Spontaneous Symmetry Breaking and Quantum Measurement

Lia den Daas
liadendaas@gmail.com

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University of Cambridge

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Contents

1 Introduction

The measurement problem of quantum theory has troubled the minds of physicists and philosophers for decades. This paper will discuss a relatively new attempt at resolving the issue, one that is rooted in the mathematics of quantum theory. It is known as the mechanism of the "flea", as put forward by Landsman and Reuvers in [13] and by Landsman in [12]. The greatest conceptual work is given by a new understanding of the measurement problem, as described in chapter 3. Then a small, asymmetric perturbation that arises from the environment is postulated. This "flea" perturbation provides the symmetry breaking required for a classical state to emerge from a quantum system, thereby bringing us one step closer to a possible solution of the measurement problem.

This approach uses the formalism of algebraic quantum theory, which allows for a unified description of classical and quantum theory. Therefore, this paper will first introduce the notion of a C*-algebra and will discuss the connections between the Hilbert space formalism and the algebraic formalism in chapter 2. Chapter 3 then describes how the measurement problem is understood by Landsman and Reuvers, setting up the context for the flea mechanism. It lays out the unresolved aspects of the process of measurement that the proposal attempts to address. The "flea", a small perturbation of the potential of a system, is then introduced in the following chapter 4. As in [13], this paper will use the symmetric double well as the model to discuss the details of the flea mechanism. Landsman also discusses other models for which the mechanism of symmetry breaking can be seen to apply in [11]. However, as these are less instructive to understanding the concept of the perturbation by the flea, these are at present not discussed.

Chapter 5 hopes to offer an evaluation of the flea mechanism. These comments focus on the limitations of the models considered, as well as the nature of the flea perturbation. In particular, the challenges in attempting to recover the Born rule are discussed. This discussion is informed by the work of Van Heugten and Wolters in [22]. Finally, the conclusion contains some general reflections on the flea proposal, portraying it as the start of a new way of making the idea of a wave function collapse mathematically precise.

2 Algebraic Quantum Theory

This paper will consider the measurement problem and Landsman's proposal of "the flea" in the formalism of algebraic (quantum) theory. Before introducing some of the main mathematical concepts of this formalism, it is important to understand where its conceptual value lies. Typically, classical mechanics is understood in terms of phase space, which forms a differentiable manifold with a symplectic structure [1]. When quantum theory is subsequently introduced, it tends to be through the method of canonical quantisation. In its most elementary form, this corresponds to promoting the variables position and momentum, as well as other observables, to be linear operators on a Hilbert space. The interplay between these observables is now no longer determined by Poisson brackets, as for the variables in the classical theory, but by the commutation relations of the operators.[2] Although some aspects of the two theories seem similar at first glance, they concern two very different mathematical structures. In a sense, the process of canonical quantisation works primarily due to the benefit of hindsight. Knowledge of the result of the process, namely the correct structure for quantum theory, is essential for quantisation to be useful.

Given this contrast between the classical and quantum mathematical structures, comparing the two theories is not trivial. Describing them algebraically allows us to bring classical and quantum theory into a shared framework, thereby formalising the transition between the two realms. This is particularly important in our case, since the measurement problem is principally the question of how to go from a quantum state to a classical state.

2.1 Structure

The algebraic structure that is shared by both classical and quantum theory consists of only two main components. Rather than taking the state of a physical system as central, we focus on the observables that we associate with a system, such as spin, energy, and momentum. Recognising an algebraic structure in this set of observables leads to the following basic axioms of the algebraic formalism of quantum theory.

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1. A physical system is defined by its associated unital C*-algebra $\mathfrak{A}$. The **observables** of the system are self-adjoint elements of that algebra.

2. The **states** of a system are taken to be normalised positive linear functionals ω on the algebra $\mathfrak{A}$, i.e. $\omega : \mathfrak{A} \to \mathbb{C}$ s.t.

$$\omega(A^*A) \geq 0 \quad \forall A \in \mathfrak{A} \quad \text{and} \quad \omega(\mathbb{1}) = 1. \tag{1}$$

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[2] Changing the relations between position and momentum, or similarly in field theory, the field and its conjugate momentum, has a deeper significance. It corresponds to changing the symplectic structure on the space of fields, which classically gives you the Poisson brackets, to a symplectic structure that gives you the commutation relations of quantum theory (page 10, [19]). In general, any new choice of symplectic structure will result in a new (quantum field) theory.

Before understanding how these concepts connect to the structure of quantum and classical physics, let us unpack the most essential mathematical terms from functional analysis. This account is by no means comprehensive, and thus the reader is referred to other sources for complete definitions and a more detailed description. For a mathematical approach, see [12, 14], or for a more philosophical angle, see [16]. This paper will introduce definitions based on [14].

Observables

We start with the concept of an algebra. For our purposes, we can consider an algebra to be an extension of a vector space with a product, i.e.

Definition 1. An algebra $\mathfrak{A}$ over the field $\mathbb{K}$ is a $\mathbb{K}$-vector space with a product $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ that is associative and distributive. An algebra is said to be unital if it contains a unit $\mathbb{1} \in \mathfrak{A}$ s.t. $\mathbb{1}A = A\mathbb{1} \quad \forall A \in \mathfrak{A}$.

In order to get a notion of what properties a C*-algebra has, and thus the structure associated with the set of observables, there are a few properties of algebras that are helpful to list.

Definition 2. A complex involutive algebra $\mathfrak{A}$, also denoted as *-algebra, is an algebra over the field $\mathbb{C}$, which also has a map $* : \mathfrak{A} \to \mathfrak{A}$ called an involution. It has the following properties $\forall A, B \in \mathfrak{A}$ and $\forall \lambda \in \mathbb{C}$:

1. $(A + B)^* = A^* + B^*$
2. $(AB)^* = B^* A^*$
3. $(\lambda A)^* = \overline{\lambda} A^*$
4. $(A^*)^* = A$.

Definition 3. A normed algebra $\mathfrak{A}$ is a normed vector space whose norm satisfies

$$\|AB\| \leq \|A\| \|B\| .$$

For a unital algebra $\mathfrak{A}$, if $\|\mathbb{1}\| = 1$ then $\mathfrak{A}$ is a normed unital algebra.

A C*-algebra has both of the above properties, but is also a complete normed space. In particular, it is special type of Banach algebra. Although it involves some notions from topology, this paper will not go into detail beyond the following definitions.

Definition 4. A Banach space is a normed vector space that is equipped with a topology induced by the norm and that is complete with respect to this topology. A Banach unital algebra is then a Banach space that is also a normed unital algebra with respect to the same norm.

Definition 5. A Banach involutive algebra is a Banach algebra with an involution that satisfies $\|A\| = \|A^*\|$.

Definition 6. Finally, a C*-algebra $\mathfrak{A}$ is a Banach involutive algebra s.t. the norm has the C*-property:

$$\|A^* A\| = \|A\| \|A^*\| = \|A\|^2 \quad \forall A \in \mathfrak{A}.$$

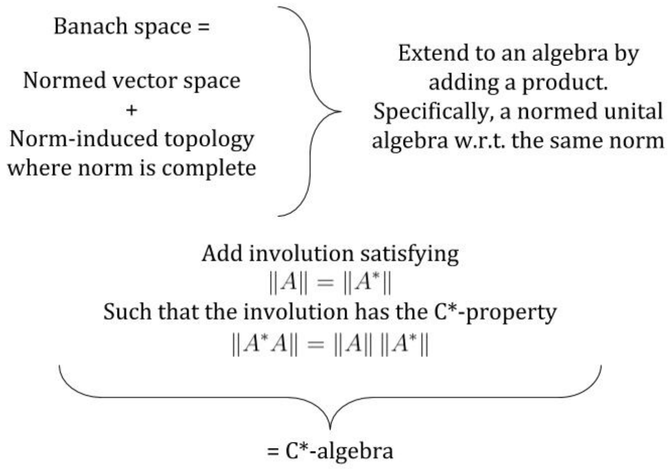

Figure 1: C*-algebra

Here is a diagram of the different properties constituting a C*-algebra, following the definitions above. Although the figure does not incorporate all information, it gives an idea of how the different definitions are related.

States

As stated above, a state of a system is fully specified by a map that assigns to each element in the algebra $\mathfrak{A}$ a complex number. The physical idea behind this number is the following. For a self-adjoint element A of $\mathfrak{A}$, this number $\omega(A)$ is real and represents the expectation value of A in the state ω. More on the physical interpretation follows in the next section. We can make a distinction here between pure states and mixed states.

Definition 7. A pure state ω can only be trivially decomposed. That is, if for some states ω_1, ω_2 and some $p \in (0,1)$ we can write $\omega = p\omega_1 + (1-p)\omega_2$, then $\omega_1 = \omega_2$.

Pure states in the algebraic formulation are the counterparts of rays in Hilbert space. Thus, mixed states are convex combinations of such pure states [6].

2.2 From Hilbert space $\mathcal{H}$ to algebra $\mathfrak{A}$

Given the above mathematical structure, how can we see that a quantum system can indeed be described in such a manner? This section will give some intuition for how this structure emerges from the Hilbert space formalism of quantum mechanics that most readers are familiar with. There are nuances to the Hilbert space formalism that for the sake of brevity I have ignored. For a more detailed description, see the first chapter of [7], or chapter four of [16]. In particular, following Landsman and Reuvers [13], I will idealise our description by assuming all operators to be bounded.

Firstly, let us briefly recall the relevant structure of the Hilbert space formalism. A state $\Psi \in \mathcal{H}$ is a wavefunction in the Hilbert space associated with a physical system, while an observable A is a self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$. The adjoint of an operator is indicated $A^\dagger$ and is defined with respect to the hermitian inner product defined on $\mathcal{H}$. We can express the abstract wave function Ψ in a orthogonal basis constructed from the eigenvectors of the operator A. To that observable we can then associate an expectation value $\langle A \rangle$ for a given state Ψ computed in the following manner

$$\langle A \rangle := \langle \Psi, A\Psi \rangle. \tag{2}$$

Observables are self-adjoint operators, meaning $A = A^\dagger$. For these, the expectation value is a real number by definition of the hermitian inner product:

$$\langle \Psi, A\Psi \rangle \equiv \overline{\langle A\Psi, \Psi \rangle} = \overline{\langle \Psi, A\Psi \rangle}. \tag{3}$$

These ideas are translated to the algebra as follows. The space of all bounded linear operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$, forms a C*-algebra. A state $\Psi \in \mathcal{H}$ is associated with an algebraic state ω on the algebra $\mathfrak{A}$, through the GNS theorem given by (1) in the next section. A norm for the operators is introduced by taking the definition in terms of a supremum over states

$$\|A\| := \sup_{\omega} |\omega(A)| \tag{4}$$

which is a finite positive number. Taking the adjoint of an operator corresponds to involution on the algebra, i.e. we identify $A^\dagger$ with involution A^*. Since observables are self-adjoint operators, they are the elements of the algebra for which $A^* = A$.

Inherent in the algebraic formalism is an operational approach to quantum theory, that is, the viewpoint that a state is fully defined in terms of the possible measurement outcomes of the observables for that state.[3] This is reflected in the interpretation of the (complex) number $\omega(A)$ that a state assigns to elements in the algebra. Namely, for observables we understand it to be the expectation value for that observable A in the state Ψ, which is associated with the algebraic state ω. Given the two following properties, this understanding is warranted.

1. Both $\omega(A)$ and $\langle A \rangle_\Psi$ are real numbers for self-adjoint A.
2. The expectation value associated to $A^\dagger A$ is $\langle \Psi, A^\dagger A\Psi \rangle = \langle A\Psi, A\Psi \rangle \geq 0$, which corresponds to the assumed positivity of states on $\mathfrak{A}$.

The assumption that quantum states can be taken as linear functionals on the algebra is not as innocent as it may seem. For two observables A and B that do not commute, it is not obvious if linearity, i.e. $\omega(A + B) = \omega(A) + \omega(B)$, still ought to hold. In his work laying the foundations for quantum mechanics [23], Von Neumann assumed that any real-linear

[3]The possible outcomes of a measurement of an observable are given - for the case of an operator with pure discrete spectrum - by the set of eigenvalues associated with the operator, which for self-adjoint operators are real valued. The spectral theorem ensures that we can write such an observable as a real linear sum of the projection operators associated with these eigenvalues. The details of how this spectral decomposition is defined for operators with a more general spectrum and is subsequently incorporated in the algebraic formulation are interesting, but not relevant for this paper. Further information on such details can be found in [16].

combination of two observables A and B itself forms an observable, for which this linearity holds. This assumption was criticised by Bell in 1966 [3], thereby questioning Von Neumann's famous proof which ruled out hidden variables in quantum theory. Notwithstanding Bell's insight, the algebraic formalism follows Von Neumann's lead, likewise assuming linearity to hold for the combination of any two observables.

2.3 From algebra $\mathfrak{A}$ to Hilbert space $\mathcal{H}$

We have seen that an algebraic structure emerges from the Hilbert space formalism. Conversely, an element A of an abstract involutive algebra $\mathfrak{A}$ can be taken to be a bounded linear operator on an appropriate Hilbert space $\mathcal{H}$, namely by choosing a representation.

Definition 8. A representation π of a unital C*-algebra $\mathfrak{A}$ is a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. A representation π is called faithful if $\ker \pi = \{0\}$. It is called irreducible if there are no non-trivial subspaces of $\mathcal{H}$ invariant under $\pi(\mathfrak{A})$.

The concept of unitarily equivalent representations will be useful when discussing the flea perturbation:

Definition 9. Two representations of a C*-algebra $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitarily equivalent if, for some unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$

$$U\pi_1(A) = \pi_2(A)U \quad \forall A \in \mathfrak{A}.$$

For clarity, since $\pi_1(A) : \mathcal{H}_1 \to \mathcal{H}_1$ and $U : \mathcal{H}_1 \to \mathcal{H}_2$, the composite map $U\pi_1(A) : \mathcal{H}_1 \to \mathcal{H}_2$. Similarly, since $\pi_2(A) : \mathcal{H}_2 \to \mathcal{H}_2$, the composite map $\pi_2(A)U$ maps also from $\mathcal{H}_1$ to $\mathcal{H}_2$.

States on C*-algebras can be used to construct a representation of the algebra on a Hilbert space. This is provided by the Gelfand-Naimark-Segal (GNS) theorem.

Theorem 1 (GNS Theorem)**.** *Let ω be a state on a C*-algebra $\mathfrak{A}$. Then there exists a representation π of the algebra on some Hilbert space $\mathcal{H}$, and a unit vector $\Omega \in \mathcal{H}$, such that*

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle \quad \forall A \in \mathfrak{A}.$$

Any such representation is called a GNS representation.

A proof of a more general version of the GNS theorem can be found in [7, 14]. Such a GNS representation is unique for a state $\omega \in \mathfrak{A}$, up to unitary equivalence. Thus, this theorem provides a clear way of representing an abstract algebraic state ω by a unit vector Ω in some Hilbert space $\mathcal{H}$, through a representation of the abstract algebra $\mathfrak{A}$.

A representation of $\mathfrak{A}$ constructed using the above theorem will in general not be irreducible; it is only irreducible if the state ω used in the construction is a pure state. For a mixed state, as opposed to a pure state, each pure state in the decomposition will induce a irreducible representation of $\mathfrak{A}$ (e.g. π_1 and π_2). Then the composite representation of any given $A \in \mathfrak{A}$ can be seen as a block-diagonal matrix constructed from the respective irreducible representations, e.g.

$$\pi(A) = \begin{bmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{bmatrix}.$$

3 The Measurement Problem

The measurement problem is a grand question in the philosophy of physics, discussed by many great minds. The aim of this section is certainly not to give a comprehensive overview of this discussion. Rather, we explain how the measurement problem is understood in the context of Landsman's work. We hope to clarify where the current description of the process of measurement is problematic, so as to make apparent how the "flea" perturbation could hope to answer these problematic aspects.

It is important to briefly mention the context in which the approach discussed in this paper, Landsman's mechanism involving a perturbation, operates. There are a few main approaches to solving the measurement problem that can be distinguished, often characterised as different interpretations of quantum mechanics. The "flea" perturbation allows for classical behaviour to emerge from the quantum through an explicit breaking of symmetry. Thus, it can be seen as a mathematical description of some type of "collapse". Namely, there is a multiplicity of possible states of a system and only one outcome is realised. In contrast, Everettian mechanics posits that all possible outcomes are realised, admitting the existence of many worlds. Lastly, the pilot wave picture singles out the position of a quantum particle as the observable (beable) that holds a definite value at all times. It can be classified as a hidden variable theory.

In terms of J.S. Bell's six possible world of quantum mechanics, as discussed beautifully in [4], we could perhaps see Landsman's work as a technical, mathematical approach to the measurement problem. It sketches an unromantic quantum world, characterised in Bell's words as

> "Surely the big and the small should merge smoothly with one another? And surely in fundamental physical theory this merging should be described not just by vague words but by precise mathematics?" (page 1210, [4])

In his discussion of asymptotic Bohrification in [12], Landsman explicitly makes the assumption that the classical behaviour indeed emerges from the quantum theory, and seeks a mathematical connection between the quantum description and apparent classicality. Thus, his endeavours can be seen to follow the rationale above.

Firstly, the algebraic formulation of both classical theory and quantum theory are introduced in Section 3.1. Then, the 'bridge' between these two theories is discussed in Section 3.2, namely the limiting procedure of taking $\hbar \to 0$. With these technical notions in hand, Section 3.3 discusses the process of measurement in more detail and the problematic aspects are made precise. Finally, before moving on to Chapter 4, some important conceptual remarks are made about the classical limit.

3.1 Two theories

Classical Let us call the algebra associated with classical systems $\mathfrak{A}_c$. We will be concerned with a point particle confined to one spatial dimension, so that its phase space is $\mathbb{R}^2$. This algebra can be represented as (a subset of) the space of essentially bounded measurable functions on the phase space $\mathbb{R}^2$, i.e.

$$\pi : \mathfrak{A}_c \to L^\infty(\mathbb{R}^2). \tag{5}$$

Thus, every element $A \in \mathfrak{A}_c$ is represented as

$$\pi(A) := f \tag{6}$$

for some $f \in L^\infty(\mathbb{R}^2)$

$$f : \mathbb{R}^2 \to \mathbb{C} \quad , \quad f : z \mapsto f(z). \tag{7}$$

Operations on the algebra, e.g. the product between two elements of the algebra, are defined pointwise. Thus, the algebra is commutative. Involution is taken as pointwise complex conjugation. The norm for algebraic elements is the supremum norm, thus

$$\|f\| := \sup \left\{ |f(z)| : z \in \mathbb{R}^2 \right\}. \tag{8}$$

States on this algebra correspond to probability measures on phase space. In particular, pure states are measures of the Dirac form, δ_z. Thus, pure states correspond bijectively to points $z \in \mathbb{R}^2$ through the δ-function. Finally, to correctly describe a classical system, we need to impose extra continuity conditions on the functions. This essentially reduces our algebra to the continuous function on the phase space, i.e. $C_0(\mathbb{R}^2)$ [13].

Quantum As previously described in section 2.2, to a quantum system we can associate an algebra $\mathfrak{A}_q$ that is represented as (a subset of) the space of bounded linear operators on a Hilbert space $\mathcal{H}$. For the quantum systems we will be concerned with, $\mathcal{H}$ is the space of square-integrable wave functions, $L^2(\mathbb{R})$. Thus,

$$\pi : \mathfrak{A}_q \to \mathcal{B}(L^2(\mathbb{R})). \tag{9}$$

Since there are operators in quantum mechanics that do not commute, this algebra will be non-commutative. Its elements obey the canonical commutation relations in their bounded form. Involution is hermitian conjugation, and the norm is defined as in equation (4). States on this algebra can be identified with density matrices on $L^2(\mathbb{R})$. By means of the GNS-construction for a state, pure states on this algebra correspond bijectively to rays $\mathbb{C} \cdot \Psi$ in the Hilbert space $L^2(\mathbb{R})$. Imposing the correct continuity conditions on these operators means we reduce the algebra to $K(L^2(\mathbb{R}))$, the space of compact operators on $L^2(\mathbb{R})$.

3.2 Taking the limit $\hbar \to 0$

In order to consider how quantum states can converge to classical states, we have to associate an element of the classical algebra, i.e. a function, with an element of the quantum algebra, i.e. an operator. This can be done by means of a quantisation map,

$$f \mapsto Q_\hbar(f) \tag{10}$$

where $f \in \pi(\mathfrak{A}_c) = L^\infty(\mathbb{R}^2)$ and $Q_\hbar(f) \in \pi(\mathfrak{A}_q) = \mathcal{B}(L^2(\mathbb{R}))$. This paper will take the Berezin quantisation map, following Landsman and Reuvers [13]. In order to define this map, we first introduce the concept of coherent states $\Phi_\hbar^{(p,q)} \in \mathcal{H} = L^2(\mathbb{R})$. These states are labelled by points $z = (p,q) \in \mathbb{R}^2$, and defined as

$$\Phi_\hbar^{(p,q)}(x) := (\pi\hbar)^{-\frac{1}{4}} e^{-\frac{ipq}{2\hbar}} e^{\frac{ipx}{\hbar}} e^{-\frac{(x-q)^2}{2\hbar}}. \tag{11}$$

These coherent states are states in the Hilbert space, and thus quantum mechanical. They are constructed so that they are Gaussian wave functions with minimum uncertainty in both position and momentum. This makes a coherent state the natural "quantum analogue" of a given classical state, hence we label coherent states with $z = (p,q)$. This indeed provides a map from an algebraic classical state to an algebraic quantum state, since

- a pure state ω_0 on the classical algebra corresponds bijectively to a point $z = (p,q) \in \mathbb{R}^2$;
- for a given value of $\hbar$, a pure state $\omega_\hbar$ on the quantum algebra corresponds bijectively (up to a phase) to the coherent state $\Psi_\hbar^{(p,q)} \in \mathcal{H}$.

Given the definition for coherent states above, the Berezin quantisation map is defined as

$$Q_\hbar(f) := \int_{\mathbb{R}^2} \frac{dpdq}{2\pi\hbar} f(p,q) \left|\Phi_\hbar^{(p,q)}(x)\right\rangle \left\langle \Phi_\hbar^{(p,q)}(x)\right|. \tag{12}$$

The construction implies that the following convergence occurs. The pure states $\omega_\hbar$ form a family of quantum states labelled by the parameter $\hbar$ with values $\in (0,1]$. Thus, for a fixed value $\hbar$,

$$\omega_\hbar : \mathcal{B}(L^2(\mathbb{R})) \to \mathbb{C}. \tag{13}$$

Recall that ω_0 is a classical state, i.e.

$$\omega_0 : L^\infty(\mathbb{R}^2) \to \mathbb{C}. \tag{14}$$

Now suppose that we take the classical limit, corresponding to taking the quantum parameter $\hbar$ to zero. Then the quantum states converge to the classical, i.e.

$$\lim_{\hbar\to 0} \omega_\hbar(Q_\hbar(f)) = \omega_0(f) : \tag{15}$$

when this convergence holds $\forall f$, we write

$$\lim_{\hbar\to 0} \omega_\hbar = \omega_0. \tag{16}$$

We introduce one more shorthand in the interest of convenience and clarity. Let ρ_0 be a pure classical state in the phase space of a classical system, thus describing a point $z \in \mathbb{R}^2$. It corresponds to the algebraic state ω_0 on the classical algebra. On the quantum side, we have already stated that the coherent states $\Phi_\hbar \in \mathcal{H}$ correspond to the algebraic states $\omega_\hbar$. Thus, rather than writing the convergence in terms of the algebraic states, we can write

$$\lim_{\hbar\to 0} \Phi_\hbar = \rho_0. \tag{17}$$

For the purposes of this paper, there is no need to go into further detail on this convergence. The curious reader is referred to chapter seven of [12] for more mathematical detail.

3.3 The process of measurement

To understand the problematic aspects of measurement in quantum theory, it is helpful to inspect the process of measurement in more detail. Two steps in this process can be distinguished. Two avoid confusion with Von Neumann's process 1 and 2 in [23], we call them $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

$$\Psi \quad \rightarrow_{[\alpha]} \quad \Phi_{\hbar} \quad \rightarrow_{[\beta]} \quad \rho_0$$

We start with a quantum state, $\Psi \in \mathcal{H}$, and we want to end up with a classical state, ρ_0. As we saw in the last section, the classical state ρ_0 is the limit of some family of coherent states, $\Phi_{\hbar}$. This limiting procedure is thought of as the second step of the measurement process, $\boldsymbol{\beta}$.

The measurement problem arises in the following case. Consider the quantum state Ψ we start out with, the ground state of the system, to be a quantum superposition. This means, if we were to perform the limiting procedure immediately on Ψ and take $\hbar \to 0$, we would find the system to be in a combination of multiple classical states. This is a mixed state, rather than a pure state. In other words, Schrödinger's cat will be 'alive' + 'dead'. Or, in the words of Landsman

> "Certain *pure* post-measurement states of an (ontologically quantum mechanical!) apparatus coupled to a microscopic quantum object induce *mixed* states on the apparatus (and on the composite) once the apparatus is described classically" (page 453, [12]).

Now, a mixed state is not the outcome we observe, nor one that properly describes a classical system. Thus, before we perform the limiting procedure, some other process has to occur. There has to be an intermediate state, such that we will eventually acquire the correct state ρ_0. This intermediate state can be thought of as the coherent state $\Phi_{\hbar}$. Some process, let us call it $\boldsymbol{\alpha}$, needs to change the quantum ground state Ψ into a state that - if not equals, then at least approximates - the coherent state $\Phi_{\hbar}$. Once we have such a state, we can perform the limiting procedure $\boldsymbol{\beta}$ to find a 'single' classical state ρ_0. This process $\boldsymbol{\alpha}$ must involve some kind of symmetry breaking, selecting between finding the cat to be either 'alive' or 'dead'. Thus,

> "At heart the problem does not lie with the (dis)appearance of interference terms (which is a red herring) but with the inability of quantum mechanics to predict single outcomes" (page 453, [12]).

The solution to this problem must be found in giving an account of the required symmetry breaking process $\boldsymbol{\alpha}$. This will be provided by the proposed perturbation, a "flea on Schrödinger's cat".

3.4 About the classical limit

Before we move on to discuss in detail Landsman's proposal of a perturbation, there are a few important notes to make about the classical limit.

The reader may be wondering how we can take a constant, $\hbar$, to vary in value. After all, isn't it simply given as, approximately, 1.05457×10^{-34}Js? The idea is that the parameter in the limit, let us call it $\tilde{\hbar}$, does not actually represent the constant quoted above. When we consider the classical limit, we mean to take $\tilde{\hbar} \to 0$. The full answer to this confusion is twofold.

Firstly, we require the parameter $\tilde{\hbar}$ that we vary to be dimensionless. Thus we can combine the constant $\hbar$ with other constants and parameters in the theory to acquire a dimensionless parameter [10]. A special case of this is where the constant $\hbar$ is divided by the typical action for the physical system under consideration. Since the constant has dimensions of physical action, this creates a dimensionless parameter $\tilde{\hbar}$.

Secondly, we take the limit of this newly constructed dimensionless parameter by considering the appropriate limit of a physical parameter that is included in creating $\tilde{\hbar}$. In the case above, the classical limit of the system may correspond to the action becoming much larger than $\hbar$, i.e. taking $\tilde{\hbar} \to 0$. For a different physical system, the parameter N may be used to construct $\tilde{\hbar}$, so that the macroscopic limit, i.e. $N \to \infty$, may correspond to taking the classical limit, i.e. $\tilde{\hbar} \to 0$. More such examples are considered in [10].

Thus, the constant $\hbar$ can be re-scaled by the action or can be combined with other constants and variables in the theory, so that taking it to zero becomes the desired classical limit. Certainly, this idea has been prevalent for a long time, as Landsman points out in the introduction of [12] by quoting both Heisenberg from 1958 and Planck from 1906. However, it must be noted that there does not seem to be a general 'proof' for why we are allowed to assume that such a construction is possible for every quantum system. In particular, it is an open question if every quantum system allows for emergent classical behaviour by taking some mathematical limit. Or, conversely, if classical behaviour can always be constructed as the limit of some quantum mechanical theory. This is often referred to as Bohr's correspondence principle. In light of the next Section on Butterfield's principle, where we describe the limiting quantum theory to be the physically real theory, in place of some classical theory, these assumptions are noted but not resolved.

3.5 Butterfield's principle

Given that this paper discusses how classical behaviour is to emerge from the quantum description of a system by taking a limit, it is important to discuss some nuances of the concept of emergence. Many of these nuances in relation to taking limits were precisely examined by Butterfield in [5]. Some relevant notions will be applied to the case at hand.

Assuming that taking the particular limit $\hbar \to 0$ is the appropriate one to consider, what does it mean for classical behaviour to 'emerge'? Butterfield takes emergence to be behaviour that is both novel and robust. In the case at hand, as we move from the quantum description to a classical description of the system, a new behaviour arises. In the example discussed in the next chapter, we can characterise this emergent behaviour as 'localisation'. This paper will not attempt to describe the general case of going from quantum to classical theory, as this would involve much more examination. A masterful survey of the technicalities that are relevant for such a project was given by Landsman in [10].

Butterfield argues that emergent behaviour is exhibited in different ways as we take the appropriate limit. Although he takes the case of $N \to \infty$, the same ideas are applicable to the $\hbar \to 0$ case. The two different versions of emergence that Butterfield identifies are as follows.

At the limit When we set $\hbar = 0$, there is a case of strong emergence: we have a fully classical theory that explicitly describes the emergent behaviour we are considering.

Before the limit When the value of $\hbar$ becomes sufficiently small, the system exhibits a weaker version of the emergent behaviour.

Butterfield's principle, as named by Landsman in [11] and [12], is the claim that the first case, where $\hbar = 0$, is not physically real. Rather, the weaker version of emergence occurs in the physical world. When we consider the limit $N \to \infty$, where N indicates the number of particles in a system, we can see how this idea makes some intuitive sense. Namely, physically, this number does not become infinite: rather it becomes very large. Thus, claiming that the limit $N = \infty$ is physically real does not seem sensible. The emergent behaviour occurs already for sufficiently large N.

For these ideas to hold, both the theory at $\hbar = 0$ and the theory at sufficiently small $\hbar$ are required to be empirically accurate. However, typically the mathematics at small values of $\hbar$ may be more complicated and inconvenient to work with. Thus, setting $\hbar = 0$ may result in a more tractable theory, which works in practice in describing the emergent, classical behaviour. Yet this is only an idealisation of reality. We will see this explicitly in the example of the symmetric double well of the next chapter.

4 Perturbation by the Flea

Finally, we arrive at Landsman's proposed description of process $\boldsymbol{\alpha}$ of the measurement process, the symmetry breaking necessary for a classical theory to emerge. It comes as the introduction of a small perturbation to the potential of a system, one that is asymmetric by construction. This idea was first proposed in a slightly different context by Simon, in a paper titled 'Semiclassical Analysis of Low Lying Eigenvalues. IV. The Flea on the Elephant' [18]. In the context of the measurement problem, Landsman refers to the small perturbation as a "flea on Schrödinger's cat" [13].

Following Landsman and Reuvers in [13], in this chapter we will consider the simple example of the symmetric double well on the real line $\mathbb{R}$. As mentioned in the Introduction, we have chosen to focus on this model rather than other models discussed in [11]. This choice is motivated by the following properties that make the symmetric double well particularly illuminating.

1. The quantum mechanical ground state of the system does not converge to a single classical ground state, and thus the measurement problem arises.
2. The situation is simple enough to provide an intuition of the process, yet complete enough to offer a mathematical description.
3. It provides a model of a quantum mechanical measurement apparatus with a pointer that can 'land' on different outcomes. Thereby it is especially appropriate for the measurement problem. More will be said about this in the next chapter.

Assuming the perturbation by the flea indeed causes a collapse, there are remaining questions on the dynamics of the perturbation: How does it arise and how does it evolve? This paper will not address these technical questions and instead refers the reader to Landsman's work.

4.1 Symmetric double well

The potential of the symmetric double well on $\mathbb{R}$ is given by

$$V(x) = -\frac{\omega^2}{2}x^2 + \frac{\lambda}{4}x^4 + \frac{\omega^4}{4\lambda} = \frac{\lambda}{4}(x^2 - a^2)^2 \tag{18}$$

where we assume ω and λ to be positive, real numbers and so also $a = \frac{\omega}{\sqrt{\lambda}} > 0$. Note that we are working in position space, so x is a point in our manifold $M = \mathbb{R}$, and V is a real scalar function, $V : M \to \mathbb{R}$. This system has very different quantum and classical low energy states associated.

Classical Since the potential has two equal minima, there exist two classical ground states. Namely, the classical particle sits still at $+a$ or at $-a$. We refer to these positions as 'right' ($x > 0$) and 'left' ($x < 0$) respectively. Let ρ_0^+ be the state on the right, and ρ_0^- the one on the left i.e.

$$\rho_0^{\pm} = (p = 0, q = \pm a) \tag{19}$$

Quantum There exists one unique ground state, let us call it $\Psi_\hbar^0$, with associated energy E_0. This state is real, positive definite and has two peaks, one above $+a$ and one above $-a$. However, the quantum mechanical system reflects the degeneracy of the classical ground states in that it has exactly two "very low" energy eigenstates, the ground state, $\Psi_\hbar^0$, and the first excited state, let us denote it $\Psi_\hbar^1$. Both of these low energy eigenstates converge to the same mixed classical state in the limit $\hbar \to 0$. The excited state $\Psi_\hbar^1$ is also real, but anti-symmetric around $x = 0$. At first glance this seems to break the symmetry inherent in the potential. However, quantum mechanical states that differ with a phase are identified as the same physical state. Thus, the state with a peak above $+a$ and below $-a$ is the *same* physical state as that with a peak above $-a$ and below $+a$. This ensures no symmetry breaking occurs. The two low energy states are displayed in the figure below, which is reproduced from Landsman and Reuvers' paper [13].

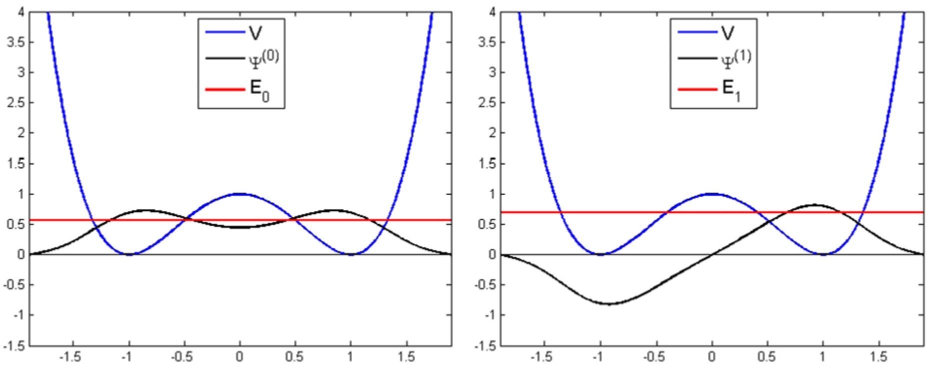

Figure 2: The ground state and first excited state of the quantum double well for $\hbar = 0.5$, from [13].

Let us consider the coherent states we associate with the degenerate classical ground states. These will be labelled by their classical (p, q) . Let $\Phi_\hbar^+$ and $\Phi_\hbar^-$ denote the coherent states associated with the particle being on the right or on the left, respectively, i.e.

$$\Phi_\hbar^\pm = \Phi_\hbar^{(p=0,q=\pm a)}. \tag{20}$$

Recall that these coherent states will converge to their respective classical states, ρ_0^+ and ρ_0^-, in the limit $\hbar \to 0$. Since they converge to classical states with a definite position, we refer to these states as localised coherent states. In this case, they are closely related to the quantum mechanical ground state and first excited state. Namely, we can create the following states

$$\Psi_\hbar^\pm = \frac{\Psi_\hbar^0 \pm \Psi_\hbar^1}{\sqrt{2}} \tag{21}$$

in which the quantum mechanical particle is confined to one of the two wells. As such, they approximate the coherent states defined above, i.e.

$$\Phi_\hbar^\pm \approx \Psi_\hbar^\pm. \tag{22}$$

The question is now: what process can get us from the quantum state $\Psi_\hbar^0$, the ground state of the system, to a state resembling either $\Phi_\hbar^+$ or $\Phi_\hbar^-$? That is, how does the quantum mechanical ground state choose a side? We seem to have a spontaneous breaking of a symmetry on the classical level, while no such breaking occurs on the quantum level.

The proposed answer will come as no surprise. The small "flea" perturbation δV of the potential V is introduced. By assumption, this flea must have the following three properties.

1 Given that the potential function is a real valued function on the manifold, we ask the added perturbation δV to be real valued as well. In addition, we ask that is has a fixed sign to ensure the right dynamics. Lastly, we ask it has some nice mathematical properties by requiring $\delta V \in C_c^\infty$. Thus, it is a smooth function and has connected support, not including the minima $x = a$ or $x = -a$.

In order to formulate the other two properties, we state some results from the WKB approximation to this wavefunction. There can be some confusion as to which factors are the correct ones. This paper follows Landsman and Reuvers in [13] who refer to [8]. The reader is directed to these for further details. The typical WKB-factor is given as

$$d_V = \int_{-a}^{a} dx \sqrt{V(x)}. \tag{23}$$

This factor determines the ground state splitting $\Delta \equiv E_1 - E_0$ in the classical limit $\hbar \to 0$, as $\Delta \sim \left(\hbar\omega / \sqrt{\frac{1}{2} e\pi}\right) \cdot e^{-d_V/\hbar}$. We can now state the second property we assume the perturbation to have.

2 We want the perturbation to be sufficiently large as we go towards the classical limit, taking $\hbar \to 0$. Thus, for sufficiently small values of $\hbar$, we ask the absolute value of the perturbation to be much larger than the fall off of the wavefunction, so as to ensure localisation takes place, i.e.

$$|\delta V| >> e^{-d_V/\hbar}. \tag{24}$$

The last assumption will ensure that our perturbation has the correct notion of asymmetry, which will allow the symmetry of the quantum system to be broken. We first introduce some further notation, extending the WKB-factor (23) to any two points y, z

$$d_V(y, z) = \left| \int_{y}^{z} dx \sqrt{V(x)} \right|. \tag{25}$$

Now, from a point to a set of points, A, we define the quantity d_V as the infimum of the set

$$d_V(y, A) = \inf\{d_V(y, z) : z \in A\}. \tag{26}$$

We then introduce two more symbols, namely d'_V and d''_V, as

$$d'_V = 2 \cdot \min\left\{ d_V\left(-a, \mathrm{supp}(\delta V)\right), d_V\left(a, \mathrm{supp}(\delta V)\right) \right\} \tag{27}$$

$$d''_V = 2 \cdot \max\left\{ d_V\left(-a, \mathrm{supp}(\delta V)\right), d_V\left(a, \mathrm{supp}(\delta V)\right) \right\} \tag{28}$$

3 The asymmetry property of the flea is then the assumption that either of the following two cases holds:

$$d'_V < d_V < d''_V; \tag{29}$$

$$d'_V < d''_V < d_V. \tag{30}$$

This excludes the possibility where $d'_V = d''_V$, where $d_V\left(-a, \mathrm{supp}(\delta V)\right) = d_V\left(a, \mathrm{supp}(\delta V)\right)$. So then the perturbation has the same 'relation' to $-a$ as to $+a$, thus this corresponds to the symmetric case. In addition it also requires the perturbation to be located sufficiently close to either of the minima, and not arbitrarily far on either the left external side or the right external side on the well. Since, if the perturbation were located too far to either side, the set value of d_V would be smaller than both d'_V and $d''V$.

The first condition (29) corresponds to the case where the perturbation is located on either of the external sides of the wells, thus $x < -a$ or $x > a$. The second condition (30) corresponds to the case where the perturbation is located in between the wells, thus in the region $-a < x < a$.

What effect does such a perturbation have on our system? Taking the potential of our double well to be $V + \delta V$, let us consider a δV that is **positive** and localised to the **right**. This can correspond to either condition (29) or (30), as long as $d_V\left(-a, \mathrm{supp}(\delta V)\right) > d_V\left(a, \mathrm{supp}(\delta V)\right)$. In terms of energy considerations, the left well will have a lower potential energy relative to the right well, so that the ground state of the system, let us call it $\Psi_\hbar^{0,\delta}$, is no longer a symmetric wave function of two peaks. Rather, the left peak (at $-a$) will be of larger magnitude than the right peak (at a), given by the ratio

$$\frac{\Psi_\hbar^{0,\delta}(a)}{\Psi_\hbar^{0,\delta}(-a)} \sim e^{-d_V/\hbar}. \tag{31}$$

For small $\hbar$, figure 3 from Landsman and Reuvers [13] shows that the wavefunction becomes almost completely localised. In particular, this means $\Psi_\hbar^{0,\delta} \approx \Phi_\hbar^-$. In the classical limit when $\hbar \to 0$, both states converge to the same classical state, ρ_0^-,

$$\lim_{\hbar \to 0} \Psi_\hbar^{0,\delta} = \lim_{\hbar \to 0} \Phi_\hbar^- = \rho_0^-. \tag{32}$$

The perturbation also effects the first excited state. However, Landsman and Reuvers find that the first excited states localises oppositely to the ground state, as can be seen in figure 3. Let $\Psi_\hbar^{1,\delta}$ be the first excited state. Then under the same small perturbation δV that is positive and on the right, and for small $\hbar$, we find that the state is localised almost fully to the right. Thus,

$$\lim_{\hbar \to 0} \Psi_\hbar^{1,\delta} = \lim_{\hbar \to 0} \Phi_\hbar^+ = \rho_0^+. \tag{33}$$

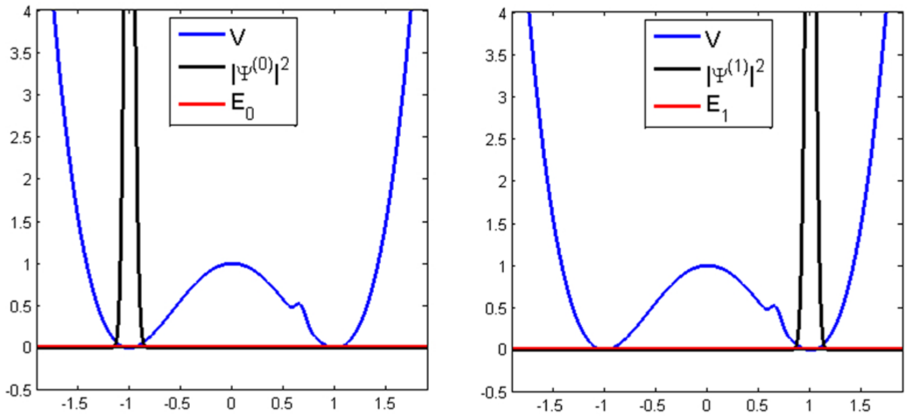

Figure 3: Localisation of the ground state and first excited state for $\hbar = 0.01$, from [13]

Of course, taking the flea δV to be localised on the right was an arbitrary choice. We can also consider a similar perturbation δV that is **positive** and localised on the **left**. Following the same reasoning as above, we then find that in the classical limit $\hbar \to 0$, the ground state localises to the right, while the first excited state localises to the left. For sufficiently small $\hbar$, we see that the quantum energy states approximate the localised coherent states:

$$\Psi_{\hbar}^{0,\delta} \approx \begin{cases} \Phi_{\hbar}^{-} & \text{for } \delta V \\ \Phi_{\hbar}^{+} & \text{for } \delta V \end{cases} \qquad\qquad \Psi_{\hbar}^{1,\delta} \approx \begin{cases} \Phi_{\hbar}^{+} & \text{for } \delta V \\ \Phi_{\hbar}^{-} & \text{for } \delta V \end{cases}$$

In summary, the full measurement process under perturbations δV and δV can be abstractly seen to happen as displayed in figure 4. As before, process $\boldsymbol{\alpha}$ represents the symmetry breaking due to the perturbation by the flea, from the energy state of the system to a state that approximates a localised coherent state. Then process $\boldsymbol{\beta}$ represents the convergence from the approximate coherent state to the classical state. The colours correspond to the perturbations described above.

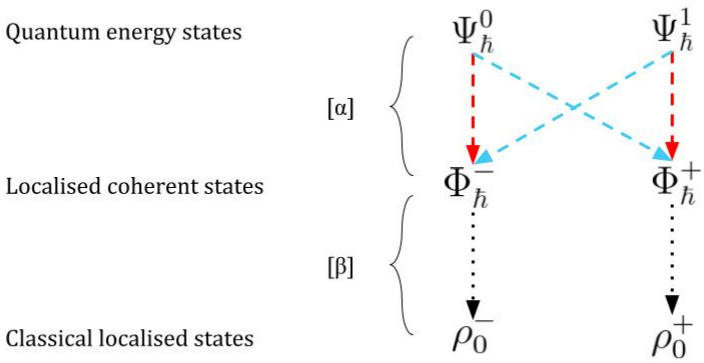

Figure 4: The steps of the measurement process

4.2 Symmetry breaking

Before we can consider how exactly the symmetry is broken in the above example, we need to recast our understanding of symmetries to fit the algebraic framework. Generally, symmetries are understood as transformations of states that leave the dynamics of the system invariant. Thus, a symmetry transformation maps $\psi \mapsto \psi'$ so as to leave the action invariant, i.e. $S(\psi) = S(\psi')$. In the algebraic formulation, we have put the algebra of observables in the centre. Thus, a symmetry is now understood not as a transformation on the states, but a transformation on the operators in the algebra $\mathfrak{A}$. A symmetry transformation is given by an automorphism of the algebra

$$\alpha : \mathfrak{A} \to \mathfrak{A} : \tag{34}$$

$$\alpha : A \mapsto \alpha(A). \tag{35}$$

that leaves the dynamics of the system invariant [17]. Any such symmetry α can be shown to be a *-automorphism on the algebra $\mathfrak{A}$, meaning it preserves the algebraic structure and the involution on the algebra [2, 15]. From the transformation on the operators in the algebra, a symmetry transformation on the states can be derived. Namely,

$$\alpha' : \omega \mapsto \omega' = \alpha'(\omega) \tag{36}$$

is given by $\omega \circ \alpha^{-1}$, such that

$$\omega'(A) = \alpha'(\omega)(A) = (\omega \circ \alpha^{-1})(A) = \omega(\alpha^{-1}(A)). \tag{37}$$

Given the GNS theorem, every state ω can be used to construct a representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. For a state ω and a second state ω' that is related to the first by a symmetry transformation, we can ask what the relation is between their respective GNS representations, $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H})$. Since the states describe the same system and are related by a symmetry, their representations will map to the same Hilbert space $\mathcal{H}$. For states that respect the symmetry of the theory, there exists a unitary mapping $U : \mathcal{H} \to \mathcal{H}$ between the two

$$U\pi(A) = \pi'(A)U = \pi(\alpha(A))U; \tag{38}$$

i.e. π and π' are unitarily equivalent representations of $\mathfrak{A}$. If this is the case, we say α is unitarily implementable in the representation π.

However, for a state ω that breaks the symmetry of the theory, it is a necessary condition[4] that the GNS representation π' generated by the transformed state $\omega' = \omega \circ \alpha^{-1}$ is not related to the GNS representation π' by a unitary mapping. Thus, these representations are unitarily inequivalent.

Let us now apply these ideas to the example at hand, the symmetric double well. For this system, before introduction of the perturbation, the dynamics and laws do not distinguish between states that differ only by the sign of the position variable, x. In the case of the symmetric double well, we see this reflected in the potential $V(x)$ in equation (18): it is invariant under $x \mapsto -x$, as it contains only even powers of x. Translating this to a symmetry

[4] Authors differ on whether this condition is sufficient, in addition to being necessary, for a symmetry to be broken (Introduction, [2]).

on the algebra, this corresponds to the automorphism α on $\mathfrak{A}$ that maps the position operator Q to $-Q$.

How is the symmetry broken in this system? In the higher level theory, the classical theory, a symmetry seems to be spontaneously broken. Namely, the theory of the symmetric double well respects a $\mathbb{Z}_2$ symmetry that the two degenerate ground states do not respect. Thus, the GNS representations generated by the states ρ^- and ρ^+ are required to be unitarily inequivalent. This means, since in a physical system only one of the ground states occurs, that the symmetry is spontaneously broken. When we look at the details of the lower level theory, namely the quantum theory, we see that this breaking of the symmetry is "foreshadowed in quantum mechanics for small yet positive $\hbar$" (page 375, [12]). By introducing the asymmetric flea perturbation of the potential, δV, we see that the theory is no longer invariant under the transformation $x \mapsto -x$. Thus, the symmetry is broken explicitly by the system (chapter 2, [20]). This explicit symmetry breaking on the lower level theory is perceived on the higher level theory as a spontaneous symmetry breaking.

5 Evaluating the Flea

Now that the main ideas of the perturbation by the flea have been introduced, this chapter will provide some assessment of the proposal in its current state. It will first discuss the model that is used by Landsman and Reuvers, the symmetric double well. Thereafter it will discuss the perturbation by the flea itself. In each case, favourable aspects and problematic aspects will be considered. The first have, mostly, been previously recognised and emphasised by Landsman in [12], while many of the latter have been raised by the recent paper of Van Heugten and Wolters in [22]. As in the previous sections of the paper, we will not consider the dynamics of the flea mechanism.

5.1 Model and measurement

In Landsman's approach, the symmetric double well is taken as the model for a measurement apparatus that is in a Cat state, in reference to Schrödinger's beautiful paper [21]. That is: a two-state quantum superposition where each outcome represents a different classical, macroscopic state. It can be thought of as the pointer of such an apparatus that can land on two distinct outcomes. Given such a state, the measurement problem arises, and thus the model is an important one to consider.

The virtue of this model is the finding that it is exceptionally sensitive to small perturbations to the potential. This was pointed out previously by Jona-Lasinio et al. in 1981 [9], and again by Simon in 1984 [18]. When discussing the model in relation to the measurement problem, Landsman and Reuvers also emphasised this property [13]. They find that, for even a tiny perturbation, when considering small enough values of $\hbar$, the system almost completely localises in one well. As we saw in equation (31), the ratio between the two peaks of the ground state goes as $e^{-d_V/\hbar}$. Thus, if we add a perturbation to the system and take $\hbar \to 0$, there is an exponential sensitivity. This exponential is not dependent on the size of the flea perturbation. As explained by Landsman, "precisely in the classical limit Cat states are destabilised even by the tiniest (asymmetric) perturbations and collapse to the 'right' states" (page 455, [12]).

Indeed, given that this collapse happens even before we reach the classical limit, this localisation behaviour emerges in the manner we would hope from Butterfield's principle (see Section 3.5). Localisation is almost complete: thus we see a weaker version of the emergent behaviour. Setting $\hbar = 0$ then means we can simplify our mathematics, but the physical behaviour occurs for small, yet non-zero, values of $\hbar$.

However, as Van Heugten and Wolters point out, this model is a simplification of the process of measurement. Namely, a description of measurement ought to start with a quantum mechanical target system and describe how the measurement apparatus interacts with this target. These systems then become entangled and can be described by a common Hamiltonian and potential. The symmetric double well is too simple to describe this entangled state. If it is meant to solely to describe the pointer of the apparatus, the flea acts on this system. Then it is only this system, that of the apparatus, that collapses due to the flea perturbation. But, if it is only the pointer that collapses, it does not seem the apparatus has actually measured the target quantum mechanical system. Thus, for a measurement, we ought to require that "the post-measurement state not only [assigns] a value to the pointer variable, but also to the system's measured observable in such a way that these values are correlated" (page 15, [22]). But how then does this collapse 'transfer' to the quantum mechanical target system? The answer to this question is, as of yet, lacking.

Alternatively, it seems more reasonable to assume the perturbation acts in some way on the complete system of apparatus and target, rather than the pointer of the apparatus alone. In this case, it is not straightforward to translate the flea perturbation to this entangled system. The proposal is still incomplete in this regard, and further exploration of how this could work could strengthen the ideas of the flea.

This can be seen as a general objection to the flea proposal in its current form. The symmetric double well that is taken is too simple to describe a realistic measurement situation, yet generalisations of the flea in other models are not unambiguous. The additional models considered by Landsman in [11], the quantum Ising chain and the quantum Curie-Weisz model, only partially address these concerns. Both of these models have a $\mathbb{Z}_2$-symmetry and thus represent physical systems in which the symmetry breaking is relatively simple. Van Heugten and Wolters further explore this aspect by considering a symmetric potential with n wells, rather than only two. In general, for a quantum mechanical system with a potential of n symmetric wells, there will be an associated ground state that has n peaks. The classical system will have n degenerate ground states, and correspondingly, the quantum system will have n low energy states. Thus, the structure is similar to the double well, and it is imaginable that the flea mechanism could cause an appropriate collapse. However, Van Heugten and Wolters find that adding a single flea perturbation does not cause such a collapse, since the wave function of the ground state does not localise in a single well (page 19, [22]). This suggests that the flea perturbation, although very effective in the symmetric double well and similar models, may be more ineffective in alternative models.

5.2 The "flea"

Concerning the flea itself, it may not be unreasonable to expect an occurrence of a small perturbation in a physical system. Landsman argues this perturbation arises externally, from the environment. After all, physical systems do interact with their environment in ways that are nearly impossible to capture. Is it reasonable to assume such a perturbation does *not* take place? Landsman even refers to the presence of asymmetric flea perturbations as "practically unavoidable" (page 378, [12]). In addition, the flea causes a perturbation of exactly the right kind, since it does not affect the quantum state in the quantum regime, i.e. it is irrelevant for relatively large values of $\hbar$. Rather, its importance arises in the classical regime, when $\hbar$ is sufficiently small.

Nonetheless, questions regarding the nature of the flea perturbation remain. These are especially pertinent when we consider that the flea perturbation ought not to 'mess up' the Born probabilities associated with quantum systems. Landsman recognises this need several times, writing that "among all remaining challenges, deriving the Born rule stands out in particular" (page 450, [12]). Even so, he leaves it to further research into the possible dynamics of the flea to address this question. Besides these dynamic concerns, the static case already informs the discussion.

In the model we have considered, the Born probabilities are equal (50/50) for the two associated classical states. Thus, the expectation of finding the particle (approximately) in the left classical state, ρ_0^-, was equal to finding it in the right classical state, ρ_0^+. Let us now consider a slightly different example, where the weights for the outcomes are slightly different. Take a simple distribution, such as (75/25), meaning that we are three times as likely to find it in the left classical state than in the right classical state. In other words, if we repeat the experiment (infinitely) many times on identically prepared systems, we ought to recover this (75/25) distribution. Given that the prepared system excludes the (uncontrollable) flea perturbation, recovering such an outcome through the flea mechanism is not unreasonable. In fact, according to Landsman, "the flea perturbation would naturally be different at each different run of an experiment under otherwise identical initial conditions" (page 456, [12]).

How would the flea perturbation have to differ at each run of the same experiment? As we saw in the example, the only factor that influences the direction of collapse was the location of the flea perturbation. If we consider it to be a 'random' perturbation from the environment, we would have no reason to expect the flea to have a higher chance of arising on the right side (so that the ground state collapses to the left) than on the left side. If the flea is assumed to be external to the system, it ought not to adhere to the Born probabilities we associate to that system. Thus, once the flea perturbation has appeared, it seems hard to maintain the Born rule, as concluded also by Van Heugten en Wolters.

> Without any relation between the flea and the initial system state there is no reason why the Born probabilities (apart from the special 50/50 case) should be replicated by the flea model (page 24, [22]).

For an independent flea, it seems that "nothing short of a conspiracy theory would be needed to replicate the Born rule" (page 14, [22]). Does this mean we ought to require the flea perturbation to be dependent on the system, in some way? Such a dependence does not seem straightforward, given the idea that the flea is part of the environment.

Alternatively, perhaps we should not assume the perturbation to be random. Rather, it could be deterministic, where we could imagine it as a missing element in quantum theory. Landsman gives a hint in that direction, by writing that "the location of the flea plays a similar role to the position variable in Bohmian mechanics, i.e. it is essentially a hidden variable" (page 397, [13]). This could open to door for determinism to re-enter quantum mechanics, although the plentiful no-go theorems are essential in this area.

In conclusion, the nature of the flea perturbation remains unclear. Firstly, although a perturbation in principle does not seem unrealistic, the specific properties of the perturbation can seem implausible. Indeed, it may have to have very particular properties if it can account for a mechanism that preserves the Born rule. Secondly, there is such a multitude of unresolved questions asking to be answered that it is difficult to have a precise discussion.

6 Conclusion

This paper has discussed the main mathematical and conceptual ideas related to the proposed solution of the measurement problem known as the 'flea mechanism', formulated by Landsman and Reuvers in [13]. Chapter 2 and 3 gave the relevant background needed to put the proposal in context. Chapter 4 discussed the workings of the mechanism for the main model considered by the Landsman and Reuvers, the symmetric double well. Both favourable and unfavourable aspects to the proposal were then discussed in chapter 5, informed by the work of Van Heugten en Wolters in [22]. To conclude this essay, some further reflections from the author are presented.

Undoubtedly, what the discussion in chapter 5 has shown is that there is more work to be done. Landsman recognizes this throughout his work, no where claiming that the proposal is complete. Philosophically, the nature of the flea perturbation is unclear, and the question of why it is reasonable to assume the occurrence of such a perturbation remains unanswered. Mathematically, the mechanism for how the flea ought to respect the Born rule is not apparent. Additionally, only limited models have been considered. Thus, it would be interesting to explore how the idea of symmetry breaking, referred to here as process α, could apply to further models. In particular, which symmetries do we expect to see in quantum systems, that such a 'flea mechanism' ought to break, in order to obtain classical outcomes? This may shed light on the flexibility of the proposed mechanism, and help clarify its essential conceptual elements.

Although the flea mechanism is currently incomplete, the work done by Landsman and Reuvers has already been illuminating. If we want to imagine a solution to the measurement problem that avoids postulating the existence of many worlds and mysterious preferred observables, as mentioned in chapter 3, the mechanism of the flea does not seem outrageous. Certainly, for those who are inclined to a collapse, the flea perturbation provides a very reasonable mathematical construction for how such a collapse could occur. In addition, it is not a great leap of faith to posit that the collapse ought to come from some small, external perturbation, which disrupts the symmetry of a quantum system and forces a single outcome. Thus, some of the problems and challenges that the flea mechanism faces could be seen as indicative of the challenges of translating the, admittedly vague, idea of collapse into a precise mathematical mechanism. Landsman and Reuver's proposal not only offers the start of a new way forward, it is also essential for understanding the work that lies ahead in giving a coherent account of a collapse of the quantum theoretical wave function.

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This paper is dedicated to Yolanda Murillo.

References

[1] V. I. Arnold. *Mathematical methods of classical mechanics; translated by K. Vogtmann and A. Weinstein.* Graduate texts in mathematics ; 60. Springer, New York ; London, 2nd ed. edition, 1989.

[2] D. J. Baker and H. Halvorson. How is spontaneous symmetry breaking possible? *ArXiv e-prints*, Mar. 2011.

[3] J. S. Bell. On the problem of hidden variables in quantum mechanics. *Reviews of Modern Physics*, 38:447–452, Jul 1966.

[4] J. S. Bell. Six possible worlds of quantum mechanics. *Foundations of Physics*, 22(10):1201–1215, 1992.

[5] J. Butterfield. Less is different: Emergence and reduction reconciled. *Foundations of Physics*, 41(6):1065–1135, June 2011.

[6] J. Butterfield. Hand out on the density matrix, measurement, improper mixtures, and decoherence, cambridge. Lectures on the Philosophy of Qauntum Field Theory, Michaelmas 2017.

[7] K. Fredenhagen and K. Rejzner. Perturbative algebraic quantum field theory. *ArXiv e-prints*, Aug. 2012.

[8] A. Garg. Tunnel splittings for one-dimensional potential wells revisited. *American Journal of Physics*, 68(5):430–437, 2000.

[9] G. Jona-Lasinio, F. Martinelli, and E. Scoppola. New approach to the semiclassical limit of quantum mechanics. *Communications in Mathematical Physics*, 80(2):223–254, Jun 1981.

[10] K. Landsman. Between classical and quantum. *ArXiv e-prints*, June 2005.

[11] K. Landsman. Spontaneous symmetry breaking in quantum systems: Emergence or reduction? *Studies in History and Philosophy of Modern Physics*, 44(4):379–394, November 2013.

[12] K. Landsman. *Foundations of quantum theory: from classical concepts to operator algebras.* Fundamental theories of physics; v. 188. Springer International Publishing, 2017.

[13] K. Landsman and R. Reuvers. A flea on schrödinger's cat. *Foundations of Physics*, 43(3):373–407, March 2013.

[14] K. Rejzner. *Perturbative algebraic quantum field theory: an introduction for mathematicians.* Mathematical physics studies. Springer International Publishing, 2016.

[15] J. Roberts and G. Roepstorff. Some basic concepts of algebraic quantum theory. *Communications in Mathematical Physics*, 11(4):321–338, December 1969.

[16] L. Ruetsche. *Interpreting Quantum Theories*, chapter Representation Without Taxation: An Unstrenuous Tour of Algebraic Notions. Oxford University Press, Oxford, 2011.

[17] L. Ruetsche. *Interpreting Quantum Theories*, chapter Interlude: Symmetry Breaking in Qsm. Oxford University Press, Oxford, 2011.

[18] B. Simon. Semiclassical analysis of low lying eigenvalues. iv. the flea on the elephant. *Journal of Functional Analysis*, 63(1):123 – 136, 1985.

[19] D. Skinner. Lecture notes on advanced quantum field theory for part iii mathematics, cambridge. www.damtp.cam.ac.uk/user/dbs26/AQFT.html, Lent 2018.

[20] C. E. Thomas. Lecture notes on the standard model for part iii mathematics, cambridge. www.damtp.cam.ac.uk/user/cet34/teaching/SM/, Lent 2018.

[21] J. D. Trimmer. The present situation in quantum mechanics: A translation of schrödinger's "cat paradox" paper. *Proceedings of the American Philosophical Society*, 124(5):323–338, October 1980.

[22] J. van Heugten and S. Wolters. Obituary for a Flea. *ArXiv e-prints.*, Oct. 2016. Page numbers refer to a revised version available from the authors.

[23] J. Von Neumann. *Mathematical foundations of quantum mechanics; translated from the German edition by Robert T. Beyer.* Investigations in physics; no. 2. Princeton University Press, Princeton, 1955.