**The role of syntactic representations in set theory**

*Abstract.* In this paper, we explore the role of syntactic representations in set theory. We highlight a common inferential scheme in set theory, which we call the Syntactic Representation Inferential Scheme, in which the set theorist infers information about a concept based on the way that concept can be represented syntactically. However, the actual syntactic representation is only indicated, not explicitly provided. We consider this phenomenon in relation to the Derivation Indicator position that asserts that the ordinary proofs given in mathematical discourse indicate syntactic derivations in a formal logical system. In particular, we note that several of the arguments against the Derivation Indicator position would seem to imply that set theorists could gain no benefits from the syntactic representations of concepts indicated by their definitions, yet set theorists clearly do.

Key words: Derivation; Proof; Mathematical practice; Set theory

**1. On the relationship between derivations and proof**

 The purpose of this paper is to contribute to the growing discussion on the relationship between ordinary mathematical argumentation, consisting of definitions, theorems, and proofs, and formal mathematical presentation in an explicit logical system, consisting of logical formulae and derivations. We follow Rav (1999) in distinguishing between *derivations*, which are formal syntactic objects in logical systems whose correctness is mechanically checkable, and *proofs*, which are the justifications that appear in journal articles and textbooks that mathematicians choose to label proofs. We add a similar distinction. We may characterize a concept with a *syntactic representation*, by which we mean a formal logical formula (*x*) in an explicit logical system that is true exactly when *x* is a member of a concept, or a *definition*, by which we mean the criterion mathematicians designate for concept membership and choose to call a definition in journal articles and textbooks.

 There are three broad positions regarding the relationship between derivations and proofs. The *derivation indicator* position affirms some version of what Rav (1999) called Hilbert’s Thesis: “the hypothesis that every conceptual proof can be converted into a formal derivation in a suitable formal system” (Rav, 1999, p. 12), along with the claim that Hilbert’s Thesis influences mathematicians’ practice in deciding which proofs to accept as valid. Hyman Bass, a mathematician, provided a compact synopsis of the derivation indicator position: “A proof is what convinces a reasonable person that a formal proof exists” (cited in Burgess, 2015, p. 91). Proponents of the derivation indicator position contend that a justification only qualifies as a proof if there exists a derivation based on that justification that can be mechanically verified (Azzouni, 2004; Burgess, 2015; Feferman, 2012; MacLane, 1986). Note that if a proof alludes to a mathematical concept, the derivation that it indicates necessarily includes a syntactic representation of that concept, for in formal mathematics, saying an object is a member of a concept is merely an abbreviation for saying that the object satisfies the logical formula indicated by the concept’s definition. Scholars have suggested that the derivation indicator position can explain some interesting descriptive aspects of mathematical practice, such as why mathematicians usually agree on whether a proof is correct and why proven claims stay proven[[1]](#footnote-1) (see Azzouni, 2004)[[2]](#footnote-2).

 The second position is what we call the *inherently semantic* position. Proponents of the inherently semantic position typically present arguments that they contend are undeniably proofs but resist translation into derivations in a formal system, on the grounds that the legitimacy of the inferences within a proof are intrinsically tied to the semantic models in which they are couched (e.g., De Toffolli & Giardino, 2014, 2016; Larvor, 2012, in press; Rav, 1999, 2007). The issue of whether these proofs are indeed inherently semantic continues to be the subject of philosophical debate and we will not adjudicate this matter here (for instance, see Azzouni, 2016, and Larvor, 2012). Rather, we observe that even if we grant that the inherently semantic proponents have been successful, what they have accomplished is demonstrating that *some* proofs resist formalization. The inherently semantic proofs that have been proposed tend to involve constructions that can be visualized and permit some kinds of continuous transformations, such as proofs in knot theory, Euclidean geometry, and low dimension topology. Further, these inherently semantic proofs usually occur in univalent structures (i.e., structures that we psychologically conceive of as being categorical) such as two- or three-dimensional Euclidean space, and not in multivalent structures that we readily conceive of us permitting non-isomorphic models. Even if many proofs within univalent structures with visual models are inherently semantic, it may still be the case that most proofs within multivalent structures or more abstract domains that resist visualization are not, allowing the derivation indicator position to be an accurate account of the practice of a substantial percentage of mathematicians. The next position rejects this possibility.

 The third position is what we will call the *no epistemic benefits* position. We will elaborate on the arguments for this position shortly, but we can succinctly state this position as follows: *Even if* we accept Hilbert’s Thesis that every proof has an accompanying derivation (or multiple derivations, as the case may be), descriptive facts about mathematical practice and the inaccessibility of these derivations ensure that these derivations can do no epistemic work for the mathematician (Pelc, 2009; Rav, 1999; Tanswell, 2015).

 In this paper, we examine the practice of one mathematical culture, set theory, with respect to members of that culture’s treatment of the relationship between ordinary argumentation and formal mathematics. We do not focus so much on the relationship between derivations and proofs but rather on the relationship between definitions and syntactic representations, a topic that has received limited attention in the literature on mathematical practice. As we have argued above, the derivation indicator position implies the analogous position that definitions indicate syntactic representations. Further, in the next section, we argue that the reasons that some scholars give for why derivations can provide no epistemic benefits to mathematicians would seem to imply that syntactic representations of concepts should provide no epistemic benefits to set theorists as well. We study set theorists because they draw strong inferences about concepts based on the existence of syntactic representations for these concepts; however, these syntactic representations are not produced, only indicated. We therefore use our study of set theorists to achieve three objectives:

1. We will provide an existence proof that one culture of mathematicians can reap epistemic benefits from formal logical objects that they never construct;
2. We will show that, at least in set theory, the epistemic benefits of knowing that syntactic representations exist go beyond the purported epistemic benefits of formalism that have hitherto been discussed in the literature;
3. We will describe the inferential schemes that permit set theorists to assume the existence of a syntactic representation of a concept, even though syntactic representations are only indicated, and would be difficult to produce.

**2. Epistemic limitations of derivations**

 We consider three arguments from the literature for why the derivation indicated by a proof, should it exist, will not yield epistemic benefits for the mathematician. The first argument is based on the simple observation that mathematicians rarely produce the derivations that purportedly are indicated by their proofs. This creates a psychological mystery of how mathematicians can imagine what these derivations look like in sufficient detail to mechanically verify them. Rav (2007) made this case in the passage below:

“But if mathematicians’ practice—so Azzouni—‘isn’t to actually execute derivations but only to indicate [them], to themselves or to others in their profession’ (p. 95)[[3]](#footnote-3), *how is one to understand that a formal derivation can be indicated without going through all the details* (and keeping in mind that ‘formal derivation’ is a highly technical concept of recent vintage)?” (Rav, 2007, p. 310, italics were our emphasis).

Later, Rav (2007) cited Kreisel (1969), who questioned, “what use is it to the mathematician to be told that his arguments *can* be formalized and thereby justified (e.g., in the Hilbert sense) if (i) he does not formalize them yet (ii) is convinced of his conclusion” (Kreisel, 1969, p. 308, italics in the original). These arguments raise the critical question of what epistemic benefits derivations can have if they are not produced and, indeed, we may even lack rational grounds to know they exist.

 The second argument we consider is Tanswell’s (2015) observation that proofs underdetermine derivations. That is, to different mathematicians, the same proof can indicate very different derivations, in part because different mathematicians may formalize their proofs in different logical systems. Tanswell thus asserted:

“The idea is not that proofs are too resistant to formalization but instead they are not resistant enough. There are multiple, equally legitimate, formal proofs corresponding to any given informal proof and *it is this multitude which throws doubt on there being any deep philosophical significance to the correspondence at all*” (Tanswell, 2015, p. 301; italics are our emphasis).

Note that Tanswell is not only questioning whether the Derivation Indicator position can account for descriptive facts about mathematical practice (such as explaining why mathematicians agree on whether an argument is a proof—e.g., Azzouni, 2004), but whether they had any philosophical significance at all.

 The third argument that we consider is from Pelc (2009), who due to the finiteness of the universe, questioned whether some derivations are realizable *in theory*, even if we had the proverbial idealized mathematician who had unlimited skill, time, and motivation to formalize a proof. To simplify Pelc’s argument slightly, he conjectures that the derivations associated with proofs of some interesting theorems would contain more characters than there are particles in the physical universe—or at least the derivation indicator proponents have given us no reason to believe that this is not the case. Hence, these derivations could not be represented in our physical universe even in principle. From this argument, Pelc not only concluded that it is impossible to gain confidence from derivations, but further added:

“We make a much stronger claim: in the case of many theorems, *derivations cannot contribute any epistemic value at all*” (p. 88; italics are our emphasis).

 We will illustrate that in set theory, these three arguments apply to syntactic representations of concepts as well as derivations. That is, we will show that set theorists often indicate syntactic representations of concepts and draw inferences from these syntactic representations, even though these syntactic representations are seldom provided, there are multiple different syntactic representations that can be produced, and some syntactic representations may be very long.

**3. Set theory**

*3. 1. Set theorists’ writing*

To document set theorists’ inferential schemes, we will refer to the two standard textbooks in graduate set theory, Jech (2000) and Kunen (2011), to see how they justify the assertions that they make. The rationale for focusing on this pair of textbooks is twofold. First, as Jech (2000) and Kunen’s (2011) textbooks are widely used, it is reasonable to assume that the justifications used in these textbooks are acceptable by the standards of the set theory community. Second, textbooks are the places in which authors are likely to be most explicit about how they are justifying their claims. Our point will be that set theorists merely indicate the existence of syntactic representations without providing them. If this is the case with standard textbooks, it is unlikely that the syntactic representations are provided in other sources.

Consider the following passage from Jech (2000):

**Definition 8.1.** Let κ be a regular uncountable cardinal. A set C ⊂ κ is a *closed unbounded* subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.

**Lemma 8.2**. If C and D are closed unbounded, then C ∩ D is closed unbounded.

***Proof.*** It is immediate that C∩D is closed. To show that C∩D is unbounded, let α < κ. Since C is unbounded, there exists an α1 > α with α1 ∈ C. Similarly there exists an α2 > α1 with α2 ∈ D. In this fashion, we construct an increasing sequence:

(8.1) α < α1 < α2 < . . . < α*n* < . . .

such that α1, α3, α5, . . . ∈ C, α2, α4, α6, . . . ∈ D. If we let β be the limit of the sequence (8.1), then β < κ, and β ∈ C and β ∈ D.

 (Jech, 2000, p. 91).

 The presentation above is typical of set theory texts in two respects. First, the new concept in question, the closed unbounded sets (or the club sets, as they are commonly called), is described using a definition, not a syntactic representation. Only one logical symbol is used (⊂) and even this symbol is not primitive in the language of set theory. The rest of the text is in ordinary English and involves previously defined set theory concepts. While a syntactic representation for club sets could perhaps be provided by unpacking the meaning of the terms “regular”, “uncountable”, “cardinal”, “unbounded”, “limit points”, and so on (which would then involve unpacking the terms used to define those terms like ordinal, cofinal, bijection, and so on, and then the terms used to define ordinal, etc.), this is rarely done and would require substantial effort. That club sets can be defined by a syntactic representation satisfying certain structural properties will be of importance later in Jech’s volume, but no explicit syntactic representation characterizing club sets is ever provided.

 Our second observation is the proof that Jech presented is clearly not a derivation. The proof is typical of ordinary proofs that we see in the mathematics literature in many respects. There is the explicit refusal to justify “obvious” facts (“It is immediate that C∩D is closed”), the justifications of other statements are left implicit (“then β < κ” follows from the fact that the cofinality of β is countable and the cofinality of κ is uncountable; “β ∈ C and β ∈ D” follows from the assumption that C and D are closed), and the reader is asked to imagine constructing a particular sequence with “…” indicating that the construction goes on ad infinitum.

 We use this example to make a more general point. In mathematical domains related to logic such as set theory, derivations and syntactic representations of concepts are often the objects of mathematical investigation. However, this does not imply that mathematicians investigating these objects define their concepts with syntactic representations or justify their claims about these concepts with derivations[[4]](#footnote-4). How these syntactic representations and derivations are indicated by normal mathematical discourse is the subject of this paper.

Finally, although this topic is not essential to the arguments in the remainder of this paper, we consider the question of whether a set theorist, or another agent such as a computer, can produce a syntactic representation of club sets via the mechanical recursive substitution of the terms in the definition that Jech provides. We contend the answer is no: formulating a syntactic representation of club sets would require judgment and contextual interpretation. To illustrate why this is so, note that Jech’s club set definition uses the word *contains*. (i.e., “C contains all its limit points less than κ”). The meaning of *contains* is not defined in Jech’s textbook and, in fact, Jech uses the word in two different senses. In the first sense, *contains* denotes set membership. For instance, “Since **Q** is dense in **R**, every nonempty open interval of **R** contains a rational number” (p. 39) is specifying that some rational number *q* is an element of the open interval, not that the sets contained in *q* (i.e., the ordered pairs of integers in the equivalence class comprising *q*) are in the open interval. In the second sense, *contains* denotes a subset relation. For instance, “Every uncountable closed set contains a perfect set” (p. 40) means that some perfect set is a subset of the uncountable closed set, not that the uncountable closed set contains a perfect set as an element. In the definition of club set, a set theorist would recognize that *contains* is clearly being used in the first sense (i.e., it is the limit points of C less than κ that are in C, not the elements of the limit points). However, one can also form a different well-formed definition using *contains* in the second sense (albeit an uninteresting definition since this would imply the only club subset of κ is κ itself). The argument in this paragraph is not to suggest that Jech’s textbook is deficient; in each case above, any knowledgeable set theorist would be able to interpret the intended meaning of *contains* from the context in which it appears. In particular, it would be far more feasible for a set theorist to provide a syntactic representation for a concept such as club sets than it would be for a mathematician to formalize a proof in their area of research. Rather we only wish to make the broader point that even in domains that attend to logical precision such as set theory, definitions often use words from natural language that are not explicitly defined and whose meaning is dependent upon context.

*3. 2. What is* **?

 One of the key classes of objects studied by set theorists is the ordinals. The ordinals are an extension of the natural numbers. The Von Neumann ordinal representation is the canonical one used by most set theorists today, where the ordinals are defined as follows:

0 = ø

1 = {0} = {ø}

2 = {0, 1} = {ø, {ø}}

3 = {0, 1, 2} = {ø, {ø}, {ø, {ø}}}

…

*n*+1 = {0, 1, 2, …, *n*}

…

 = {0, 1, 2, 3, …}

 + 1 = {0, 1, 2, 3, …, }

 + 2 = {0, 1, 2, 3, …,  + 1}

…

Before the first infinite ordinal, the only ordinals are the natural numbers studied by schoolchildren. Perhaps the key factor that distinguishes set theory from arithmetic is the focus on transfinite ordinals—the ordinals that are greater than or equal to . But what, exactly, is ? Informally, we think of “ as the set of natural numbers”. In Jech’s (2000) textbook, he defined  by: “we denote the least non-zero limit ordinal ” (p. 20)[[5]](#footnote-5). However, translating this to a sentence expressible in the language of first-order ZFC is not trivial. The difficulty does not involve *including* all the natural numbers—for that we can simply say that 0 is an element of  and the successor of each element in  is also in . The challenge lies in ensuring that we *exclude* any set that is not a natural number. (For instance, Peano arithmetic lacks the expressive power to do this, which accounts for the existence of non-standard models of Peano arithmetic). Here is one syntactic representation of  in the first-order language of set theory assuming the ZFC axioms:

ϕ*(x)=(∃a((a∈x)⋀(¬(∃b(b∈a)))))⋀((¬(∃c((c∈x)⋀(¬(∃d((d∈x)⋀((c∈d)⋀((¬(∃e((e∈c)⋀(¬(e∈d)))))⋀(¬(∃f((f∈d)⋀((¬(f=c))⋀(¬(f∈c))))))))))))))⋀(¬(∃g((g∈x)⋀((∃h(h∈g))⋀(¬(∃i((i∈x)⋀((i∈g)⋀((¬(∃j((j∈g)⋀(¬(j∈i))))⋀(¬(∃k((k∈i)⋀((¬(k∈g)))))))))))))))))*

The only *x* such that ϕ*(x)* is true is [[6]](#footnote-6).

Here one may ask whether this syntactic representation provides any epistemic benefits over the more comprehensible description that  is the least non-zero limit ordinal. The answer is yes. Note that all 11 variables introduced in this syntactic representation are *bounded*, meaning that they are declared to be elements of another set (either *x* or one of the previously introduced bounded variables). A property that can be syntactically represented in this way is called a  property. These properties are said to be *absolute* between transitive models of set theory, meaning the set that stands for  in one model will play the same role in any transitive sub-model of that model. This is not true for other infinite cardinals. A clear epistemic benefit is provided by knowing that a  syntactic representation exists, although it differs from the epistemic benefits often attributed to the existence of derivations, such as providing certainty and inducing consensus.

 This is an instance of a general inferential scheme that is used in set theory that we label the Syntactic Representation Inferential Scheme (SRIS). The inferential scheme can be abstracted as follows:

* *Premise 1:* Any concept that can be characterized by a syntactic representation with syntactic property P has property Q.
* *Premise 2:* A particular concept C can be characterized by a syntactic representation with syntactic property P.
* *Conclusion:* Therefore, concept C has property Q.

In the case above, P is having the  property, Q is being absolute between transitive models, and C is being . However, the premises can vary. For instance, if a concept has a syntactic representation that is a $Π\_{1}^{1}$, then a weakly compact cardinal cannot be the least ordinal having this property. In the case of  being , we justified that  being  representation by actually providing the representation[[7]](#footnote-7). However, in most instances of SRIS, the syntactic representation in Premise 2 is not provided, but is merely *indicated*. For instance, consider the following theorem and proof, which is typical:

**Corollary 17.19.** *Every weakly compact cardinal κ is a Mahlo cardinal*[[8]](#footnote-8).

*Proof*. Let C ⊂ κ be a closed unbounded set. Since κ is inaccessible, (Vκ, ∈, C) satisfies the following sentence:

¬∃F (F is a function from some λ < κ cofinally into κ) and C is unbounded in κ.

By $Π\_{1}^{1}$-indescribability, there exists a regular α < κ such that C ∩ α is unbounded in α; hence α ∈ C. Thus κ is Mahlo.

 (from Jech, 2000, p. 297-298).

This proof contains an instance of SRIS:

* *Premise 1:* If a concept can be characterized by a syntactic representation that is a $Π\_{1}^{1}$-property, then a weakly compact cardinal  cannot be the least cardinal with that property.
* *Premise 2:* For a given club set C[[9]](#footnote-9), the property that “ is a regular cardinal and C is unbounded in ” is a $Π\_{1}^{1}$-property that is true of κ.
* *Conclusion:* Therefore, there must be an α < κ such that  is a regular cardinal and C is unbounded in .

Note that in this case, a syntactic representation for the property “κ is a regular cardinal and C is unbounded in κ” is not provided. It is only *indicated* by the phrase “¬∃F (F is a function from some λ < κ cofinally into κ) and C is unbounded in κ”. Our understanding is that the reader is not expected to produce the syntactic representation of this property and then verify that it is a $Π\_{1}^{1}$-property, but rather to recognize that she could in principle produce such a representation. The production of the syntactic representation may not be mechanical, but verifying that the syntactic representation explicitly expressed a $Π\_{1}^{1}$-property could be accomplished. In the next section, we discuss inferential schemes that set theorists use to determine if the informal criteria listed in set theory texts actually indicate the syntactic representation that it claims to indicate.

**4. Bridging gaps in mathematical proofs**

 In Jech’s proof of Corollary 17.19, his proof contained a gap that is common in the SRIS. He gave an indication that a syntactic representation of a certain type is possible, but he did not provide the representation. We will focus on set theorists’ rationales for how those gaps can be bridged. We begin by emphasizing that having a gap like this in a proof is not necessarily problematic. Indeed, Fallis (2003) claimed that gaps in ordinary proofs are commonplace and sometimes desirable. Fallis distinguished between three types of gaps that can appear in a proof. The first type of gap occurs when the proposition that the mathematician is trying to prove does not follow from the premises in the manner that the mathematician has in mind. In other words, the mathematician made a mistake and failed to produce a correct proof. The second type of gap is an enthymematic gap. These gaps occur when a mathematician leaves out some of the details of the proof, which Fallis said is often done with the aim of saving space. The third type of gap that Fallis alluded to is an untraversed gap. An untraversed gap occurs when the mathematician leaves a gap in a proof that she has not verified herself. In the case that no mathematician has verified the gap (i.e., not the author, referee, or any readers), this gap is called a universally untraversed gap.

 For at least some claims that syntactic representations of a certain type exist, like the syntactic representation that we will discuss in section 5.4, we doubt that any mathematician has written the representation out in full[[10]](#footnote-10). Consequently, some enthymematic gaps using the SRIS are universally untraversed gaps. Andersen’s (in press) recent analysis of how referees analyze untraversed gaps can offer a possible account for how set theorists bridge the gap between definition and syntactic representation. According to Andersen, one method that referees use to address a gap in a proof is a line-by-line check, where the referee fills in every detail of this part of the proof herself or, in the case where it is not clear how to do so, asks the author for more information. This would be tantamount to writing out the syntactic representation in question or showing that the syntactic representation can be formed by property-preserving composition of other syntactic representations, which as we noted above, is generally not done. Andersen also provided a second way that mathematicians check proofs:

“She checks whether each subresult in a proof seems reasonable in light of what she already knows and, at least, for most of the subresults, whether it seems reasonable that this type of result can be proved in this type of way, with these types of tools. If so, she will usually not go on to check the subproof line-by-line” (Andersen, in press).

 We suggest that set theorists *justify* that syntactic representations exist in a similar way. The following analogy may be useful. Chess analysts and chess puzzle solvers typically study chess positions with the goal of demonstrating that the position is a winning position for the player with the White pieces, where a *winning position* for White is one where a theoretical *winning strategy* or algorithm exists that White can use that will ensure that White can checkmate the Black king, regardless of the moves that Black plays. If no such winning strategy existed, then (because chess is a closed game played in finite time) Black would necessarily have a strategy or algorithm that she could apply to achieve at least a draw. One way that White could demonstrate the existence of a winning strategy is to actually provide the strategy in the form of a tablebase or algorithm showing White’s responses to any legal black move, where all chains of moves culminate with White checkmating the Black king. These types of analyses are sometimes provided, such as when White has a forced checkmate of the Black king in a small number of moves. What is far more common, however, is for the analyst to show how to *reduce the chess position* to a subsequent position that all experts would agree is clearly winning. What does it mean to be clearly winning? We propose that a position is clearly winning when: (a) experts would all agree that White has sufficient resources to force a win (usually in the form of having an extra piece) and (b) applying these resources successfully would be a “matter of technique” that expert chess players would possess. Note that just because a problem solver solved a chess puzzle by reducing the position to one that is clearly winning in no way implies that the solver actually possessed the technique to convert the win. Hence, Emmanuel Lasker’s famous adage that the hardest thing to do in chess is win a won game! We would still give the novice credit for solving a puzzle if she showed that White could win a rook, even if she would probably go on to lose the clearly winning position that she achieved against a skilled player. We also note that chess players and analyses generally do not think in terms of tablebases when solving a puzzle.

The claim that we advance is that set theorists use similar reasoning to see whether their proofs indicate derivations. Set theorists will acknowledge that a colleague has succeeded in showing a derivation exists if the following conditions hold:

1. experts would all agree that the prover has reduced the problem to a situation in which sufficient resources are available to produce a derivation;
2. producing this derivation would be regarded as “a matter of technique”.

In Corollary 17.19, Jech (2000) reduced the claim that “ is a regular cardinal and C is unbounded in ” to the statement “¬∃F (F is a function from some λ < κ cofinally into κ) and C is unbounded in κ”. What is left for the expert to do is verify that “(F is a function from some λ < κ cofinally into κ) and C is unbounded in κ” was *n* for some integer *n*, which (presumably) experts would agree was a “matter of technique”. We now proceed to demonstrate that this account describes set theorists’ practices, discuss the warrants they use to determine what translations to syntactic representations are a matter of technique, and explain how they gain epistemic benefits from syntactic representations that are not provided.

**5. Inferential schemes of set theorists**

*5. 1. Set theorists assume that every finitistic fact can be represented and derived in set theory*

 Set theorists frequently make use of number theoretic facts when writing their proofs. For instance, the Fundamental Theorem of Arithmetic is necessary to show that the canonical Gödel coding map from formulae to integers is injective. Yet we rarely see these claims stated or proven in ZFC. There is a good reason for this. Formal derivations of even basic statements are astonishingly long. For instance, Norman Megill’s derivation in ZFC that 2+2=4 is 27,426 steps long[[11]](#footnote-11). In *some* systems, even the syntactic representation of elementary concepts can require trillions of characters[[12]](#footnote-12). Kunen (2011) described what warrants set theorists to simply assume such derivations exist.

“Facts:

1. x+y = y+x

2. 2+8=10

3. Every natural number > 1 is a product of primes.

These statements about natural numbers may be viewed as part of the finitistic metatheory. Then, when you learn about ZFC, you realize that these, and thousands of other such facts can be expressed as formal theorems in ZFC. For most of these facts, you don’t ever write out a detailed proof. You are essentially using:

**Thesis I.1.2.** *Every finitistic fact can be formalized and proved within ZFC.*

This is similar in spirit to the Church-Turing Thesis I.17.6 in recursion theory, that every function that is computable in the informal sense can be proved to satisfy your favorite mathematical definition of ‘computable’. Neither thesis is a precise mathematical statement, and so can never be given a precise mathematical proof, but no one doubts the correctness of this thesis” (Kunen, 2011, p. 9; italics were the author’s formatting).

Kunen goes on to note that while proofs in ZFC of statements (1) and (2) from the excerpt above are sometimes provided in set theory textbooks (although, of course, not derivations), no set theory textbook provides a proof of (3) because “no one doubts that the inductive proof that most people learn before studying set theory can be replicated in ZFC” (p. 9).

We interpret Kunen’s Thesis I.1.2 as follows: ZFC has sufficient resources to provide syntactic representations and derivations for the standard theorems of number theory. That is, syntactic representations for finitistic statements exist[[13]](#footnote-13). Kunen does not explicitly define what a “finitistic statement” is, but what he seems to have in mind is a statement that would be meaningful to a mathematician espousing a finitist philosophy of mathematics and that is regarded as proven by the mathematical community[[14]](#footnote-14). One (implicit) rationale for Kunen’s Thesis is that ZFC has the resources to model proofs in weaker logical systems such as Peano arithmetic. Consequently, if a statement is proven using standard methods from number theory, we may presume that the proof can be translated into ZFC. That these resources exist is the opinion of every expert (“No one doubts the correctness of this thesis”). Consequently, in any proof that reduces a set theoretic statement to justifying a known fact of number theory, one is permitted to simply assume such a derivation exists.

*5. 2. Assuming statements have syntactic representations*

 In the following passage, we illustrate how set theorists assume that some statements that they express have syntactic representations.

“When we said , we are talking about forcing one specific sentence, but we never said explicitly what that sentence is. We have in mind that one writes a formula f in L = {$\in $} saying that x is a function from y to z and we intend to assert that “p forces (f, A, B)”. But what exactly is ? You could write down an explicit  by tracing through Definition I.6.8 and earlier definitions, but no one ever does, and if they did, probably no two people would come up with exactly the same . It is important for the theory that there is such a , so that we can talk about forcing it” (Kunen, 2011, p. 265).

 We discuss two points that Kunen raised in the passage above. The first is Kunen’s insistence that f:A→B must indicate a syntactic representation in the language *L*={∈}. Indeed, the forcing relation is *defined* only on formulae in this language. More importantly, when working with forcing, set theorists routinely appeal to fundamental theorems of forcing such as the Truth Theorem and the Definability Theorem. These theorems are justified for all  in the language *L*={∈} by induction on the logical complexity of . Hence the scope of these theorems only include formulae in *L*={∈}.

 Our second point is that Kunen recognized that although f:A→B must indicate a syntactic representation, no syntactic representation is provided. Kunen further acknowledged Rav’s (2007) point that set theorists rarely go through the effort of producing the syntactic representation (“no one ever does this”) and Tanswell’s (2015) point that the statement in question will indicate different syntactic representations to different mathematicians (“probably no two people would come up with exactly the same ”). The syntactic representation is merely indicated. Kunen justified why we have the grounds to believe that syntactic representations exist by stating resources that mathematicians have to produce the syntactic representation should they desire to do so (“you could write down an explicit  by tracing through Definition I.6.8 and earlier definitions”), even though the method that Kunen described is not prescriptive enough to indicate a specific derivation.

*5. 3. Variations of definition choice*

 Tanswell (2015) made a compelling argument that the same proof might indicate different derivations to different mathematicians, in part because the formal theory underlying their derivations might be different. A non-logician might be tempted to think that this would not occur amongst set theorists operating in ZFC; the whole point of agreeing to use ZFC is that the community of set theorists would be using the same canonical theory. However, this is not so. Perhaps surprisingly, there is variation in how the ZFC axioms are presented in standard texts. For instance, Jech (2000) and Kunen’s (2011) versions of the Pairing Axiom, Union, Axiom, and Power Set Axiom are not logically equivalent statements; their equivalence can only be established with the Separation Axiom. More significantly, fundamental constructs of set theory are defined in different ways. Kunen (1980, 2011) and Jech (2000) have written the standard introductory graduate texts on set theory. To define the constructible universe *L*, Kunen (1980) originally defined *L* in terms of the so-called Gödel operations. Several decades later, Kunen (2011) followed Jech (2000) in defining *L* using definability and a model theoretic approach. These different ways of defining *L* are logically equivalent, but differ semantically and would lead to different syntactic representations for a particular level of the constructible hierarchy or the statement *V*=*L[[15]](#footnote-15)*.

 The constructible universe is just one instance of this phenomenon. Kunen (1980, 2011) defined forcing in terms of the countable models while Jech (2000) defined forcing over the universe. Any proof about *L* or forcing will necessarily lead to different derivations, depending on whether the mathematician reading the proof learned about these concepts from Kunen, Jech, or someone else. Kunen (2011) addressed this point in his text:

“The sentence |**R**| = |**C**| is meaningless in the metatheory, so this could only denote a sentence in the formal theory, where **R** and **C** are constants. The precise definitions of these constants are different in different texts, but all definitions will satisfy ZFC can derive |**R**|=|**C**|” (Kunen, 2011, p. 15).

“Under any reasonable definition of **R** (such as Definition I.14.2), L[**R**]=L[*P*()]. It is easy to see L[**R**] ⊂ HOD**R**” (Kunen, 2011, p. 313).

We interpret how the variance between formal treatments can be dealt with in derivations as follows: Variation will inevitably exist (“precise definitions… are different in different texts”). However, we can trust that whichever expert defined these constructs or developed the theory provided the mathematician with the resources to derive the standard facts about that construct or theory in ZFC. All definitions of the real numbers and complex numbers will allow one to prove they are equinumerous in ZFC. “Any *reasonable* definition” (our emphasis) of the reals will satisfy L[**R**]=L[*P*()]. The point here is that although the exact syntactic representation of a concept will differ depending on how their sub-terms were defined, we can nonetheless assume that if the constructs were defined reasonably, the syntactic characterizations will be logically equivalent in all important respects.

*5. 4. A remarkable claim about remarkability*

 In this final example, we offer a technical example illustrating the complexity of set theorists’ reasoning using the SRIS. Specifically we look at how Jech (2000) justified that being a remarkable set can be expressed with a  property syntactic representation. If we have a model **U**, I is a set of indiscernibles for **U** if I is a set of ordinals such that for all *n* and all formulae , if  andare all strictly increasing elements of I, **U** satisfies (if and only if **U** satisfies (. If I is a set of indiscernables for a model **U**, let (**U**, I) denote the set of formulae such that ( is true for all increasing  in I. A set of formulae  is called an Ehrenfeucht-Mostowski (EM) set if there exists a model **U** elementarily equivalent to some *L, * is a limit ordinal, and *I* is an infinite set of indiscernibles for **U** such that  =  (**U,** *I*). The preceding uses technical notions of model theory, but the main theme to take away is that EM sets are sets of logical formulae that are complicated to define.

 An important property that EM sets can satisfy in some models of ZFC is the property of remarkability. Remarkability is a more technical concept still that Jech (2000) defined over several pages (p. 316-318); due to the length and complexity of the definition, we omit it here. In chapter 18, Jech demonstrated that the property of remarkability is equivalent to an EM set  satisfying the following conditions:

’ consists of formulae in the language *L*={c1, c2,…} such that:

* + If (*v*1, *v*2, …, *vn)*∈, then (*c*1, *c*2, …, *cn)*∈’
	+ “*c*1 is an ordinal”∈’ and “*c*1<*c*2”∈’
	+ For all *i*1<*i*2<*i*3<…<*i*n and *j*1<*j*2<*j*3<…<*j*n and all (*v*1, *v*2, …, *vn)*∈, “ (*ci1*, *ci2*, …, *ci1)* if and only if (*cj*1, *cj2*, …, *cj1)”*∈’
	+ All of the axioms of ZFC + V=L are in ’.

’ is consistent (i.e., there is no proof of the statement 0=1 from the formulae in ’).

(3) For every Skolem term *t*(*v*1, *v*2, …, *vn*), ’ contains the formula:
“If *t*(*v*1, *v*2, …, *vn*) is an ordinal, *t*(*v*1, *v*2, …, *vn*) < *vn+1*”.

1. For every Skolem term *t*(*x*1, …, *xm, y*1, …, *yn*), ’ contains the formula:
“If *t*(*x*1, …, *xm, y*1, …, *yn*) is an ordinal smaller than *y*1, then
*t*(*x*1, …, *xm, y*1, …, *yn*) = *t*(*x*1, …, *xm, z*1, …, *zn*)”.

From this equivalence, Jech concluded that, “as remarkability can be expressed as a syntactic property, ‘Σ is a remarkable E.M. set’ can be written as a Δ0 property (with parameters *V*ω and *Form*). As such, it is absolute for transitive models” (Jech, 2000, p. 322).

The claim that remarkability is absolute for transitive models is far more complex than the claim that  is absolute for transitive models that we presented earlier in this paper. Nonetheless, both use the same SRIS argumentative scheme:

* *Premise 1:* Any concept that can be syntactically represented with a Δ0 property (with absolute parameters) is absolute for transitive models of ZFC.
* *Premise 2:* We have provided a set of criteria for remarkability that indicates a syntactic representation with a Δ0 property (with absolute parameters).
* *Conclusion:* Therefore, an EM set being remarkable is absolute for transitive models.

The difference between the argument for and the argument for remarkability are based on how Premise 2 is justified. Through a series of theorems, Jech reduced the complicated problem of characterizing remarkable sets to a list of criteria that can all be shown to be Δ0 as a “matter of technique”. Note that it is highly doubtful that Jech or a given reader has any specific syntactic representation in mind! It would also be extremely time-consuming and difficult to actually produce such a representation, which would involve (amongst other things) expressing the following in the first-order logic:

1. the axioms of ZFC,
2. the axiom V=L,
3. a way of representing proofs and consistency in ZFC (which will inevitably involve some type of Gödel numberings using the Fundamental Theorem of Arithmetic),
4. a means of defining Skolem terms, which involves the global well-ordering of the constructible universe.

Of course, this list is not exhaustive. If we were to attempt to produce a syntactic representation indicated by the criteria above, we see all the challenges raised about the implausibility of linking proofs with derivations. First, to our knowledge, such a syntactic representation has never been produced. Indeed, set theory texts do not provide syntactic representations for many of the sub-tasks involved with producing the syntactic representation for remarkability, such as a syntactic representation for the statement V = L, a definition or coding for proofs, or for Skolem terms. Second, as the definitions of the constructability, the global well-ordering of L, the specific Gödel numbering for logical formulae, and even the axioms of ZFC vary across texts, different mathematicians would produce different syntactic representations. Finally, we are aware of no reasonable upper bound on the number of characters that a syntactic representation of remarkability would have.

 Why should we believe that the syntactic representation with the Δ0 property exists? Why do we believe that Jech has reduced the theoretical complexities involved in remarkability to a situation that producing the indicated syntactic representation is a matter of technique for an expert set theorist? First, determining whether a particular set of formulae is consistent (criterion (2) above) is equivalent to determining whether any finite sequence of formulae using those initial formulae as axioms is a proof of a known contradiction. This is a finitistic criterion so we assume that we can create a logical formula that evaluates whether or not that finitistic criterion holds. (Note that this reasoning relies on the fact that any proof of the inconsistency of ‘ would indicate a derivation that formally demonstrates this inconsistency, illustrating how the Derivation Indicator position impacts set theorists’ reasoning). Second, although statements like V=L would be difficult to express syntactically, we assume that such syntactic representations exist and can be found as a matter of technique by recursively unpacking the terms in those statements. Third, even though variation will exist in how V=L, the global well-order of L, and the Gödel numbering of formulae are defined, we presume that any “reasonable” definitions of these concepts will be logically equivalent so it does not matter which definitions we choose.

**6. Discussion**

*6.1. Summary of main arguments: Set theorists gain epistemic benefits from the existence of syntactic representations that they do not produce*

 In this paper, we introduced the Syntactic Representation Inferential Scheme as a method by which set theorists derive information about the concepts that they are studying. When using this scheme, set theorists derive information based on the assumption that a syntactic representation of a concept of a particular type exists. The syntactic representation is merely indicated in the set theorist’s argument, but not provided. Based on our analysis of popular set theorist textbooks, we found three ways in which set theorists warrant their claim that syntactic representations of the desired type exist:

1. Any finitistic statement can be represented syntactically and any finitistic theorem can be proven in ZFC.
2. Ordinary set theoretic statements indicate syntactic representations.
3. We can assume that the concepts and structures referred to in a statement were defined “reasonably”, meaning that standard statements about those concepts and structures are both true and derivable.

In the remainder of this paper, we discuss the implications of our analysis for the epistemological status of the claims that set theorists produce, for the viability of the Derivative Indicator position, and finally for those who have argued that mathematicians cannot reap epistemic benefits from syntactic objects that they do not produce.

*6.2. Implications for the Cartesian story and* a priori *reasoning*

 We now critically examine purported virtues that the consideration of indicated derivations can provide in the case of set theorists. Fallis (2003) presented what he called the *Cartesian story* for how mathematical knowledge is justified. Fallis described the Cartesian story as follows:

“We can tell a fairly simple story of what a mathematician has to do in order to know a proof. A mathematician knows a proof if and only if he has verified directly that each proposition in a sequence of propositions follows from previous propositions in the sequence by a truth preserving inference. As Descartes put it, a mathematician has to ‘trace the whole chain of intermediate conclusions’ and have ‘a clear vision of each step in the process’” (Fallis, 2003, p. 46).

However, we clearly see that this is not the case in set theory. Recall that in Kunen, in his Thesis that every finitistic fact can be formalized and proven in ZFC, he acknowledged that this thesis is not, and cannot be, proven. Instead, he justified this method of inference on the grounds that no one doubts its truth—essentially an appeal to collective expert opinion and the inductive evidence that no contradiction to this claim has been found. Consequently, proofs in set theory that rely on finitistic facts have not been verified according to the Cartesian story. For a similar reason, we cannot say that claims sanctioned by such proofs represent *a priori* knowledge since set theorists’ appeal to the collective expertise of the set theory community to assume certain types of syntactic representations exist.

*6.3. An alternative for proofs providing recipes*

The notion of recipes with regard to derivations was raised and critiqued by Larvor (2016), who quoted MacLane as follows:

A mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language L(∈) as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules. . . [P]ractically no one actually bothers to write out . . . formal proofs. In practice, a proof is a sketch, in sufficient detail to make possible a routine translation of this sketch into a formal proof. . . . [T]he test for the correctness of a proposed proof is by formal criteria and not by reference to the subject matter at issue”. (MacLane, 1986, p. 377–378; quoted in Larvor, 2016, p. 402).

Larvor interpreted this passage as saying a proof that a mathematician produces provides a *recipe* or a prescription for developing a derivation. Extending our argument from the opening section, if a Derivation Recipe is provided, then a Syntactic Representation Recipe must be provided for all definitions involved in the proof/derivation. What we have shown is that for set theorists, these recipes do not exist. We would modify MacLane’s quote by replacing the phrase “routine translation” with the more cumbersome, but more accurate, “a matter of technique that expert set theorists agree they have the resources to perform”. Actually performing the translation from a definition to a syntactic representation might not be routine, and *pace* Tanswell (2015), the translation mechanisms are certainly not sufficiently prescriptive to qualify as a recipe.

Following Larvor (2016), we can interpret set theorists’ use of the SRIS’ Premise 2 (that a syntactic representation of a certain form exists for a statement C) as follows: If C is a statement, we can use the inferential schemes presented in this sub-section to warrant the following claim: “there is a suitable formal system S such that  represents C”. There is no ‘recipe’ for producing . This is merely the 1 claim that  exists and has certain desired properties.

In the beginning of the paper, we noted that Hyman Bass expressed a common viewpoint when he said, “A proof is what convinces a reasonable person that a formal proof exists” (as cited in Burgess, 2015), suggesting that a mathematician’s acceptance of a proof was dependent on her beliefs about the existence of a derivation. Critics (e.g., Kreisel, 1969; Rav, 2007) have questioned the plausibility of this position, wondering how mathematicians can infer the existence of a derivation, especially as most mathematicians have had minimal experience producing derivations of non-trivial proofs. The preceding arguments offer an interpretation of Bass’ quotes that sidesteps these criticisms but raises a different set of challenges. One interpretation of Bass’s quote is that mathematicians believe a proof is a justification for which the collective expertise of the mathematical community would have the resources to transform the proof into a derivation, even if the mathematician lacked the expertise herself. Burgess (2015), who contended that each rigorous proof has a corresponding derivation in ZFC, seemed to have this in mind. Burgess noted that many mathematicians do not know the ZFC axioms and most of mathematics would be unchanged if the mathematical community collectively opted to use category theory, rather than first-order ZFC, as their foundational theory. Borrowing a computer science analogy from Gil Kalai, Burgess claimed that mathematicians viewed the foundational theory in mathematics as the “machine language” in which their proofs were compiled. According to Burgess, mathematicians consider if their proofs were “codifiable” in a formal logical system—regardless of whether the mathematician could perform the coding or what the actual codification would be. Perhaps mathematicians who hold Bass’ view or a similar Derivation Indicator viewpoint have this interpretation in mind. If so, this account raises several challenging questions: How do mathematicians estimate if experts would be able to codify a proof? Why is considering whether the mathematical community would have the resources to codify a proof a more trustworthy judgment, or a judgment more likely to induce consensus, than considering only the semantic content of the proof?

*6. 4. Revisiting the no epistemic benefits position*

 In the beginning of the paper, we cited three scholars who argued that the derivation indicated by a proof cannot provide any epistemic benefits for a mathematician. This is because mathematicians cannot be certain that a derivation exists without producing it (Rav, 2007), the same proof may indicate multiple derivations (Tanswell, 2015), and the lengths of the derivations may be too long to express (Pelc, 2009). We suggest the arguments need to be qualified. By studying the culture of set theorists, we have documented how set theorists can and do draw conclusions about concepts by indicating syntactic representations that they do not produce, even though the same statement may be represented syntactically in multiple ways and we do not have an upper bound on how long such a representation may be.

 We nonetheless believe that the no epistemic benefits perspective is plausible. There are important differences between the scope of our investigation (the consideration of syntactic representations by set theorists) and proponents of no epistemic benefits (the consideration of derivations by ordinary mathematicians). In set theory, the language of set theory is the object language and the properties of this language are sometimes the mathematical objects being investigated. Consequently, set theorists’ expectation that a syntactic representation with certain desired properties will exist might be analogous to, say, a topologist’s expectation that a manifold with certain properties will exist. For most mathematicians, logical languages are not the objects of their study and we would not expect them to have reliable intuitions about what properties syntactic representations or derivations would have. In this sense, generalizing set theorists’ abilities to indicate syntactic representations to the broader mathematical community is problematic. A further difference is that there may be a large difference between representing mathematical statements and mathematical inferences in a formal logical system. Opponents of the Derivation Indicator position can sensibly argue that a mathematician may well be able to represent the premises and the conclusion of an inference syntactically, but if the inference was based on kinesthetic or intuitive reasoning, it would not be possible capture the warrant of that inference in a logical system (Larvor, 2012). In summary, the arguments in this paper do not challenge the no epistemic benefits perspective globally, but they do suggest that several defenses of the no epistemic benefits may be overstated and can be strengthened by focusing on mathematicians’ familiarity with formal logical systems (or lack thereof) and the nature of mathematical inferences within mathematical proofs.

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1. Many scholars contest the claim that “proven claims stay proven” by citing examples of proofs that were later retracted (e.g., Nathanson, 2009). [↑](#footnote-ref-1)
2. In his more recent writings, Azzouni differed from other proponents of the Derivation Indicator position in that he did not require that the proofs be translatable to a formal logical system, but only a system that can be mechanically checked whose inferences are conservative with respect to other accepted inferential techniques (e.g., Azzouni, 2016). [↑](#footnote-ref-2)
3. The quotation appears in Azzouni (2004). [↑](#footnote-ref-3)
4. Investigations of computability provide a sharp illustration of the distance between the objects being investigated and the discourse used to investigate these objects. The primary objects investigated in computability theory are Turing machines and their associated Gödel numbers. Yet actual Turing machines and specific Gödel numbers are almost never given. The Turing machines are indicated by informal descriptions of algorithms and a Gödel number for the machine is presumed to exist. [↑](#footnote-ref-4)
5. We can be pedantic and note that Jech’s definition of  as “the least non-zero limit ordinal” cannot be characterized syntactically by replacing “least”, “non-zero”, “limit”, and “ordinal” with syntactic definitions that Jech provided earlier. This is because until this point in Jech’s volume, “least” is only defined on *sets*, where in “the least non-zero limit ordinal”, least is modifying a class. Jech does not discuss minimal elements of classes until 44 pages later. It is interesting to note that Jech’s definition is clearly comprehensible, even though the syntactic characterization indicated by it is unclear. [↑](#footnote-ref-5)
6. The syntactic characterization can be thought of as representing three conditions on *x*: (a) the empty set is in *x*, (b) if *a* is in *x*, then the successor of *a* is in *x*, and (c) if *a* is in *x*, then either *a* is the empty set or *a* is the successor of another element in *x*. This is enough to ensure that *x* =  in ZFC because of the Foundation Axiom. It might not be sufficient in other systems without the Foundation Axiom, such as Peano arithmetic. [↑](#footnote-ref-6)
7. Jech (2000) did not provide such a representation, but the reader can construct the representation through direct substitution (see p. 164). [↑](#footnote-ref-7)
8. For simplicity, we omit the claim and proof that the set of Mahlo cardinals below  is stationary in  that Jech also stated in Corollary 17.19 and proved. The same phenomenon that we discuss in the current manuscript occurs in the justification of this claim as well. [↑](#footnote-ref-8)
9. Note here that the sentence that Jech asserted was $Π\_{1}^{1}$ referenced club sets, which we noted earlier were never explicitly provided with a syntactic representation. [↑](#footnote-ref-9)
10. We note that in principle, the set theorist could use closure theorems about the property in question (e.g., the conjunction of syntactic representations that are 0 is also 0). No doubt this plays a role in understanding why some relations are 0. However, in most cases, the individual components that are composed are never written out formally. [↑](#footnote-ref-10)
11. http://us.metamath.org/mpegif/mmset.html#trivia [↑](#footnote-ref-11)
12. Matthias (2002) claimed that Bourbaki’s definition of 1 exceeded four trillion characters. The situation is not so severe in ZFC; indeed, Matthias’ point was Bourbaki’s formalism was excessively cumbersome. [↑](#footnote-ref-12)
13. A further claim is that any hereditarily finite object can be represented in a sensible way in *V* a structure that is contained in and absolute for any model of ZFC. [↑](#footnote-ref-13)
14. Kunen (2011) treats theorems about the existence of formal derivations as finitistic statements on the grounds that derivations are finite strings of symbols and consequently finite objects. For example, “Finitists would agree that [ZFC proves 2 since the formal proof is finite and is something that they can believe in” (p. 6) and “The basics of logical syntax and formal proofs… are at the same philosophical level as elementary arithmetic; they are both equally finitistic , so that Thesis I.1.2 applies equally to them” (p. 10). [↑](#footnote-ref-14)
15. Note that this can be used to strengthen Tanswell’s (2015) point that the same proof can indicate multiple derivations. If a proof is to indicate the same derivation, all of the facts appealed to in the proof must have themselves been proven in the same way. Many independence proofs about cardinal arithmetic use the relative consistency of *V=L* to show the existence of a model where the General Continuum Hypothesis (GCH) holds (assuming the consistency of ZFC). No matter how explicit these proofs are, they must point to different derivations, since the proof that *L* satisfies GCH depends on how *L* was defined. [↑](#footnote-ref-15)