Epistemology of quasi-sets

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Abstract

I briefly discuss the epistemological role of quasi-set theory in mathematics and theoretical physics. Quasi-set theory is a first order theory, based on Zermelo-Fraenkel set theory with *Urelemente* (ZFU). Nevertheless, quasi-set theory allows us to cope with certain collections of objects where the usual notion of identity is not applicable, in the sense that $x = x$ is not a formula, if x is an arbitrary term. Basically, quasi-set theory offers us some sort of logical apparatus for questioning the need for identity in some branches of mathematics and theoretical physics. I also use this opportunity to discuss a misunderstanding about quasi-sets due mainly to Nicholas J. J. Smith, who argues, in a general way, that sense cannot be made of vague identity.

Key words: quasi-sets, identity, indistinguishability, epistemology

1 Introduction

Francisco "Chico" Doria was my doctoral co-advisor. More than that, he is one of the greatest influences on my professional and even personal life. The most important lesson I learned from him is this: be honest with yourself! If you want to publish papers, you go ahead and publish papers. But if you want to find unconventional results in science, you should not submit yourself to what other people expect from you. He never said that to me in so many words. But his actions through theoretical physics and mathematics said that to me in a very clear way. I found this hidden message in his successful works on undecidability and incompleteness in physical theories and in his "exotic" results on the P versus NP problem in computer science as well, both in collaboration with Newton da Costa, my doctoral advisor. So, that is what I try to do since then. All I want from this life is to be honest with myself. If all my efforts fail, that will be my sole responsibility. But this journey I started almost three decades ago, thanks to da Costa and Doria, is truly worthwhile in itself. So, let's talk about quasi-sets and their curious epistemological role.

Quasi-set theory $\mathcal{Q}$ [1] [10] [11] is a first order theory without identity which allows the existence, among its terms (the objects of $\mathcal{Q}$), of collections (sets and q-sets) and atoms (*Urelemente*). Some of those collections correspond to ZFU sets, in the sense that a binary predicate letter of *extensional equality* $=_E$ is explicitly defined in $\mathcal{Q}$ and a given translation from ZFU to $\mathcal{Q}$ guarantees that every translated axiom of ZFU is a theorem in $\mathcal{Q}$ (where ZFU equality is translated as the extensional equality in $\mathcal{Q}$). In other words, extensional equality in quasi-set theory has all the usual properties of standard identity in ZFU. Nevertheless, the axioms of $\mathcal{Q}$ do not allow that $x =_E x$ is necessarily a formula, for any term x. Concerning atoms in $\mathcal{Q}$, there are two kinds of *Urelemente*, termed m-atoms and M-atoms, which are identified by two unary predicates $m(x)$ and $M(x)$, respectively. M-atoms correspond (in a precise sense) to standard atoms of ZFU theory, while m-atoms are something else (or less, if the reader allows our poetic view). A weaker binary relation of "indistinguishability" (denoted by $\equiv$), is used instead of identity, and it is postulated that $\equiv$ has the standard properties of an equivalence relation. The defined binary predicate letter of extensional equality $=_E$ cannot be applied to m-atoms, since no expression of the form $x =_E y$ is a formula if either x or y denote m-atoms. Hence, there is a precise sense in saying that m-atoms can be indistinguishable without being identical. In standard mathematics, when we say that $x = y$ (x is identical to y) we are talking about the very same object, with two different labels: x and y. In quasi-set theory $\mathcal{Q}$, the formula $x \equiv y$ does not entail we are necessarily talking about the very same object. Axioms of quasi-set theory are a very natural extension of the axioms of Zermelo-Fraenkel set theory with Urelemente (ZFU). So, no one here is abandoning standard mathematics. Actually, quasi-set theory provides a specific methodology for a better understanding of identity and its role in mathematics. Geometry was better understood when its postulates were questioned in the last two centuries. Something similar happens here, concerning identity.

The development of quasi-set theory was motivated mostly by certain striking phenomena in quantum mechanics, all related to the well known *problem of non-individuality* among elementary particles [1]. According to any standard statistical mechanics textbook [3], Maxwell-Boltzmann (MB) statistics gives us the most probable distribution of N *distinguishable* objects into, say, boxes with a specified number of objects in each box. By "box", in this context, we mean "state". Thus, MB statistics can be easily described through a classical mathematical picture for ensembles of individual particles. Nonetheless, in quantum mechanics the very notion of state of a particle differs considerably from that one in classical mechanics. Besides, quantum statistics are supposed to give us the most probable distribution of N *indistinguishable* objects into distinguishable boxes with a specified number of objects in each box. The usual and physically meaningful quantum statistics are Bose-Einstein and Fermi-Dirac.

Hence, it has been argued that classical particles are "individuals" of some sort. Even when classical particles share the very same set of intrinsic properties, their individuality must be ascribed by something which "transcends" such intrinsic properties [1] [19] [22]. In quantum statistics, on the other hand, the Indistinguishability Postulate (IP) asserts that "If a permutation is applied to any state for an assembly of particles, then there is no way of distinguishing the resulting permuted state function from the original one by means of any observation at any time" [2]. IP is one of the most basic principles of quantum mechanics and it implies that permutations of quantum particles are not usually regarded as observable.

The non-individuality problem in quantum mechanics is not limited to quantum statistics. The helium atom, e.g., is probably the simplest realistic situation where the problem of individuality plays an important role. With the non-individuality question put aside, the wave function of the helium atom would be just the product of two hydrogen atom wave functions with Z = 1 changed to Z = 2. Nevertheless, the space part of the wave function for the case where one of the electrons is in the ground state (100) and the other one is in excited state (nlm) is:

$$\phi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{100}(x_1)\psi_{nlm}(x_2) \pm \psi_{100}(x_2)\psi_{nlm}(x_1)]$$

where the $+$ and $-$ signs are designated for the spin singlet and spin triplet, respectively, while x_1 and x_2 denote the vector positions of both electrons.

For the ground state, however, the space function must necessarily be symmetric. In that case, the problems regarding non-individuality have no physical effect. The most interesting case, however, is the excited state of helium. The equation stated above reflects our ignorance about which electron is in position x_1 and which one is in position x_2 . Nevertheless, in the same equation there are terms like $\psi_{100}(x_1)$, which corresponds to a specific physical property of an individual electron.

A unified quasi-set-theoretic approach to quantum distributions and the helium atom is introduced in [12]. And a comprehensive quasi-set-theoretic approach to quantum statistics is presented in [25]. Notwithstanding, a quasi-set theoretic approach to the quantum interference produced by two light beams has not yet been developed, despite the fact that a profound mathematical link between indistinguishability and coherence was presented almost three decades ago by Leonard Mandel [16].

From an epistemological and even methodological point of view, quasi-set theory presents some peculiar features, if we compare it to other mathematical tools usually developed and employed in theoretical physics. And that is a good reason to raise some questions about the theoretical and philosophical role of quasi-sets.

Mostly, mathematical approaches to cope with physical problems are developed with one simple purpose in mind: to solve problems! In order to do that, mathematical tools are created and developed. But what is the usual procedure to develop a mathematical tool which is useful in theoretical physics? The standard answer to this question emerges from *stronger* concepts, *stronger* models, and *stronger* theories.

Usually, the cornerstone of any physical theory is either a differential equation or a class of differential equations. Newtonian mechanics is based on Newton's Second Law, classical electromagnetism is based on Maxwell's equations, quantum mechanics is based on Schrödinger's equation, and so on. And differential equations simply establish the *limits* to what extent physical phenomena are supposed to be, at least from a theoretical point of view.

Quasi-set theory does not seem to work that way! Quasi-set theory is not supposed to be a problem solver, in the usual sense. And that is why I wrote this paper (in honor to my friend and former Ph.D. co-advisor Francisco Doria). Quasi-set theory allows us to discuss about a mathematical universe which is, in a sense, weaker than ZFU (since $x \equiv y$ does not entail that x and y denote the very same object). And such a feature may cause some surprise to many people. A good example to illustrate this last claim is Nicholas J. J. Smith's misunderstandings concerning quasi-sets and their role in the scientific enterprise.

So, let's elaborate the main points of this paper.

2 Epistemological character of $\mathcal{Q}$

Most of mathematics used in theoretical physics is based on set theories. Despite the fact that physicists in general are not usually concerned with set theories *per se*, there seems to be no major problem if I claim that any common mathematical scenario for theoretical physics is based on Zermelo-Fraenkel-like axioms, including those found in ZFC and ZFU. An excellent defense for such a claim may be found in Patrick Suppes' magistral book *Representation and Invariance of Scientific Structures* [29]. Other mathematical approaches are possible and even necessary in theoretical physics, like category theory [4] [15]. But ZFU set theory is powerful enough for dealing with plenty of important and well-known applications.

But, what is set theory after all? Since there is a huge myriad of set theories in the literature (ZF, von Neumann's, NBG, NF, among many others), someone could argue that the common element among them is the formal study of collections of objects, in some intuitive way which partially rescues the main original ideas due to Georg Cantor (regarding *Mengenlehre*). But that is hardly an accurate description. The most popular set theories in literature (ZF, ZFU, and NBG) are simply the formal study of two predicative letters, namely, membership ($\in$) and identity ($=$). Of course, any set theory seems to be associated to an intended interpretation of collection, as it was originally proposed by Cantor. But, from a purely formal point of view, the standard axioms of popular set theories simply state how those standard binary predicative letters ($\in$ and $=$) behave, in a very general way.

Nevertheless, physicists need much more than a simple knowledge concerning two binary predicative letters (I am not suggesting that either ZF or other formal set theories are simple theories!). Physicists need mostly functions rather than sets, and whatever they can do with such functions [27]. A function, in set-theoretic terms, is usually a particular case of a set (a very specific set of ordered pairs). And that works as some sort of mathematical constraint: physicists need specialized sets called functions. But, physicists do need specialized functions as well, namely, those functions which are solutions of very specific differential equations submitted to very specific boundary conditions. And so we have one more level of mathematical constraint.

By mathematical constraints we mean, in the present context, mathematical formulas that somehow can be intuitively translated into either physical laws or physical principles. Such a specialized mathematical apparatus provides the necessary impulse to raise and develop physical theories. After all, it is far from enough to say the position of a given particle in a given space can be described by a function. Physicists do need to assert special conditions for a function in order to describe position: like the Second Law of Classical Mechanics, namely, a specific differential equation!

On the other side of this mathematical spectrum, however, there are some few works in the literature which follow a different philosophical approach. It is something that has more to do with a *revisionist perspective* than with plain mathematical applications. For example, it is easy to show that in many natural axiomatic formulations of physical and even mathematical theories, there is a substantial list of superfluous concepts usually assumed as primitive [27], like time, space and spacetime. That happens mainly when those theories are formulated in a set-theoretic language, such as Zermelo-Fraenkel's. On the other hand, in 1925, John von Neumann created a set theory where sets are definable as special cases of functions. And in [27] it is provided a reformulation of von Neumann's set theory where it is demonstrated that such an axiomatic framework can be used to formulate physical and mathematical theories with less primitive concepts in a very natural fashion.

Another historical example is Hertz's mechanics [6]. Despite the fact that Padoa's method guarantees that force is an indispensable concept in certain axiomatic formulations for non-relativistic classical particle mechanics [17], nothing prevents us from introducing a formal framework for non-relativist classical particle mechanics without any notion of force [24]. Even Kepler's Laws for planetary orbits may be derived within this "forceless" framework [26]. Concerning the standard concepts of force and mass, a fascinating and enlightening discussion may be found in [8].

All this means that mathematics may play a twofold epistemological role in theoretical physics: application and revision. Applications are achieved by means of stronger assumptions than those naturally offered by formal set theories axioms (like specialized sets called functions, and specialized functions which are solutions of specific differential equations under specific boundary conditions). Revision is achieved when we use foundational tools in order to answer to the following natural question: Do we really need all standard mathematical apparatus in order to do mathematical physics? More specifically: Do we really need the mathematical counterpart which describes space? Do we need time? Do we need spacetime? Are we doomed to always use identity in all theoretical physics? What do we really mean by usual concepts like space, time, spacetime, mass, force, and identity?

Quasi-set theory is just a useful mathematical tool which allows us to answer rather important questions concerning mathematical methods in physics.

From a formal point of view, identity is usually associated to the predicate letter $=$ in standard set theories. One of the major mathematical applications of identity is the statement of symmetry principles. How can we say, for example, the angular momentum of the Earth-Moon system is invariant? The usual answer is provided by a formula grounded on the standard notion of identity:

$$\mathbf{H} = \mathbf{H}_E + \mathbf{H}_M,$$

where $\mathbf{H}$ is the total angular momentum of the Earth-Moon system, and $\mathbf{H}_E$ and $\mathbf{H}_M$ are the angular momenta of Earth and Moon, respectively.

Nevertheless, do we really need identity in order to state symmetry principles? That is a vague and, so, a tricky question! Therefore, allow me to rephrase my doubt. Can we rewrite the last equation as

$$\mathbf{H} \equiv \mathbf{H}_E + \mathbf{H}_M?$$

Quasi-set theory has been used for a better understanding (in a philosophical fashion) of some phenomena in quantum mechanics. Within this context, the indistinguishability predicate $\equiv$ has been mostly used among *Urelemente* of $\mathcal{Q}$. Nevertheless, the same predicate letter may be used among collections as well, including quasi-functions (the quasi-set theoretic counterpart of functions). One thing is to say the total angular momentum of the Earth-Moon system is identical to the vector sum of the angular momenta of Earth and Moon. Another thing is to say both functions are simply indistinguishable. That sort of philosophical perspective may inspire new and relevant ideas. In the specific case of the last equation, it seems reasonable to consider both functions $\mathbf{H}$ and $\mathbf{H}_E + \mathbf{H}_M$ as standard ZF-sets, even within $\mathcal{Q}$. Thus, as a consequence we have the following:

$$\mathbf{H} =_E \mathbf{H}_E + \mathbf{H}_M,$$

where $=_E$ stands for the extensional equality mentioned in the Introduction. Nevertheless, quasi-set theory allows us to state symmetry principles in a more relaxed way.

By using category theory, Shahn Majid proposed a very general symmetry principle inspired on Einstein's formulation for equivalence between gravitation and acceleration [15]. His idea was to propose a mathematical tool for coping with quantum gravitation. Analogously, if macroscopic objects are, in some sense, composed of microscopic objects, why can't we abandon identity right at start? After all, microscopic particles (usually described by quantum theories) seem to be devoid of any classical notion of identity.

In standard set theories, an arbitrary permutation of the elements of a set x is a classical example of an automorphism. And the automorphism group of x is the symmetric group on x. Notwithstanding, in quasi-set theory the automorphism group simply presents indistinguishable elements. That's all! Nevertheless, such indistinguishable elements offer a new way to examine quantum statistical mechanics. And that is a revisionist perspective that mathematics can effectively offer to theoretical physics.

That revisionist perspective offered by quasi-set theory may be responsible for some misunderstandings among physicists and even philosophers. We illustrate this point in the next Section.

3 Nicholas Smith's criticisms

Nicholas J. J. Smith published ten years ago a paper about the philosophical dispute concerning any clear notion of vague identity [28]. According to him:

> [T]o make clear sense of something, one must at least model it set-theoretically; but due to the special place of identity in set-theoretic models, any vague relation that one does model set-theoretically will not be identity, for real identity will already be there, built into the background of the model, and perfectly precise.

More specifically, Smith argues that Krause and collaborators [13] made no progress towards the problem of clarifying any notion of vague identity. According to Smith, Krause and colleagues

> "simply present quasi-set theory as an *axiomatic theory* − i.e. as a list of definitions and axioms involving a primitive relation of indistinguishability, symbolised as $\equiv$. This approach leads directly to the dilemma that Priest and van Inwagen faced. If we

> understand this axiomatic theory in the usual way − i.e. as picking out the class of (standard set-theoretic) models in which all the axioms are true then we can only understand ≡ as a new relation in addition to the precise identity relation which inheres in all standard models. We have then done nothing to remove precise identity in favour of the vague relation of indistinguishability − i.e. we have gone no way towards achieving Krause et als stated goal of developing a framework which does not treat objects as individuals in the standard sense."

The works cited by Smith are [21] and [7] (Priest and Inwagen, respectively).

Well, let's analyse Smith's arguments. After all, it seems to me that no one did this until now, at least from the perspective I wish to explore here.

First of all, it is easy to find a manifold prejudice to the interests of philosophy, when someone reads a statement like this: "to make clear sense of something, one must at least model it set-theoretically". Is this statement supposed to be a critical one? If that is the case, what is set theory supposed to be? Smith argues he is not talking about a particular formulation for set theory, like ZFC or NBG (the most popular examples in specialized literature). He refers to set theory as "a way of thinking with which we need to be inculcated if we are to understand any mathematical theory". That is quite puzzling! After all, there are many well known mathematical theories that require no particular notion of set whatsoever: classical propositional calculus [18], non-classical propositional calculi [9], certain formulations for category theory [5], mereology [14], type theory [23], and, of course, set theory itself [18] [30]!

Besides, ZFC and NBG are first order theories whose primitive concepts are equality $=$ and membership $\in$. And that is all! There is no need to talk about sets (in the intuitive sense of a collection of objects), when we are dealing with ZFC and NBG. Both ZFC and NBG are well known examples of formal theories in which the main purpose is to deal with two binary predicate letters, namely, equality and membership. The objects of ZFC and NBG are simply *terms*, from a first order language point of view. If a mathematician refers to such terms as *sets*, that happens because both ZFC and NBG are defined through axioms somehow associated to an *intended and merely intuitive interpretation* of what a set (collection) is supposed to be. Somehow, ZFC and NBG were erected in order to capture an intended interpretation that started with the work of Georg Cantor in the late 19th century. Nevertheless, there is no way to guarantee which formulation is better suited for Cantor's ideas about sets.

Cantor said "a set is any collection of definite, distinguishable objects of our intuition or of our intellect to be conceived as a whole". That is a really vague notion about what a set is supposed to be! That is why there is a huge myriad of set theories in specialized literature!

In von Neumann set theory, e.g., there are in fact sets [27]! A set may be defined as a quite particular term in von Neumann's axiomatic framework. Not all terms are sets in von Neumann's theory. But in ZFC and NBG, *all* terms are usually called sets (except in ZFU and its variations). For all that matters, those terms could be called *things*! Even if that was the case, still ZFC and NBG should be seen as first order theories whose primitive concepts are just equality and membership. So, what makes Smith think he knows for sure what a set should be?

During the early development of formal set theories, many mathematicians displayed serious prejudice against certain set-theoretic ideas. The best known example is the Axiom of Choice. Banach-Tarski theorem was used sometimes as an attempt to refute the Axiom of Choice [20]. The argument was simple (yet naive!): since the theorem is intuitively false, the Axiom of Choice cannot be true! That happened because many mathematicians of the early 20th century had their own ideas about what a set is supposed to be (like Smith is trying to do, with his own sense of what a clear sense is supposed to be).

Mathematics is not the science of clear sense! Mathematics is a social enterprise which fundamentally depends on fresh ideas and, of course, intense and critical discussion. Despite the fact that mathematics can be done in a formal way, formal languages cannot hide mathematics from social criticism. Yuri Manin proposed in 1974:

> We should consider the possibilities of developing a totally new language to speak about infinity. Classical critics of Cantor (Brouwer *et al.*) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.
>
> I would like to point out that this is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even

> ‘sets’ os photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the ‘set’ of grains of sand. In general, a highly probabilistic ‘physical infinity’ looks considerably more complicated and interesting than a plain infinity of ‘things’.

That point raised by Manin was one of the major motivations for developing quasi-set theory. The point here is not the concept of set *per se*, but it is rather a formal way to cope with infinities, specially infinite collections in some sense. If quasi set theory answers to Yuri Manin questions, that is naturally debatable. Nevertheless, quasi-set theory is as licit as any usual standard set theory, if we are trying to talk about infinite collections in a rigorous and formal way.

Smith argues quasi-set theory is simply an axiomatic theory. Well, we can obviously say exactly the same about any standard set theory, like ZFC and NBG, among others! So, why quasi-set theory should be seen in any different way?

The point raised by Smith seems to be directly connected to (standard) model theory. Nevertheless, what prevents us from developing a quasi-set-theoretic version of model theory? Few people in the world work with quasi-sets. And there is a lot to do if we really want to get some answers good enough to resist to qualified criticisms. On the other hand, some shy examples of quasi-set-theoretic models already exist in the literature, like the ones presented in [12] and [25] (apparently ignored by Smith).

Georg Cantor himself once said: “The essence of mathematics lies in its freedom”. So, let us be free to create! Manin wrote [1] “In accordance with Hilbert’s prophecy, we are living in Cantor’s paradise. So, we are bound to be tempted.” That non-Cantorian proposal is in a poetic agreement to the notion of freedom advocated by Cantor himself!

Allow me to extrapolate Smith’s ideas (at my own risk) to another realm: geometry. What about a thesis like the following one? “To make clear sense of geometrical concepts, one must at least model it in Euclidian geometry; but due to the special place of the parallel postulate in geometry, any vague axiom that one does model geometrically will not be the parallel postulate, for real parallelism will already be there, built into the background of the geometrical model, and perfectly precise.” Well, nowadays non-euclidian geometry is as good geometry as euclidian geometry! So, what seems to be the problem?

There is no precise definition for what geometry is! Analogously, there is no precise definition for what set theory is supposed to be!

The real issue here, mostly from a philosophical point of view, is this: can we cope with infinite collections (in a precise sense) without appealing to identity? If the answer is negative, why? And if the answer to this question is positive, what is the advantage of this kind of formal approach? That is why I wrote this paper! Any discussion about the importance of quasi-set theory should take epistemological issues under consideration. And the epistemological character of quasi-set theory is quite unusual, from a general scientific perspective. That happens as a consequence of the revision power quasi-set theory provides. We do not need to live in a world enslaved into identity. One of the most valuable lessons of twentieth century mathematics was this: there is no room for prejudice.

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