Interpretive Implications of the Sample Space

Abstract

In this paper I claim that Kolmogorov's probability theory has other basic notions in addition to 'probability' and 'event'. These notions are described by the sample space component of his probability space structure. This claim has several interesting consequences, two of which I discuss in this paper. The major consequence is that the main interpretations of probability theory are in fact *not* interpretations of Kolmogorov's theory, simply because an interpretation of a mathematical theory in a strict sense must explicate *all* of the theory's basic notions, while the main interpretations of probability do *not* explicate all of Kolmogorov's theory's basic notions. In particular, the main interpretations *only* explicate 'probability' and 'event' and do not explicitly address the additional basic notions which I claim Kolmogorov's theory includes. The other important consequence of my claim concerns the relation between 'probability' and 'event'. Very roughly, contrary to the common conception of 'events' as independent of 'probabilities', I claim that in some cases they do depend on them!

1. The main claims

Kolmogorov's probability theory is almost universally accepted as *the* mathematical probability theory. It is commonly considered to be highly important due to its significant roles in mathematics and in the sciences: "[it] has been widely accepted as the mathematical theory of probability: all the major mathematical results on probability theory are obtained in this framework, and the successful probabilistic models in the sciences also are created in terms of measure theoretic probability theory [...]" (Gyenis & Rédei, 2014, p. 4). The overwhelming acceptance of Kolmogorov's theory is interesting since there are known objections to some of its parts[1] and also since there are several alternative mathematical probability theories[2]. However, as Gillies points out, "There is an enormous body of theorems based on the Kolmogorov axioms. The mathematical community is unlikely to give up this formidable structure and substitute another for it unless there are very considerable gains in so doing." (Gillies [2000], p. 136). Hence, despite the objections and alternative theories, Kolmogorov's theory retains its status of orthodoxy.

The term "interpretations of probability" commonly refers to the different theories which provide different definitions for the notions: 'probability' and 'event'[3]. This term gives the impression that these theories are different interpretations of the same formal system, in this case Kolmogorov's probability theory. However, according to some scholars (specifically Lyon (2016)) this is a false impression. Roughly, the argument is that since the main interpretations do not satisfy Kolmogorov's *axioms*, they are not interpretations of his theory: "Even though Kolmogorov’s axioms are the orthodox probability axioms, they appear to be incompatible with the most common so–called “interpretations” of probability. Finite actual frequencies, infinite hypothetical frequencies, propensities, degrees of entailment, and even rational partial belief all appear to fail to satisfy Kolmogorov’s axiomatisation of probability." (Lyon, 2016, p 155). Lyon's claim is based on the idea that an interpretation of a mathematical theory is a way of

[1] See Lyon (2010) for a good survey of the main objections. Also see Hájek (2003) for a specific objection to Kolmogorov's definition of conditional probability, but see Z. Gyenis, Hofer-Szabó, & Rédei (2016) for a defense of Kolmogorov's definition.

[2] (Goosens, 1979; Popper, 1938, 1955, 1959, Chapter 8; Rényi, 1955).

[3] For good surveys of the various interpretations of probability theory see Gillies (2000); Hájek (2012). For a more historical perspective see Von Plato (1994).

ascribing meanings to its basic notions and axioms. Hence, since the various interpretations fail to satisfy Kolmogorov's axioms, strictly speaking they are not interpretations of his theory[4].

Kolmogorov's theory is commonly said to be based on the two basic notions: 'event' and 'probability'. Kolmogorov himself explains that "The postulational basis of the theory of probability can be established by different methods in respect to the selection of axioms as well as in the selection of basic concepts and relations. However, if our aim is to achieve the utmost simplicity both in the system of axioms and in the further development of the theory, then the postulational concepts of a *random event* and its *probability* seem the most suitable." (Kolmogorov, 1933, pp. 1–2; my emphasis).

The fact that Kolmogorov's theory has reached a status of orthodoxy is a good reason for analyzing it, including its basic notions. In this paper I argue for the claim that Kolmogorov's probability theory has in fact other basic notions in addition to 'probability' and 'event'. The major one, which I call 'possibilities', is defined by the members of the sample space component of Kolmogorov's probability space. I explain this notion in section 4. Briefly, the idea that Kolmogorov's theory has basic notions in addition to 'probability' and 'event' relies on the distinction between the sample space and the σ-algebra components of Kolmogorov's probability space. I argue that this distinction plays a central role in the explications of the basic notions of this theory, whereas in the standard literature this distinction is largely overlooked and is not considered central.

I discuss two consequences of my claim. The first, major consequence is that the main interpretations of probability are *not* interpretations of Kolmogorov's theory. Roughly, the reason is that an interpretation of a mathematical theory in a strict sense should explicate *all* of the theory's basic notions. However, the main interpretations of probability explicate only the notions of 'probability' and 'event', but not the additional notions I claim Kolmogorov's theory has. As a result, they are not interpretations of Kolmogorov's theory in a strict sense.

[4] Lyon's argument even appears in the Stanford Encyclopedia of Philosophy's entry about interpretations of probability: "Normally, we speak of interpreting *a formal system*, that is, attaching familiar meanings to the primitive terms in its axioms and theorems, [... Kolmogorov's axiomatization] has achieved the status of orthodoxy, and it is typically what philosophers have in mind when they think of 'probability theory'. Nevertheless, several of the leading 'interpretations of probability' fail to satisfy all of Kolmogorov's axioms [...]" (Hájek [2012]). Interestingly, despite this argument, the main interpretations are still commonly considered interpretations of Kolmogorov's theory.

Furthermore, I argue that the task of explicating the additional notion of 'possibilities' affects our understanding of the notions 'probability' and 'event'. This suggests that amending the current interpretations of probability to become interpretations of Kolmogorov's theory requires more than just explicating 'possibilities'. In other words, an explication of 'possibilities' may also call for a change in the definitions of the other two notions. This conclusion is important because the task of defining probability is at the heart of the philosophy of probability.

The second consequence of my claim that Kolmogorov's theory has basic notions in addition to 'probability' and 'event' is that in some cases events depend on probabilities. In other words, I claim that, according to Kolmogorov's formalism, there are cases where the fact that some events have particular probability values determines whether or not some other collections of possibilities can be events. Briefly, the idea is that according to the definition of a probability space there can be non-probability-measurable sets of members of the sample space component. More precisely, given a probability space, a set of members of the sample space component is a non-probability-measurable set iff it is non-measurable by the probability measure component. By definition, non-probability-measurable sets are not events. Moreover, the fact that these sets are non-probability-measurable means that they cannot be added to the given σ-algebra component and hence *cannot* be events![5] This implies roughly that the probabilities of some sets of members of the sample space (which are events) determine whether some other sets of members of the sample space can have probability values and thus be events. I further explain this consequence in sections 5 and 6.

This consequence is important since it goes against the view of events as independent of probabilities. Commonly, the fact whether something is an event does not seem to depend on any probability value. Hence the second consequence, that in some cases events do depend on probabilities, is significant for clarifying the relation between them and thus the explication of these notions.

[5] More precisely, given a probability space $<\Omega,\Sigma,P>$ in which the sample space Ω has non-measurable subsets by the probability measure P, the σ-algebra Σ cannot be expanded to include these sets without changing the probabilities P assigns to the members of Σ. In other words, these sets cannot be events in the sense that P cannot be expanded to assign them probabilities. Given a different probability measure, these sets may well be events.

In the next section I briefly discuss Kolmogorov's theory and its relation to the various interpretations of probability. I present the interpretations in a general way which is somewhat different than the way they are commonly presented in the standard literature. This enables me to focus on the essential difference between them and Kolmogorov's theory, which is important for my argument.

2. Kolmogorov's Probability Theory and Interpretations of Probability

In the philosophy of probability there is a commonly made distinction between mathematical probability theories (such as Kolmogorov's) and interpretations of probability. According to Gillies, "The theory of probability has a mathematical aspect and a foundational or philosophical aspect. There is a remarkable contrast between the two. While an almost complete consensus and agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy." (Gillies [2000], p. 1). Lyon explains this distinction between the two types of probability theories (mathematical theories and interpretations) by claiming that they aim to answer different questions regarding the notion of 'probability': "In philosophy of probability, there are two main questions that we are concerned with. The first question is: what is the correct mathematical theory of probability? Orthodoxy has it that this question was laid to rest by Andrei Kolmogorov in 1933 [...] this is far from true; there are many competing formal theories of probability, [...] These formal theories of probability tell us how probabilities behave, how to calculate probabilities from other probabilities, but they do not tell us what probabilities *are*. This leads us to the second central question in philosophy of probability: just what are probabilities? [...] philosophers have tried to answer this question. Such answers are typically called *interpretations of probability*, or philosophical theories of probability." (Lyon [2010], p. 93).

Roughly, according to Lyon, mathematical probability theories such as Kolmogorov's are concerned with the behavior of probabilities while interpretations of probability are concerned with all other aspects of this notion. This explanation seems to me to rely on the implicit idea that the definitions of 'probability' (and of 'event') given by the two types of probability theories (mathematical theories and interpretations) are only *partial* definitions. This means that Kolmogorov's theory does not provide complete definitions of the pre-theoretical notions of 'probability' and 'event'[6] but only the mathematical parts of some plausible complete definitions. Similarly, each of the different interpretations provides different non-mathematical parts of different complete definitions of 'probability' and 'event'. Complete definitions of these notions can be obtained *only* by a combination of Kolmogorov's theory (or any other mathematical probability theory) and an interpretation of it.

However, it is important to realize that the two parts of any complete definition of 'probability' and 'event' are connected and affect each other. The mathematical parts have implications for the non-mathematical (the interpretive) parts[7] and vice versa. This means that Kolmogorov's theory *explicitly* describes the mathematical parts of complete definitions of 'probability' and 'event' and also *implicitly* describes some aspects of the interpretive parts. The interpretive aspects which are implicitly described by Kolmogorov's theory are common to all interpretations of this theory, simply because these aspects stem from the restrictions imposed by the mathematical parts. For example, according to the definition of Kolmogorov's probability space, mathematically events are sets and every probability space includes the empty set as an event (commonly known as the "empty event"). This means that anything that is correctly described by a Kolmogorov's probability space has to include something correctly described by an empty event. This restriction on the type of things describable by Kolmogorov's theory is in fact an implicit interpretive implication of his probability space. It means that any interpretation of Kolmogorov's theory must explain the meaning of the fact that there is always an empty event and indeed, many of them do so. For example, subjective interpretations of Kolmogorov's theory, according to which events are propositions, must explain what an empty proposition is (it is commonly considered an impossible proposition or a contradiction). Similarly, objective interpretations of the theory, according to which events are possible states of the world, must explain what an empty state of the world is (it is commonly considered an impossible state of the world), etc. I hope that this way of portraying the relation between Kolmogorov's theory and the interpretations of probability will become clearer as the current paper progresses.

[6] Here "pre-theoretical" refers to the way these notions are used in ordinary language.
[7] At least in the sense that they impose restrictions on them.

My argument does not focus on the distinctions between the different interpretations (or types of interpretations) of Kolmogorov's theory. On the contrary, my aim is to emphasize features common to all of them. I claim that some of these features are implicitly described by Kolmogorov's theory. According to Gyenis and Rédei, "Interpretations of probability are typical classes of applications of probability theory, classes consisting of applications that possess some common features, which the interpretation isolates and analyses." (Gyenis & Rédei, 2014, p. 19). Hence, it can be said that the main common feature of all interpretations of Kolmogorov's theory is that each of them tries to characterize in non-mathematical terms things which are correctly described by this theory.

For convenience I shall use the term "probabilistic states" to denote the things Kolmogorov's theory aims to describe. Hence, an interpretation of Kolmogorov's theory provides a general non-mathematical description of a collection of probabilistic states. The main characteristic of a probabilistic state is that it involves at least two different possibilities[8] (or alternatives, or options). For example, a probabilistic state of an ideal coin toss involves the possibility of the coin landing on 'heads' and the possibility of it landing on 'tails'. Similarly, a probabilistic state of an agent picking an integer between 0 and 10 (inclusive) includes each of the eleven options of picking one of these integers and perhaps also the option of not picking any of them. Commonly, a state which includes only one possibility is not what we normally consider a probabilistic one[9]. The possibilities which characterize a probabilistic state are mutually exclusive and exhaustive, which means that there are no other alternatives which are relevant to it. Moreover, it is assumed that necessarily one of these possibilities manifests (or occurs, or happens). For example, an ideal coin toss *does not* include the possibility of the coin landing on its side (and hence landing not on 'heads' or 'tails') - this option is simply not part of the ideal state. Notice that this characterization of the possibilities of a probabilistic state is not a full description of them. It is only a rough sketch. Complete definitions of them include a mathematical part given by

[8] Here "possibilities" denotes a pre-theoretical notion. In this paper, I claim that this notion is mathematically described by the members of the sample space component of Kolmogorov's probability space.

[9] Interestingly, Kolmogorov's probability space can in fact describe probabilistic states which include only one possibility. However, such a state by definition has exactly one event with probability 1 and hence does not seem to mathematically describe anything that is normally considered probabilistic.

Kolmogorov's theory and an interpretive part which must be given by any interpretation of the theory.

It is also important to explain that possibilities are not events. Roughly, an event is a set of possibilities with a certain probability. Mathematically, events are described by the members of the σ-algebra component of Kolmogorov's probability space, while, according to my claim, possibilities are described by the members of the sample space component. I explain the difference between possibilities and events further in the following sections. In the meantime, this description of a probabilistic state seems to me to capture features which are common to all interpretations of Kolmogorov's probability theory.

This paper accords special attention to some of the interpretive aspects that stem from Kolmogorov's theory because such aspects are common to all interpretations of the theory. This enables me to discuss issues concerning interpretive aspects of 'probability' and 'event' without committing myself to a specific interpretation. This also means that the conclusions of this discussion are relevant to all interpretations claimed to be interpretations of Kolmogorov's theory. Somewhat ironically, my main conclusion is that the main interpretations of probability are not interpretations of Kolmogorov's theory in a strict sense. Furthermore, I raise the concern that it might be impossible to amend any of the existing interpretations to become interpretations of Kolmogorov's theory without having to change the definitions of 'probability' and 'event'.

In the next section I present a definition of Kolmogorov's probability space. I highlight the parts that are relevant to my claim that 'possibilities' are a basic notion of Kolmogorov's theory, in addition to 'probability' and 'event'.

3. Kolmogorov's definition of a probability space

My claim that Kolmogorov's theory has basic notions in addition to 'probability' and 'event' is based on the way these notions are defined in the theory. Roughly, 'probabilities' are mathematically defined by the probability measure component of Kolmogorov's probability space structure, while 'events' are defined by the σ-algebra component. I claim that the additional basic notions are defined by the sample space component of the probability space structure. In this section I present a definition of a probability space[10] and highlight some points that are important for my claim.

A probability space is defined as a triple $<\Omega,\Sigma,P>$ consisting of the following three components: a sample space (Ω), a σ-algebra (Σ) and a probability measure (P).

The probability space's components are defined as follows:

1. A sample space (Ω) - a nonempty set.

The members of the sample space are sometimes called "elementary events"[11]. Nevertheless, these members are *not* a mathematical formalization of 'events'. 'Events' are mathematically defined by the σ-algebra component[12] (which is defined just below). However, the sample space and its members are indeed elementary in the sense that events (and probabilities) depend on them. Roughly, as I show shortly, all events[13] are composed of members of the sample space, which means that any change in the sample space causes change in the events. And since probabilities are the values of a function from events to the unit interval, any change in the events causes a change in the probabilities as well. Hence probabilities depend on events, which in turn depend on the members of the sample space.

My main claim in this paper is that the members of the sample space describe a new basic notion of Kolmogorov's probability theory which I call "possibilities". In the next section I elaborate on this notion. Roughly, I show that since there can be sets of members of the sample space which

[10] The following definition is one of several standard ways of defining Kolmogorov's probability space. See Billingsley (1995, p. 23) for a similar but more rigorous definition.
[11] For example, Kolmogorov calls the members of the sample space "elementary events" and the members of the σ-algebra "random events" (See Kolmogorov (1933, p. 2)).
[12] More precisely, since the σ-algebra is defined over the given sample space, Kolmogorov's mathematical definition of 'events' requires both components.
[13] Except perhaps for the empty event.

are not events, the members of the sample space mathematically describe a notion which is not just "parts of events" but rather a new basic notion.

In the previous section I described the pre-theoretic notion of 'possibilities'. I characterized possibilities as being mutually exclusive and exhaustive. This fits well with the fact that they are mathematically defined by the members of the sample space, because these members are elements of a set and as such mutually exclusive and exhaustive by definition.

2. A σ-algebra (Σ) (defined over the sample space) - a subset of the power set[14] of the sample space (i.e. a set of subsets of Ω) which satisfies the following three conditions:
   2.1. Σ is not empty (or, equivalently, Ω is in Σ)
   2.2. Σ is closed under complementation (i.e. if A is in Σ, then so is Ω\A).
   2.3. Σ is closed under countable unions (i.e. if A_1, A_2, A_3 ... are in Σ, then so is $A = A_1 \cup A_2 \cup A_3 \cup \ldots$)

The members of Σ are the mathematical formalization of the pre-theoretic notion of 'events'. In other words, according to Kolmogorov's definition, the mathematical part of a complete definition of events describes them as the members of the σ-algebra of a given probability space. This means that whatever events may be according to the interpretive part of their definition, mathematically they are sets that stand in certain relations to one another and together form a σ-algebra. Moreover, since this σ-algebra is part of a given probability space, it is defined over a specific sample space, which means that mathematically events are *sets of members of a sample space*[15]. The members of the σ-algebra are commonly referred to as "events".

Two facts are important for my argument later on. The first is that the σ-algebra is defined as a *subset* of the power set of the sample space. Hence it is not necessarily the power set of the sample space. In fact, it can be a proper subset of the power set of the sample space. For example, the σ-algebra $\Sigma_3 = \{\emptyset, \Omega_3, \{1\}, \{2,3\}\}$ defined over the sample space $\Omega_3 = \{1,2,3\}$ is not the power set of Ω_3. (The power set of Ω_3 is $\Sigma_P = \{\emptyset, \Omega_3, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$).

[14] The power set of a set (S) is the set containing all the subsets of S.
[15] The given σ-algebra is also connected to a specific probability measure, which means that mathematically events are sets of members of a sample space that have specific probability values.

The second is that a σ-algebra always contains the sample space event and the empty event (i.e. $\emptyset, \Omega \in \Sigma$). Thus, these two events may be called "the mandatory events".

3. A probability measure (P) - a real-valued function defined over Σ which satisfies the following conditions:
   3.1. P is non-negative
   3.2. $\mathrm{P}(\emptyset) = 0$
   3.3. P is countably additive (which means that for all countable collections $\{E_i\}$ of pairwise disjoint sets, $\mathrm{P}(\bigcup_i E_i) = \sum_i \mathrm{P}(E_i)$)
   3.4. The codomain of P is the unit interval [0,1] and $\mathrm{P}(\Omega) = 1$

The value assigned to a member of the σ-algebra (*A*) by the probability-measure function (i.e. P(*A*)) is called "the probability of *A*". This means that mathematically the pre-theoretic notion of 'probabilities' is defined as the values of a function from a σ-algebra to the unit interval which satisfies certain conditions (described by the definition of a probability measure[16]). In other words, according to Kolmogorov's definition, the mathematical part of a complete definition of probabilities describes them as the values of the probability-measure function of a given probability space. (The interpretive parts of different complete definitions of probabilities are provided by the different interpretations).

Notice that the probability measure is a function whose domain is the σ-algebra. In other words, it assigns a value to each of the members of the σ-algebra, which means that each event has a probability. Hence it can be said that an interpretive implication of this fact is that an event is a thing which *necessarily* has a probability. This implication is important for my argument that the members of the sample space mathematically describe a basic notion of Kolmogorov's theory which is not 'events'. Very briefly, since the members of the sample space do not have probabilities, they are not events.

[16] Notice that conditions 3.1-3.3 of the probability measure's definition are just the formal definition of the mathematical notion of measure. Condition 3.4 is what makes the probability measure a special kind of measure. Any measure that satisfies condition 3.4 is a probability measure.

In the literature of the philosophy of probability it is quite common to find other mathematical definitions of the notions 'event' and 'probability' which seem to be different from Kolmogorov's definition given above. More specifically, events are commonly mathematically defined using an algebra without the explicit mention of a sample space. However, an algebra is a type of an algebraic structure[17] and as such it is defined as a set with operations defined on it. This means that an algebra *always* has an underlying set which plays the same role as the sample space plays in Kolmogorov's definition. Thus, the common way of mathematically defining events and probabilities using only an algebra and a probability measure, without an explicit mention of a sample space, is actually just a *partial* description of the relevant Kolmogorovian probability space. Realizing this is very important. It means that the consequences I put forward in this paper are relevant even to those who do not explicitly mention Kolmogorov's sample space component in their mathematical definitions of the notions 'event' and 'probability'.

A word about notation. In the rest of this paper I use "event" and "probability" (without inverted commas) to denote the two fundamental pre-theoretic notions which Kolmogorov's theory and the different interpretations aim to describe. Recall that a given mathematical probability theory (such as Kolmogorov's) provides the mathematical parts of complete definitions of these notions, while the non-mathematical (or interpretive) parts are provided by the different interpretations of the given mathematical theory. I use "K-event" and "K-probability" (without inverted commas) to denote the mathematical parts given by Kolmogorov's probability theory, meaning that a K-event is a member of the σ-algebra of a given probability space, and a K-probability is a value of the probability measure of a given probability space. Similarly, I use "K-possibilities" to refer to the members of the sample space and "possibilities" to denote the new fundamental pre-theoretic notion which I claim is mathematically defined by the K-possibilities.

In the literature of the philosophy of probability the pre-theoretic notions which I call "events" and "probabilities" are commonly referred to by different names in accordance with the writer's assumed interpretation. For example, events are referred to as propositions or states of the world, and the like, and probabilities are referred to as credences, degrees of belief, chances,

[17] "A set, together with one or more operations on the set, is called an **algebraic structure**. The set is called the **underlying set** of the structure." (Gilbert & Nicholson, 2004, p. 4).

propensities and the like. These names denote different complete definitions of event and probability. They suggest that the different definitions share the same mathematical parts (given by Kolmogorov's theory) and differ only in their interpretive parts. The terms I use in this paper enables me to distinguish between the pre-theoretic notions that Kolmogorov's theory and the various interpretations aim to define, and the mathematical parts of their complete definitions given by Kolmogorov's theory.

4. The basic notions described by the members of the sample space

In this section I argue for my claim that Kolmogorov's theory mathematically describes basic notions in addition to probability and event. I argue that the sample space component of the probability space structure describes these new basic notions, and I claim that the K-possibilities (i.e., the members of the sample space) mathematically describe the possibilities (or the alternatives or options) which are the main characteristic of every probabilistic state.

My claim that Kolmogorov's probability space mathematically describes basic notions in addition to probability and event is based mainly on the fact that there can be sets of K-possibilities which are not K-events. Such sets, by definition, do not have a K-probability value, which means that such sets mathematically describe things that do not have probabilities. And since events are things which *necessarily* have probabilities, the things described by sets of K-possibilities which are not K-events are not events. Moreover, the fact that there can be non-probability-measurable sets of K-possibilities means that there are cases where there are sets of K-possibilities which cannot be K-events (I explain this claim shortly). This implies that there are probabilistic states in which the things which are mathematically described by non-probability-measurable sets of K-possibilities, not only that they are not events, but also that they *cannot* be events!

I now address two possible objections to my claim. Roughly, the two objections are that the sample space does not mathematically describe a *new* basic notion but rather only an existing basic notion, namely the events.

The first objection is roughly that the K-possibilities mathematically describe elementary events which are a particular type of events. My response is that the possibilities they describe indeed have a one-to-one correspondence with *potential* elementary events, but the latter are not necessarily events and in fact can (and perhaps, should) be thought of as a different notion.

I start by clarifying the difference between K-possibilities and elementary K-events. A K-event is elementary if it contains exactly one K-possibility (i.e. it is a singleton[18] subset of the sample space). It is elementary in the sense that it cannot be decomposed into simpler sets of K-possibilities. More precisely, it does not include any other set of K-possibilities. Any event which is mathematically described by such an elementary K-event is called an elementary event. Each K-possibility has a corresponding singleton of which it is a member. However, these singletons are not necessarily included in the σ-algebra. Only those which are included in the σ-algebra are elementary K-events.

One may argue that the difference between the singleton subsets of the sample space and the K-events is not a conceptual one but rather a pragmatical or technical matter. In other words, that any such singleton *can be* a K-event. However, this modal distinction between them (that a singleton can be but is not necessarily a K-event) plays a crucial role when there are infinitely many K-possibilities. In such cases, it is common to distinguish between all potential (or possible) events and those which are focused on. In particular, the potential events and especially their putative probabilities (assuming that they can have probabilities) are not explicitly mentioned in the probabilistic state description.

The two distinctions mentioned above, between elementary and nonelementary events and between potential and focused-on events (whether or not they are elementary), are important. Together they suggest that in fact there are two *different* notions: potential elementary events (i.e. singletons of possibilities) and focused-on events (i.e. "regular" events)[19]. The key difference

[18] A singleton is a set that contains exactly one element.
[19] Notice that in many cases all possibilities have corresponding events. In particular when the sample space is finite, the σ-algebra usually contains all its singleton subsets.

between them lies in the modal characterization of potential elementary events as potential. As such, they are treated differently than events. For example, according to subjective interpretations when there are infinitely many potential propositions, it is commonly assumed that an agent holds only a small finite number of them. Hence there is a difference between all possible propositions and those held by the agent. Similarly, according to objective interpretations, when there are infinitely many states of the world, it is commonly assumed that their relative frequencies can be measured, which means that they are distinguishable by observation. However, this latter consequence is commonly considered too strong when there are infinitely many world-states (and even wrong when there are uncountably many such states). Thus there is a difference between world states which are distinguishable by observation and those which are not.

The distinction between potential and regular elementary events is not always clear. For example, when Lewis discusses what he calls "a reasonable initial credence function" (*C*) he claims: "[…] *C* was to be a probability distribution over (at least) the space whose points are possible worlds and whose regions (sets of worlds) are propositions. *C* is nonnegative, normalized, finitely additive measure defined on all propositions." (Lewis [1980], p. 267). Here *C* is defined over possible worlds (points in space) and propositions (regions in space). This seems to imply that Lewis' possible worlds refer to regular elementary events. However, Lewis also emphasizes that propositions and possible worlds are distinct and that only propositions have credences since credences are degrees of belief in propositions. This suggests that possible worlds are potential elementary events. Another example, can be found in Laplace's classical interpretation of probability: "The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible […]" (Laplace [1902], p. 6). Here it is unclear whether "cases" are potential elementary events or not mainly because they are said to be "equally possible" which seems to suggest that they have (equal) probabilities and thus are regular elementary events.

In short, K-possibilities have a one-to-one correspondence the singleton subsets of the sample space which mathematically describe potential elementary events. However, the latter are not necessarily "regular" events. In particular, when there are infinitely many K-possibilities, potential elementary events are treated differently than events in a way which suggests that they are two distinct notions. This means that the K-possibilities do not describe a particular type of events but rather a different notion.

The second possible objection to my claim that Kolmogorov's probability space mathematically describes basic notions in addition to probability and event is the claim that the K-possibilities mathematically describes the elementary parts (or "building blocks") of the events. And since a notion of the form "a part of *X*" where *X* is a basic notion is not necessarily a basic notion itself, possibilities are not a new basic notion (at least not necessarily). Such a claim is based on the assumption that K-possibilities are nothing more than parts of K-events, which implies that possibilities are nothing more than the parts constituting the events.

However, Kolmogorov's definition of a probability space provides a good reason for rejecting this objection. Recall that according to his definition, the σ-algebra does not have to be the power set of the sample space but may well be a proper subset of this power set. And when the σ-algebra is a *proper* subset of the sample space, some sets of K-possibilities are *not* K-events. This roughly means that there can be collections of possibilities which are not events. In other words, possibilities can constitute things which are not events (as well as events). Hence, K-possibilities mathematically describe notions which are not merely "parts of events". For example, given the σ-algebra $\Sigma_3 = \{\emptyset, \Omega_3, \{1\}, \{2,3\}\}$ defined over the sample space $\Omega_3 = \{1,2,3\}$, the sets $\{2\}$,$\{3\}$,$\{1,2\}$ and $\{1,3\}$ are all sets of members of Ω_3 which are not K-events. If these sets mathematically describe anything then those things by definition are not events because they do not have probability values.

The above conclusion relies on the facts that according to Kolmogorov's definition events are mathematically described by K-events, and that not all sets of K-possibilities are K-events. Naively, it seems that one can claim that this definition is wrong and that events should be mathematically defined as *any* set of K-possibilities, regardless of whether or not this set is a K-event. Such a change to this definition would imply that the members of the sample space always describe parts of events and nothing else. In this case, assuming that a part of an event is not a basic notion would lead to the conclusion that the K-possibilities do not describe any new basic notion.

However, the above claim is not really a valid option, at least unless Kolmogorov's formalism is changed even more - roughly because of the issue of non-measurable sets (which is discussed in the following sections). Very briefly, due to the non-measurability issue, there are cases where not every set of K-possibilities *can* mathematically describe an event[20]. In particular, there are cases where not every set of K-possibilities can be assigned a K-probability by the given probability measure. This is a known fact which Van Fraassen summarizes as follows: "It will now be quite clear, therefore, that the requirement to have probability defined everywhere, would be unacceptable. We must accept as genuine probability measures also those which *cannot be extended to measures on all subsets of their domain*." (Van Fraassen, 1989, p. 329, my emphasis). Van Fraassen claims that there are cases when not every set of K-possibilities can have a K-probability. Due to this inability, such sets cannot be K-events. This fact stands in contrast to the pre-theoretic assumption that in any given probabilistic state, any collection of possibilities can be an event[21]. Hence the fact that there can be cases where there are sets of K-possibilities that cannot mathematically describe events means that the K-possibilities can describe notions which are not just "parts of events".

In conclusion, the K-possibilities mathematically describe a notion which I have called “possibilities”. Possibilities are neither elementary events (since K-possibilities do not have K-probabilities) nor merely "parts of events" (since there can be sets of K-possibilities which are not K-events). This supports the claim that the K-possibilities mathematically describe *additional* basic notions to probability and event.

5. Sets of members of the sample space which are not K-events

The fact that according to the definition of K-events, there can be sets of K-possibilities which are not K-events also seems to suggest that perhaps there are other new basic notions in addition

[20] At least not an event that behaves as events "normally" behave. I elaborate on this point later on.
[21] Some philosophers object to this assumption, (see for example: Lewis, 1980).

to possibilities. In particular, it is not clear whether *a set of K-possibilities which is not a K-event* mathematically describes a basic notion of Kolmogorov's theory. As with K-possibilities, this putative basic notion is also described by the sample space component. The main problem with deciding whether a set of K-possibilities which is not a K-event mathematically describes a basic notion of Kolmogorov's theory is that the main interpretation of probability do not explicitly address such sets. Hence it is not clear what is mathematically described by such sets.

To try and clarify this issue, it is important to understand Kolmogorov's reason for his definition of a probability space and in particular, the explicit distinction between the sample space and the σ-algebra components, which allows such sets. The mathematical reason for this distinction is that it is a way (perhaps the most common way) of dealing with the issue of non-measurability[22]. This issue is relevant to Kolmogorov's theory because the probability measure is a particular kind of measure[23].

Non-measurability belongs to the mathematical field of measure theory. In this theory it is well-known that given certain pairs of a set and a measure function, the given set can have subsets which are non-measurable by the given measure[24]. Loosely speaking, in such cases the non-measurable subsets do not "behave" as expected because their "sizes" (i.e. their measure values) change when they undergo certain transformations that are not supposed to change them. The fact that non-measurable sets do not "behave" as expected is commonly considered a problem that needs to be dealt with partly because there are as many non-measurable sets as they are measurable ones, in fact "[…] every set of positive Lebesgue measure contains a nonmeasurable subset." (Bogachev [2007a], p. 460).

There are several ways to deal with non-measurable subsets of a given set. The most common is to have a collection of subsets of the given set which excludes all the non-measurable subsets and includes only measurable ones. Such collections are good because they only include sets which "behave" as expected and are sufficient for many mathematical tasks. A σ-algebra is such a collection.

[22] See Bogachev (2007, p. 31,58-59).
[23] See footnote n.16 above.
[24] See Billingsley (1995, p. 45) for one proof of the existence of non-measurable sets.

The main alternative ways which do not try to exclude the non-measurable sets are 1) accepting that the measure of some subsets are not invariant to rotation or translation; 2) accepting that some subsets do not satisfy the countable additivity condition in the definition of measure[25]; 3) making a more fundamental change in the assumptions of measure theory or set theory (such as accepting an alternative to the ZFC axioms[26]).

The mathematical fact that there are non-measurable sets has the consequence that there are cases when a σ-algebra cannot be the power set of a given set. More precisely, given a set (*S*) and a measure (*m*), if *S* has subsets that are non-measurable by *m*, any σ-algebra defined over *S* cannot be the power set of *S*[27]. This is because the power set of a given set includes *all* its subsets by definition regardless of their measurability, but a σ-algebra includes *only* measurable subsets of the set it is defined over. Hence any σ-algebra defined over *S* has to be a *proper* subset of the power set of *S* which includes only measurable subsets of *S*. As a result, a σ-algebra over a given set (*A*) does not necessarily include all subsets of *A* but every subset of *A* it does include is a measurable set.

The fact that a σ-algebra defined over a set (*A*) does not necessarily include *every* subset of *A* is the reason for the fact that in Kolmogorov's theory there can be sets of K-possibilities which are not K-events. Recall that K-events are Kolmogorov's mathematical definition of events. In this definition, Kolmogorov uses a σ-algebra as a way of preventing all non-probability-measurable sets of members of a sample space from being K-events. More precisely, Kolmogorov assumes that the σ-algebra component is a Borel field (a specific kind of σ-algebra) and explains: "Only in the case of Borel fields of probability do we obtain full freedom of action, without danger of the occurrence of events having no probability." (Kolmogorov, 1933, p. 16). This definition guarantees that all K-events are subsets of the given sample space which are measurable by the given probability measure. In short, the usage of a σ-algebra in Kolmogorov's definition of a probability space guarantees that all K-events have K-probabilities, which implies that all events have probabilities.

[25] In which case the measure will no longer be a measure according to the current definition.
[26] ZFC is the standard axiom system of set theory. See Bagaria (2016, sec. 2) for a list of the axioms of ZFC.
[27] As long as the measure (*m*) is not changed to a different measure (m_2) such that all subsets of *S* are measurable by m_2, in which case it would be possible to define a σ-algebra over *S* that would be the power set of *S*.

However, Kolmogorov's definition of a probability space is too "loose" in the sense that it does not guarantee that *all* probability-measurable subsets of the sample space are K-events[28]. From a mathematical perspective, there is nothing special about this particular feature of the definition. However, it does have implications for the different interpretations of Kolmogorov's theory as it means that any such interpretation must address the issue of sets of K-possibilities which are not K-events. In other words, such interpretations must explain what is a set of possibilities which is not an event.

To sum up, the usage of a σ-algebra in Kolmogorov's probability space's definition has two interesting results. One is that all K-events have K-probabilities. This seems to be a desirable result simply because it means that all events have probabilities[29]. The other result is that there can be sets of K-possibilities which are not K-events. This means that not all collections of possibilities are events. Moreover, due to the non-measurability issue (which I discuss in the next section), there are cases in which some collections of possibilities cannot have probabilities and thus be events. These results should be addressed by any interpretation of Kolmogorov's theory. Currently, the main interpretations do not explicitly address the latter consequence. They do not explain what a collection of possibilities that cannot be an event might be. This supports the claim that they are not interpretations of Kolmogorov's probability theory in a strict sense.

6. Non-measurable subsets of the sample space: Interpretive implications

So far, I have argued that Kolmogorov's theory mathematically describes basic notions in addition to events and probabilities. I have claimed that these additional basic notions which I called possibilities are described by the members of the sample space. I now focus on a specific kind of sets of K-possibilities that are not K-events, namely non-probability-measurable sets of

[28] More precisely, given a probability space PS=<Ω,Σ,P>, Σ is not necessarily "maximal" in the sense that there can be subsets of Ω which can be added to it without changing the K-probabilities of the current K-events. In other words, there can be a probability space PS_1=<Ω,Σ_1,P_1> such that $\Sigma \subset \Sigma_1$ and $\forall e \in \Sigma, \mathrm{P}(e) = \mathrm{P}_1(e)$ which means that it expends *PS* in the sense that it includes all *PS*'s K-events and more.

[29] Even if the probability of an event is zero it is still the case that the event *has* a probability value.

K-possibilities. Given a probability space in which there are such sets, not only that they are not K-events, but also that they *cannot* be added to the given σ-algebra component and thus be K-events. I claim that the major interpretive implication of these sets is that there are cases where events depend on probabilities and that any interpretation of Kolmogorov's theory must explain what this dependency means. In particular, the main interpretations of probability cannot be interpretations of Kolmogorov's theory without providing such an explanation. This suggests that perhaps fixing the mismatch between the main interpretations and Kolmogorov's theory will result with changing the complete definitions of events and probability, either the mathematical parts of these definitions or the interpretive parts or both.

There seem to be two ways of dealing with non-probability-measurable sets of K-possibilities. The first way is to accept Kolmogorov's mathematical definition of events as K-events and the second is to reject it. According to the first way, non-probability-measurable sets of K-possibilities do not mathematically describe events because they are not K-events, which means that not every collection of possibilities is an event. The second way is to claim that non-probability-measurable sets of K-possibilities *do* mathematically describe events, even though they are not K-events, which means that every collection of possibilities is indeed an event, but also that not all events are "normal" events. (I explain what I mean by "normal" shortly. Loosely speaking, these events are abnormal in the sense that they do not have probabilities in the way events described by K-events have).

The first way accepts Kolmogorov's definition and thus does not call for a change in it. However, it does raise the following interpretive questions: Does a non-probability-measurable set of K-possibilities mathematically describe any notion? If so, what does it describe? Is this notion a basic notion? Why not every collection of possibilities is an event, and what exactly a collection of possibilities is, if it is not an event? Currently, none of the main interpretations address these questions or any other interpretive aspect of non-probability-measurable sets of K-possibilities.

According to the second way of dealing with non-probability-measurable sets of K-possibilities, they do describe events. However, the events they describe do not behave in the "normal" way in which events described by K-events behave. Specifically, they do not have a probability value in the way that events described by K-events have. (For convenience, in the rest of this section, I refer to this putative new notion of events (which covers both normal and abnormal events) as N-events). A complete definition of N-events should have a mathematical part and an interpretive part. As mentioned, N-events are mathematically defined as any set of K-possibilities. The interpretive part of the definition of N-events should addresses the following points: What does it mean for an N-event to not behave normally? (For example, what does it mean that the probabilities of some events are not countably additive?) Why do some N-events behave normally while other do not? Why not all of them behave "normally"? In particular, what does it mean that an N-event *cannot* have a probability value like those described by K-events have?

In contrast to the first way, the second way of dealing with non-probability-measurable sets of K-possibilities calls for a change in Kolmogorov's theory. Treating all sets of K-possibilities, including non-probability-measurable ones, as mathematically describing events requires making one of the abovementioned changes[30] to Kolmogorov's theory. In addition, the second way also raises issues concerning the definition of these N-events which need to be addressed by any interpretation of this modified Kolmogorov's theory. Currently none of the main interpretations address these issues, since they do not address non-probability-measurable sets of K-possibilities at all.

Both ways call for interpretations of probability to address non-probability-measurable sets of K-possibilities. The fact that the second way also involves changes in Kolmogorov's formalism and, in particular, changes in the mathematical parts of the definitions of events and probabilities, seems to me to make it worse than the first.

The main reason that any interpretation of Kolmogorov's theory should address the fact that there can be non-probability-measurable sets of K-possibilities is that it implies that in some cases events depend on probabilities. More precisely, it implies that there are probabilistic states in which the probabilities determine whether or not some collections of possibilities can be events.

[30] Either the measure of some subsets of the sample space would not be invariant to rotation or translation, or some (other) subsets would not satisfy the countable additivity condition in the definition of measure, or some other fundamental changes would have to be made (such as an alteration to the axioms of ZFC).

This implication is important because it goes against the common conception of events as independent of probabilities.

The key point in regard to non-probability-measurable sets of K-possibilities is their dependence on the probability measure. Given a sample space (S) and a probability measure (P), the fact that some subsets of S are non-probability-measurable depends on the particular probability measure P. Given a different probability measure (P_2), the same subsets may well be probability-measurable by P_2. Hájek emphasizes this point: "I want to stress that non-measurability is a *relation* that a set may bear to one probability measure, while not bearing it to another." (Hájek [2003], p. 302). In other words, the fact that some subsets of a sample space are non-probability-measurable depends on the fact that some other subsets of the sample space have particular K-probabilities (attributed to them by the given probability measure). This means roughly that there are cases where some collections of possibilities cannot have any probability value[31] because some events have particular probabilities.

This implication - that in some cases events depend on probabilities - seems to me to be a very strong, peculiar addition to the relation between them. Events are commonly thought of as independent of their probabilities in the sense that they can have any probability value, under the restriction that all their probabilities together satisfy the definition of Kolmogorov's probability measure. (This restriction can be seen as implying that the probabilities of some events determine the probabilities of other events[32]). More precisely, according to Kolmogorov's definition there can be infinitely many probability measures defined over the same σ-algebra[33], which means roughly that the same events can have different probabilities. Whether or not a collection of possibilities is an event is commonly thought of as independent of any probability, and in particular, this fact does not depend on some events having *particular* probability values.

The fact that there can be non-probability-measurable sets of K-possibilities implies that there are cases where some collections of possibilities are not events because some other collections of possibilities (which are events) have particular probabilities. In other words, there are cases

[31] At least not in the "normal" way.

[32] For example, if the probability of event e is p, then the probability of its complementary event (e^c - the case that e does not occur) must be $1-p$. In this sense the probability of e determines the probability of e^c (and vice versa).

[33] Except for when the σ-algebra is a trivial one (a σ-algebra that contains only the sample space K-event and the empty K-event). The only probability measure that can be defined over a given trivial σ-algebra is its corresponding trivial probability measure (which assigns 1 to the sample space K-event and 0 to the empty K-event).

where the probabilities of the events determine that some other collections of possibilities are not events because cannot have probabilities. Furthermore, if the probabilities of these events were different, then those other collections of possibilities might well be events. For example, in a fair lottery over the natural numbers, all events corresponding to one result (i.e. a natural number n) should have the same probability. However, strictly speaking, this lottery cannot be described by Kolmogorov's theory because the K-probability of the putative K-events describing these events must be a real number that is either zero or strictly positive. Thus, the K-probability of their union (which is the sample space K-event) is either 0 or ∞, and not 1 as it should be. In other words, the probability measure (P_{fair}) is supposed to be uniform over the singleton subsets of the sample space, however these subsets are non-probability-measurable by it. Notice that not all subsets of the sample space are non-probability-measurable. For example, P_{fair} can assign 1/2 to each of the K-events: $e_{even} = \{n|n \; is \; even\}$ and $e_{odd} = \{n|n \; is \; odd\}$, or 1/3 to $e_{\%3=0} = \{n|n\%3 = 0\}$ and the like. However, when the lottery is not a fair one, the singleton subsets may have probability values and thus be events. For example, the K-events describing them can have different K-probabilities (for example $p(\{n\}) = \frac{1}{2^n}$) and thus be probability-measurable by the given non-uniform probability measure. Notice that when the lottery is unfair, the K-probabilities of e_{even}, e_{odd} and $e_{\%3=0}$ will not be as described above (1/2, 1/2 and 1/3 respectively). This is important because put differently it shows that the fact whether the singletons corresponding to one result can be events, depends on the probabilities of other events (such as those described by e_{even}, e_{odd} and the like).

There is another sense in which events depend on probabilities which concerns the fact that the range of K-probabilities is the *real* numbers between 0 and 1[34]. This range does not include infinitesimals simply because they are not real numbers. The idea is roughly that non-probability-measurable sets of K-possibilities are subsets of a given sample space which cannot have K-probabilities because they are real numbers. If Kolmogorov's definition is changed so that the range of K-probabilities includes infinitesimals, then these sets can have infinitesimal K-probabilities and thus be probability-measurable. For example, Benci, Horsten, & Wenmackers propose an alternative approach to probability theory in which they use a non-Archimedean field

[34] I would like to thank an anonymous referee for pointing this out.

as the range of the probability function. This range includes infinitesimals. As a result, "[…] unlike in classical probability theory, all subsets of an infinite sample space are measurable […]" (Benci *et al.* [2013], p. 121). This means that the *range* of K-probabilities affects whether certain sets of K-possibilities are non-probability-measurable and thus whether or not these sets can be K-events. This suggests that events depend on the range of probabilities.

The fact that this dependence of events on probabilities stems from Kolmogorov's definition means that it should be addressed by any interpretation of his theory. Moreover, it implies that if events are characterized by a given interpretation of Kolmogorov's theory as independent of their probabilities, then that interpretation is in fact not an interpretation of the theory. Any interpretation of Kolmogorov's theory should define event and probability in a way that reflects the dependency relation between them. This suggests that the main interpretations of probability cannot be amended to become interpretations of Kolmogorov's theory without redefining event or probability or both.

7. Conclusion

In this paper I have claimed that Kolmogorov's probability theory mathematically describes other basic notions in addition to probability and event. These notions are described by the sample space component of Kolmogorov's probability space. In particular, I have claimed that the members of the sample space mathematically describe a notion which I call possibilities. Roughly, possibilities are the main characteristic of probabilistic states (which is the term I use to denote the things Kolmogorov's theory aims to describe). Any probabilistic state involves at least two different possibilities (or alternatives or options). These possibilities are mutually exclusive and exhaustive, and necessarily one of them manifests. According to Kolmogorov's definition, possibilities compose events. However, possibilities can also compose things which are not events, which means that possibilities are more than just "elementary parts of events". More importantly, according to Kolmogorov's definition possibilities are neither events nor probabilities and hence are a distinct basic notion described by his theory. This description of possibilities is an interpretive implication of Kolmogorov's definition. It is not a full description of possibilities, however, and the exact details should be filled by any interpretation of Kolmogorov's theory.

The major consequence of this claim is that the main interpretations of probability theory are *not* interpretations of Kolmogorov's theory in a strict sense. An interpretation of a mathematical theory should explicate all of the theory's basic notions. However, the main interpretations do not explicitly ascribe meaning to K-possibilities, which means that they do not explicate all the basic notions of Kolmogorov's theory.

In this paper I also discussed another interesting consequence of this claim. Very roughly, I claimed that in some cases events depend on probabilities. This claim stems from the fact that according to Kolmogorov's definition there can be sets of K-possibilities which are not K-events. More precisely, it stems from that fact that there can be non-probability-measurable sets of this sort. This means that there are probabilistic states in which not all collections of possibilities can be events. These collections which are mathematically described by non-probability-measurable sets of K-possibilities, cannot have probabilities and thus be events without changing the probabilities of other events. In other words, this impossibility depends on the given probability measure, meaning that whether or not a collection of possibilities is an event depends on the probabilities of other collections of possibilities which are events. Hence it turns out that, contrary to the common conception of events as independent of probabilities, in some cases events do depend on probabilities.

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