Do ‘classical’ space and time provide identity to quantum particles?

Décio Krause*
Department of Philosophy
Federal University of Santa Catarina
Research Group on Logic and Foundations of Science – CNPq

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Abstract

Non-relativistic quantum mechanics is grounded on ‘classical’ (Newtonian) space and time (NST). The mathematical description of these concepts entails that any two spatially separated objects are necessarily *different*, which implies that they are *discernible* (in classical logic, identity is defined by means of indiscernibility) — we say that the space is T_2, or "Hausdorff". But quantum systems, in the most interesting cases, sometimes need to be taken as indiscernible, so that there is no way to tell which system is which, and this holds even in the case of fermions. But in the NST setting, it seems that we can always give an identity to them, which seems to be contra the physical situation. In this paper we discuss this topic for a case study (that of two potentially infinite wells) and conclude that, taking into account the quantum case, that is, when physics enter the discussion, even NST cannot be used to say that the systems do have identity.

Keywords: identity of quantum particles, spatial identity, space and time in quantum mechanics.

1 The problem

In discussing the idea that physics comprises two languages, Roland Omnès [9] addresses that

> "Physics, being both an empirical and a theoretical science can only be fully expressed by means of two distinct languages, or two different kinds of propositions. There is a mathematical language for theory. In the case of quantum mechanics, the framework of this

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language is provided by the theory of Hilbert spaces. Some of its main propositions express how wave functions evolve in time; other propositions may state the value of matrix elements for observables, from which one derives spectra, probabilities and therefrom cross sections and the statistics of measurements.

There is also another language dealing with empirically meaningful propositions. These propositions describe an experimental set-up; they state which events occur; they indicate the reading on a voltmeter or a measuring device. They are concerned directly with experiments, expressing: how these experiments are performed and their results. The existence of two languages in physics may look trivial, but it raises the question of their relation and of their mutual consistency, which is the backbone of *interpretation*."

Of course Hilbert spaces are not the only way to erect a quantum mechanics (see [13] for *nine* different alternatives), but he is right concerning the standard way of considering the quantum formalism. The most important point to our concerns is the alleged possible *consistency* of these two languages. But, first, let me try to elaborate the distinction in order to appropriately deal with the idea (to which we agree almost *in totum* with Omnès, as the reader san see in [8]). The first language, let us call it $\mathcal{L}_T$, is the object language of the theory properly speaking, and here we suppose that the theory is axiomatized. The second, let us call it $\mathcal{L}_M$, is the metalanguage, the one which provides, according to Omnès, the empirical meaningful propositions and by means of which we start thinking about the subject; see [8] again for more discussions on this topic.

Consistency is a term that has an intuitive appeal, and it seems that it is in this sense that it is used in physics, but adquires a precise meaning only inside a formal system whose language comprises a negation symbol $\neg$. Thus, we can say that a theory T is consistent in two ways: *syntactical consistency* means that there is no formula α such that $T \vdash \alpha$ (α is deduced in T) and $T \vdash \neg\alpha$; *semantic consistency* (stronger that the syntactic version) means that T has a model, that is, there is an interpretation of its non-logical symbols that make the non-logical axioms true or, alternatively, theorems of the metatheory (which without loss of generality can be assumed to be a set theory, such as the system ZFC — Zermelo-Fraenkel set theory with the axiom of choice. For the differences between these alternatives, see [8]).

But these definitions apparently are much for Omnès requirement. What he is supposed to require is that there should be no discrepancies between the language of the theory and the language which expresses its empirical propositions. And it will be in this sense that we shall consider the case study of this paper. We take $\mathcal{L}_T$ as the standard Hilbert space formalism for non-relativistic quantum mechanics, and for $\mathcal{L}_M$ the metamathematical language of ZFC, where we express the space-time counterpart of this theory (see the Appendix to understand how space and time enter the quantum formalism).

Non-relativistic quantum mechanics (QM) is grounded on 'classical' concepts of space and time. In short, and without going to a precise description

[10, Chap.17], [12], the space and time setting can be identified with the $\mathbb{R}^4$ Euclidean manifold. In this framework, due to its topological characteristics, if we have two distinct objects, they can be located into two disjoined open sets and this provides them an *identity*: we can call Peter the first object and Paul de second one, and if it is Peter who is in the first open set, then Paul is not there, so they present a property not shared by the another and, so, by one of the fundamental rules of classical logic, namely, Leibniz's Principle of the Identity of Indiscernibles, they are *different* (not equal, not the same object).[1] Furthermore, if we move Peter, say by an orthogonal transformation or if we translate it, or both,[2] we can always recognize that it was Peter who has being moved, for we can trace back the motion (the transformations are invertible) and identify the object by the open set it belonged to. To make an analogy which will be useful later, we can say that every object in this manifold (that is, every object *represented* in such a mathematical framework) has an identity cart, a document that enables us to identify the object *as such* in whatever situation or context. We shall say that it is an *individual*. An individual, thus, is something that has identity, in the sense of possessing an identity card. The particular object is the unique one with such an identity card, and every *other* object presents a distinction from it, for this object has at least one property shared by no other object, namely, its identity. The underlying logic is Leibnizian: there are no distinct indiscernible objects. Indiscernible objects (objects partaking all their properties) are identical, are the very same object. This is classical logic, this is classical mathematics, this is NST.

In other words, classical logic, standard mathematics (say, that mathematics that can be developed in a set theory such as the Zermelo-Fraenkel (ZF) set theory) and classical mechanics are Leibnizian in this sense. In classical physics, even entities like electrons, yet having the same characteristics, don't lose their identity, for they have different and impenetrable trajectories which serve to identify them in every instant of time. They have that what Post has termed *transcendental individuality* (see [11] and [3, p.11]).

Quantum mechanics, it is agreed by most physicists and philosophers, is different. As Landau and Lifshitz have said (adapting), due to the uncertainty principle, if the position of an electron is exactly known in a given instant, its coordinates have no definite values even at the next instant: "by localizing and numbering the electrons at some instant, we make no progress towards identifying them at subsequent instants; if we localize one of the electrons, at some other instant, at some point of space, we cannot say which of the electrons has arrived at this point." [6, p.227] (see also the quotation at the end of this paper).

But wait! How can we not be able to identify the particles if they are represented in classical space and time? How can we explain this fact? Here we propose a situation that serves for the analysis of this puzzle. The core idea is

[1] By the way, this is the meaning of *numerical* identity: numerically identical objects are not *different* objects, but *the very same* one.

[2] The composition of a translation with an orthogonal transformation is called a rigid motion, and they are typical of Euclidian geometry and of classical mechanics.

that once we assume that quantum systems are not mere mathematical entities, there cannot be identity (in the standard sense of numerical identity) provided to them by NST. In other terms, when physics enter the scenario and provide situations which are not purely mathematical (and I suppose we all agree that physics *is not* mathematics),[3] we need to take into account the 'second language' mentioned by Omnès, namely, the language of physics properly speaking (in distinction from its mathematical language), the situation changes so that we cannot say that the distinctly located quantum systems possesses identity.

To analyse the situation, we consider what is perhaps the most limit case where apparently there would be no doubt about the identity of the quantum systems, namely, two infinite potential wells with one particle each, the particles being of the same species ('identical' in the physicists's jargon). By supposing that the particles cannot scape the wells, their positions in space would serve to provide them an identity card, an identity. We shall see that this conclusion cannot be reached so easily in 'real' physical situations, so there is an apparent contradiction between that what QM says and that what its underlying mathematics enables us to do (specially 'classical' space and time). So, Omnès two languages are, in this case, not mutually consistent. We conclude with foundational considerations of both physics and logic.

2 Two wells and 'classical' identity

The situation we shall consider is formed by two infinite potential wells located at a great distance from one another so that we can suppose that there are no interactions between them. Inside the wells there are two particles of identical kind, say two electrons. That is (see Figure 1),

$$V(x) = \begin{cases} 0 & x_1 - \epsilon < x < x_1 + \epsilon \text{ and } x_2 - \epsilon < x < x_2 + \epsilon \\ \infty & \text{otherwise,} \end{cases}$$

where x_1 and x_2 ($x_1 \neq x_2$) are the centers of the wells (each of length $2\epsilon > 0$) in the x axis. We can separately solve the independent of time Schrödinger equations for the two wells and get the corresponding wave-functions that describe the energies of the particles in the wells, one for each well [4, p.24ff]. But these wave-functions can also be used for granting us the existence of the two particles at those distinct locations (being $x_1 + \epsilon \lll x_2 - \epsilon$). Due to the topological structure of this T_2 space-time manifold, we can find two disjoined open sets A and B containing the space regions corresponding to the wells, so that we can surely say that the particles belong to disjoined open sets in the manifold. *Thus*, we conclude grounded in classical logic, they are different.

[3] But we may recall Max Tegmark's suggestion that the universe is a mathematical structure, the *mathematical universe hypothesis*, or MUH (see the Wikipedia entry on MUH). It should be observed, despite this subject is out of the aims of this paper, that this thesis has problems as the following one: *if* MUH is correct, then the mathematical structure needs to be constructed in some mathematical apparatus. Which one? Furthermore, which mathematical structure is the universe?

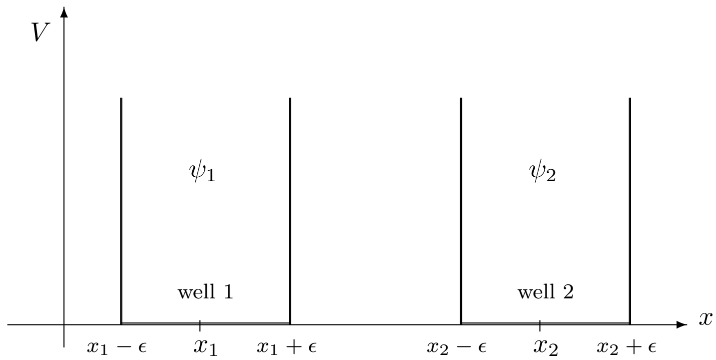

Figure 1: Two infinite potential wells. Are they conferring identity to the particles described by ψ_1 and ψ_2?

But, what does *different* mean? Let us have a look on the underlying concept of identity before we continue with the discussion about the wells.

3 Identity

We have an intuitive idea of identity and difference. Identical things are the very same thing; two or more things are different things. This belief is grounded on a metaphysical assumption that goes back at least to the Stoics (cf. [5, pp.339]) but had its fortification with Leibniz's principle mentioned above. The main philosophical question is this: if there is a difference between two objects, where this difference resides? The answers belong to one of the two main 'theories of individuation', *substratum theories* and *bundle theories*. The first ones assume the existence of some kind of substratum underlying the thing's properties. This mysterious substratum receives different names, but with the same main characteristics: *haecceity*, *primitive thisness*, *quid*, etc. So, two things may partake all their properties or characteristics, but differ in what respects their substratum, something that cannot be described by (or reduced to) properties. The problems regarding such a view in quantum mechanics can be seen in [15], the main one being the difficulty of explaining what should be such a substratum, since it cannot be described by means of properties. Bundle theories dispense any kind of substratum; the identity of a thing is given only by its properties, either one property or a collection of them. This view also presents some problems; let us mention one, perhaps the most important one. The question is this: how can we be sure that there are no two or more things partaking the same collection of characteristic properties? The answer is that we don't know; we need to assume this idea or reject it based on metaphysical groundings. Leibnizian metaphysics takes this hypothesis, and it has been incorporated to our preferred pantheon of assumptions.

Classical logic and standard mathematics are Leibnizian in this sense, which means that the thesis that distinct things have distinct properties is a logical fact. Really, take a standard set theory (the mathematical framework where NST is developed). Give any object a (described in such a set theory), we can define the 'property' of being identical to a as follows: $I_a(x) \leftrightarrow x \in \{a\}$, by considering that the unitary set of a can be formed in the theory for any a.[4] Thus, the only object with this property is a itself, so it presents a property shared with no *other* object.

This Leibnizian notion of identity is *extensional* in the following sense: two sets are identical (are the very same set) if and only if they have the same elements. Two ur-elements are identical if and only if they belong to the same sets. In other terms, a set, or a context (whatever it is) changes if a thing belonging to it is exchanged by a thing not belonging to it. We can express this idea by the following theorem of extensional set theories (theories encompassing an Axiom of Extensionality):

$$x \in A \wedge y \notin A \to \Big((A - \{x\}) \cup \{y\}) \neq A\Big). \tag{1}$$

In quantum mechanics, apparently things don't run this way. Think of ionization. Let A be an atom whatever in its fundamental state (for instance, an Helium atom). We can ionize the atom by realizing an electron and getting a positive ion, A^+. After some time, we can make the ion absorbs an electron again, turning to be a neutral atom once more. Questions: is the 'new' neutral atom the same as the 'old' one? Are the realized electron and the absorbed electron the same electron? Obviously that these questions cannot be answered out of serious conceptual doubts. In fact, we cannot say that we are realizing this or that electron, but just *one* electron; the same concerning the absorbed electron. The intuitive notion of identity seems to be not applicable in this domain. Electrons, atoms and other *quanta* don't have identity cards. And, more importantly, they don't acquire identity even after being realized, for once the electron merges the environment, it[5] becomes tangled up with 'other' electrons, so that we cannot identify it ever more.

It seems to me, as we have discusses elsewhere (see [1, 3]), that the better and fair idea is to say that the neutral atoms are *indiscernible* or *indistinguishable* from one another (see below), so as are the two electrons. Physicists surely will agree, saying that this is the obvious conclusion to be made. But we remark that this is not *so* obvious as it seems: in classical logic (set theory included), indiscernibility entails identity, for the very notion of identity is introduced by means of indiscernibility (Leibniz's Law): two things are identical if and only if they share all their properties. But, in the permutation of quantum entities, what we are looking for is something like the equivalence indicated bellow, where $\equiv$ stands for a relation of indiscernibility and x and y stand for the realized and the absorbed electrons and A is the neutral atom:

$$(A - \{x\}) \cup \{y\}) \equiv A, \tag{2}$$

[4]Thus result holds even if a is an ur-element, that is, an entity that is not a set but which can be member of sets.

[5]In so far as we can speak of *it* and *others* — see below how to make language precise.

but we neither have true conditions for asserting that electron x belongs to the atom nor that electron y does not. In A we just have a kind of *weight* of energy which enables us to say that there is a certain number of electrons there (in the He atom, for instance, there are two electrons, but we cannot count them[6]), but never that this or that electron belongs to it.[7] Interesting enough, there is a mathematics grounding in the *theory of quasi-sets* which makes the trick of separating the notions of identity and indiscernibility so that the equivalence (2) is a theorem of the theory [3, §7.2.6]. In this theory, we can speak at most in collections having a certain cardinal (termed *quasi-cardinal*) but without providing them an identity. So, to make the above informal language precise, we can say that the collection (quasi-set) of the electrons of the neutral atom A is two, while the quasi-cardinal of the ion A^+ is one. But this theory is not the story to be told here. So, let us go back to the wells.

4 Back to the wells

Let us consider again the wave functions ψ_1 and ψ_2 of the two wells, which we call well 1 and well 2, and of course suppose that the experience can be performed (infinite potential wells are idealizations). As said before, these wave functions describe the stationary states of the wells, but shall be read also as marking that there are two particles, one in each well.

The question is: representing the two wells in the standard space and time setting, does this attribute identity to the particles? I don't think so, and let me explain why. As said before, to have identity means to have an identity card, a something which enables us to distinguish the entity in other situations. Furthermore, this identity is *extensional* in the sense explained earlier: the contexts change when different (here this notion makes perfect sense) things are interchanged. Quantum particles don't have identity in this sense. Although trapped in the infinite wells, they have only what Toraldo di Francia termed *mock individuality*, and individuality (and 'identity') that is lost as soon as the wells are open or when another similar particle is added to the well (if this was possible). There is no identity card for quantum particles. They are not individuals, yet can be isolated by trapping them for some time.

Some time ago I wrote a paper contesting the idea that Hans Dehmelt's positron Priscilla is an individual on the same grounds [7]. Any positron could act as Priscilla as well, while no other person could substitute Donald Trump (I suppose that no one could say what he says and think as he thinks). Trump is supposed to be an individual, Priscilla surely is not.

But let us explore the argumentation a little bit deeper. Suppose we aim at to describe the two particles at once, one in each well. Since they are 'identical',

[6] By 'counting', we mean the definition of a bijection between the collection of the two electrons and the von Neumann ordinal number $2 = \{0, 1\}$. Really, to which electron should we attribute the number 0?

[7] Some authors think that since the cardinal of the collection is greater than one, the elements of the collection are necessarily distinct. The existence of quasi-set theory shows that this hypothesis can be discussed.

we need to use (anti-)symmetric wave functions, something like (let us consider the anty-symmetric case)

$$\psi_{12}(a,b) = \frac{1}{\sqrt{2}}\Big(\psi_1(a)\psi_2(b) - \psi_1(b)\psi_2(a)\Big) \tag{3}$$

in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ where the sub-indexes name the wells and a and b 'name' the particles. Important to recall that the use of (anti-)symmetric vectors intend precisely to make these labels not conferring identity to the particles.

Thus,

$$|\psi_{12}(a,b)|^2 = \frac{1}{2}\Big(|\psi_1(a)\psi_2(b)|^2 + |\psi_1(b)\psi_2(a)|^2 - 2Re\Big\langle \psi_1(a)\psi_2(b)\Big|\psi_1(b)\psi_2(a)\Big\rangle\Big).$$

Let us consider the interference term. We have that

$$\Big\langle \psi_1(a)\psi_2(b)\Big|\psi_1(b)\psi_2(a)\Big\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} = \langle\psi_1(a)|\psi_1(b)\rangle_{\mathcal{H}_1}\cdot\langle\psi_2(b)|\psi_2(a)\rangle_{\mathcal{H}_2} = 0$$

since $\psi_1(b) = \psi_2(a) = 0$ for b is out of well 1 and a is out of well 2. Hence the interference term is null and this is interpreted as saying that there is no interaction between the particles. But note that this reasoning demands that we are able to identify the particles a and b, and we can do this only by the reference to the wells: particle a is that one which is in well 1. But, if the particles are indistinguishable, how can we known which particle is in well 1? Without identity, the most we can say is that we have two wells (mathematically seen as disjoined sets) with cardinal 1 each, and so that their elements are indistinguishable. So far, so good. Some remarks are in order, for physics need to enter the discussion.

As said before, infinite potential wells are idealizations. They don't exist and cannot be constructed. The most we can say is that we have two very hight wells, which despite the great potential involved, do not avoid tunneling and perhaps nonlocal interactions between the wells. So we *really* cannot say that the particles are in fact non interacting. But let us continue with the mathematical description as a *Gedankenexperiment*.

Since the interference term is null, the state (3) is separable, so the probabilities, then, result from the probabilities of the wells separately. That is, we have

$$|\psi_{12}(a,b)|^2 = \frac{1}{2}\Big(|\psi_1(a)\psi_2(b)|^2\Big) + \frac{1}{2}\Big(|\psi_1(b)\psi_2(a)|^2\Big). \tag{4}$$

Interpretation Let's read the second member of this equality as indicating that we have the sum of the probability of particle a be in well 1 *and* particle b in well 2 (let us call α this case), and the probability of particle b be in well 1 *and* particle a be in well 2 (call β this assumption).

What about the first member? To give it an interpretation, we recall that one of the postulates of the calculus of probability says that $P(\alpha\cup\beta) = P(\alpha)+P(\beta)$

if α and β are mutually exclusive events, which is the case, since we have assumed that the interference term is null. So, the first member of (4) can be read as indicating that we have a probability of particle a be in well 1 and particle b in well 2, *or* particle b be in well 1 and particle a be in well 2. In reading the equality this way, it results that there is no way to know which particle is which! Summing up, there is no identity attributed to the particles. I recall once more that one thing is to individuate an object, put it in isolation from others. Another one is to confer it an identity.

In other words, the physics does not enable us to identify the particles, even if metamathematically they can be separated and provided with an identity by their space location. Thus, we have here a typical case where the language of the mathematics of the theory and the language of physics do not conform each other. And, as far as a logic comprises a semantics [2], a physical theory cannot rest on pure mathematics. And in the quantum case, when this happens, there appears a conflict between the mathematics used (NST) and the physics.

To reinforce that, perhaps we can recall how space and time enters in the formalism of orthodox quantum mechanics. We shall see in the Appendix that it does not enter in the formalism (axioms) directly, but in the standard formalism it runs in parallel with it, as a kind of *step theory*, being subsumed into the set theoretical framework, in the sense of Suppes' axiomatization [14, Chap.12], [8].

5 The (supposed) right answer

I shall propose here a very short answer to the question posed in the title of this note, namely, "Do 'classical' space and time provide identity to quantum particles?". As for the wells, from the physical point of view the right answer would be NO, for the experiment of confining a particle in an infinite potential well cannot be performed in practice. So, even by assuming that quantum particles exist, either in orthodox quantum mechanics or as field excitations in a quantum field theory, infinite potential wells do not confer identity for the particles, for they do not exist. But, by considering the problem as a *Gedanken-experinent*, in considering space and time location, mainly spatial location, the answer is NO again. The reasons are, first, that particles (seen either as point particles in non-relativistic quantum mechanics or in quantum field theories) don't have well defined positions, but position-states, and the position operator doesn't have eingenfunctions in the standard sense. The use of distributions in a rigged Hilbert space just provide an approximation for the positions, and in reality we should go to quantum field theories, where space and time become (Minkowskian) spacetime but the problem of identity doesn't desapear. To see why this is so in this case, perhaps the best thing is to read the quotation below, taken from Viktor Toth's answer to "Do subatomic particles have solid surfaces?" on Quora [16], which helps in enlighten the point:

> Question: Do subatomic particles have solid surfaces?
>
> Answer: "No. Subatomic particles are not like anything you expe-

rience in the classical world. In particular, they are not miniature cannonballs or planets or whatever.

In a quantum particle theory, elementary particles are point-like. However, these points do not usually have a classically defined location, not unless they interact with other things (e.g., an instrument) that confines them to a location. This is why, instead of a position coordinate, the particle's location is described by its wave-function, which basically provides a probability field, assigning to each location of space a probability of finding the particle at that location.

But quantum *particle* theories are, in fact, inadequate when it comes to describing particles: they have trouble dealing with relativity, and they are especially incapable of describing the creation and annihilation of particles (e.g., when an electron emits or absorbs a photon.) That's why, somewhat paradoxically, our best particle theory is not a particle theory at all: it is quantum *field* theory.

In a quantum field theory, there are no particles, only ever-present fields. For instance, there is the one and only electromagnetic field. But this field cannot just be in any state. It being a quantum field, its *excitations* come in set units (quanta). It is these unit excitations that we associate with the concept of particles. So you see, 'particles' are really not particles at all! These excitations, however, may be confined to a small volume, in which case they indeed exhibit particle-like behavior (which can be seen, e.g., in particle colliders that can trace particle paths.)

But no matter how you look at it, there are no solid surfaces or indeed, other classical behavior or properties in the quantum world. Don't even try to imagine them? If you do, you are already on the wrong track. Sadly, this also means that intuition can be an obstacle (or worse yet, can be grossly misleading) when it comes to the quantum world; for a solid understanding, the mathematics is unfortunately essential."

In short, contrary to Quine, we can say that quantum mechanics presents us entities with no identity, and quasi-se theory is a mathematics able to deal with them.

6 Appendix: How space and time enter the quantum schema

We shall use the Hilbert space formalism, most common in most philosophical discussions. From a technical point of view, we can assume that we shall be working with the resources of ZFC. So, let us introduce the following definition (more details in [8]), which follows the style presented in [14, Chap.12]:[8]

Definition 6.1 A non relativistic quantum mechanics is a 5-tuple of the form

$$\mathcal{Q} = \langle S, \{\mathcal{H}_i\}, \{\hat{A}_{ij}\}, \{\hat{U}_{ik}\}, \mathcal{B}(\mathbb{R})\rangle, \text{ with } i \in I, j \in J, k \in K$$

where:

1. S is a collection whose elements are called *physical objects*, or *physical systems*.
2. $\{H_i\}$ is a collection of mathematical structures, namely, complex separable Hilbert spaces whose cardinality is defined by the particular application of the theory.
3. $\{\hat{A}_{ij}\}$ is a collection of self-adjunct (or Hermitian) operators over a particular Hilbert space H_i.
4. $\{U_{ik}\}$ is a collection of unitary operators over a particular Hilbert space H_i
5. $\mathcal{B}(\mathbb{R})$ is the collection of Borel sets over the set of real numbers.

In order to connect the formalism with experience, we construct a mathematical framework for representing experience. Important to remark that this is another theoretical (abstract) construction: there is no connection, out of the informal one, of the formalism with reality *per se*. In order to do it, we need to elaborate reality, turning it a mathematical construct too. So, to each quantum system $s \in \mathcal{S}$ we associate a 4-tuple

$$\sigma = \langle \mathbb{E}^4, \psi(\mathbf{x}, t), \Delta, P\rangle,$$

where $\mathbb{E}^4$ is the Newtonian spacetime [10, chap.17], where each point is denoted by a 4-tuple $\langle \mathbf{x}, t\rangle$ where $\mathbf{x} = \langle x, y, z\rangle$ denote the coordinates of the system and t is a parameter representing time, $\psi(\mathbf{x}, t)$ is a function over $\mathbb{E}^4$ called the *wave function* of the system, $\Delta \in \mathcal{B}(\mathbb{R})$ is a Borelian, and P is a function defined, for some i (determined by the physical system s), in $\mathcal{H}_i \times \{\hat{A}_{ij}\} \times \mathcal{B}(\mathbb{R})$ and assuming values in $[0, 1]$, so that the value $P(\psi, \hat{A}, \Delta) \in [0, 1]$ is the probability that the measurement of the observable A (represented by the self-adjunct operator $\hat{A}$) for the system in the state $\psi(\mathbf{x}, t)$ lies in the Borelian set Δ.[9]

Summing up: using standard mathematics, and classical space and time setting, this is all we can do, and it works! Physics can be made within such a framework, but philosophically we can raise questions such as those posed above (and below). Let us see something more on this respect.

[8] Here we introduce a set $\mathcal{S}$ for the quantum systems being considered, which is not useful in the standard presentations. The aim is to question either such a collection can be considered as a *set* of standard set theories, since the quantum systems may be indiscernible. But this point will be not discussed here — see [1].

[9] Of course if we have a system with n elements, the dimension of the space must be $3n$ (here, roughly, $\mathbb{E}^4 = \mathbb{R}^3 \times \mathbb{R}$, and in the case with n systems, we shall have $\mathbb{R}^{3n} \times \mathbb{R}$).

7 Foundations

What should we do when we realize that the theory does not conform with the (perhaps *Gedanken*) experiments? What did Einstein when he realized that the Newtonian space and time did not conform with the proposed ideas of the special relativity? What did von Neumann and Birkhoff when they realized that the 'logic' of quantum mechanics didn't conform to classical logic? Cases like these could be mentioned to exhaustion. Well, Einstein proposed to change the notions of space and time, and we got the sole notion of spacetime (Minkowski); von Neumann and Birkhoff proposed that quantum logic should be non-distributive, hence not classical. The reasons they had are of deep nature, and we all know about the consequences.

What to say about the above discussion involving classical space and time and the very nature of quantum particles, at least in what concerns their properties mentioned in the previous sections? It seems to me that in ignoring physics and paying attention just to the underlying mathematics, for example in saying that the standard Newtonian space and time framework provide identity to quantum objects when they are separated in space (let me for a moment call this thesis Thesis of Identity, TI), conflicts with the physics, for as we have seen that from the physical point of view there are no perfect separated quantum objects. To assume TI without considering physics is similar to consider a logic without paying attention to its semantic aspects. In the same vein that a logic comprises not only its syntactical aspects, but also its semantics, a physical theory cannot be grounded only in its mathematics; physics, that is, physical suppositions, need to be also considered.

This was what motivated above mentioned authors (Einstein, von Neumann and Birkhoff) to depart from classical frameworks and propose radically different bases. The same seems to be happing here. Newtonian space and time and the TI thesis, at least to me, conflict with quantum mechanics. I still don't know what kind of space and time or spacetime should be the right one, perhaps a Minkowskian is enough, but I still don't know. But it seems clear to me that there is a contradiction, at least at the metalevel, between non-relativistic quantum mechanics and its underlying mathematics also in this respect (for another kind of discrepancies, see [3], [1] where the discrepancies of the classical mathematical and logic notion of individual conflicts with the non-individuality of quantum objects).

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