

# Hawking's area theorem with a weaker energy condition

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November 16, 2017

## Abstract

Hawking's area theorem is a fundamental result in black hole theory that is universally associated with the null energy condition. That this condition can be weakened is illustrated by the formulation of a strengthened version of the theorem based on an energy condition that allows for violations of the null energy condition. This result tightens the conventional wisdom that quantum field theoretic violations of the null energy condition account for why the conclusion of the area theorem can be bypassed in the semi-classical context. Shown here is that violations of the null energy condition, though necessary, are not sufficient to violate the conclusion of the area theorem. As an added benefit, the specific form of the energy condition used here suggests that the area non-decrease behavior described by the area theorem is a quasi-local effect that depends, in large measure, on the energetic character of the relevant fields in the vicinity of the event horizon.

## 1 Introduction

Many classic theorems of relativity are obtained by positing a number of local conditions on the geometry of spacetime. These geometric conditions are inequalities imposed, by fiat, on certain contractions of the Einstein or Ricci tensor. With the use of Einstein's equations, these geometric conditions become energy conditions that, supposedly, represent certain energetic characteristics of matter residing in spacetime. It is now understood, however, that these local energy conditions are violated by a number of classical matter models and, moreover, that violations are ubiquitous in the context of quantum field theory in both flat and curved spacetime.<sup>1</sup> To put it another way, the classic theorems aforementioned rely on assumptions not satisfied in contexts considered physically relevant. In view of this, we might wish to ask, for certain theorems of interest, whether they can be formulated with weaker energy conditions. The purpose of this paper is two-fold. To show that Hawking's area theorem can be strengthened as such, and, in view of the theorem's relevance to Hawking radiation, to interpret the result presented.<sup>2</sup>

It is instructive, before delving into the area theorem, to consider the well known singularity theorems.<sup>3</sup> There is, it seems, a common template to these theorems. Their assumptions usually include an energy condition, a restriction on the causal properties of the spacetime, and an initial or boundary condition, and their conclusions almost always involve no more than the failure of non-spacelike geodesic completeness. Useful in many proofs of such theorems is Raychaudhuri's equation describing geodesic congruences, which, in the null case, reads

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^ak^b$$

with  $\lambda$  the affine parameter,  $\sigma$  the shear,  $\theta$  the expansion and  $R_{ab}k^ak^b$  the Ricci tensor twice contracted with a null vector  $k^a$  tangent to the geodesic. Assuming the null convergence condition,  $R_{ab}k^ak^b \geq 0$ , or, with Einstein's equations in four dimensions, the null energy condition (NEC),  $T_{ab}k^ak^b \geq 0$ , it follows that if the expansion  $\theta$  satisfies  $\theta(\lambda_0) < 0$ , then  $\theta \rightarrow -\infty$  within finite affine parameter  $\lambda \in (\lambda_0, \infty)$ . This behavior is sometimes referred to as geodesic focusing. The onset of geodesic focusing signals the failure of certain geodesics to satisfy certain properties which, in other circumstances, are associated with them. In the null case, for instance, a null geodesic focusing to the future of a point signals the geodesic's failure to remain on the boundary of the causal future of that point.<sup>4</sup> Many proofs of singularity theorems work by setting up a contradiction under the assumption that all null or timelike geodesics are complete. In

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<sup>1</sup>See [19] for a foundational perspective on the status of energy conditions.

<sup>2</sup>The definitions used in this article are as in Wald [14].

<sup>3</sup>See [12], [14] for an introduction.

<sup>4</sup>In the timelike case, focusing signals the geodesic's failure to maximize proper time.

particularly straightforward cases, arguments run like this. Assume that all null (timelike) geodesics are complete and deduce, under the energy and boundary or initial conditions, the onset of geodesic focusing. Then, combining causal restriction and initial or boundary condition, show that the focusing produced leads to a contradiction. Deduce, therefore, that not all null (timelike) geodesics can be complete.

Tipler [2] was among the first to show that singularity theorems may be strengthened by way of weaker energy conditions. He defined these weaker energy conditions as non-local restrictions on the integral, along certain types of null or timelike geodesics, of various contractions of the Ricci or Einstein tensor. He showed that these conditions were sufficient to cause focusing. This made it possible to strengthen certain singularity theorems without major amendments to the original arguments. His observation was developed, refined and generalized by a number of authors [3, 4, 5, 6] and, in time, there arose a number of weaker energy conditions falling under the umbrella term of average energy conditions. One example that continues to generate interest is the average null energy condition (ANEC), which, roughly speaking, is the requirement that

$$\int_{\gamma} R_{ab}k^ak^bd\lambda \geq 0$$

for some suitable class of null geodesics  $\{\gamma\}$  with  $k^a$  a null vector tangent to the geodesic.<sup>5</sup>

Though mathematically weaker, the physical interpretation of these average energy conditions remains, to this day, murky at best. Over the course of a long list of studies, it has been found that many of the average energy conditions allowing for theorem-strengthening are violated by classical matter models, and, less straightforwardly, that they are also violated in the context of quantum field theory in curved spacetime (QFTCS).<sup>6</sup> Put another way, many classic theorems of relativity do not, at present, cover a whole host of classical and quantum matter models otherwise considered physically relevant. And, as such, efforts to better this situation are ongoing.

In the QFTCS context, there is a growing suspicion that the extent of the violation of certain energy conditions is, in some sense, restricted. There is a whole body of work dedicated to making this more precise. The idea is to produce certain kinds of inequalities that represent the spatiotemporal constraints that (contractions of) renormalized stress energy tensors in various contexts of QFTCS obey.<sup>7</sup> These inequalities are known as quantum energy inequalities (QEI), and, in the best of cases, they have been used to constrain the properties of certain spacetime scenarios otherwise associated with violations of certain more standard energy conditions. An early and famous example of this kind is Ford and Roman's study of the properties of traversable wormholes [10].

Despite providing valuable insights, QEI have not yet permitted the extension of certain classic relativity theorems to the semi-classical context. Galloway and Fewster [1] recently proposed a study aiming to provide a step in this direction. They formulated versions of Hawking's cosmological and Penrose's collapse singularity theorems based on energy conditions allowing for, respectively, violations of the strong and null energy conditions. Though inspired by QEI methods, the conditions that stand in as energy conditions in their singularity theorems are not, strictly speaking, QEI. It is worth noting, on this point, that the relationship between QEI and energy conditions is not particularly well understood, with perhaps the only exception being Fewster and Roman's result that the existence of QEI is not necessary for the ANEC to be satisfied [11].

Galloway and Fewster's arguments rely on lemmas that establish sufficient conditions for focusing. One of these reads as follows.

**Lemma 1.1.** *Consider the initial value problem for  $z(t)$*

$$\dot{z} = \frac{z^2}{s} + r$$

where  $r(t)$  is continuous on  $[0, \infty)$ , and  $s > 0$  is constant. If there exists  $c \geq 0$  such that

$$z_0 - \frac{c}{2} + \liminf_{T \rightarrow \infty} \int_0^T e^{-2ct/s} r(t) dt > 0$$

then the initial value problem has no solution on  $[0, \infty)$ , where 'no solution' means  $z(t) \rightarrow \infty$  as  $t \rightarrow t_*^- < \infty$ .

Using such lemmas, they are able to strengthen the singularity theorems of, respectively, Penrose and Hawking. Moreover, they do so without having to make any essential amendments to the original spirit of the proof of these theorems. Their version of the former is as follows.

<sup>5</sup>This is not a precise definition; more care is taken when we come to formulate the main result.

<sup>6</sup>See [19, 15, 9] for an expression of this fact and links to some of the relevant references.

<sup>7</sup>See [9] for an introduction and links to various relevant references.

**Theorem 1.2.** *Let  $M$  be a spacetime of dimension  $n \geq 3$  with a noncompact Cauchy surface  $\Sigma$ . Let  $S$  be a smooth compact acausal spacelike submanifold of  $M$  of codimension two, with null expansion scalars  $\theta_{+(-)}$  associated to the future directed null normal vector fields  $l_{+(-)}$ . Suppose along each future complete affinely parametrised null geodesic  $\gamma : [0, \infty) \rightarrow M$ , issuing orthogonally from  $S$  with initial tangent  $l_{+(-)}$ , there exists  $c \geq 0$  such that,*

$$\liminf_{T \rightarrow \infty} \int_0^T e^{-2ct/(n-2)} r(t) dt > \theta_{+(-)}(p) + \frac{c}{2}$$

where  $p = \gamma(0)$  and  $r(t) := R_{ab} \dot{\gamma}^a \dot{\gamma}^b(t)$ . Then  $M$  is future null geodesically incomplete.

They then illustrate how a matter model can violate the NEC whilst satisfying the assumptions of theorem 1.2.

## 2 Strengthening the area theorem

Hawking's area theorem [12] is often considered to be one of the most important results in black hole theory. It describes a fundamental property of dynamical black holes, it underlies black hole thermodynamics and it bounds the amount of radiation that can be emitted upon black hole collisions. Consider Ashketar and Krishnan's remarks in their recent very well cited review on generalized black holes [13].

For fully dynamical black holes, apart from the 'topological censorship' results which restrict the horizon topology [...], there has essentially been only one major result in exact general relativity. This is the celebrated area theorem proved by Hawking in the early seventies [...]: If matter satisfies the null energy condition, the area of the black hole event horizon can never decrease. This theorem has been extremely influential because of its similarity with the second law of thermodynamics.

The precise statement of the area theorem depends on a number of definitions for which there are a number of possible choices. The definitions and the proof used in theorem below are identical to those used by Wald [14]. This choice is made for reasons of expediency. Note, however, that the following arguments apply to other available formulations of the area theorem.<sup>8</sup> Consider now the main result of this article.

**Theorem 2.1.** *Let  $(M, g_{ab})$  be a strongly asymptotically predictable spacetime such that along each future complete affinely parametrized null geodesic  $\gamma : [0, \infty) \rightarrow M$ , there exists a non-negative constant  $c \geq 0$  such that*

$$\liminf_{T \rightarrow \infty} \int_0^T e^{-ct} R_{ab} \dot{\gamma}^a \dot{\gamma}^b dt - \frac{c}{2} > 0$$

where  $p = \gamma(0)$  and  $\dot{\gamma}^a$  is tangent to the null geodesic  $\gamma$ . Let  $\Sigma_1$  and  $\Sigma_2$  be spacelike Cauchy surfaces for the globally hyperbolic region  $\tilde{V}$  with  $\Sigma_2 \subset I^+(\Sigma_1)$  and let  $\mathcal{H}_{1(2)} = H \cap \Sigma_{1(2)}$ , where  $H$  denotes the event horizon, i.e., the boundary of the black hole region of  $(M, g_{ab})$ . Then the area of  $\mathcal{H}_2$  is greater or equal to the area of  $\mathcal{H}_1$

*Proof.* The argument is a straightforward application of Galloway and Fewster's lemma 1.1 with the original arguments by Hawking, which in this formulation, are as in [14]. Namely, first establish that the expansion  $\theta$  of the null generators of  $H$  is everywhere non-negative. This proceeds in the same way as in [14] but the relevant contradiction is now provided by applying lemma 1.1 to the null geodesics generator of  $\partial J^+(K)$ . Namely, if  $\theta < 0$  and the energy condition of theorem 2.1 is satisfied, then, by lemma 1.1, there develops a focal point within finite affine parameter on the null geodesic generator of the achronal boundary  $\partial J^+(K)$ . The rest of the argument proceeds as in [14].  $\square$

*Remark 1.* The energy condition used in this theorem stems from the work of Galloway and Fewster. This choice is made because their energy condition is both state of the art and neatly amenable to classical field models violating the NEC. An appendix in their paper [1] contains a construction of such a model. Inessential modifications to their construction make it straightforwardly applicable to theorem 2.1. Other conditions similar in form to the ANEC for semi-complete geodesics could have been used - eg. Roman's condition [6] in his version of Penrose's theorem.

*Remark 2.* The exponential damping in the condition shows that boundary effects are crucial in determining whether area non-decrease obtains. Provided there occurs positive contributions to the null contractions of the Ricci tensor near the horizon, the condition may be satisfied even if negative contributions persist for arbitrarily long segments of the null geodesic away from the horizon. That is, the area may be non-decreasing even if there are large violations to the standard ANEC for semi-complete geodesics. The result can also be used in the negatively. Namely, if QFTCS effects are responsible for

<sup>8</sup>Recent studies [20, 21] have shown that there are, in both Hawking and Wald's formulation of the area theorem, lacunas in the form of unstated or undesirable assumptions of differentiability of the horizon. These deficiencies have been overcome in the formulation of more precise area theorems [20, 21]. Though important, these developments are tangential to the current argument which is, after all, about energy conditions.

black hole area decrease, then it is within a neighborhood of the horizon that the energetic character of the relevant fields deviate from the expected classical behavior. This also suggests that the singularity theorems of Galloway and Fewster, though sophisticated from an energy conditions point of view, are classical results.<sup>9</sup>

*Remark 3.* A number of other standard theorems about black holes can be strengthened in the way described. The list includes results describing the location of trapped surfaces, marginally trapped surfaces, apparent horizons and so on. The area theorem is particularly prominent because of the obvious tension it generates with the idea of Hawking radiation.

There is now a certain conventional wisdom regarding the tension between Hawking radiation and the area theorem. Consider, for instance, the remarks made by Visser [15] and, respectively, Bousso and Engelhart [16].

The very fact that Hawking evaporation occurs at all violates the area increase theorem...for classical black holes. This implies that the quantum process underlying the Hawking evaporation process must also induce a violation of one or more of the input assumptions used in proving the classical area increase theorem. The only input assumption that seems vulnerable to quantum violation is the assumed applicability of the null energy condition.

Hawking's theorem holds in spacetimes obeying the null curvature condition,  $R_{ab}k^ak^b \geq 0$  for any null vector  $k^a$ . This will be the case if the Einstein equations are obeyed with a stress tensor satisfying the NEC,  $T_{ab}k^ak^b \geq 0$ . The NEC is satisfied by ordinary classical matter, but it is violated by valid quantum states (e.g., in the Standard Model). In particular, the NEC fails in a neighborhood of a black hole horizon when Hawking radiation is emitted. Indeed, the area of the event horizon of an evaporating black hole decreases, violating the Hawking area law.

The real upshot of theorem 2.1 is that it offers a precisification of such remarks. It shows that violating the NEC is necessary but not sufficient to negate the conclusion of the area theorem. Or, to put it another way, that if black hole areas decrease due to QFTCS, the underlying mechanism must induce a violation of an energy condition that is strictly weaker than the NEC, and, moreover, the question of whether areas increase or decrease concerns the energetic character of the relevant fields in the vicinity of the horizon.

As a final point, this conclusion seems to dovetail nicely with what emerges from the generalized black holes framework.<sup>10</sup> In that framework, area non-decrease results are obtained for trapping, isolated and dynamical horizons upon assuming the NEC. Moreover, these results only need the NEC to hold within a neighborhood of the relevant horizon. That being said, given that the relation between global and quasi-local types of horizons is still not fully understood, the area non-decrease behaviors should not, strictly speaking, be identified. The parallel is nevertheless worth underlining.

## Acknowledgements

I thank the Ruth and Nevil Mott scholarship and the AHRC for funding this research. I thank Prof. Erik Curiel for his guidance and for writing the paper that provided the initial inspiration for this work [19]. I also thank my supervisor, Prof. Harvey Brown, for his support.

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<sup>9</sup>Note that neither author claim otherwise.

<sup>10</sup>See [22] for an introduction to horizons in this framework.

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