in principle* visualize non-Euclidean geometries, then they do not exist. In this sense, NAFL vindicates Kant's position that Euclidean geometry must be unavoidably true, although Kant did not rule out (as does NAFL) the logical existence of non-Euclidean geometries [11].

In conclusion, we state an argument for why it is impossible to visualize non-Euclidean geometries in principle. Our contention is that 'curvature' of a line at a point is a notion that can exist if and only if the associated center and radius of curvature in the associated Euclidean space exist. It is tempting to assume that one can 'draw' a curve and find it to be 'really' an arc of a circle without ever making use of a compass. But the moment we assert that the drawn curve is 'really' an arc of a circle, we are also unavoidably asserting the Platonic existence of the center and radius of the circle and of the associated Euclidean space. In the BK model of HG noted above, one can certainly draw arbitrary curved lines inside γ; this immediately entails the Platonic existence of the entire Euclidean plane outside γ and means that circles of arbitrarily large radius can take the place of γ in the BK model; as noted earlier, this amounts to an explicit validation of Euclid's fifth postulate. Similarly, in elliptic (Riemannian) geometry associated with a sphere, geodesics (great circles) are taken to be 'intrinsically' straght lines; however, the associated Platonic existence of the center of the sphere and radii of curvature of the great circles makes the truth of Euclidean geometry inevitable. In short, 'true' non-Euclidean geometries cannot be visualized even in principle because of the unavoidable associated Platonic truth of Euclidean geometry; the 'models' of non-Euclidean geometries are unavoidably Euclidean objects in Euclidean space.

Dedication

The author dedicates this research to his son R. Anand and wife R. Jayanti.

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