

# Foundations of a Probabilistic Theory of Causal Strength: Proofs

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**Proof of Theorem 1:** Suppose that for mutually exclusive  $E, E' \in \mathcal{E}$ ,  $p^*(E'|C) = p^*(E'|C')$ . Then, we infer with the help of Generalized Difference-Making

$$\begin{aligned}\eta(C, E) &= f(p^*(E|C), p^*(E|C')) \\ \eta(C, E \vee E') &= f(p^*(E \vee E'|C), p^*(E \vee E'|C')) \\ &= f(p^*(E|C) + p^*(E'|C), p^*(E|C') + p^*(E'|C')) \\ &= f(p^*(E|C) + p^*(E'|C), p^*(E|C') + p^*(E'|C))\end{aligned}$$

Applying Separability of Effects implies  $\eta(C, E \vee E') = \eta(C, E)$  and leads to the equality

$$f(p^*(E|C), p^*(E|C')) = f(p^*(E|C) + p^*(E'|C), p^*(E'|C') + p^*(E'|C))$$

Since we have made no assumptions about the values of these conditional probabilities,  $f$  satisfies the formula  $f(x, x') = f(x + y, x' + y)$  in full generality. It is then easy to see (e.g., by looking at the indifference curves of  $f$ ) that there

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must be a function  $g$  such that  $f(x, x') = g(x - x')$ . Hence,

$$\eta(C, E) = f(p^*(E|C), p^*(E|C')) = g(p^*(E|C) - p^*(E|C'))$$

showing the desired ordinal equivalence claim. q.e.d.

**Proof of Theorem 2:** By Generalized Difference-Making with  $C' = \neg C$  we can focus on the function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  such that  $\eta(C, E) = f(p^*(E|C), p^*(E|\neg C))$ . We would like to derive the equality

$$f(\alpha, \bar{\alpha}) \cdot f(\beta, \bar{\beta}) = f(\alpha\beta + (1 - \alpha)\bar{\beta}, \bar{\alpha}\beta + (1 - \bar{\alpha})\bar{\beta}) \quad (1)$$

for a causal strength measure that satisfies Multiplicativity. To this end, recall the single-path Bayesian network reproduced in Figure 1.

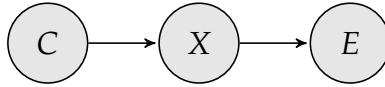


Figure 1: The Bayesian Network for causation along a single path.

We know by Multiplicativity that for  $C \in \mathcal{C}$ ,  $E \in \mathcal{E}$ , and  $X \in \mathcal{X}$ ,

$$\begin{aligned} \eta(C, E) &= \eta(C, X) \cdot \eta(X, E) \\ &= f(p^*(X|C), p^*(X|\neg C)) \cdot f(p^*(E|X), p^*(E|\neg X)) \\ &= f(p^*(X|C), p^*(X|\neg C)) \cdot f(p^*(E|X), p^*(E|\neg X)) \end{aligned}$$

and at the same time,

$$\begin{aligned} \eta(C, E) &= f(p^*(E|C), p^*(E|\neg C)) \\ &= f\left(\sum_{\pm X} p^*(X|C)p^*(E|C, X), \sum_{\pm X} p^*(X|\neg C)p^*(E|\neg C, X)\right) \\ &= f\left(\sum_{\pm X} p^*(X|C)p^*(E|X), \sum_{\pm X} p^*(X|\neg C)p^*(E|X)\right) \end{aligned}$$

Combining both equations yields

$$\begin{aligned} & f(p^*(X|C), p^*(X|\neg C)) \cdot f(p^*(E|X), p^*(E|\neg X)) \\ = & f\left(\sum_{\pm X} p^*(X|C) p^*(E|C, X), \sum_{\pm X} p^*(X|\neg C) p^*(E|\neg C, X)\right) \end{aligned}$$

With the variable settings

$$\begin{aligned} \alpha &= p^*(X|C) & \beta &= p^*(E|X) \\ \bar{\alpha} &= p^*(X|\neg C) & \bar{\beta} &= p^*(E|\neg X) \end{aligned}$$

equation (1) follows immediately.

Second, we are going to show that for any extension of  $f$  to  $\mathbb{R}^2$ ,

$$f(x, y) = -f(y - x, 0) \tag{2}$$

To this end, we first note a couple of facts about  $f$ .

**Fact 1**  $f(\alpha, 0)f(\beta, 0) = f(\alpha\beta, 0)$ . Follows immediately from Equation (1) with  $\bar{\alpha} = \bar{\beta} = 0$ .

**Fact 2**  $f(0, 1) \cdot f(\beta, \bar{\beta}) = f(\bar{\beta}, \beta)$ . Follows immediately from Equation (1) with  $\alpha = 0, \bar{\alpha} = 1$ .

**Fact 3**  $f(1, 0) = 1$ . With  $\beta = 1$ , Fact 1 entails that  $f(\alpha, 0)f(1, 0) = f(\alpha, 0)$ . Hence, either  $f(1, 0) = 1$  or  $f(\alpha, 0) \equiv 0$  for all values of  $\alpha$ . However, the latter would also imply  $f \equiv 0$  and trivialize  $f$ .

**Fact 4**  $f(0, 1) = -1$ . Equation (1) (with  $\alpha = \beta = 0, \bar{\alpha} = \bar{\beta} = 1$ ) and Fact 3 entail that  $f(0, 1) \cdot f(0, 1) = f(1, 0) = 1$ . Hence, either  $f(0, 1) = -1$  or  $f(0, 1) = 1$ . If the latter were the case, then the monotonicity requirement in Generalized Difference-Making would be violated. Thus,  $f(0, 1) = -1$ .

These facts will allow us to derive Equation (2). Note that (2) is trivial if  $y = 0$ . So we can restrict ourselves to the case that  $y > 0$ . We choose the variable

settings

$$\begin{aligned}\alpha &= \frac{y-x}{y} & \beta &= 0 \\ \bar{\alpha} &= 0 & \bar{\beta} &= y\end{aligned}$$

Then we obtain by means of Equation (1) and the previously proven facts

$$\begin{aligned}f(x, y) &= f((y-x)/y, 0) \cdot f(0, y) \\ &= f(y-x, 0) \cdot f(1/y, 0) \cdot f(0, y) && \text{(Fact 1)} \\ &= f(y-x, 0) \cdot f(1/y, 0) \cdot f(y, 0) \cdot f(0, 1) && \text{(Fact 2)} \\ &= f(y-x, 0) \cdot f(1, 0) \cdot f(0, 1) && \text{(Fact 1)} \\ &= -f(y-x, 0) && \text{(Fact 3+4)}\end{aligned}$$

This implies

$$\eta(C, E) = f(p^*(E|C), p^*(E|\neg C)) = -f((-1) \cdot (p^*(E|C) - p^*(E|\neg C)), 0)$$

Hence,  $\eta(C, E)$  can be represented as a function of  $p^*(E|C) - p^*(E|\neg C)$  only. From Generalized Difference-Making we infer that  $f$  must be non-decreasing in  $p^*(E|C) - p^*(E|\neg C)$ . This concludes the proof of Theorem 2. q.e.d.

**Proof of Theorem 3:** The proof relies on a move from the proof of Theorem 1 in Schupbach and Sprenger (2011). Consider three variables  $C$ ,  $E_1$  and  $E_2$  with  $E_2 \perp\!\!\!\perp C$  and  $(E_2 \perp\!\!\!\perp E_1)|C$ . Let  $C \in \mathcal{C}$ ,  $E_1 \in \mathcal{E}_1$ , and  $E_2 \in \mathcal{E}_2$  be propositions about the values of these variables. Then, No Dilution for Irrelevant Effects implies that

$$\begin{aligned}p^*(E_1 \wedge E_2|C) &= p^*(E_1|C) p^*(E_2|C) \\ p^*(E_1 \wedge E_2|\neg C) &= p^*(E_1|\neg C) p^*(E_2|\neg C) \\ p^*(E_2) &= p^*(E_2|\neg C) = p^*(E_2|C)\end{aligned}$$

In particular, it follows that

$$\begin{aligned} p^*(E_1 \wedge E_2|C) &= p^*(E_2) p^*(E_1|C) \\ p^*(E_1 \wedge E_2|\neg C) &= p^*(E_2) p^*(E_1|\neg C) \end{aligned}$$

According to Generalized Difference-Making with  $C' = \neg C$ , the causal strength measure  $\eta$  can be written as  $\eta(C, E_1) = f(p^*(E_1|C), p^*(E_1|\neg C))$  for a continuous function  $f$ . From No Dilution and the above calculations we can infer that

$$\begin{aligned} f(p^*(E_1|C), p^*(E_1|\neg C)) &= \eta(C, E_1) \\ &= \eta(C, E_1 \wedge E_2) \\ &= f(p^*(E_1 \wedge E_2|C), p^*(E_1 \wedge E_2|\neg C)) \\ &= f(p^*(E_2) p^*(E_1|C), p^*(E_2) p^*(E_1|\neg C)) \end{aligned}$$

Since we have made no assumptions on the values of these probabilities, we can infer the general relationship

$$f(x, y) = f(cx, cy). \quad (3)$$

for all  $0 < c \leq \min(1/x, 1/y)$ . Without loss of generality, let  $x > y$ . Then, choose  $c := 1/x$ . In this case, equation (3) becomes

$$f(x, y) = f(cx, cy) = f(1, y/x).$$

This implies that  $f$  must be a function of  $y/x$  only, that is, of the ratio  $p^*(E|\neg C)/p^*(E|C)$ . Generalized Difference-Making then implies that all such functions must be non-increasing, concluding the proof of Theorem 3. q.e.d.

**Proof of Theorem 4:** We write the causal strength measure  $\eta_{cg}$  as

$$\eta_{cg}(C, E) = \begin{cases} \eta^+(C, E) & \text{for positive causation} \\ \eta^-(C, E) & \text{for causal preemption} \end{cases}$$

We know from the previous theorem that  $\eta^-(C, E)$  must be ordinally equivalent to  $\eta_r(C, E)$ . Now we show that all  $\eta^+(C, E)$ -measures are ordinally equivalent to  $\eta_g(C, E) = p^*(\neg E|\neg C)/p^*(\neg E|C)$ . Since we have already shown that  $\eta_g$  and  $\eta_c$  are ordinally equivalent, this is sufficient for proving the theorem.

Because of Generalized Difference-Making, we can represent  $\eta^+$  by a function  $f(x, y)$  with  $x = p^*(E|C)$  and  $y = p^*(E|\neg C)$ . Suppose that there are  $x > y$  and  $x' > y' \in [0, 1]$  such that  $(1 - y)/(1 - x) = (1 - y')/(1 - x')$ , but  $f(x, y) \neq f(x', y')$ . (Otherwise  $\eta^+$  would just be a function of  $\eta_g$ , and we would be done.) In that case we can find a probability space such that  $p^*(E_1|C) = x$ ,  $p^*(E_1|\neg C) = y$ ,  $p^*(E_2|C) = x'$ ,  $p^*(E_2|\neg C) = y'$  and  $C$  screens off  $E_1$  and  $E_2$  (proof omitted, but straightforward). Hence  $\eta^+(C, E_1) \neq \eta^+(C, E_2)$ . By Weak Causation-Prevention Symmetry, we can then infer  $\eta^-(C, \neg E_1) \neq \eta^-(C, \neg E_2)$ .

However, since  $\eta^-$  is ordinally equivalent to  $\eta_r$ , there is a function  $f$  such that

$$\begin{aligned}\eta^-(C, \neg E_1) &= f\left(\frac{p^*(\neg E_1|C)}{p^*(\neg E_1|\neg C)}\right) = f\left(\frac{1-x}{1-y}\right) \\ \eta^-(C, \neg E_2) &= f\left(\frac{p^*(\neg E_2|C)}{p^*(\neg E_2|\neg C)}\right) = f\left(\frac{1-x'}{1-y'}\right)\end{aligned}$$

By assumption,

$$\frac{1-x}{1-y} = \left(\frac{1-y}{1-x}\right)^{-1} = \left(\frac{1-y'}{1-x'}\right)^{-1} = \frac{1-x'}{1-y'}$$

and so we can infer  $\eta^-(C, \neg E_1) = \eta^-(C, \neg E_2)$ , leading to a contradiction. Hence  $\eta^+(C, E)$  can be represented by a non-decreasing function of  $p^*(\neg E|\neg C)/p^*(\neg E|C)$ , completing the proof of Theorem 4. q.e.d.

**Proof of Theorem 5:** By Generalized Difference-Making, we have that  $\eta(C, E) = f(p^*(E|C), p^*(E|\neg C))$  for some continuous function  $f : [0, 1]^2 \rightarrow \mathbb{R}$ . Assume that  $\eta(C, E_1) = \eta(C, E_2) = t$ , that  $C$  screens off  $E_1$  and  $E_2$  and that  $p^*(E_1|C) = p^*(E_2|C) = x$ ,  $p^*(E_1|\neg C) = p^*(E_2|\neg C) = y$ , for some  $x, y \in \mathbb{R}$ . By

the Conjunctive Closure Principle, we can infer

$$\eta(\mathbf{C}, \mathbf{E}_1 \wedge \mathbf{E}_2) = \eta(\mathbf{C}, \mathbf{E}_1) = f(x, y)$$

Moreover, we can infer

$$\begin{aligned} \eta(\mathbf{C}, \mathbf{E}_1 \wedge \mathbf{E}_2) &= f(p^*(\mathbf{E}_1 \wedge \mathbf{E}_2 | \mathbf{C}), p^*(\mathbf{E}_1 \wedge \mathbf{E}_2 | \neg \mathbf{C})) \\ &= f(p^*(\mathbf{E}_1 | \mathbf{C}) \cdot p^*(\mathbf{E}_2 | \mathbf{C}), p^*(\mathbf{E}_1 | \neg \mathbf{C}) \cdot p^*(\mathbf{E}_2 | \neg \mathbf{C})) \\ &= f(x^2, y^2) \end{aligned}$$

Taking both calculations together, we obtain

$$f(x^2, y^2) = f(x, y) \tag{4}$$

as a structural requirement on  $f$ , since we have not made any assumptions on  $x$  and  $y$ .

Following Atkinson (2012), we now define  $u = \frac{\log x}{\log y}$  and define a function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $g(x, u) := f(x, y)$ . Equation (4) then implies the requirement

$$g(x^2, u) = f(x^2, y^2) = f(x, y) = g(x, u)$$

and by iterating the same procedure, we obtain

$$g(x^{2^n}, u) = g(x, u)$$

for some  $n \in \mathbb{N}$ . Due to the continuity of  $f$  and  $g$ , we can infer that  $g$  cannot depend on its first argument. Moreover, taking the limit  $n \rightarrow \infty$  yields  $g(x, u) = g(0, u)$ . Hence, also

$$f(x, y) = g(0, u) = g(0, \log x / \log y)$$

and we see that

$$\eta(\mathbf{C}, \mathbf{E}) = h \left( \frac{\log p^*(\mathbf{E} | \mathbf{C})}{\log p^*(\mathbf{E} | \neg \mathbf{C})} \right)$$

for some continuous function  $h : \mathbb{R} \rightarrow \mathbb{R}$ . It remains to show that  $h$  is

non-decreasing. Generalized Difference-Making implies that  $\eta(C, E)$  is a non-decreasing function of  $p^*(E|C)$  and a non-increasing function of  $p^*(E|\neg C)$ . So it must be a non-decreasing function of  $\log p^*(E|C)/\log p^*(E|\neg C)$ , too. This implies that  $h$  is a non-decreasing function. Hence, all measures of causal strength that satisfy Generalized Difference-Making and the Conjunctive Closure Principle are ordinally equivalent to

$$\eta_{cc}(C, E) = \frac{\log p^*(E|C)}{\log p^*(E|\neg C)}. \quad \text{q.e.d.}$$

**Proof of Theorem 6:** We know by assumption that any measure that satisfies Generalized Difference-Making with  $C' = \Omega_C$  is of the form

$$\eta(C, E) = f(p^*(E|C), p^*(E)).$$

Suppose now that there are  $x, y, y' \in [0, 1]$  such that  $f(x, y) \neq f(x, y')$ . In that case, we can choose propositions  $C, E_1,$  and  $E_2$  and choose a probability distribution  $p^*$  such that  $x = p^*(E_1|C), y = p^*(E_1)$  and  $y' = p^*(E_2)$  and  $C \wedge E_1 \models E_2,$  and  $C \wedge E_1 \not\models E_2$ . Then,  $p^*(E_{1,2}|C) = p^*(E_1 \wedge E_2|C)$  and

$$\begin{aligned} \eta(C, E_1) &= f(p^*(E_1|C), p^*(E_1)) = f(p^*(E_1 \wedge E_2|C), p^*(E_1)) \\ &= f(p^*(E_2|C), p^*(E_1)) \end{aligned}$$

and by Conditional Equivalence, also

$$\eta(C, E_1) = \eta(C, E_2) = f(p^*(E_2|C), p^*(E_2))$$

Taking both equations together leads to a contradiction with our assumption  $f(p^*(E_2|C), p^*(E_1)) \neq f(p^*(E_2|C), p^*(E_2))$ . So  $f$  cannot depend on its second argument. Hence, all causal strength measures that satisfy Generalized Difference-Making with  $C' = \Omega_C$  and Conditional Equivalence must be ordinally equivalent to  $\eta_{ph}(C, E) = p^*(E|C)$ . q.e.d.



## References

- Atkinson, D. (2012). Confirmation and Justification: A Commentary on Shogenji's Measure. *Synthese*, 184:49–61.
- Schupbach, J. N. and Sprenger, J. (2011). The Logic of Explanatory Power. *Philosophy of Science*, 78:105–127.