UNBOUNDED EXPECTATIONS AND THE SHOOTING ROOM

Abstract. Several treatments of the Shooting Room Paradox have failed to recognize the crucial role played by its involving a number of players unbounded in expectation. We indicate Reflection violations and/or Dutch Book vulnerabilities in extant "solutions" and show that the paradox does not arise when the expected number of participants is finite; the Shooting Room thus takes its place in the growing list of puzzles that have been shown to require infinite expectation. Recognizing this fact, we conclude that prospects for a "straight solution" are dim to nonexistent.

1. Introduction

Several well-known puzzles and paradoxes philosophers have been discussing in the previous couple decades, including the the St. Peterburg Paradox (see e.g. Martin 2001), the Two Envelopes Problem (see e.g. Chalmers 1994), the so-called Pasadena Game (Nover and Hájek 2004), etc., crucially involve quantities unbounded in expectation. The Shooting Room (Leslie 1996) is a puzzle that clearly involves a quantity unbounded in expectation. Heretofore, however, published treatments have failed to indicate in a clear way that this involvement is crucial. In this paper we pay infinite expectation the respect it is due. In particular, we won't try to surmount it.

William Eckhardt (1997) reworks the puzzle to his comfort zone:

> Successive groups of individuals are brought into a room and given the same highly favorable wager, say, betting $100.00 that the "house," with fair dice, rolls anything but double sixes. (In the original formulation, losing players are shot, but this added gruesomeness, if nothing else, complicates the question of how one should bet.) Whenever the room occupants win their bets, ten times as many people are recruited for the next round. Once the house wins, the game series is over. So the house can truthfully announce before any games are played that, in spite of the highly favorable odds, at least 90% of all players will lose. The puzzle is that these bets appear to be both favorable and unfavorable: favorable because double sixes are rare, unfavorable because the overwhelming majority of players lose.

We will make a few cosmetic changes/addenda to the above scenario. First, one person bets in the first round, nine in the second round, ninety in the third, then nine hundred, nine thousand, etc. Next, the players are stipulated to have epistemically similar backgrounds and bet in isolation from each other; they can't tell how many others are betting in that round. Finally, the results of the bets are only announced at an open (players can count the number of attendees) debriefing attended by all the players from all the rounds of betting.

Eckhardt argues for a credence of $\frac{1}{36}$ in *I lose* prior to the roll one is betting on, but is also comfortable, so far as we can tell, in a credence $\approx \frac{9}{10}$ in *I lose* once the bets are concluded:

> "...it is an error to consider yourself a random or typical player until you lose your bet. Before then, you have only about a 3% chance of belonging to the 90% majority. ... This means *you should not consider yourself random until the game series is over.* ...if a player about to bet were truly random among all players, then he would have better than a 90% chance of losing....""

What troubles us is the emphasized (in the original) clause, which appears to sanction a belief, by the player, that she has lost with 90% probability once "the game series is over" (at the debriefing, in our version). We don't accept that any "solution" endorsing this has dissolved the paradox; a player having credence $\frac{1}{36}$ in *I lose* at the time the bet is placed, knowing that with probability 1 the game will end and she will have credence of at least $\frac{9}{10}$ at the debriefing, violates Reflection (not merely in its naive formulations but in its apparently valid ones; see Schervish et. al. 2004).

In an enlightening paper, Paul Bartha and Christopher Hitchcock (1999) offer a way out of the seeming paradox. Assigning "draft positions" to the potential participants that determine the order in which they will be called to the room, they show that if an individual has *any* countably additive probability distribution over her possible draft position, her expected credence in *I lose* at the debriefing will be precisely $\frac{1}{36}$.

To see this let p_n be the probability of a "draft round" equal to $n+1$ (e.g. if your "draft position" is in $\{11, 12, \ldots, 100\}$, your "draft round" is 3) and note that:

(1) Prior probability in *I play* is $\sum_{n=0}^{\infty}(\frac{35}{36})^n p_n$.
(2) Posterior in *I lose* conditional on 10^x present at debriefing is $\frac{p_x}{p_0+p_1+\cdots+p_x}$.
(3) Probability of 10^x present conditional on *I play* is

$$\frac{Pr(10^x)Pr(play|10^x)}{Pr(play)} = \frac{(\frac{35}{36})^x(\frac{1}{36})(p_0+p_1+\cdots+p_x)}{\sum_{n=0}^{\infty}(\frac{35}{36})^n p_n}, \text{ hence}$$

(4) Expectation of posterior in *I lose* at debriefing conditional on *I play* is

$$\sum_{x=0}^{\infty}\frac{(\frac{35}{36})^x(\frac{1}{36})(p_0+p_1+\cdots+p_x)}{\sum_{n=0}^{\infty}(\frac{35}{36})^n p_n}\Big(\frac{p_x}{p_0+p_1+\cdots+p_x}\Big) = \frac{1}{36}.$$

This solves the Reflection problem but introduces a new one in its place. Since we assume that the participants have similar epistemic experiences prior to being assigned draft numbers, it seems reasonable that we should (or indeed that we must) also assume that all participants have the same prior distribution over their own numbers. This, however, implies that any large enough finite subset of the participant pool is subject to a group Dutch Book.[1] Indeed, choosing an n such that $p_n > 0$, offer to each participant in a group having cardinality $K > \frac{10^n}{p_n}$ a bet paying $1 - p_n$ if that

[1]We see no reason to think that group Dutch Books aren't incriminating in the way that individual ones are in cases where a given participant views her fellow group members as peers. That much can be assumed here, as the participants are evidential counterparts having identical credence functions.

participant has a draft round equal to $n+1$ and $-p_n$ otherwise. The participants view these bets as fair, but since at most $9 \cdot 10^{n-1}$ participants can win their bet and at least $K - 9 \cdot 10^{n-1}$ must lose, accepting them ensures a net payoff of at most

$$9 \cdot 10^{n-1}(1-p_n) - p_n(K - 9 \cdot 10^{n-1}) = 9 \cdot 10^{n-1} - p_n K < -10^{n-1}.$$

Bartha and Hitchcock sum up the problems with a countably additive prior thus: "...the weakness of this analysis is its inability to accomodate the intuition that (the participant) is equally likely to have any draft position." They then offer another analysis in which participants have a merely finitely additive distribution over draft positions assigning equal infinitesimal weight to each natural number n.

Without going into the specifics of how, Bartha and Hitchcock conclude that, although conditional on the participant being selected to enter the room and the experiment having ended with a losing round, the probability that she lost is indeed $\frac{9}{10}$, it is however the case that conditional just on the participant being selected, the probability that she lost is $\frac{1}{36}$. The explanation, roughly, as for why this is so is that, conditional on the participant being selected to enter the room, the probability that the experiment ends isn't equal to 1, but rather is equal to the probability of obtaining a large straight on the first roll of a Yahtzee turn[2], namely $\frac{5}{162}$. Thus

$$\begin{aligned}
&Prob(Lose|Selected)\\
=&Prob(End|Selected)Prob(Lose|End \,\&\, Selected)\\
&\quad + Prob(No\ End|Selected)Prob(Lose|No\ End \,\&\, Selected)\\
=&\left(\frac{5}{162}\right)\left(\frac{9}{10}\right) + \left(\frac{157}{162}\right)\left(\frac{0}{1}\right) = \frac{1}{36}. \qquad (1.1)
\end{aligned}$$

But this merely pushes the paradox back a round. Indeed, Bartha and Hitchcock look to be committed to a probability of the participant being in a final winning round, conditional on the participant being selected to enter the room, of $\frac{1}{360}$; she is one-tenth as likely to be in a final winning round as in a losing round. That is:

$$\begin{aligned}
&Prob(Final\ Winning|Selected)\\
=&Prob(End|Selected)Prob(Final\ Winning \,\&\, Selected)\\
&\quad + Prob(No\ End|Selected)Prob(Final\ Winning|No\ End \,\&\, Selected)\\
=&\left(\frac{5}{162}\right)\left(\frac{9}{100}\right) + \left(\frac{157}{162}\right)\left(\frac{0}{1}\right) = \frac{1}{360}. \qquad (1.2)
\end{aligned}$$

More generally, they should say that conditional on her entering the room, the probability that there are exactly n winning rounds after hers is $(\frac{1}{10})^{n+1}\frac{1}{36}$. The fact that $\sum_{n=-1}^{\infty}(\frac{1}{10})^{n+1}\frac{1}{36} = \frac{5}{162}$ would appear to provide evidence that this is in fact what Bartha and Hitchcock would assent to.

[2]Coincidentally, of course.

And that's just as puzzling as the original paradox. Indeed, Bartha and Hitchcock owe an explanation[3] for why it isn't the case that

$$Prob(Final\ Winning|Selected) = \frac{35}{36} \cdot \frac{1}{36} = \frac{35}{1296}.$$

If no satisfactory explanation is forthcoming (it is hard to imagine that one is), one must conclude that Bartha and Hitchcock's argument fails.

2. Infinite repeated Shooting Room

At the beginning of the paper we promised an analysis of the Shooting Room paradox showcasing the role of infinite expectation. To this end it is instructive to contrast a *finite* expectation variant of the Shooting Room in which players lose their bets with probability $\frac{35}{36}$; e.g., they win on double sixes and lose otherwise. Rounds with identically mounting betting populations are again conducted until some group loses. Call the venue for these bets the *Brutal Room.* The Brutal Room game again requires an infinity of potential participants in order to ensure, with probability one, that one can complete the requisite series of rounds, and we'd again like to assign draft position priors in such a way that, at a postgame debriefing attended by N participants, each comes to have a draft position posterior that is uniform on $\{1, \ldots, N\}$.

As things are currently set up, however, this entails a uniform draft position prior over the infinite pool of potential participants. Merely finitely additive distributions (as we've just seen evidence of) are seldom, if ever, the way out of a paradox; more typically, they give rise to them. A more useful first step is to recognize that one *can* identify a countably additive draft position distribution that, when taken as our participant's distribution conditional on *I play*, yields a uniform debriefing posterior.

We identify the distribution. Denote by x the probability, conditional on *I play*, of draft round $D = 1$. Since 9 times as many enter with draft round $D = 2$ (namely positions 2-10), but round 2 takes place on only $\frac{1}{36}$ of iterations, the probability, again conditional on *I play*, of draft round $D = 2$ is $x(9)(1/36) = \frac{1}{4}x$. Similar reasoning shows the probability, conditional on *I play*, of draft round $D > 2$ is $\frac{1}{4}x(\frac{5}{18})^{D-2}$.

Setting the sum of these probabilities equal to 1 and solving for x we get:
$Pr(D = 1) = x = \frac{26}{35}$,
$Pr(D = 2) = \frac{13}{70}$,
$Pr(D = 3) = \frac{13}{252}$, etc.

If now a participant shows up at the debriefing and there are (for example) 10^3 participants present, her posterior draft position (not round, but position) distribution will be uniform on $\{1, 2, \ldots, 100\}$. This is because she knows one of (a)-(c) holds:
(a) $D = 1$ and two double sixes rolled after her arrival (probability $q_1 = \frac{26 \cdot 35}{35 \cdot 36 \cdot 36 \cdot 36}$),
(b) $D = 2$ and one double six rolled after her arrival (probability $q_2 = \frac{13 \cdot 35}{70 \cdot 36 \cdot 36} = 9q_1$),
(c) $D = 3$ and zero double sixes rolled after her arrival (probability $q_3 = \frac{13 \cdot 35}{252 \cdot 36} = 10q_2$).

[3]All the more so in that they cite the Principal Principle in support of $Prob(Lose|Selected) = \frac{1}{36}$; see their footnote 3. (Thanks to an anonymous referee for this point.)

This holds generally...regardless of how many players our participant encounters at the debriefing, she will come to have uniform draft position posterior over the corresponding initial segment of $\mathbf{N}$. Of course one might think that there is still a potential complication...to achieve the draft position distribution we found conditional on *I play* would still require a uniform prior, should we continue to assume that the game is played exactly once and that players are informed as to their status as pool members prior to learning that they have been selected to enter the room.

There is no such complication. Simply do not tell members of the pool that they are such, or even that there is any such game. Then, there is no point in time at which participants who are selected need to entertain credences that fail countable additivity. Another way (our preference) is to assume that the Brutal Room game has been played infinitely many times throughout an infinite past, and will continue to be played throughout an infinite future. Indeed, we may assume for convenience here that every being is chosen to participate in exactly one iteration of the game. (Perhaps they are selected in order of birth.) Once a given participant has entered their iteration of the game, the first participant in that iteration identifies an "origin" and the participant may revert to our former talk of "draft position" and 'draft round". Participants now may be told about the game; since *I play* is now a certainty, however, there is still no need for the problematic uniform prior.

A Brutal Room participant suffers no Reflection violation. During her betting session, her credence in *I will be alone at the debriefing*, is $Pr(D = 1)$ times the probability that the first roll is not double six, i.e. $(\frac{26}{35})(\frac{35}{36}) = \frac{13}{18}$. Credence in *I win* at the debriefing is 0 if they are alone and $\frac{1}{10}$ otherwise, with expectation $(\frac{5}{18})(\frac{1}{10}) = \frac{1}{36}$.

Buoyed by this result, one might think to analyze the Shooting Room along similar lines. We start by attempting to find a countably additive draft position distribution that, if taken as a Shooting Room participant's distribution conditional on *I play*, yields a uniform debriefing posterior. Denote then by x, as before, the probability, conditional on *I play*, of draft round $D = 1$. Since 9 times as many enter with draft round $D = 2$ (namely positions 2-10), and round 2 takes place on $\frac{35}{36}$ of iterations, the probability, conditional on *I play*, of draft round $D = 2$ is $x(9)(35/36) = \frac{35}{4}x$. Similarly the probability, conditional on *I play*, of draft round $D > 2$ is $\frac{35}{4}x(\frac{175}{18})^{D-2}$. But in order for these not to sum to ∞, one must have $x < \alpha$ for every positive real α, so our participant is again saddled with a merely finitely additive prior. A Reflection violation, too: the participant's credence in *I win* is $\frac{1}{36}$ during her betting session, while her credence in *I win* at debriefing is 0 if she is alone and $\frac{1}{10}$ otherwise, with expectation $(1 - x)\frac{1}{10} \approx \frac{1}{10}$.

More generally, if p is the (objective) probability that bettors win in a Shooting Room style game (with round sizes fixed at 1, 9, 90, etc.) then the (subjective) probability x of draft round $D = 1$, conditonal on *I play*, is equal to the multiplicative inverse of the expected number of bettors in a given iteration of the game. x is therefore infinitesimal (and the bettors' credences conditional on *I play* merely finite additive) if and only if the expected number of players, namely $1 + 9p\sum_{n=0}^{\infty}(10p)^n$, is infinite (that is, precisely when $p \geq \frac{1}{10}$). Since Reflection violations don't arise when the participants have countably additive distributions, then, such games require infinite expectation in order to generate a paradox.

3. Conclusion

Our purpose here hasn't been to "solve" the Shooting Room Paradox, but rather to call attention to the fact that the number of players participating in a single iteration of the game is infinite in expectation, and that if the conditions of the game are altered to remove this feature, the paradox evaporates. The puzzle therefore takes its place in a larger group of similar puzzles depending crucially on infinite expectation.

No extant treatment of any such puzzle should give one reason to be optimistic about prospects for a fully satisfying straight solution[4] in the case of the Shooting Room; proposals that take the setup of such problems at face value invariably involve Reflection violations, failures of dominance reasoning, merely finitely additive credence functions, vulnerability to Dutch Books, etc. It might be argued that some concessions are more benign than others, and that the inherent contentiousness of determining which is part of what places such problems under the purview of philosophers (rather than, say, economists or mathematicians). Even if that's right, however, infinite expectation is what constitutes the common character of these puzzles; not to at least flag it looks, to us, tantamount to missing the point.

References

Arntzenius, Frank, Adam Elga and John Hawthorne. 2004. Bayesianism, Infinite Decisions and Binding. *Mind* 113:251-283.

Bartha, Paul and Christopher Hitchcock. 1999. The Shooting-Room Paradox and Conditionalization on Measurable Challenged Sets. *Synthese* 118(3):403-437.

Chalmers, David. 1994. The Two-Envelope Paradox: A Complete Analysis? Manuscript. Available at http://consc.net/papers/envelope.html

Eckhardt, William. 1997. A Shooting-Room View of Doomsday. *Journal of Philosophy* 94(5):244-259.

Jeffrey, Richard. 1983. *The Logic of Decision, Second Edition.* Chicago: University of Chicago Press.

Leslie, John. 1996. *The End of the World.* New York: Routledge.

[4]One sees other approaches. One is to take the setup of such a problem at face value and lament the ensuing rational conundrum. Authors taking this line have penned such provocative passages as "there are contexts in which full rationality is impossible" and even "rational decision making is a lost cause". These quotes are taken from Ross (2010), who considers a Sleeping Beauty experiment with a number of awakenings having infinite expectation, and McGee (1999), who constructs a Dutch Book out of individually favorable payoffs X_i such that $\sum_{i=1}^{\infty} \min\{X_i, 0\} = -\infty$. (Neither discusses infinite expectation's role.) Another is to explore which restrictions must be placed on decision theory so as to avoid paradox. (Compare: type theorists and the developers of modern set theory explored which restrictions must be placed on, e.g., set formation so as to avoid paradox.) Richard Jeffrey (1983), for example, wrote "anyone who offers to let the agent play the St. Petersburg game is a liar, for he is pretending to have an indefinitely large bank," while Chalmers (1994) is content to observe that the Two Envelopes Paradox fails to arise in the finite expectation case.

Martin, R. 2001. The St. Petersburg Paradox. in E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Fall 2001 Edition)* [online encyclopedia], Fall 2001. Available at http://plato.standford.edu/archives/fall2001/entries/paradox-stpetersburg/

McGee, Vann. 1999. An airtight Dutch Book. *Analysis.* 59:257-265.

Nover, H. and A. Hájek. 2004. Vexing Expectations. *Mind* 113:237-249.

Ross, Jacob. 2010. Countable additivity, Dutch books, and the Sleeping Beauty problem. *Philosophical Review* 119:411-447.

Schervish, M.J., T. Seidenfeld and J.B. Kadane. 2004. Stopping to Reflect. *Journal of Philosophy* 101(6):315-322.