Comment on "On the logical consistency of special relativity theory and non-Euclidean geometries: Platonism versus formalism"

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Abstract

As observed in the PhilSci preprint ID Code 1255 [1], consistency in the author's proposed non-Aristotelian finitary logic (NAFL) demands that Euclid's fifth postulate must necessarily be provable from the first four, and that diagrammatic reasoning with Euclidean concepts must necessarily be admitted into the rules of inference for plane neutral geometry in order to argue for the said provability. Two important consequences, namely, the indispensable role of diagrams as formal objects of Euclidean geometry in NAFL and the resulting NAFL concept of 'line' as an infinite proper class of line segments are highlighted and elaborated upon. A misleading comment in Remark 6 of [1], regarding negation for undecidable propositions in the theory of special relativity (SR), is corrected. This comment is unrelated to the main argument in [1] and the resulting analysis reinforces the conclusions of [1] that negation and implication are problematic concepts for undecidable propositions in SR.

1 The essential role of diagrammatic reasoning in NAFL and the resulting concept of 'line' as an infinite proper class in Euclid's geometry

Consider plane neutral geometry (NG) as formalised in classical first-order predicate logic (FOPL). Our goal in this section is to highlight and further elaborate upon two important results deduced in [1], namely, the *essential* role of diagrams as formal geometric objects in the non-Aristotelian finitary logic (NAFL) proposed by the author [2, 3] and the consequent NAFL concept of 'line' as an infinite proper class of line segments. In contrast, classical logic admits a line as a mathematical object and considers diagrams to be informal (and sometimes unreliable) reasoning tools.

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An excellent reference for the formalization of diagrammatic reasoning is the work of Miller [4]. The modern attitude towards diagrams is neatly summarized by the following quote from Chapter 1 of [4]:

> It has often been asserted that proofs like this, which make crucial use of diagrams, are inherently informal. The comments made by Henry Forder in *The Foundations of Geometry* are typical: 'Theoretically, figures are unnecessary; actually they are needed as a prop to human infirmity. Their sole function is to help the reader to follow the reasoning; in the reasoning itself they must play no part.' ... The transition from mathematics with geometry at its core to mathematics with arithmetic at its core had a profound influence on the way in which people viewed geometric diagrams. When geometric diagrams were seen as the foundation of mathematics, the geometric diagrams used in these proofs had an important role to play. Once geometry had come to be seen as an extension of arithmetic, however, geometric diagrams could be viewed as merely being a way of trying to visualize underlying sets of real numbers. It was in this context that it became possible to view diagrams as being "theoretically unnecessary" and mere "props to human infirmity".

Miller [4] proceeds to develop the thesis that diagrams can indeed be considered as formal objects in plane Euclidean geometry and that diagrammatic proofs are as rigorous as their sentential counterparts. NAFL takes this a step further: diagrams are *necessarily* formal objects and diagrammatic reasoning must *necessarily* be admitted into the rules of inference of classical NG in order to argue for the provability, required by consistency in NAFL, of Euclid's fifth postulate from the first four; see Sec. 2 of [1].

The theory NG(NAFL) is the NAFL version of NG with diagrammatic reasoning (admitting only Euclidean concepts) introduced formally into the rules of inference. For an idea of how this can be done, see [4]. In order to argue for the provability of the fifth postulate in NG(NAFL), 'parallel' was re-defined in [1] as follows:

Definition 1. Two distinct coplanar lines are parallel if and only if they are equidistant at all points, where distance between the lines at a point (on either line) is defined as the length of the perpendicular to the other line dropped from that point. Similarly, a line segment AB that does not lie on a line l is parallel to l if and only if AB is equidistant from l at every point of AB.

It was stated in [1] that Definition 1 is first due to Posidonius in the first century B.C.; however, it appears that Archimedes (287–212 B.C.) in his treatise *On parallel lines* had already defined parallelism similarly. Definition 1 can be extended in an obvious manner to define parallelism between two coplanar line segments and we assume that this has been done. Proposition 1 of [1] is stated below for convenience:

Proposition 1. *Given a line l and a point* P *at an arbitrary non-zero distance D from l, there exists a unique line segment* M *through* P *parallel to l, such that* P *is at the midpoint of* M *and* M *is of a given arbitrary non-zero length L. Here D and L are (standard) finite lengths. The line segment* M *will remain parallel to l when extended by an arbitrary (standard) finite length such that* P *continues to remain at the midpoint of M. Here 'parallel' is defined in Definition 1.*

A diagrammatic proof of Proposition 1 was given in [1], with a scaling argument to represent line segments M of arbitrary length in a single diagram drawn to scale; different line segments will imply different scale factors for the diagram. Here we emphasize that Tarski's axioms for NG do not permit the consideration of 'arbitrary' entities, such as, line segments; if NG is suitably modified to permit this, it becomes classically incomplete, since NG would then interpret classical Peano Arithmetic [5]. NAFL, however, rejects Gödel's incompleteness theorems [3] and so NG(NAFL) can admit arbitrary line segments without becoming incomplete (equivalently in NAFL, inconsistent). In fact consistency *demands* that NG(NAFL) interpret Peano Arithmetic, for the notion of 'arbitrary' line segment is essential in the above diagrammatic proof.

The rationale for the diagrammatic proof is that NAFL permits only constructive Euclidean concepts, and the diagram represents precisely the constructions we must have in mind when we think of 'line' or 'line segment'. The scaling argument is simply justified by noting that the ideal 'lengths' of line segments in NG(NAFL) have nothing to do with the *physical* length of the line segment drawn in the diagram; we are completely free to attribute an arbitrary scale factor to represent arbitrary *ideal* lengths, which are *mental* constructions, by a single *physical* diagram. Let '*idlg*' represent the unit ideal length of a line segment in NG(NAFL), and let the (usual, real-world) 'inch' be the corresponding unit length in the diagram. One might take the scale factor for the line segment M of Proposition 1 in the diagram as "one inch = one hundred *idlgs*"; one could also take the scale factor as, for example, "one inch = one hundred thousand *idlgs*", etc. It should be emphasized that these two scale factors correspond to two *different* line segments M in NG(NAFL), represented in a single diagram. The point here is that there is no reality for the question of how many *idlgs* are 'actually' contained in one inch, since NAFL does not require our mental constructions of line segments and their unit length scale *idlg* to have any connection whatsoever with the real world; formally, this question is undecidable in NG(NAFL), and the Main Postulate for NAFL [2, 3] makes truth for such a proposition purely *axiomatic* in nature. It follows that in the absence of any axiomatic declaration as to the value of the length L (in *idlgs*) of the line segment M represented in the diagram, *all* ideal length scales are present in one inch in a superposed state, as required by NAFL. It is this fact that permits the scaling argument in NAFL, and the consequent representation of arbitrarily long (mentally constructed) line segments superposed in a single (physical) diagram. In fact this scaling argument is absolutely essential for consistency of NG(NAFL), as will be argued shortly. In contrast, FOPL requires that the line segment M in the diagram represents one and only one instance of Proposition 1, which prevents the diagram from being a direct proof of Proposition 1 and amounts to a Platonic assertion that the line segments of NG 'really' exist. Clearly, FOPL requires that there is a fact of the matter as to the conversion factor between the ideal length scale *idlg* and the physical length scale of one inch in the diagram, despite the undecidability of this proposition in NG; different conversion factors only succeed in redefining *idlg* in terms of inches from the FOPL point of view, and we are looking at one and only one line segment M of a fixed length. Of course, similar diagrammatic proofs can be given in NG(NAFL) for each of the other four postulates of Euclid which must also be asserted as tautologously true in NG(NAFL). The fifth postulate was perceived to be 'counter-intuitive' precisely because FOPL does not permit such a diagrammatic proof, as will be explained below.

Playfair's postulate, logically equivalent to Euclid's fifth postulate ψ, asserts that for every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l. We use Playfair's postulate interchangeably with ψ. It was argued in [1] that Proposition 1 provides a *direct, constructive* proof of ψ in NG(NAFL). An indirect proof of ψ may also be obtained in NG(NAFL) by dropping perpendiculars from the end-points of the line segment M to the line l of Proposition 1 and exhibiting diagrammatically the rectangle bounded by M, l and the two perpendiculars. Since only Euclidean concepts are permitted in diagrammatic proofs, the existence of such a rectangle is equivalent to and proves ψ in NG(NAFL). It was noted in [1] that such a diagrammatic proof is not legitimate in classical NG, because FOPL requires that even infinitely many instances of Proposition 1 (with Euclidean concepts) do *not* prove ψ in NG; none of these instances represent the line m, which has infinite length and is a separate mathematical object in classical logic. Thus we have the classical anomaly that infinitely many instances of Proposition 1 should not prove ψ in NG, but if the diagrammatic proof (with Euclidean concepts) is permitted, ψ can be inferred *indirectly* from a single such instance. It was concluded in [1] that consistency in FOPL necessarily requires the uninterpreted entities of NG to have a non-constructive existence, and consequently, ψ must necessarily be undecidable in NG with non-Euclidean concepts permitted. The above classical scenario is diametrically opposite to the notion of consistency in NG(NAFL), as noted in [1].

The first observation is that the scaling argument in the diagrammatic proof of Proposition 1 in NG(NAFL) neatly removes the classical anomaly noted above; a single instance of Proposition 1, as represented in the diagram, can be *interpreted* via the scaling argument to represent infinitely many instances and hence provides a direct as well as indirect proof (via the rectangle construction noted above) of Proposition 1. Hence it is absolutely essential to incorporate the scaling argument into the rules of inference of NG(NAFL). Once this is done, we might as well rename the resulting theory as EG(NAFL), where 'EG' stands for Euclidean geometry; all the axioms of classical EG are theorems of EG(NAFL), provable by diagrammatic constructions. Here we have in mind a finitely axiomatizable version of EG proposed by Tarski [5], all of whose theorems are provable by elementary ruler and compass constructions. EG(NAFL) has only diagrammatic rules of inference and no axioms. It follows that diagrams of EG(NAFL) are formal objects that come prior to and are more fundamental than the axioms of Tarski's elementary EG. Indeed, since all of the theorems of EG are provable by ruler and compass constructions, it is obvious that the axioms of EG must also be so provable. It is striking that diagrammatic reasoning, which plays a crucial and indispensable role in preventing inconsistency in EG(NAFL), would be considered as 'inconsistent' and 'unreliable' in classical neutral geometry. Note that the diagrams are physical (real-world) devices by which we communicate our mental geometric constructions as 'proofs' of the Euclidean postulates to our fellow-human beings. As shown in Sec. 2 of [1], we can only have these Euclidean constructions in mind; the classical non-Euclidean models neccessarily have to assume the metamathematical (Platonic) truth of the Euclidean postulates and so are not models at all by the NAFL truth definition. If there were only one human being in the whole world, such an individual need not construct any diagrams as formal devices to prove the Euclidean postulates; the mental constructions would suffice for this purpose.

The second observation is that the line segment M of Proposition 1 is of *arbitrary* length L, which is to be interpreted in NAFL as being in a superposed state of (quantification over) all possible *standard* values for L. As noted in [1], this amounts to an explicit construction for the line m because there are no nonstandard models for arithmetic (and hence, for EG) in NAFL. Thus the line m of Proposition 1 is modelled as the union of an infinite proper class of line segments M, as represented by the superposed state. Here we wish to make a few remarks on the notion of infinite class, which is a *proper class* and not a mathematical object in NAFL [2].

In NAFL, whenever infinitely many mathematical objects identified by a given property (such as, that of being a natural number) exist within a theory, that theory must also necessarily admit the corresponding infinite class of such objects. The class comprehension scheme is necessarily a theorem of NAFL theories which admit infinitely many objects in the universe of discourse; quantification is restricted to be only over objects that belong to classes. NAFL interprets the axiom of extensionality for classes (which is also necessarily a theorem of such NAFL theories) to mean that an infinite class must be identified by *all* and *only* its elements; the infinite proper class is by itself not a mathematical object. The existence and uniqueness of an infinite class can be inferred from, and is equivalent to, the existence and uniqueness of every element of that class. In the example of Proposition 1, EG(NAFL) requires it to be universally quantified over all possible lengths L; that there are infinitely many such line segments M of arbitrarily large lengths results from Euclid's second postulate, which is a theorem of EG(NAFL). It immediately follows in NAFL that an infinite proper class of such line segments, identified constructively by a given property, exists; call this class C. The assertion that m is parallel to l is to be interpreted in NAFL as meaning that every line segment of the infinite class C is parallel to every line segment of the corresponding infinite class constituting l. The 'existence' and 'uniqueness' of the line m is to be identified with, and may be inferred from, the existence and uniqueness of every element of C.

At this stage the reader might wonder *why* NAFL requires quantification to be restricted to classes when infinitely many objects are involved. This has been formally established in [3]; here we will give an intuitive explanation as follows. Let us take the example of a line segment of initial length L_0, and successively extended (equally on both sides of the segment) to lengths $2L_0$, $3L_0$, $4L_0$, etc. NAFL asserts that the process of extending the line segment may be said to have been 'completed', or equivalently, reached an 'arbitrary' length nL_0, if and only if the infinite proper class $C = \{L_0, 2L_0, 3L_0, \ldots, nL_0, (n+1)L_0, \ldots,\}$ exists. This infinite class represents *all* extensions of the original line segment *at the same time.* If C did not exist, one can only imagine the line segment being extended, say, one instance (in general, at most finitely many instances) at a time, but by induction, such a process can never ever be 'completed' and it would be wrong to quantify over 'all' such extensions. Equivalently, one can never exhaust the class N of natural numbers by counting them one at a time; induction says that there will always be infinitely many natural numbers left to be counted. Therefore quantification over infinitely many natural numbers automatically implies the existence of N in NAFL [3]. However, N is a paradoxical entity because *any* element of N is clearly accessible by counting one at a time. It is clear that 'any' in the preceding sentence does not translate to 'all', for then the above induction (that N cannot be exhausted by counting one at a time) would be violated. This is the intuitive explanation for why N must be a proper class in NAFL; for the formal argument that results from the axiomatic nature of NAFL truth, see Sec. 3 of [2]. To summarize, for the purposes of quantification over infinitely many mathematical objects, NAFL requires that N must necessarily exist as a 'completed' infinite class, but the 'incomplete' or 'potential' nature of N is recognized by not admitting it as a mathematical object (i.e., a set) in NAFL theories. Thus NAFL provides the correct logical framework for assertions denying the existence of a 'completed' infinity by a long list of famous logicians/mathematicians/philosophers, including Aristotle, Gauss, Kronecker, Poincaré and Brouwer; it is somewhat ironic that, in order to accomplish this feat, NAFL has to deny two of the most fundamental laws of classical logic enunciated by Aristotle, namely, the laws of non-contradiction and the excluded middle [2].

Classical logic (FOPL), however, insists that we can talk about 'all' natural numbers or 'all' elements of C without ever invoking the existence of N or C; the induction mentioned above that such a counting process is always incomplete then leads us to nonstandard models of arithmetic and the corresponding Platonism which is rejected by NAFL as inconsistent and unacceptable [3]. That NAFL should consider a 'line' to be the union of the infinite proper class C (rather than a separate mathematical object as in classical logic) is also entirely natural and consistent with the axiomatic nature of NAFL truth. Indeed, we do not have any mental picture (i.e., a 'construction') of a *completed* line as a separate mathematical object; we can only conceive of an infinitely long line as obtained by the process represented in C of extending a (finite) line segment infinitely many times. This is an inherent human limitation of being unable to conceive of the infinite and is intuitively another reason why N and C must be proper classes in NAFL; the classical position that infinite entities nevertheless 'exist' as mathematical objects (i.e., as sets) independently of human limitations violates the axiomatic nature of NAFL truth and leads to Platonism and inconsistency (from the NAFL viewpoint).

In conclusion, we have demonstrated in this section the importance of diagrams as formal objects in EG(NAFL) and that the provability of Euclid's fifth postulate in EG(NAFL) (as demanded by consistency) poses severe restrictions on the concept of 'line', which cannot be treated as a mathematical object; consequently, quantification over 'lines' is banned in EG(NAFL). It was stated in [1] that a problem for future research is to figure out how the classical continuum of real numbers can be handled in EG(NAFL). Indeed, not only 'lines', but even 'points' and 'line segments' must be *analytically* represented in any continuum theory by (collections of) real numbers, which are also infinite proper classes in NAFL. So the question of how points and/or line segments may be treated *analytically* (rather than diagrammatically) as mathematical objects in EG(NAFL) must be resolved. The author believes that *physically meaningful* statements about (sets/classes of) real numbers, as represented by diagrams, could possibly be treated analytically in NAFL by some sort of 'translation' procedure into Peano Arithmetic (or equivalently, finite set theory); such a procedure will have to differ radically from its classical equivalents. The resulting NAFL theory, if accomplished, should satisfactorily resolve classical paradoxes, such as, the Banach-Tarski paradox or Zeno's paradoxes of motion, without having to deny the existence of precise positions in space or instants in time as done by Lynds [6, 7]. Indeed, it is already clear that 'points' (real numbers) must exist in NAFL theories as infinite proper classes and these can certainly represent precise positions in space or instants in time. It is only quantification over sets/classes of real numbers that is problematic in NAFL, for the real numbers constitute neither a set nor a class (being themselves proper classes); here justification of any formal translation procedure of statements that involve quantification over real numbers into finite set theory must necessarily come from diagrams as formal objects. Thus the formal existence of diagrams would still play an essential and indispensable role in justifying such an analytical treatment of geometry/analysis. The author also believes that any continuum theory of space, time and matter, even if possible to formulate consistently in NAFL, is nevertheless an approximation to reality and will fail at quantum scales. As noted in the concluding remarks of [3], the ultimate NAFL theory that describes reality must be one in which *everything*, including space, time and matter, is discrete (quantized).

2 Relativistic determinism – the clash with logic

As in [1], we consider the theory of special relativity (SR) formalized in classical first-order predicate logic (FOPL). The context for this section is best explained by the following extensive quote from Sec. 1 of [1]:

"Let A and B be relatively moving inertial observers who happen to coincide in space at a given instant defined by $t = 0$ in A's frame and $t' = 0$ in B's frame. Let C be an instantaneous event that is localized in space and distant to both A and B. Let $U(IBC)$ define a non-trivial universe of material objects with certain well-posed initial-boundary conditions IBC. Define the proposition P as "From A's point of view, C occurs in $U(IBC)$ when A's local clock reads $t = 0$" and the proposition Q as "From B's point of view, C occurs in $U(IBC)$ when B's local clock reads $t' = T$". Here $T > 0$ is a constant obtained from the Lorentz transformations as applied to the event C in A's and B's inertial frames. Relativistic determinism asserts that if P is true then Q must be true (or $P \Rightarrow Q$); in other words, B's future at time $t' = 0$ is determined by the fact that A has observed C at precisely that instant (when A and B coincided) and so B must necessarily observe C at $t' = T$."

"In order to obtain a logical contradiction from the above scenario, let us further stipulate that the proposition "Event C occurs in $U(IBC)$" is undecidable in SR, i.e., in particular, neither A nor B can either prove or refute this proposition. Such undecidability could occur in many ways, for example, as a result of Gödel's incompleteness theorems; alternatively, C could be a probabilistic event, such as, the outcome of a coin toss experiment or some quantum phenomenon; or else, C could be completely unpredictable as a result of being decided by the instantaneous free will of a human being. It immediately follows that P and Q are undecidable in SR; see the ensuing paragraph for the definition of such undecidability. Note however, that SR requires $P \Leftrightarrow Q$ to be a theorem despite the undecidability of P and Q; this fact immediately makes SR inconsistent in the non-Aristotelian finitary logic (NAFL) proposed by the author in [4] and [5] (in particular, see Remark 5 of [4] and Section 2.2 of [5]). This argument for inconsistency of SR in NAFL is simpler than the one given using inertial frames in [6]. It follows that the philosophy of formalism as embodied by NAFL [5] immediately rejects relativistic determinism. The goal of this paper is to show that an inconsistency can be deduced in SR even within FOPL, if one insists on formalism."

"Henceforth, whenever we refer to A (B), it is to be understood that our argument may apply equally well to any observer in A's (B's) set of inertial frames. Note that we require the following restrictions regarding propositions involving P and Q. The truth of P (Q) can be *asserted* (via an observation, for example) or *deduced* in SR *only* by A (B). However, B (A) can consider and either accept or refute in SR any assertion/deduction of the truth of P (Q) made by A (B); but B (A) cannot *assert* or *deduce* the truth of P (Q). The undecidability of P (Q) in SR means that A (B) can neither prove nor refute P (Q) in SR. $P \Rightarrow Q$ is a theorem in B's (and not A's) frame;

in other words, only B has the right to deduce Q in SR from an assertion of P made by A (if B happens to agree with A's assertion). Similarly, $Q \Rightarrow P$ is a theorem in A's (and not B's) frame. In fact $P \Rightarrow Q$ and $Q \Rightarrow P$ are illegitimate propositions in A's and B's frames respectively. The idea behind these restrictions is to allow A (B) to consider the truth of Q (P) without undermining the Lorentz transformations."

"In particular, Q is undecidable in SR, which means, as noted above, that B can neither prove nor refute Q in SR. The question we wish to consider is as follows. Given that A has asserted the truth of P, and given that $P \Rightarrow Q$ is a theorem of SR in B's frame, can B accept A's assertion and conclude Q? In the metatheorem that follows, we argue that B in fact has a formal refutation of A's assertion; i.e., B has a proof of $\neg P$ in SR and hence B has no way to conclude Q despite A's assertion of P. However, B does not have the right to use $Q \Rightarrow P$ along with the said proof of $\neg P$ to deduce $\neg Q$, because, as noted above, $Q \Rightarrow P$ is a theorem of A's (and not B's) frame. Hence Q continues to remain undecided in SR (in B's frame) despite A's assertion of P. See Remark 6 below for further clarifications."

"Before proceeding to the main result in the metatheorem below, we observe that an additional restriction is necessary, as follows. A and B accept each other's observations/theorems as true/valid if and only if there is no disagreement with (or a refutation of) the observations or any step used in the proof of the said theorems, including the theorems themselves. As an example, suppose A asserts $\neg P$ and concludes $\neg Q$ from the theorem $Q \Rightarrow P$ of A's frame. Then B accepts A's assertion $\neg P$ as true and A's inference $\neg P \Rightarrow \neg Q$ as valid despite that fact that such an inference is illegal in B's frame. Thus B accepts A's conclusion $\neg Q$ as true; i.e., B does not insist that because of the illegality of the inference $\neg P \Rightarrow \neg Q$ in B's frame, there must exist a model for SR in which A asserts $\neg P$ and B asserts Q."

As explained in the above quote, we considered in [1] the situation when A asserts P and we concluded that B cannot accept this assertion because B has a formal refutation of P; consequently, B concludes that Q continues to remain undecided in SR despite A's assertion, as noted in the metatheorem of [1]. Conversely, we argued in Remark 3 and Remark 4 of [1] that A cannot likewise refute an assertion of Q made by B; hence A accepts that P follows from the theorem $Q \Rightarrow P$ of A's frame.

In the present note, we wish to focus on the misleading claim made in the final paragraph of the above quote (and also in Remark 6 of [1]) that B will automatically accept any assertion of $\neg P$ made by A, because B has a formal proof of $\neg P$ in SR. The problem with this claim is that B's concept of the negation of P is not necessarily the same as that of A and so the said claim leads to the incorrect consequence that B must necessarily accept A's conclusion of $\neg Q$ from $\neg P$. This is best illustrated with the following example.

Let the event C in the above quote denote the outcome 'heads' in an instantaneous coin toss experiment E that is distant to both A and B; the definitions of P and Q follow. Further, let R denote "From A's point of view, the coin toss E occurs in $U(IBC)$ when A's local clock reads $t = 0$", and let S denote "From B's point of view, the coin toss E occurs in $U(IBC)$ when B's local clock reads $t' = T$"; here T is the same positive constant obtained from the Lorentz transformations as in the definition of Q. For clarity, information about the spatial locations of events is suppressed from all propositions defined here and in [1], although strictly speaking, such information must be considered to be tacitly present. Finally, let U denote "From A's point of view, the outcome of the coin toss E in $U(IBC)$ is 'tails' when A's local clock reads $t = 0$", and let V denote "From B's point of view, the outcome of the coin toss E in $U(IBC)$ is 'tails' when B's local clock reads $t' = T$ ". It follows that P and Q are undecidable in SR (in the sense noted in the above quote) and so are U and V; further, U and V must satisfy the same restrictions as P and Q respectively, as quoted above, and may be substituted for P and Q in the metatheorem of [1].

Remark 1. In the first instance, suppose that A can prove R in SR; consistency demands that IBC must be such that B must also be able to prove S in SR. Note that B does not have to rely on A's claim of R in order to prove S; indeed B has a refutation of R by the metatheorem of [1]. Hence $R \Leftrightarrow S$ is a theorem of SR and decidable propositions by themselves do not pose any problem for relativistic determinism. It immediately follows that

$$U \Leftrightarrow \neg P \tag{1}$$

is a theorem of SR in A's frame, and

$$V \Leftrightarrow \neg Q \tag{2}$$

is a theorem of SR in B's frame. Next suppose A claims U; from A's point of view, (1) implies that a claim of U is equivalent to a claim of $\neg P$. Since $\neg P \Rightarrow \neg Q$ is a theorem of A's frame, A concludes $\neg Q$. However, by the metatheorem of [1], B has a refutation of U *and* a proof of $\neg P$ in SR. Clearly, B does not accept the equivalence in (1). Since B has a refutation of R in SR (again, by the metatheorem of [1]), B only accepts $\neg P$ in the sense that the coin toss did not happen at all when A and B coincided at ($t = 0$, $t' = T$). Hence B concludes that A has *deduced* $\neg P$ from a false assertion of U, and consequently *does not accept* the validity of A's conclusion $\neg Q$. So from B's point of view there must still exist a model for SR in which Q is the case, despite A's claim of U; equation (2) shows that this is in agreement with the metatheorem of [1], which asserts that from B's point of view, there must exist a model for SR in which $\neg V$ is the case despite A's claim of U. Conversely, suppose B claims V, or equivalently, from (2), $\neg Q$. A accepts this claim and concludes U from the theorem $V \Rightarrow U$ of A's frame. A also accepts the validity of B's conclusion of $\neg P$ from the theorem $\neg Q \Rightarrow \neg P$ of B's frame, despite its illegality in A's frame, and one sees that A is acting consistently with (1). There is no clash with the metatheorem of [1] because, as noted in Remark 3 and Remark 4 of [1], A cannot refute any claim of Q or V made by B. It follows that A accepts the equivalence in (2) and one again sees the asymmetry noted in Remark 3 and Remark 4 of [1].

Remark 2. Secondly, suppose that A does not have a proof or refutation of R in SR. Consistency demands that B likewise does not have a proof or refutation of S in SR. By our definition of undecidability, R and S are undecidable in SR. Next suppose that A claims $\neg P$; it is easy to see that, by our definitions, B will accept this claim *if and only if* it is also accompanied by a claim of $\neg R$ by A. Consequently, B accepts the truth of A's conclusion $\neg Q$ in this instance, despite the illegality of A's inference $\neg P \Rightarrow \neg Q$ in B's frame. But if A claims $R\&\neg P$, the conclusions of Remark 1 will again hold. Conversely, A will accept an assertion of $\neg Q$ by B irrespective of the status of B's claim with respect to S; consequently, A unconditionally accepts B's conclusion of $\neg P$ from the theorem $\neg Q \Rightarrow \neg P$ of B's frame, despite its illegality in A's frame.

In summary, we have established here and in [1] that the concepts of negation and implication for undecidable propositions in SR are problematic in FOPL and lead to inconsistency from the point of view of formalism. The non-Aristotelian finitary logic (NAFL) [2, 3] proposed by the author also demands that these concepts be handled in a different manner for undecidable propositions; the problems with FOPL should encourage further investigation of these concepts in NAFL, despite that fact that NAFL refutes SR as noted in [1].

Dedication

The author dedicates this research to his son R. Anand and wife R. Jayanti.

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