Brain Functors:
A mathematical model of intentional perception and action

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Abstract

Category theory has foundational importance because it provides conceptual lenses to characterize what is important and universal in mathematics - with adjunctions being the primary lens. If adjunctions are so important in mathematics, then perhaps they will isolate concepts of some importance in the empirical sciences. But the applications of adjunctions have been hampered by an overly restrictive formulation that avoids heteromorphisms or hets. By reformulating an adjunction using hets, it is split into two parts, a left and a right semiadjunction. Semiadjunctions (essentially a formulation of universal mapping properties using hets) can then be combined in a new way to define the notion of a brain functor that provides an abstract model of the intentionality of perception and action (as opposed to the passive reception of sense-data or the reflex generation of behavior).

Keywords: category theory, adjunctions, brain functors, heteromorphisms, intentional perception and action.

1. Introduction

Category theory has foundational importance because it provides conceptual lenses to characterize what is important and universal in mathematics—with an adjunction (or pair of adjoint functors) being the primary lens. The mathematical importance of adjunctions is now well recognized. As Steven Awodey put it in his recent text:

> The notion of adjoint functor applies everything that we have learned up to now to unify and subsume all the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics. (Awodey, 2006, p. 179)

If a concept, like that of a pair of adjoint functors, is of such importance in mathematics, then one might expect it to have applications, perhaps of some importance, in the empirical sciences. Yet this seems to be rarely the case, particularly in the life sciences. Perhaps the problem has been finding the right level of generality or specificity where non-trivial applications can be found, i.e., finding out "where theory lives."

This paper argues that the application of adjoints has been hampered by an overly specific formulation of the adjunctive properties that only uses homomorphisms or homs[1] (object-to-object morphisms within a category). A reformulation of adjunctions using heteromorphisms or hets (object-to-object morphisms between objects of different categories) allows an adjunction to be split into two "semiadjunctions." The argument is that a semiadjunction (essentially a reformulation of a universal mapping property using hets) turns out to be the right concept for applications. Moreover, by "splitting the atom" of an adjunction into two semiadjunctions, the semiadjunctions can be recombined in a different way to define the cognate notion of a "brain functor"–which, as the name indicates, may have applications in cognitive science.

[1] "Hom" is pronounced to rhyme with "Tom" or "bomb."

There is already a considerable but widely varying literature on the application of category theory to the life sciences–such as the work of Robert Rosen (1991) and his followers[2] as well as Andrée Ehresmann and Jean-Paul Vanbremeersch (2007) and their commentators.[3] But it is still early days, and many approaches need to be tried to find out "where theory lives."

The approach taken here is based on a specific use of the characteristic concepts of category theory, namely universal mapping properties, to define a general schema of determination through universals. The closest approach in the literature (but without the hets and semiadjunctions) is that of François Magnan and Gonzalo Reyes (1994) which emphasizes that "Category theory provides means to circumscribe and study what is universal in mathematics and other scientific disciplines." (Magnan & Reyes, 1994, p. 57). Their intended field of application is cognitive science.

We may even suggest that universals of the mind may be expressed by means of universal properties in the theory of categories and much of the work done up to now in this area seems to bear out this suggestion....
By discussing the process of counting in some detail, we give evidence that this universal ability of the human mind may be conveniently conceptualized in terms of this theory of universals which is category theory. (Magnan and Reyes, 1994, p. 59)

The approach of determination through universals, like any approach, should be judged on how well it isolates and describes the essential and important features of biological and cognitive systems.

2. Adjunctions, hets, and semiadjunctions

In the body of this paper, I will try to keep the mathematics at a minimal conceptual level– which the mathematical formulations restricted to the Appendix. Category theory lends itself to visualization in diagrams so that non-mathematical style of presentation is emphasized by an abundant use of diagrams.

A *category* is intuitively a set of objects of the same type. Morphisms between objects should be thought of as a type of determining relation or cause-effect relation between the objects. When a morphism is between objects of the same category, it is called a homomorphism or hom, and when between objects of different categories it is a heteromorphism or het.[4] One of the problems in the conventional treatment of category theory[5] is that it tries to ignore heteromorphisms even though hets are a natural part of working mathematics. This leads to certain definitions being rather contrived (to avoid mentioning hets), the usual treatment of the universal mapping properties in adjunctions being the case in point.

Adjunctions will be introduced informally and in the natural manner using hets. The general setting is how the objects in one category (e.g., the "environment" in a life sciences context), the "sending" category, will "affect" or "determine" objects in another category (e.g., "organisms"), the "receiving" category. We start with an object X in the sending category, an object A in the receiving category, and a specific het determination $d : X \rightarrow A$ from X to A.[6]

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[2] See (Zafiris, 2012) and that paper's references.

[3] See (Kainen, 2009) for Kainen's comments on the Ehresmann-Vanbremeersch approach, Kainen's own approach, and a broad bibliography of relevant papers.

[4] See (Ellerman, 2006; 2015) for the introduction of heteromorphisms in category theory.

[5] The standard presentation is Mac Lane's text (1971) but Lawvere and Schanuel's book (1997) is a more conceptual introduction. Magnan and Reyes (1994) give an excellent informal treatment of the main universal constructions.

[6] Hets are represented by thin arrows $\rightarrow$ and homs by thick arrows $\Rightarrow$.

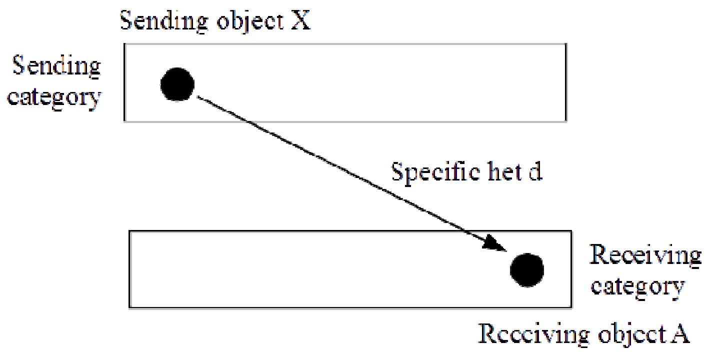

Figure 1: Het d: X→A

One of the most important concepts isolated by category theory (more basic than adjunctions) is the concept of a universal mapping property or UMP[7] which models an important type of determination, determination through universals. With each sending object X in the sending category, there is an associated receiving universal F(X) in the receiving category, and there is a universal receiving het $h_X : X \to F(X)$. The universal mapping property is: for every het $d : X \to A$, there is a unique hom $f(d) : F(X) \Rightarrow A$ in the receiving category such that:

$$f(d)h_X = X \to F(X) \Rightarrow A = X \to A = d,$$

i.e., such that the determination through the receiving universal het $h_X: X \to F(X)$ followed by the hom $f(d): F(X) \Rightarrow A$ is the same as the original het $d: X \to A$.

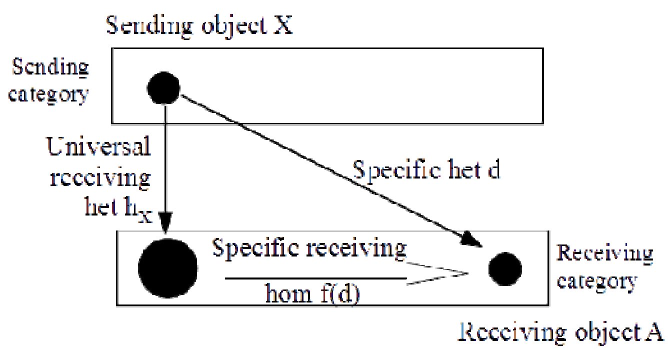

Figure 2: Scheme for determination by a receiving universal F(X)

Moreover given any hom $f : F(X) \Rightarrow A$ in the receiving category, preceding it by the universal receiving het h_X would give a het $f\,h_X : X \to A$ which would play the role of $d : X \to A$ in the above equation. Hence there is a one-to-one correspondence (or isomorphism) between the hets $d : X \to A$ and the homs $f : F(X) \Rightarrow A$. This isomorphism is also canonical or natural (in the category-theoretic sense), and can be represented as:

$$Hom_{receiving}(F(X),A) \cong HetX,A).$$

Even before completing our definition of an adjunction, we can define the above situation, given by the association of the receiving universal F(X) with each object X in the sending category (and similarly for morphisms so it is a "functor") along with the canonical isomorphism $Hom_{receiving}(F(X),A) \cong Het(X,A)$ as a left semiadjunction.

Then we return to the specific het d and take the dual case which will define the dual notion of a "right semiadjunction." With each receiving object A in the receiving category, there is an associated sending universal G(A) in the sending category, and there is a *sending universal het* e_A :

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[7] A UMP illustrates the philosophical notion of a concrete or self-participating universal (Ellerman, 1988).

G(A)→A. The universal mapping property is: for every het d : X→A, there is a unique hom g(d) : X⇒G(A) in the sending category such that:

$$e_A g(d) = X \Rightarrow G(A) \rightarrow A = X \rightarrow A = d$$

i.e., such that the determination through the universal sending het eA : G(A)→A preceded by the hom g(d) : X⇒G(A) is the same as the original het d : X→A.

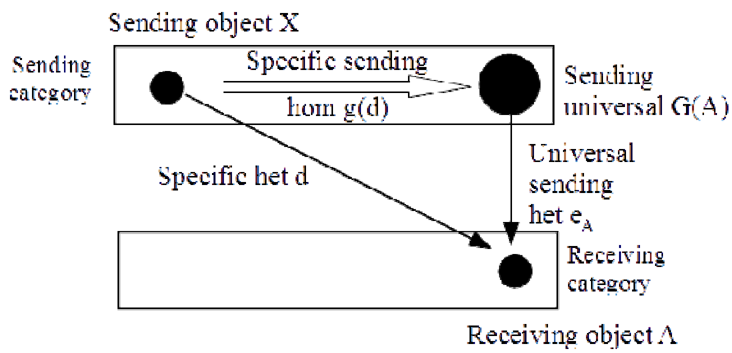

Figure 3: Scheme for determination by a sending universal G(A)

Moreover given any hom g : X⇒G(A) in the sending category, following it by the universal sending het e_A would give a het $e_A g$: X→A which would play the role of d : X→A in the above equation. Hence there is a one-to-one correspondence (or isomorphism) between the hets d : X→A and the homs g : X⇒G(A). This isomorphism is also canonical or natural, and can be represented as:

$$\text{Het}(X,A) \cong \text{Hom}_{\text{sending}}(X,G(A)).$$

Dually, we can define the above situation, given by the association of the sending universal G(A) with each object A in the receiving category along with the canonical isomorphism Het(X,A) ≅ $\text{Hom}_{\text{sending}}$(X,G(A)), as a *right semiadjunction*.

Now we are prepared to define an *adjunction* essentially as:

$$\text{adjunction} = \text{left semiadjunction} + \text{right semiadjunction}$$
$$\text{Hom}_{\text{receiving}}(F(X),A) \cong \text{Het}(X,A) \cong \text{Hom}_{\text{sending}}(X,G(A)).$$

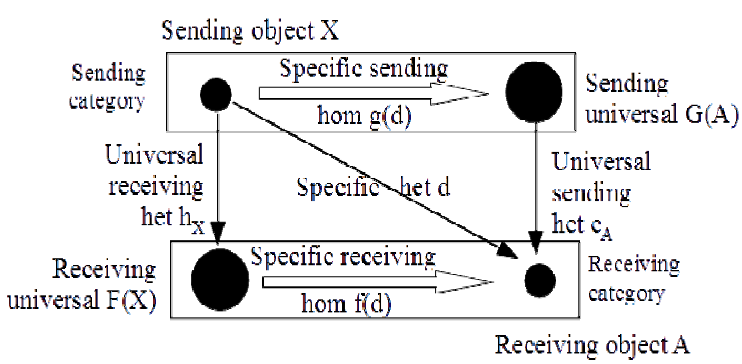

Figure 4: The Adjunctive Square Diagram

Recall that the conventional "heterophobic" treatment of an adjunction keeps the hets "in the closet," so the defining natural isomorphism just leaves out the middle het term Het(X, A) to give the het-free notion of an adjunction: Hom(F(X),A) ≅ Hom(X,G(A)) where F( ) is the left adjoint and G( ) is the *right adjoint*.

In almost all adjunctions, one of the adjoints is rather trivial and contrived so the principal universal mapping property is expressed by the other adjoint, but one needs the trivial adjoint in order to avoid the hets. This has hampered applications since it is the main receiving or sending universal that is important, not the trivial other adjoint needed to formulate the het-free adjunction. Moreover since determinations between quite different type of objects are important in applications, hets are an essential part of the story, and thus the treatment of adjunction leaving out the hets also hampers applications. But by bringing the hets "out of the closet," we can "fission" the adjunction into two semiadjunctions, one of which is typically appropriate in an application. And we can recombine the two semiadjunctions in a different way to define a "brain functor"–which combines two non-trivial dual semiadjunctions in one scheme of two-way determination.

3. Determinations through receiving universals

3.1. Active/indirect versus passive/direct learning

In the most generic example of determination through a receiving universal (i.e., the determinative scheme of a left semiadjunction), the sending category or domain is the "environment" while the receiving category or domain is an "organism" and the factoring of a het determination $d : X \rightarrow A$ through the receiving universal $F(X)$ is a type of selective "recognition."
This recalls a much older theme of "recollection." One of the tell-tale signs of a process of determination through universals is the indirectness of the factorization through a universal. The interplay between these two accounts dates back at least to the Platonic-Socratic account of learning not as the result of direct external instruction but as a process of catalyzing internal recollection. One of the striking epigrams of neo-Platonism is the thesis that "no man ever does or can teach another anything" (Burnyeat, 1987, p. 1). In the early fifth century, Augustine in *De Magistro* (The Teacher) made the point contrasting "outward" passive instruction with active learning "within."

But men are mistaken, so that they call those teachers who are not, merely because for the most part there is no delay between the time of speaking and the time of cognition. And since after the speaker has reminded them, the pupils quickly learn within, they think that they have been taught outwardly by him who prompts them. [Chapter XIV]

In the nineteenth century, Wilhelm von Humboldt made the same point even recognizing the symmetry between listener and speaker (a symmetry captured by the right semiadjunctions).

Nothing can be present in the mind (Seele) that has not originated from one's own activity. Moreover understanding and speaking are but different effects of the selfsame power of speech. Speaking is never comparable to the transmission of mere matter (Stoff). In the person comprehending as well as in the speaker, the subject matter must be developed by the individual's own innate power. What the listener receives is merely the harmonious vocal stimulus. (Humboldt 1997, p. 102)

A similar theme has been a mainstay in active learning theories of education. As John Dewey put it:

It is that no thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told, it is, to the one to whom it is told, another given fact, not an idea. The communication may stimulate the other person to realize the question for himself and to think out a like idea, or it may smother his intellectual interest and suppress his dawning effort at thought. (Dewey 1916, p. 159)

3.2. Language understanding

The ordinary understanding of language is an example well-modeled by determination through a receiving universal. If the auditory input is in a language that the listener understands, that means there is an internal process triggered by the auditory signals that recognizes, interprets, and understands the input.

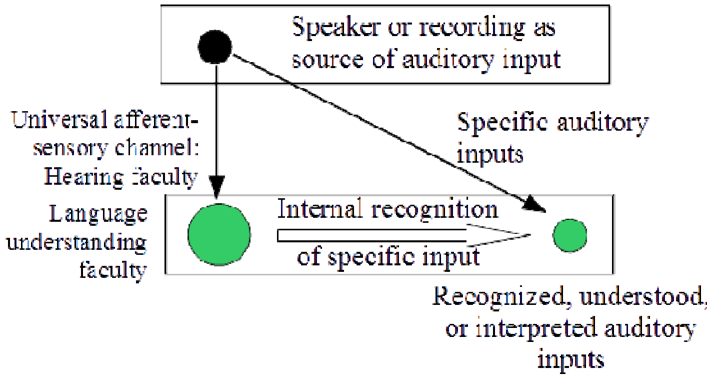

Figure 5: Language recognition as a receiving universal

The determination through the receiving universal thus adds a second level to the input which is variously called recognition, understanding, or the *intentionality of perception*. This second level is often indicated by saying that the sensory input is "perceived as", "recognized as", or "understood as" something further.

3.3. Generic "recognition" or "perception"

Before turning to right semiadjunctions, it might be useful to present a rather generic version of determination through a receiving universal as model of "recognition" or "perception" that captures many of the common features of the various examples.

The determination through the receiving universal is the active internal process that supplies the "interpretation" or "intentionality" to the raw sense data. The red blotch is seen as a tomato; the sound "ya" is understood as indicating agreement, and so forth. In the passive/direct alternative, the raw sensory input supplies Lockean "perception" like writing on a blank slate or a stamp making an impression on wax.

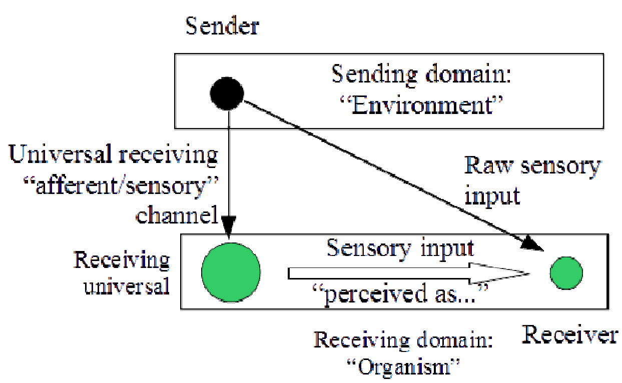

Figure 6: "Perception" as determination through a receiving universal

4. Determinations through a sending universal

4.1. Generic "action"

Dual to the generic model of "perception" is the generic model of "action"–which is the determinative scheme given by a right semiadjunction. In the model of perception, there is the uninterpreted message as just a sensory input (the external het), and then there is the second level where the factorization (the internal hom) through the receiving universal recognizes the interpretation, meaning, or intentionality of the message. In the dual model of "action," the external het specifies the external behavior (which could be even a reflex behavior) while internal hom factoring through a sending universal that supplies the "intentionality" of the "action" (where an "action" is a "behavior" plus the second level of "intentionality"). In each case, we end up with a certain behavior but determined by two different means.

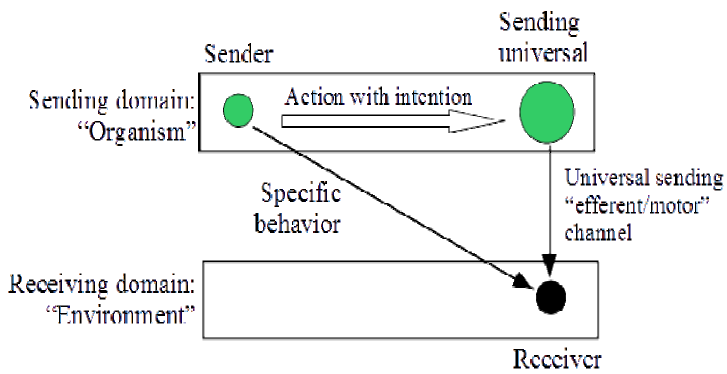

Figure 7: "Action" as determination through a sending universal

4.2. Language action

The dual to "language understanding" is language production or linguistic action (e.g., "speech acts"). The role of the specific het is played by some auditory output such as utterances (Humboldt's "vocal stimulus"). But the corresponding internal specific hom is the speech act (i.e., internal speech with intentionality) that through the language faculty produces the same outputs but as intentional speech.

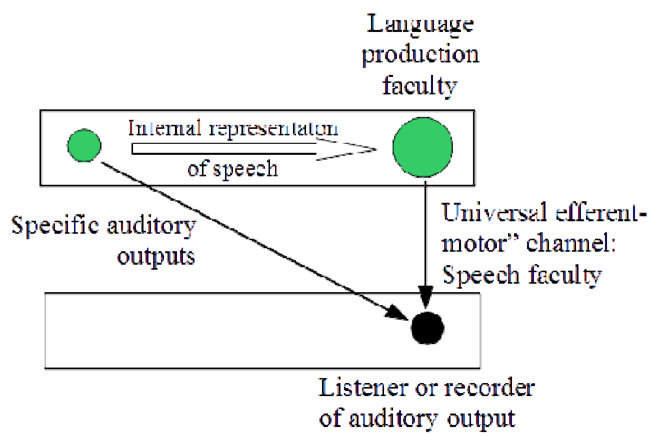

Figure 8: Language production through a sending universal

5. Brain Functors

5.1. Recombination of left and right semiadjunctions

It might be noted that the same faculty, e.g., the language faculty, may play both the role of the receiving or perception universal and the sending or action universal. Hence it should be possible to recombine the semiadjunctions so that the two universals coincide, and that yields the concept of the brain functor (the "brain" being that universal for both perception and action). From the mathematical viewpoint, the key to this was using the hets to split an adjunction into left and right semiadjunctions which can then be recombined in the opposite way.

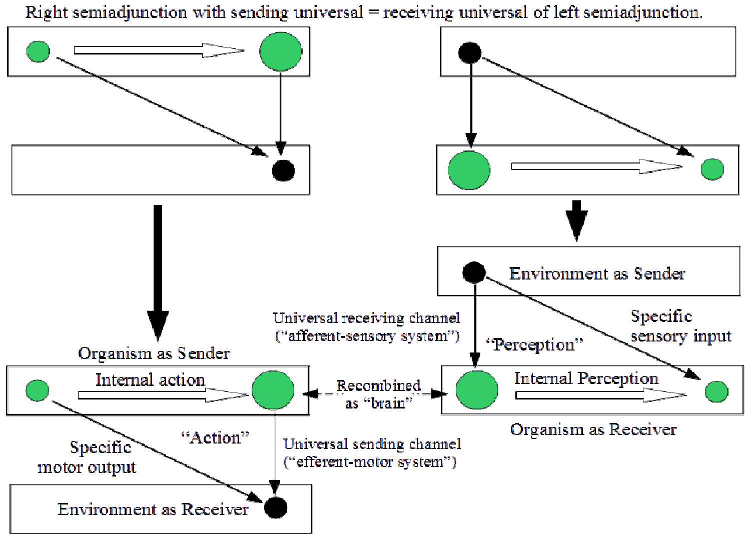

Figure 9: Different way to combine semiadjunctions to make a brain functor.

The recombined semiadjunctions then form a *brain functor*. In the following *butterfly diagram* for a brain functor, we use labels appropriate to the "brain" label.

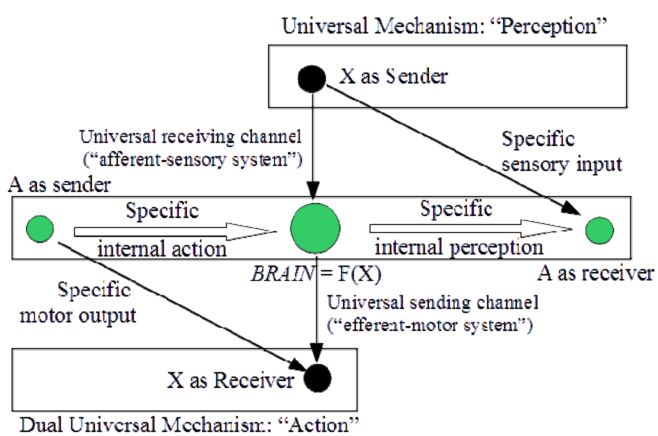

Figure 10: Brain Functor: Scheme for receiving and sending through one universal

Mathematically, for each het X→A there is a unique internal "perceiving" hom F(X)⇒A so there is a canonical isomorphism:

$$\mathrm{Hom}(F(X), A) \cong \mathrm{Het}(X, A).$$

And for each het A→X in the other direction, there is a unique internal "action" hom A⇒F(X) so there is also a canonical isomorphism:

$$\mathrm{Het}(A, X) \cong \mathrm{Hom}(A, F(X)).$$

The concept of a brain functor is the natural cognate or associated concept to an adjunction. For an adjunction, there are two functors that represent on the left and right the hets going one way between the categories:

$$\mathrm{Hom}(F(X), A) \cong \mathrm{Het}(X, A) \cong \mathrm{Hom}(X,G(A)).$$

For a brain functor, there is one functor that represents on the left and right the hets going the two ways between the categories:

$$\mathrm{Hom}(F(X), A) \cong \mathrm{Het}(X, A) \text{ and}$$
$$\mathrm{Het}(A, X) \cong \mathrm{Hom}(A, F(X)).$$

Hence the general scheme given by a brain functor is receiving and sending determination through one universal (the "brain").

5.2. Simple two-way determinations through one universal

The simplest form of a brain "functor" is just a two-way representation or coding system that constructs and implements a set of codes. Given some set of objects, it is encoded using some isomorphic set of representations or codes for the objects, and then given an instance of the code, it is decoded to determine the object.

Coordinatizing is a form of coding. The geometrical plane is a collection of points, and the Cartesian coordinate system represents each point P by a pair (x_P, y_P) of coordinates. Given a point P , the "coordinate" function selects the coordinates (x_P, y_P) of the point which is the recognized or coded output, and given the coordinates or code for a point (x_P, y_P) as an input, the "plot" function designates the point.

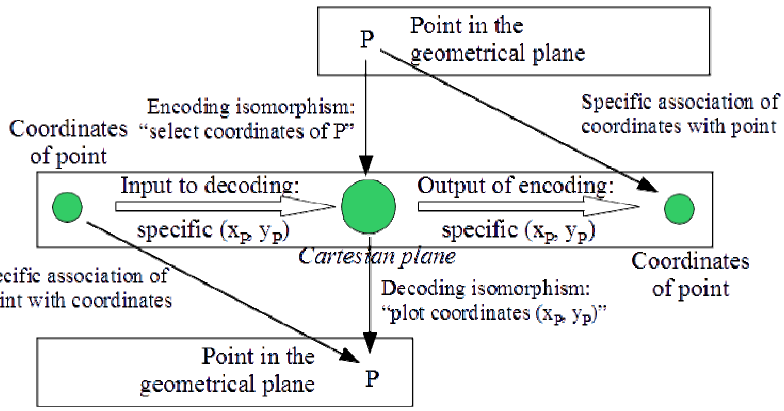

Figure 11: Coding and decoding Cartesian coordinates of geometrical points

5.3. The language faculty

A brain functor, broadly put, is any universal mechanism of determination that can factor determination either way through a universal–rather than an adjunction that factors one way determination through two (receiving and sending) universals. In some contexts in the life sciences, determination is strictly one way so one might expect to find a semiadjunction but not a two-way system like a brain functor.

An application of the scheme for a brain functor in the cognitive sciences is to model the language faculty where there is two way determination between vocal stimuli and internal representations. The previous semiadjunctions for language understanding and language action can be merged to arrive at the brain-like function of the language faculty.

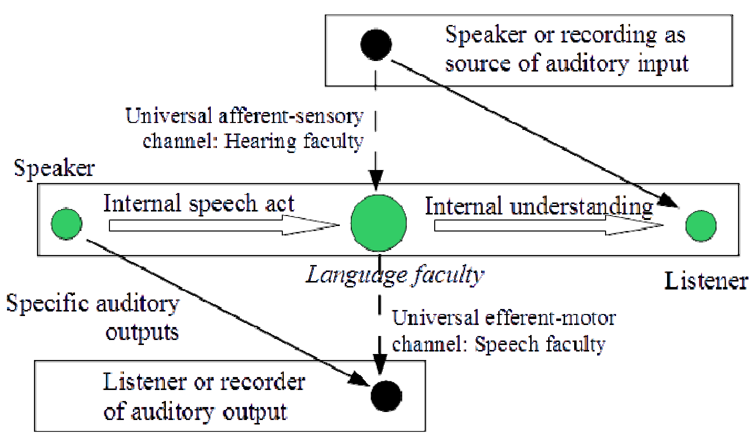

Figure 12: Language faculty as two-way determination through a universal

6. Summary

The following table gives the principal exact category-theoretic concepts to describe the corresponding schemes of determination through universals as well as the main generic examples.

Table 1: Principal forms of determination through universals

CT concept	Determination through universals	Generic example
Left semiadjunction	Det. through a receiving universal	Intentional perception
Right semiadjunction	Det. through a sending universal	Intentional action
Brain functor	Two-way det. through a universal	Perception + Action

The importance of category-theoretical universals in isolating the important concepts in pure mathematics suggests that the universals may play a similar role in the empirical sciences. Our results suggest that this is indeed the case in the biological and cognitive sciences. The category-theoretic schemes of determination through universals are at a high level of abstraction, but, in this case at least, that seems to be where some significant theory lives. Note particularly how the physiological "duality" for "afferent versus efferent" or "sensory versus motor" comes out natually as an instance of category-theoretic duality. Regardless of the great differences in the underlying substrate processes, many of the most important mechanisms and faculties of the cognitive sciences not only seem to fit into, but also to have their key features characterized by, the schemes for determination through universals.

7. Appendix: Defining hets in category theory

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects. Since the morphisms are between objects of similar structure, they are ordinarily called "homomorphisms."

But there have always been other morphisms which occur in mathematical practice that are between objects with different structures (i.e., in different categories) such as the insertion-of-generators map from a set to the free group on that set. Indeed, the working mathematician might well characterize the free group $F(X)$ on a set X as the group such that for any set-to-group map $f : X \rightarrow G$, there is a unique group homomorphism $f^*: F(X) \Rightarrow G$ that factors f through the canonical insertion of generators $i: X \rightarrow F(X)$, i.e., $f = f^*i$. In order to contrast these morphisms such as $f : X \rightarrow G$ and $i : X \rightarrow F(X)$ with the homomorphisms between objects within a category such as $f : F(X) \Rightarrow G$, the former are called *heteromorphisms* or *hets* (for short). Hets are like chimeras since they have a tail in one category and a head in another category.

We assume familiarity with the usual machinery of category theory (bifunctors, in particular) which can be adapted to give a rigorous treatment of heteromorphisms (and their compositions with homomorphisms) that is parallel to the usual bifunctorial treatment of homomorphisms.

The cross-category object-to-object hets $d : X \rightarrow A$ will be indicated by thin arrows ($\rightarrow$) rather than thick arrows ($\Rightarrow$). The first question is how do heteromorphisms compose with one another? But that is not necessary. Chimera do not need to 'mate' with other chimera to form a 'species' or category; they only need to mate with the intra-category morphisms on each side to form other chimera.[8]

Given a het $d : X \rightarrow A$ from an object in a category **X** to an object in a category **A**, and homs $h : X' \Rightarrow X$ in X and $k : A \Rightarrow A'$ in A, the composition $dh : X' \Rightarrow X \rightarrow A$ is another het $X' \rightarrow A$ and the composition $kd : X \rightarrow A \Rightarrow A'$ is another het $X \rightarrow A'$.

[8] The chimera genes are dominant in these mongrel matings. While mules cannot mate with mules, it is 'as if'mules could mate with either horses or donkeys to produce other mules.

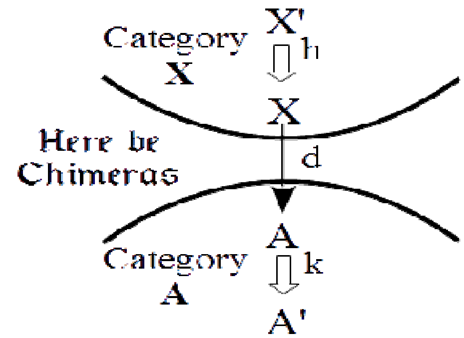

Figure 13: Composition of hets and homs

This action is exactly described by a bifunctor Het : $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$ (functors and bifunctors between categories are represented by ordinary $\rightarrow$ arrrows and are not hets) where Het(X,A) = {f:X→A} and where **Set** is the category of sets and set functions. The natural machinery to treat object-to-object morphisms between categories are het-bifunctors Het : $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$ that generalize the hom-bifunctors Hom : $\mathbf{X}^{op}\times\mathbf{X}\rightarrow\mathbf{Set}$ used to treat object-to-object morphisms within a category.[9] For any **A**-hom k : A⇒A'and any het d:X→A in Het(X, A), there is a composite het kd: X→A⇒A' = X→A', i.e., k induces a map Het(X, k) : Het(X,A)→Het(X,A'). For an **X**-hom h: X'⇒X and het d:X→A in Het(X,A), there is the composite het hd:X'⇒X →A = X'→A, i.e., h induces a map Het(h,A) : Het(X,A)→Het(X',A) (note the reversal of direction). The induced maps would respect identity and composite morphisms in each category. Moreover, composition is associative in the sense that (kd)h = k(dh). This means that the assignments of sets of chimera morphisms Het(X, A) = {f:X→A} and the induced maps between them constitute a bifunctor Het: $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$ (contravariant in the first variable and covariant in the second).

With this motivation, we may turn around and define heteromorphisms from **X**-objects to **A**-objects as the elements in the values of a given bifunctor Het : $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$. This would be analogous to defining the homomorphisms in **X** as the elements in the values of a given hom-bifunctor $\mathrm{Hom}_{\mathbf{X}}$: $\mathbf{X}^{op}\times\mathbf{X}\rightarrow\mathbf{Set}$ and similarly for $\mathrm{Hom}_{\mathbf{A}}$:$\mathbf{A}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$.

Given any bifunctor Het : $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$, it is *representable on the left* if for each **X**-object X, there is an **A**-object F(X) that represents the functor Het(X,-), i e , there is an isomorphism $\psi_{X,A}$: $\mathrm{Hom}_{\mathbf{A}}(F(X),A) \cong \mathrm{Het}(X,A)$ natural in A.This defines a functor F: **X**→**A**, and any such functor with natural isomorphisms $\mathrm{Hom}_{\mathbf{A}}(F(X),A) \cong \mathrm{Het}(X,A)$ is a left semiadjunction.

Given a bifunctor Het : $\mathbf{X}^{op}\times\mathbf{A}\rightarrow\mathbf{Set}$, it is *representable on the right* if for each **A**-object A, there is an **X**-object G(A) that represents the functor Het(-,A), i.e., there is an isomorphism $\phi_{X,A}$: $\mathrm{Het}(X,A) \cong \mathrm{Hom}_{\mathbf{X}}(X,G(A))$ natural in X. This defines a functor G : **A**→**X**, and any such functor with natural isomorphisms $\mathrm{Het}(X,A) \cong \mathrm{Hom}_{\mathbf{X}}(X,G(A))$ is a *right semiadjunction*.

An *adjunction* is given by two functors F : **X**→**A** and G : **A**→**X** that form left and right semiadjunctions, i.e.,

$$\mathrm{Hom}_{\mathbf{A}}(F(X),A) \cong \mathrm{Het}(X,A) \cong \mathrm{Hom}_{\mathbf{X}}(X,G(A))$$

natural in X and A. The identity hom 1F(X) in HomA(F(X), F(X)) is associated by the left isomorphism with the universal receiving het hX : X→F(X), and the identity hom 1G(A) in HomX(G(A), G(A)) is associated by the right isomorphism with the universal sending het eA : G(A)→A. Then for any het d in Het(X, A), the uniquely associated hom f(d) in HomA(F(X), A)

[9] Such het-bifunctors and generalizations replacing **Set** by other categories have been studied by the Australian school under the name of *profunctors* (Kelly, 1982), by the French school under the name of *distributors* (Bénabou, 1973), and by F. William Lawvere under the name of *bimodules* (Lawvere, 2002). However, the guiding interpretation has been interestingly different. "Roughly speaking, a distributor is to a functor what a relation is to a mapping" [Borceux, 1994, p. 308] (and hence the name "profunctor" in the Australian school).

gives: d = f(d)hX. And for the same d, the uniquely associated g(d) in HomX(X, G(A)) gives: d = eAg(d). Combining the two gives the adjunctive square diagram (figure 4) used previously in the text.

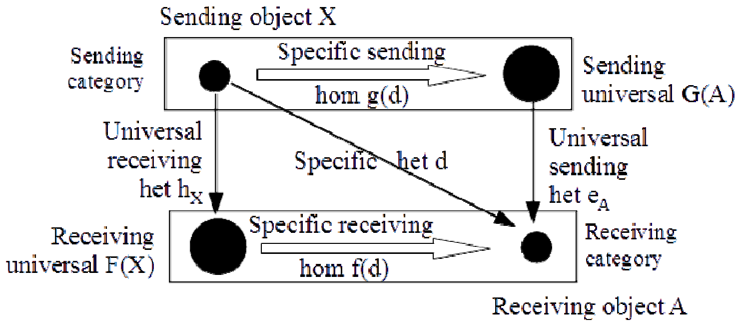

Figure 14: Adjunctive square diagram

Finally, a brain functor is a functor F: **X**→**A** that is a left semiadjunction for Het(X, A) and a right semiadjunction for Het(A, X), i.e.,

$$Hom_{\mathbf{A}}(F(X),A) \cong Het(X,A)$$

and

$$Het(A,X) \cong Hom_{\mathbf{A}}(A,F(X)).$$

For each d in Het(X, A), there is a unique hom f(d) in $Hom_{\mathbf{A}}$(F(X), A) so that the upper triangular 'wing' in the butterfly diagram commutes. For each d' in Het(A, X), there is a unique hom g(d') in $Hom_{\mathbf{A}}$(A, F(X)) so that the lower triangular 'wing' commutes.

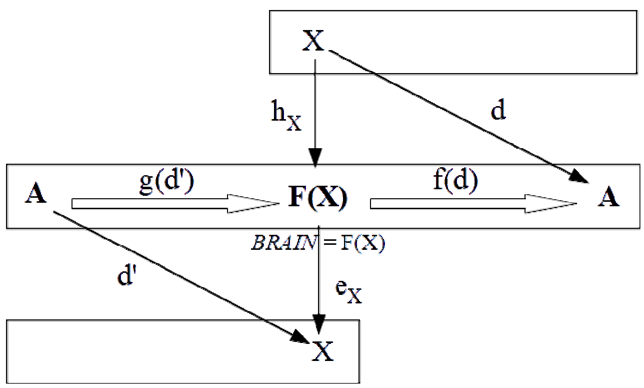

Figure 15: Mathematical butterfly diagram for a brain functor

If a functor F : **X**→**A** has a right adjoint G : **A** →**X**, then:

$$Hom_{\mathbf{A}}(F(X),A) \cong Het(X,A) \cong Hom_{\mathbf{X}}(X,G(A)).$$

If the functor F also has a left adjoint H : **A**→**X**, then:

$$Hom_{\mathbf{X}}(H(A),X) \cong Het(A,X) \cong Hom_{\mathbf{A}}(A,F(X)).$$

Then taking the isomorphisms that do not involve G or H gives:

$$Hom_{\mathbf{A}}(F(X),A) \cong Het(X,A)$$

and

$$Het(A,X) \cong Hom_{\mathbf{A}}(A,F(X)),$$

i.e., F is a brain functor. Hence all functors that have both right and left adjoints are brain functors.

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