Deciphering the Algebraic CPT Theorem

Noel Swanson*

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Abstract

The CPT theorem states that any causal, Lorentz-invariant, thermodynamically well-behaved quantum field theory must also be invariant under a reflection symmetry that reverses the direction of time (T), flips spatial parity (P), and conjugates charge (C). Although its physical basis remains obscure, CPT symmetry appears to be necessary in order to unify quantum mechanics with relativity. This paper attempts to decipher the physical reasoning behind proofs of the CPT theorem in algebraic quantum field theory. Ultimately, CPT symmetry is linked to a reversal of the C^*-algebraic Lie product that encodes the generating relationship between observables and symmetries. In any physically reasonable relativistic quantum field theory, it is always possible to systematically flip this generating relationship while preserving the dynamics, spectra, and localization properties of physical systems. Rather than the product of three separate reflections, CPT symmetry is revealed to be a single global reflection of the theory's state space.

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*Department of Philosophy, University of Delaware, 24 Kent Way, Newark, DE 19716, USA, nswanson@udel.edu

1 Introduction: Explaining CPT Symmetry

Virtually every serious candidate for a fundamental physical theory from Newtonian gravitation to classical electrodynamics has been time-reversal-invariant. For every nomologically possible world, there is another nomologically possible world where the direction of time is reversed. Surprisingly, this is not true for relativistic quantum field theories (QFTs). It is possible to write down physically reasonable QFTs which are not time-reversal-invariant, and as James Cronin and Val Fitch experimentally demonstrated in 1964, weak nuclear interactions in the actual world are described by such a theory.[1]

While QFTs may fail to be symmetric under simple time reversal, the *CPT theorem* ensures that there is always a more complicated time reversal symmetry present. The theorem loosely states that any causal, Lorentz-invariant, thermodynamically well-behaved QFT must be invariant under a combined symmetry operation that reverses the direction of time (T), flips spatial parity (P), and conjugates all charges present in the theory (C). Since particles and antiparticles carry opposite charge, the net effect of charge conjugation is to swap matter and

[1] In work that would win them the 1980 Nobel Prize, Cronin and Fitch observed that neutral kaons transform into their antiparticle partners at a different rate than the reverse process.

antimatter. In a CPT-invariant theory, every nomologically possible world has a dopplegänger where the future is the past, left is right, and you and I are made out of antiparticles.[2]

The historical development of QFT is closely tied to the CPT theorem. Initial attempts to relativize quantum theory in the late 1920s ran aground on a cluster of problems stemming from a conflict between relativistic causality and energy positivity.[3] Plugging the relativistic dispersion relation, $E = \sqrt{p^2 + m^2}$, directly into the Schrödinger equation yields a Lorentz-invariant wave equation, however, the resulting dynamics are non-hyperbolic — initially well-localized wavepackets spread faster than the speed of light, raising the specter of faster-than-light signaling and other causality-violating paradoxes. Hyperbolic wave equations, like the Klein-Gordon and Dirac equations, avoid these immediate problems, however such equations have non-physical negative energy solutions. These can be cut off by hand, but only at the cost of ruining Lorentz invariance, hyperbolicity, or both.[4] QFT effectively sidesteps this problem by dropping the requirement that the theory describe a finite, fixed number of particles. The negative energy states never really go away. Rather they are reinterpreted as positive energy states with opposite charge. This trick only works to restore Lorentz invariance and hyperbolicity, though, if there is an exact correspondence between particles and antiparticles; they must be indistinguishable except for their charge. The CPT theorem accounts for this, explaining why particle/antiparticle pairs have the same mass, spin, and lifetime. Viewed from this angle, CPT symmetry plays a fundamental explanatory role in QFT. It is only because the theory is CPT-invariant that we can reinterpret negative energy states as describing antiparticles in a manner consistent with the requirements of relativistic causality.

But what explains the origins of CPT symmetry itself? Why is this seemingly ad hoc combination of reflections always a symmetry of nature? Despite its importance, the physical basis for the CPT theorem remains notoriously obscure. As Bain (2016) emphasizes, part of the problem is that there are several different versions of the theorem with different starting assumptions. Many of the more technical assumptions do not have a clear physical interpretation, making comparisons between various proofs challenging. To compound this difficulty, the the-

[2]As in the case of T invariance, CPT invariance is often interpreted as indicating that these apparently distinct possibilities are in fact different representations of the same physically possible world. For present purposes, I will set aside this interpretive question.

[3]See Weinberg (1995, Ch. 1) and Strocchi (2013, Ch. 1) for surveys of these problems and the various attempts to circumvent them that lead to the development of QFT.

[4]These obstacles can be turned into a rigorous no-go theorem. See Strocchi (2013, Prop. 2.2).

orems are couched within different mathematical frameworks, Lagrangian QFT, Wightman QFT, S-matrix QFT, and algebraic QFT. Lagrangian and S-matrix proofs, while more physically transparent, lack mathematical rigor, whereas the rigorous axiomatic proofs in the Wightman and algebraic frameworks are more physically opaque. This state of affairs has prompted Greaves (2010) and Greaves and Thomas (2014) to search for a rigorous Lagrangian version of the CPT theorem. In this paper I attack the problem from the opposite direction, by looking for a more physically perspicuous interpretation of proofs in algebraic QFT, the framework perhaps most familiar to philosophers of physics.

In §2, I give an overview of algebraic QFT, focusing on the features most essential for understanding the CPT theorem. Algebraic proofs rely on the following main idea: in the vacuum representation, certain local algebraic invariants associated with spacelike wedge regions act geometrically as elements of the Poincaré group. One of these invariants, the *modular conjugation* operator, J_W, represents a full CPT transformation of the theory when combined with a spatial rotation. But why do modular invariants play such a pivotal role, and what makes wedge regions so special? Most proofs simply begin by positing critical geometric or analytic properties of the modular objects, and existing mathematical surveys, Borchers (2000) and Borchers and Yngvason (2000), do not broach these deeper interpretive questions.

In order to answer them, we will have to dig down into the algebraic foundations of QFT. The central portion of the paper, §3, takes the form of a mathematical physics whodunit. If we suspect that a generic QFT must have some generalized time-reversal symmetry, where might we look for it in the structure of algebraic QFT? By tracing the theorem's starting point back to the physically-motivated Haag-Kastler axioms and carefully dissecting its logical structure piece by piece, we will discover why the wedge modular conjugation must be the culprit.

Our detective work points towards an intriguing geometric explanation for CPT invariance: ultimately, it is linked to a reversal of the C^*-algebraic Lie product that encodes the generating relationship between observables and symmetries. In any causal, Lorentz-invariant, thermodynamically well-behaved QFT, it is always possible to systematically flip this generating relationship while preserving the dynamics, spectra, and localization properties of physical systems. Rather than the product of three separate reflections, CPT symmetry is revealed to be a single global reflection of the theory's state space.

In §4, I explore the ramifications that this story has for existing philosophical debates about the explanation of CPT invariance. Recently, Bain (2016) has issued an important skeptical challenge, arguing that major divergences between proofs of the CPT theorem couched in different frameworks preclude any of them from providing an explanation for the CPT symmetry observed in nature. Contra Bain, I argue that the present investigation reveals that the algebraic proof shares a great deal of structure with proofs in Lagrangian, S-matrix, and Wightman QFT, suggesting convergence towards a core set of explanatory ideas. The algebraic proof offers some of the greatest insight into this core that we have at present.

Meanwhile, Greaves (2010) offers a different geometric story about the origins of CPT invariance in Lagrangian field theory. A surprising corollary of this explanation is that the CPT theorem is an essentially relativistic result; quantum mechanical assumptions do not play a major role. I argue that the algebraic CPT theorem provides a better explanation for the origins of CPT invariance incorporating a more unified picture of antimatter captured by algebraic superselection theory. Although there are intriguing similarities between the algebraic proof and the more recent Lagrangian proof given by Greaves and Thomas (2014), I argue that the case for a purely classical explanation of CPT symmetry is not convincing.

Although these counterarguments are not decisive, they significantly advance both debates and illustrate the potential fruitfulness of further philosophical investigation into algebraic QFT. I conclude in §5 by highlighting several open questions and framing a conjecture relating state space, spatiotemporal, and charge orientation structures that will be the subject of future work.[5]

2 The Algebraic CPT Theorem

One of the most mathematically rigorous approaches to QFT currently on the table, algebraic QFT (AQFT) serves as a natural framework for investigating the conceptual underpinnings of relativistic quantum theories. Rather than beginning with the specification of a Hilbert space, AQFT starts with an abstract character-

[5] "CPT, Spin-Statistics, and State Space Geometry," (in preparation). This paper is based on previous dissertation work (Swanson, 2014, ch. 3). The main conclusions drawn and the broad structural account of the algebraic CPT theorem are the same, but some of the central details are different. In particular, the distinction between *-isomorphisms, anti-isomorphisms, and conjugate-isomorphisms are more clearly drawn by lemma 1 and directly connected to modular theory by lemma 2. Lemma 3 is also new. Rather than starting from modular covariance, the present account uses Borchers's auxiliary analyticity assumptions to more clearly link central steps in the proof back to the Haag-Kastler axioms, thereby reinforcing the arguments in Swanson (2018). The discussion of charge conjugation in §3.6 is also different, highlighting the importance of modular inclusions and hewing more closely to existing proofs in the mathematical physics literature.

ization of the algebraic properties of gauge-invariant physical quantities known as *observables*. It is typically assumed that the observables of a quantum system form the self-adjoint part of a noncommutative C^*-*algebra*, $\mathfrak{A}$, an abstract collection of elements isomorphic to a subalgebra of bounded Hilbert space operators.[6]

States are given by normalized, positive linear functionals, $\phi : \mathfrak{A} \to \mathbb{C}$, whose values represent the expectation values of observables in $\mathfrak{A}$. Given a state, the Gelfand-Naimark-Segal (GNS) construction determines a unique representation, $\pi_\phi(\mathfrak{A})$, of $\mathfrak{A}$ as a concrete subalgebra of operators acting on a Hilbert space, $\mathcal{H}_\phi$. Within a representation, the closure of $\pi_\phi(\mathfrak{A})$ with respect to the Hilbert space weak topology defines a von Neumann algebra equivalent to the double commutant, $\pi_\phi(\mathfrak{A})''$. Such algebras have a complete lattice of projection operators, and in AQFT, this procedure allows for the definition of additional representation-dependent observables including energy-momentum operators and superselected charges. Two representations $\pi_{\phi_1}, \pi_{\phi_2}$ are *quasiequivalent* iff $\pi_{\phi_1}(\mathfrak{A})''$ and $\pi_{\phi_2}(\mathfrak{A})''$ are *-isomorphic. (This generalizes the more familiar notion of *unitary equivalence* to reducible representations.) Because field systems in AQFT have infinitely many degrees of freedom, GNS representations of a given algebra will typically not be quasiequivalent (unlike the situation in non-relativistic quantum mechanics).

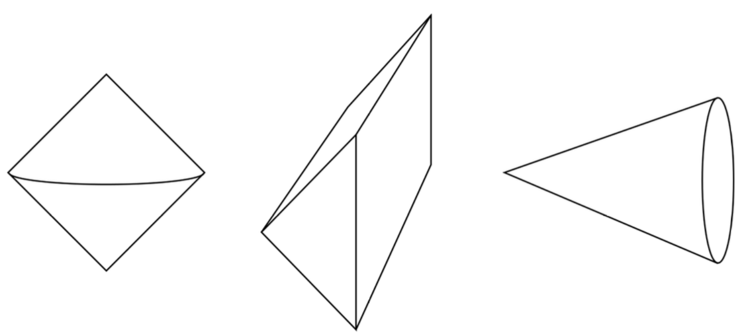

Figure 1: A doublecone, a spacelike wedge, and a spacelike cone.

We will focus on AQFT in flat spacetime. Throughout, O will denote an open region of Minkowski spacetime and O' the interior of its *causal complement*, the set of all points spacelike separated from all points in O. Certain special regions will be important to keep track of. A *doublecone* is a compact region formed by

[6]See Kadison and Ringrose (1997) and Blackadar (2006) for a thorough introduction to the mathematics of C^*-algebras and Haag (1996) for the application of these ideas to AQFT. Halvorson and Müger (2006) and Ruetsche (2011) represent more philosophically-oriented surveys of AQFT.

the intersection of a past and future lightcone at two timelike separated points. A *spacelike wedge* is an unbounded wedge-shaped region whose two defining planes are tangent to the edges of some lightcone. A *spacelike cone* is a cone-shaped subset of a spacelike wedge, infinitely extended in one spacelike direction.

2.1 Assumptions

A model of AQFT is given by an assignment, $\{\mathfrak{A}(O)\}$, of C^*-algebras to regions of spacetime satisfying the Haag-Kastler axioms, along with a set of physically possible states, $\{\phi\}$. Each state in $\{\phi\}$ determines a GNS representation and a corresponding assignment of local von Neumann algebras, $\{\mathfrak{R}_\phi(O)\} = \{\pi_\phi(\mathfrak{A}(O))''\}$. The self-adjoint elements of these algebras represent locally measurable physical quantities, while the Haag-Kastler axioms specify the dynamics and enforce the joint requirements of relativity and quantum mechanics. There are five standard axioms, and they all play a crucial role in the algebraic CPT theorem:

Isotony: *If $O_1 \subset O_2$, then $\mathfrak{A}(O_1) \subset \mathfrak{A}(O_2)$.* This gives the assignment $\{\mathfrak{A}(O)\}$ the structure of a net and allows us to define the *quasilocal algebra*, $\mathfrak{A}$, as its upwards inductive limit. (The family of physical states $\{\phi\}$ is formally defined as a set of states of $\mathfrak{A}$.)

Microcausality: *If $O_1 \subset O_2'$, then $\mathfrak{A}(O_1)$ and $\mathfrak{A}(O_2)$ commute.* This enforces relativistic no-signaling constraints, ruling out act-outcome correlations at spacelike separation. (It is also sometimes called the *locality* axiom.)

Covariance: *The net $\{\mathfrak{A}(O)\}$ transforms covariantly under a faithful representation of the connected Poincaré group (or more generally its covering group) as automorphisms of $\mathfrak{A}$.* The full group of isometries of Minkowski spacetime is the Poincaré group. Its covering group has the same Lie algebra and is used to represent symmetries of spinor fields. The connected Poincaré group is the subgroup topologically connected to the identity, consisting of translations, rotations, and boosts. (It does not include orientation-reversing isometries like P, T, or PT reflections.) The dynamical laws of the theory are encoded in the translation subgroup of this representation and are guaranteed to be Lorentz-invariant.

Vacuum: *There exists at least one translation-invariant state, $\omega \in \{\phi\}$.* This is a necessary condition for ω to be interpretable as a vacuum state.

In the corresponding GNS representation, the translation subgroup is implemented by a strongly continuous 1-parameter group of unitary operators, $U(a)$. The group generators are the energy-momentum observables and are affiliated with the global von Neumann algebra, $\mathfrak{R}_\omega = \pi_\omega(\mathfrak{A})''$.

Spectrum Condition: *In each vacuum GNS representation, the energy-momentum observables have spectral support in the same lightcone lobe in momentum space.* This ensures that the energy spectrum is bounded from below in all Lorentz frames and that the vacuum is thermodynamically stable.[7]

In addition to the Haag-Kastler axioms, the algebraic CPT theorem relies on five other assumptions:

Additivity *For any family of bounded open regions, $\{O_i\}$, the local algebra $\mathfrak{A}(\cup O_i)$ is the C^*-algebra generated by the family of local algebras $\{\mathfrak{A}(O_i)\}$.* This is a technical condition relating the algebras of bounded and unbounded regions. It is used in the analysis of charge superselection structure, and it entails *weak additivity* in vacuum representations. Weak additivity ensures that the global von Neumann algebra $\mathfrak{R}_\omega$ can be generated by translations of any local algebra $\mathfrak{R}_\omega(O)$. It is an important ingredient in the Reeh-Schlieder theorem and several crucial lemmas in the algebraic CPT theorem.

Wedge Intersection Property: *For any doublecone D, in any vacuum representation, $\mathfrak{R}_\omega(D) = \bigcap \mathfrak{R}_\omega(W_i)$ for all spacelike wedges $W_i \supset D$.* This is another technical condition allowing vacuum doublecone algebras to be defined by the intersection of families of wedge algebras. It is used in the proof of the Bisognano-Wichmann property and to construct the minimal Poincaré representation in §3.5. If a model of AQFT does not satisfy the wedge intersection property it is always possible to expand the net of local algebras so that it is satisfied, although the extension will not typically be unique.

Split Property: *If regions O_1 and O_2 are spacelike separated and not tangent, then in the vacuum representation $\mathfrak{R}_\omega(O_1)$ and $\mathfrak{R}_\omega(O_2)$ can be "split,"*

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[7]As usually formulated, the spectrum condition requires that the spectral support of $U(a)$ lie in the closed forward lightcone, $\overline{V^+}$, in momentum space. The apparent reference to a distinguished temporal orientation is eliminable. It is only required that $U(a)$ must have spectral support in a closed convex set $\overline{V}$ which is asymmetric under taking additive inverses: $\{\overline{V}\} \cap \{-\overline{V}\} = \{0\}$.

i.e., they generate a tensor product of von Neumann algebras. Along with the spectrum condition, the split property is part of the characterization of thermodynamically well-behaved QFTs. It entails that the family $\{\phi\}$ includes well-defined thermal equilibrium states satisfying the *Kubo-Martin-Schwinger (KMS) condition* and is a necessary condition for a model of AQFT to have an emergent particle interpretation (Haag, 1996, ch. V.5). Existing algebraic proofs of the CPT theorem rely on the weaker *distal split property*, which only requires the existence of some pair of spacelike separated wedges such that $\mathfrak{R}_\omega(W_1)$ and $\mathfrak{R}_\omega(W_2)$ can be split.

Analyticity: At certain critical stages, proofs of the algebraic CPT theorem, like proofs in the Wightman framework, rely on tricky analytic continuation arguments. As we will go on to see, in AQFT many important analyticity properties are derived from the Haag-Kastler axioms and weak additivity. It remains an open question if these assumptions along with the split property are sufficient to derive all of the analyticity needed for the CPT theorem. Existing algebraic proofs require auxiliary analyticity assumptions, and the choice of which assumptions to make marks a place where different algebraic proofs diverge. In our presentation, two closely related assumptions, *B-analyticity* and *B-reality*, will be introduced in §3.5 once we have developed the necessary technical machinery.

DHR/BF Selection Criteria: *For every physical state $\phi \in \{\phi\}$, the GNS representation π_ϕ is quasiequivalent to the vacuum representation in the causal complement of some doublecone or spacelike cone (i.e., for some doublecone or spacelike cone, O, the restrictions $\pi_\phi(\mathfrak{A})|_{O'}$ and $\pi_\omega(\mathfrak{A})|_{O'}$ are quasiequivalent).* This final assumption is the key to the algebraic analysis of charge structure. Its physical motivation is the subject of the next section.

2.2 Charges and Superselection Structure

Rather than a single Hilbert space, the state space of a model of AQFT is a collection of different GNS representations, grouped into unitary equivalence classes called *sectors*. Each representation in a given sector has the same folium of density operators, representing states with the same global boundary conditions, characterized by the values of representation-dependent observables in the global algebra, $\mathfrak{R}_\phi = \pi_\phi(\mathfrak{A})''$. In different models of AQFT, different families of global states and their corresponding GNS representations carry physical significance. The analysis of *charge representations* initiated by Doplicher, Haag, and Roberts (1969a,b) is one of the crowning achievements of AQFT and plays a central role in the algebraic CPT theorem.

Charges are gauge-invariant, conserved quantities associated with particular force laws. Electric charge is the conserved quantity that couples to the electromagnetic force; color charge is the conserved quantity that couples to the strong force.[8] Besides satisfying global conservation laws, they obey superselection rules that forbid states which are superpositions of different charges. In addition, charges can be localized within some region of spacetime, and every charge has a well-defined conjugate charge. Particles carrying conjugate charges can annihilate. Conversely, particle/antiparticle pairs can spontaneously spring from the vacuum state.

In AQFT, these features are captured using special mappings called *localized transportable morphisms*. Formally, these are injective *-homomorphisms $\varrho : \mathfrak{A} \to B(\mathcal{H}_\omega)$, where $\mathcal{H}_\omega$ is the vacuum GNS Hilbert space. Each morphism must be localized in some region, O (i.e., it acts as the identity on $\mathfrak{A}(O')$), and it must be possible to transport ϱ to any other similarly shaped region in spacetime using unitary mappings (i.e., for any similar region, O_2, there is a localized morphism, ϱ_2, and a unitary operator, U, such that $U\varrho(A) = \varrho_2(A)U$ for all $A \in \mathfrak{A}$).

The collection of localized transportable morphisms has a rich mathematical structure, that of a symmetric tensor *-category.[9] In particular, the category has a natural tensor product which allows us to define notions of charge composition and conjugate charges. Each morphism induces a corresponding mapping on global states over $\mathfrak{A}$. If ω is a vacuum state, $\omega \circ \varrho$ describes a state with charge Q localized in region O. Its conjugate is defined as the unique morphism, $\bar{\varrho}$, such that $\omega \circ \varrho \circ \bar{\varrho}$ is a mixed state containing a component in the vacuum sector. This captures the necessary condition for pair creation/annihilation.[10]

[8] The notion of color charge discussed here is not the same as the more familiar quark color labels *red*, *blue*, and *green*. These labels do not have a gauge-invariant meaning and can be superimposed. Color charge is a $\mathbb{Z}_3$-valued gauge-invariant superselected quantity in the center of SU(3) constructed from functions of local Casimir invariants. See Kijowski and Rudolph (2003) for a discussion of the superselection structure of quantum chromodynamics on a finite lattice.

[9] In the original analysis of Doplicher, Haag, and Roberts, Haag duality entails that the charge morphisms are actually endomorphisms of the quasilocal algebra, $\varrho : \mathfrak{A} \to \mathfrak{A}$. This greatly simplifies the mathematical analysis. In the more general case considered by Buchholz and Fredenhagen, this is no longer true, and considerably more work is required to prove that the charge morphisms have nice categorical properties.

[10] An example of a common annihilation event is $e^+ + e^- \to \gamma + \gamma$. Since charge is globally conserved and photons are chargeless, any interaction of this kind requires that particles and antiparticles have conjugate charge. If $\omega \circ \varrho \circ \bar{\varrho}$ is a mixed state with a component in the vacuum

Doplicher, Haag, and Roberts analyze charges described by morphisms localized in compact spatiotemporal regions. Such charges couple to forces like the strong force whose strength falls off sharply as a function of distance. They prove that the relevant category of localized transportable morphisms is equivalent to the category of GNS representations of states satisfying the *DHR selection criterion* — ϕ satisfies the DHR selection criterion if its GNS representation is quasiequivalent to the vacuum representation in the causal complement of some doublecone. The corresponding charge sectors are labeled by the value of the total charge observable, and conjugate sectors are defined by the condition that $\pi(\mathfrak{A})'' \otimes \bar{\pi}(\mathfrak{A})''$ contains a copy of the vacuum representation, $\pi_\omega(\mathfrak{A})''$. According to the DHR picture, matter and antimatter states are represented by states in conjugate sectors, giving rigorous mathematical content to the idea that such states have opposite charge quantum numbers (Baker and Halvorson, 2010).

For theories with compactly localized charges like quantum chromodynamics, the DHR selection criterion is a physically plausible constraint on the family of possible global states, $\{\phi\}$. Buchholz and Fredenhagen (1982) extend the DHR picture to include topological charges localized in spacelike cones. In theories with a mass gap, there is a 1-1 correspondence between particle representations (i.e., any representation in which the translation subgroup is unitarily implemented and satisfies the spectrum condition) and states satisfying the *BF selection criterion* — ϕ satisfies the BF selection criterion if its GNS representation is quasiequivalent to the vacuum representation in the causal complement of some spacelike cone. While impressive, the analysis of charge structure in AQFT is still incomplete. Because of the infrared problem, we currently lack a full understanding of the localization properties of charges in theories involving massless particles, and thus the algebraic proof of the CPT theorem cannot be applied to theories like quantum electrodynamics at this stage.[11]

2.3 CPT Symmetry and the Bisognano-Wichmann Property

Algebraically, we can view a CPT transformation as an automorphism, $\theta : \mathfrak{A} \to \mathfrak{A}$, satisfying the following constraints:

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sector, then the probability of a creation/annihilation event is nonzero according to the Born rule.

[11] See Buchholz and Roberts (2014) for recent work on extending the tools of DHR/BF analysis to massless theories.

(a) $\theta^2 = \text{id}$.

(b) If $O_1 \subset O_2$, then $\theta(\mathfrak{A}(O_1)) \subset \theta(\mathfrak{A}(O_2))$.

(c) $\theta(\mathfrak{A}(O)) = \mathfrak{A}(-O)$, where $-O$ is the region obtained from O via a full spatiotemporal inversion in both the space and time coordinates.

(d) $\theta \circ \alpha_{a,\Lambda} = \alpha_{-a,\Lambda} \circ \theta$, where $\alpha_{a,\Lambda}$ is the representation of the connected Poincaré group (or its covering group) posited by the covariance axiom.

(e) For any DHR/BF charge morphisms, $\theta \circ \varrho = \bar{\varrho} \circ \theta$.

The first two conditions require θ to be an involution that preserves the localization structure of the net $\{\mathfrak{A}(O)\}$. Conditions (c)-(d) ensure that θ represents a full spatiotemporal inversion (corresponding to the element $-\text{id}$ in the Lorentz group) and has the right commutation relations with the Poincaré transformations. (Technically, (c), requires the choice of a privileged origin, but in light of (d), this choice does not matter.) The final condition tells us that θ conjugates all charges present in the theory. (Note, the term "automorphism" here is intended to encompass the various generalizations of *-automorphisms canvassed in §3.1.)

Roberts and Roepstorff (1969) characterize symmetries in AQFT as automorphisms of the net that preserve transition probabilities and permute superselection sectors. Generalizing Wigner's famous theorem from non-relativistic quantum mechanics, they prove that any such symmetry can be represented by the adjoint action of a unitary or antiunitary operator affiliated with the universal enveloping von Neumann algebra, $\mathfrak{A}^{**}$.[12] Note that for any GNS representation, π_ϕ, and any automorphism, $\alpha : \mathfrak{A} \to \mathfrak{A}$, $\pi_\phi \circ \alpha(A) := \pi_\phi(\alpha(A))$ is a GNS representation for $\mathfrak{A}$ generated by the state $\phi \circ \alpha^{-1}(A) := \phi(\alpha^{-1}(A))$. Roberts and Roepstorff prove that there always exists a unitary or antiunitary operator, $W : \mathcal{H}_\phi \to \mathcal{H}_{\phi\circ\alpha^{-1}}$, *representing* α, meaning that $W\pi_\phi(\mathfrak{A})W^* = \pi_\phi \circ \alpha(\mathfrak{A})$. Since W is an isometry, it is guaranteed to preserve transition probabilities. Therefore, in order for α to be a symmetry of a model of AQFT, for every physical state $\phi \in \{\phi\}$, the symmetry-transformed state must also be a physical state, $\phi \circ \alpha^{-1} \in \{\phi\}$.[13]

[12] The universal enveloping algebra, $\mathfrak{A}^{**}$, is isomorphic to the direct sum of all GNS representations of $\mathfrak{A}$.

[13] As Baker and Halvorson (2013) emphasize, the Wigner unitary/antiunitary, W, will not in general implement a unitary/antiunitary equivalence between π_ϕ and $\pi_\phi \circ \alpha$. This will be true just in case W intertwines the two representations, $W\pi_\phi(A)W^* = \pi_\phi \circ \alpha(A)$ for all $A \in \mathfrak{A}$ (i.e., W maps $\pi_\phi(\mathfrak{A})$ to $\pi_\phi \circ \alpha(\mathfrak{A})$ pointwise). The distinction allows us to accommodate the phenomenon of spontaneous symmetry breaking. According to one standard definition, a

Thus if CPT is a symmetry of a model of AQFT, there exists a unitary or antiunitary operator, Θ, such that for any physical representation π_ϕ, $\Theta\pi_\phi(\mathfrak{A})\Theta^* = \pi_\phi \circ \theta(\mathfrak{A})$, where $\pi_\phi \circ \theta$ is also a physical representation. In this case, (a) entails that $\Theta = \Theta^* = \Theta^{-1}$ (since any isometric involutive operator is self-adjoint). Meanwhile, (c) ensures that $\Theta U(a, \Lambda)\Theta = U(-a, \Lambda)$, in any sector carrying a unitary representation of the connected Poincaré transformations, $U(a, \Lambda)$. This ensures that CPT reflection commutes or anticommutes with the Hamiltonian and preserves the mass and spin properties of particles which are given by the Casimir invariants of the representation $U(a, \Lambda)$. In addition, (e) guarantees that CPT reflection maps conjugate charge sectors onto one another.[14]

This leads to a statement of the main theorem:

CPT Theorem. *Given a model of AQFT satisfying the Haag-Kastler axioms, additivity, the wedge intersection property, the distal split property, and the DHR/BF selection criterion, if the model also satisfies analyticity conditions sufficient to entail the Bisognano-Wichmann property, then there exists an antiunitary operator, Θ, (unique up to unitary equivalence), whose adjoint action represents an algebraic CPT transformation preserving the set of physical states, $\{\phi\}$, and satisfying (a)-(e).*

Algebraic proofs of the CPT theorem are based on generalizations of a lesser known result from constructive QFT. Bisognano and Wichmann (1975, 1976) prove that for QFTs satisfying the Wightman axioms, local algebras associated with spacelike wedges in the vacuum representation contain special invariants that generate particular Poincaré transformations. The *modular unitaries*, Δ_W^{it}, associated with $\mathfrak{R}_\omega(W)$ generate the unique 1-parameter group of W-preserving Lorentz boosts,

symmetry is unbroken iff W implements a unitary/antiunitary equivalence. Otherwise it is spontaneously broken. I thank two anonymous referees for drawing my attention to the subtleties surrounding the Roberts-Roepstorff theorem.

[14]Since conjugate charge representations are typically not quasiequivalent, it appears that CPT symmetry will generally be broken in charge sectors, i.e., the adjoint action of Θ will not intertwine π and $\bar{\pi}$. This is not the case. Even if there are no unitary intertwiners between π and $\bar{\pi}$, there may still exist antiunitary intertwiners. As we will go on to see in §3.6, each irreducible charge sector is equivalent to an irreducible representation of some compact gauge group G. Varadarajan (1968, Lem. 3.8) proves that a necessary and sufficient condition for the existence of antiunitary intertwiners between irreducible representations of G is that the representations in question are conjugate to one another. So there will always exist antiunitary intertwiners between charge sectors in the DHR/BF picture. If CPT symmetry is unbroken in the vacuum sector (as we expect is the case in many, if not all, physical models), then the adjoint action of Θ will intertwine conjugate charge representations.

$\Lambda_W(t)$.[15] The antiunitary *modular conjugation*, J_W, represents a P_1T reflection that reverses the direction of time and flips one spatial direction perpendicular to the edge of the wedge. Interestingly, it turns out that J_W also conjugates charge. The CPT theorem is then an immediate corollary of rotational covariance.

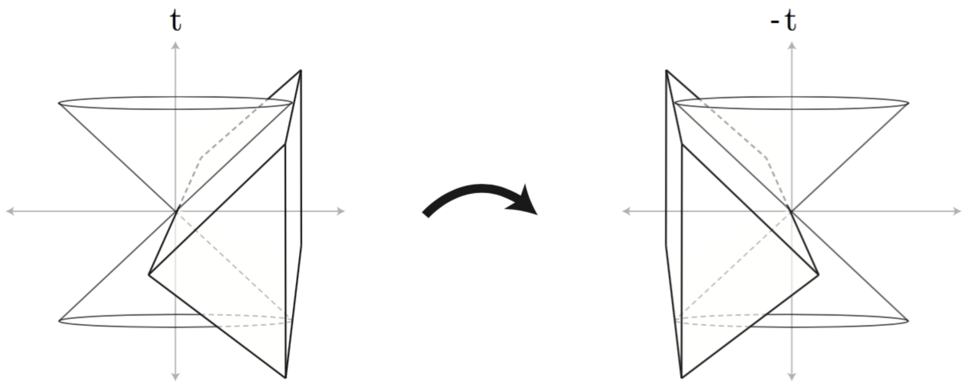

Figure 2: The P_1T reflection represented by J_W

The original proof of the Bisognano-Wichmann theorem uses extensive analytic continuation techniques relying on the special properties of gauge-dependent Wightman field operators. It thus does not directly apply to AQFT. Nonetheless, mathematical physicists have long suspected that the theorem is actually a more general consequence of the structure of gauge-independent local observables. As we will see in §3.3, the existence of local modular invariants is a consequence of the Haag-Kastler axioms and weak additivity, and their geometric interpretation is tightly constrained. This motivates the following:

[15] A spacelike wedge is the region of Minkowski spacetime causally connected to an immortal, uniformly accelerating observer, the so-called *Rindler wedge*. If the observer is accelerating in the x_1 direction, their trajectory can be written in standard coordinates as

$$x^0(\tau) = a^{-1}\sinh(\tau)$$
$$x^1(\tau) = a^{-1}\cosh(\tau)$$
$$x^2(\tau) = x^3(\tau) = 0,$$

where τ is proper time. The wedge region is defined by the condition $x^1 > |x^0|$. The Bisognano-Wichmann theorem tells us that in the vacuum representation, $\Delta_W^{it} = e^{2\pi i t K_1}$ (where K_1 is the generator of an x_1-boost). This is a simple rescaling of proper time translations along the observer's worldline.

Bisognano-Wichmann Property: *In the vacuum representation, for any spacelike wedge W, the wedge modular unitaries generate $\Lambda_W(t)$.*

The Bisognano-Wichmann property holds iff J_W represents a $\mathrm{CP_1T}$ reflection. Algebraic proofs of the CPT theorem therefore attempt to isolate analyticity assumptions that are sufficient for establishing the Bisognano-Wichmann property.

In 2-dimensional theories, no additional assumptions are needed. The first algebraic proof of the CPT theorem, Borchers (1992), inventively uses the analyticity properties entailed by covariance and the spectrum condition to establish the Bisognano-Wichmann property for 2-dimensional models of AQFT. In higher dimensions the situation is less clear. Haag (1996) conjectures that the Haag-Kastler axioms and the split property should be sufficient to entail the Bisognano-Wichmann property on their own, but this problem remains unsolved. Although there are models of the Haag-Kastler axioms in which the Bisognano-Wichmann property fails (Yngvason, 1994; Buchholz et al., 2000), there are none that also satisfy the split property.

The first proof for 3- and 4-dimensional theories, Guido and Longo (1995), drops the covariance axiom and spectrum condition in favor of a geometric constraint on Δ_W^{it}:

Modular Covariance: *In the vacuum representation, for any spacelike wedge W,*

$$\Delta_W^{it}\mathfrak{R}_\omega(O)\Delta_W^{-it} = \mathfrak{R}_\omega(\Lambda_W(t)O) \ .$$

This requires that the adjoint action of the modular unitaries maps arbitrary local algebras in the vacuum representation onto the algebras of $\Lambda_W(t)$-boosted regions. This covariance entails the Bisognano-Wichmann property, but showing that each Δ_W^{it} acts geometrically as a boost requires a detailed argument exploiting the algebraic and analytic properties of the modular invariants. Bain (2016) draws a number of philosophical conclusions about the algebraic CPT theorem (e.g., that it does not assume Lorentz invariance) based on a direct reading of the Guido-Longo proof. Swanson (2018) cautions against such a direct reading, arguing that modular covariance essentially bundles together the covariance axiom, spectrum condition, and additional analyticity properties, obscuring the physical justification behind various steps in the proof.

Here I will focus on the approach of Borchers (1995, 1996a, 1998, 2000), which seeks to identify precisely which analyticity conditions are needed in higher dimensions in addition to those already implicitly encoded in the Haag-Kastler axioms.

Although in their present form these conditions are quite technical and their physical motivation is poorly understood, pursuing this strategy will enable us to trace the clearest possible chain of argument back to the Haag-Kastler axioms.[16]

3 Deciphering the Theorem

Summing up the mathematical philosophy behind AQFT, Halvorson and Müger (2006, p. 740) observe:

> AQFT proceeds by isolating some structural assumptions that hold in most known QFT models. It formalizes these structural assumptions, and then uses "abstract but efficient nonsense" to derive consequences of these assumptions.

Prima facie, the algebraic CPT theorem is a paradigm example of this approach. With the exception of technical conditions like additivity, the wedge intersection property, and analyticity, its main structural inputs are reasonably physically transparent. The chain of argument is anything but. Somehow, using the geometric properties of wedge-localized modular invariants, we can construct an extended representation of the Poincaré group which miraculously includes an antiunitary CPT operator. To make the situation even more challenging, presentations of the CPT theorem in the mathematical physics literature typically begin by trying to establish the Bisognano-Wichmann property without providing physical motivation for this starting point. The proofs are spread over many separate papers, and often appeal to more abstract assumptions than the Haag-Kastler axioms, aiming for the highest level of mathematical generality possible. (Frequently they seek to prove the spin-statistics theorem at the same time.)

The goal of this section is to decipher the chain of physical reasoning behind the CPT theorem, starting from a more elementary algebraic foundation than existing presentations. The groundwork for the theorem is laid in §3.1-3 by connecting the modular conjugation to a reversal of the C^*-algebraic Lie product (lemmas 1-2). In §3.4, this connection is used to illuminate why time-reversal symmetry must

[16] There are several other approaches that deserve mention. Kuckert (1997) proves that if the wedge modular invariants map open regions onto open regions, then the Bisognano-Wichmann property follows. Buchholz et al. (2000) employs an alternative geometric constraint, the *condition of geometric modular action*, on the family of wedge modular conjugations $\{J_W\}$. Mund (2001) proves the Bisognano-Wichmann property for asymptotically complete QFTs with a mass gap using elementary algebraic assumptions and tools from Haag-Ruelle scattering theory.

be antiunitary and to motivate the focus on wedge regions (lemma 3). Proofs of these lemmas are given in the appendix. We then proceed through the heart of the theorem in §3.5-6 with the aim of using these ingredients to clarify the physical justification for each mathematical step. While the argument is certainly abstract, it is less nonsensical than it first appears.

3.1 The Canonical Involution

We suspect that a generic model of AQFT should contain a hidden CPT reflection symmetry, but the Haag-Kastler axioms only require covariance with respect to connected Poincaré symmetries. How can a reflection like CPT get into the mix?

Our first important observation is that there is already a passel of algebraic reflection symmetries hiding in plain sight. Even though the Haag-Kastler axioms do not explicitly mention reflections, they do so implicitly. By definition, every C^*-algebra employed by a model of AQFT is equipped with a canonical involution mapping, $^* : \mathfrak{A} \to \mathfrak{A}$, satisfying

$$\begin{aligned} &(A^*)^* = A, \quad (A+B)^* = A^* + B^*, \\ &(cA)^* = \bar{c}A^*, \quad (AB)^* = B^*A^*, \end{aligned} \tag{1}$$

for all $A, B \in \mathfrak{A}$, $c \in \mathbb{C}$ (where $\bar{c}$ denotes complex conjugation). The canonical involution can be viewed as a reflection of the algebra across its self-adjoint subspace, $\mathfrak{A}_{SA}$. Every operator in $\mathfrak{A}$ can be uniquely written in "complex form," $A = H + iK$, where $H = \frac{1}{2}(A + A^*)$ and $K = \frac{i}{2}(A^* - A)$ are self adjoint. A quick calculation reveals that the canonical involution acts as "complex conjugation," sending $A = H + iK$ to $A^* = H - iK$ and leaving $\mathfrak{A}_{SA}$ pointwise invariant. Thus just like the complex numbers, a C^*-algebra is self-similar; there is a conjugation operation that reflects the algebra across its "real axis," $\mathfrak{A}_{SA}$.

Strictly speaking, the canonical involution is not an automorphism of $\mathfrak{A}$. It reverses the order of operator multiplication, $(AB)^* = B^*A^*$, and it is conjugate-linear on the underlying vector space, $(c_1A + c_2B)^* = \overline{c_1}A^* + \overline{c_2}B^*$. It restricts to the identity on $\mathfrak{A}_{SA}$, however, and since physical quantities are represented by self-adjoint operators, this suggests that we should interpret it as a symmetry. Moreover, while the order of multiplication and the difference between i and $-i$ matters within the algebra, from the outside looking in, the choice of an operator product and complex unit looks like an arbitrary convention. We could choose right instead of left operator multiplication and $-i$ rather than i as a complex unit and still be able to encode the same algebraic relations.

We can capture this intuition as follows. Let $\mathfrak{A}^{op}$ denote the *opposite algebra* relative to $\mathfrak{A}$, consisting of the same underlying vector space, involution, and norm as $\mathfrak{A}$, but with the opposite C^*-product, $(AB)^{op} = BA$. Similarly, let $\mathfrak{A}^c$ denote the *conjugate algebra*, consisting of the same involution, norm, and operator product as $\mathfrak{A}$, but whose underlying vector space is conjugate, $i^c = -i$. Finally, let $\mathfrak{A}^{cop}$ denote the analogously defined *conjugate-opposite algebra*.

We say that two C^*-algebras are **-isomorphic* if there exists a linear bijection between them that preserves the identity, involution, and the operator product. (These conditions entail that the norm is also preserved.) An *anti-isomorphism* is similarly defined but reverses the order of the operator product, while a *conjugate-isomorphism* acts conjugate linearly on the underlying vector space. A *conjugate-anti-isomorphism* does both. It follows from this cluster of definitions that $\mathfrak{A}$ is *antiautomorphic*, *conjugate-automorphic*, or *conjugate-antiautomorphic* (to itself) iff $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^{op}$, $\mathfrak{A}^c$, or $\mathfrak{A}^{cop}$ respectively.

Lemma 1. *Let $\mathfrak{A}$ be any C^*-algebra:*

(i) $\mathfrak{A}$ is naturally isomorphic to $\mathfrak{A}^{cop}$,

(ii) $\mathfrak{A}^{op}$ is naturally isomorphic to $\mathfrak{A}^c$,

(iii) $\mathfrak{A}$ is naturally anti-isomorphic to $\mathfrak{A}^{op}$ and $\mathfrak{A}^c$,

(iv) $\mathfrak{A}$ is naturally conjugate-isomorphic to $\mathfrak{A}^{op}$ and $\mathfrak{A}^c$,

with the relevant isomorphisms defined by the involution structure common to all four algebras.

This lemma precisely characterizes the sense in which a C^*-algebra is self-similar: $\mathfrak{A}$ is naturally conjugate-antiautomorphic to itself, with the canonical involution defining the relevant reflection symmetry. Furthermore, it reveals that there is an entire family of related isomorphisms linking C^*-algebras with opposite choices of operator product and complex unit.

3.2 The Lie-Jordan Product and State Space Orientation

Do these formal algebraic symmetries have any physical consequences? If the observables are all contained in $\mathfrak{A}_{SA}$, what role does the "imaginary" part of the algebra play? Our second important observation, following Alfsen and Shultz (2001, 2003), is that observables have double roles — they represent physical quantities and they act as infinitesimal generators of symmetries. The operator product is in fact two products in disguise:

Theorem (Alfsen-Shultz). *In any C^*-algebra, the operator product has a natural decomposition,*

$$AB = A \bullet B - i(A \star B) \ , \tag{2}$$

where $A \bullet B := \frac{1}{2}(AB + BA)$ is a commutative, non-associative Jordan product, and $A \star B := \frac{i}{2}(AB - BA)$ is a noncommutative, associative Lie product.[17]

The self-adjoint subspace, $\mathfrak{A}_{SA}$, is closed under the Jordan product which encodes all spectral information about the observables.[18] The Jordan product therefore captures the way in which observables represent physical quantities. The Lie product, on the other hand, captures the way in which observables generate symmetries. Each element $A \in \mathfrak{A}_{SA}$ defines a 1-parameter group of automorphisms of $\mathfrak{A}$, given by

$$\alpha_t(X) := e^{itA} X e^{-itA}, \tag{3}$$

for all $t \in \mathbb{R}$, $X \in \mathfrak{A}$. Infinitesimally, this can be rewritten in terms of the Lie

[17]See Alfsen and Shultz (2003, ch. 6). The Jordan product satisfies the Jordan identity,

$$(A^2 \bullet B) \bullet A = A^2 \bullet (B \bullet A) \ ,$$

where $A^2 := A \bullet A$, while the Lie product satisfies the well-known Jacobi identity,

$$A \star (B \star C) + C \star (A \star B) + B \star (C \star A) = 0 \ .$$

They also satisfy two important compatibility conditions, the Leibniz rule and the associator identity:

$$A \star (B \bullet C) = (A \star B) \bullet C + B \bullet (A \star C) \ , \quad \text{and} \quad (A \bullet B) \bullet C - A \bullet (B \bullet C) = (A \star C) \star B \ .$$

The first tells us that that the map $B \mapsto A \star B$ is a derivation on $(\mathfrak{A}_{SA}, \bullet)$ viewed as a real Jordan algebra. The second quantifies the departure from associativity of the Jordan product and is linked to the Heisenberg uncertainty relations. This structure allows us to canonically view the original C^*-algebra, $\mathfrak{A}$, as a dual Lie-Jordan algebra defined on the complexified space $\mathfrak{A}_{SA} + i\mathfrak{A}_{SA}$. See Zalamea (2018) for an analysis of the physical significance of this dual structure in both quantum and classical mechanics.

[18]The spectrum of $A \in \mathfrak{A}_{SA}$ is defined as the set of real numbers λ such that $A - \lambda I$ is not invertible. The invertibility of $A - \lambda I$ is equivalent to the existence of $B \in \mathfrak{A}_{SA}$ such that $(A - \lambda I) \bullet B = I$ and $(A - \lambda I)^2 \bullet B = (A - \lambda I)$ (Alfsen and Shultz, 2003, Lem. 1.16–Cor. 1.19).

product:

$$\frac{d\alpha_t(X)}{dt}|_{t=0} = i(AX - XA) \\ = 2(A \star X) \tag{4}$$

Thus the Lie product $A \star X$ represents the tangent vector of the flow associated with the group of symmetries defined by A at $t = 0$. Unlike the Jordan structure, $\mathfrak{A}_{SA}$ is not closed under the Lie product. In fact the closure of $\mathfrak{A}_{SA}$ with respect to the Lie product is the entire C^*-algebra. This reveals that the imaginary part of $\mathfrak{A}$ algebraically encodes the generating relationship between observables and symmetries.

Putting this result together with lemma 1, we see that the four algebras we have introduced, $\mathfrak{A}$, $\mathfrak{A}^{op}$, $\mathfrak{A}^c$, and $\mathfrak{A}^{cop}$, all have the same Jordan product, so they agree on spectral properties of observables. The choice of a C^*-operator product, Lie product, and complex unit are constrained — specifying any two naturally defines a choice for the third:

Algebra	C^*-Product	Lie Product	Complex Unit
$\mathfrak{A}$	AB	$A \star B$	i
$\mathfrak{A}^{op}$	BA	$B \star A$	i
$\mathfrak{A}^c$	AB	$B \star A$	$-i$
$\mathfrak{A}^{cop}$	BA	$A \star B$	$-i$

The algebras $\mathfrak{A}$ and $\mathfrak{A}^{cop}$ adopt one possible convention linking observables to symmetries, the Lie product $A \star B$, while $\mathfrak{A}^{op}$ and $\mathfrak{A}^c$ adopt the opposite convention, the opposite Lie product $(A \star B)^{op} = B \star A$. Thus, when we specify a model of AQFT by choosing a net of C^*-algebras we are implicitly choosing one of these conventions.[19] Since the Lie product is antisymmetric, $A \star B = -(B \star A)$, if we choose the opposite convention, the tangent vectors defined by (4) will point in opposite directions. The canonical anti-isomorphism and conjugate-isomorphism linking $\mathfrak{A}$ with $\mathfrak{A}^{op}$ and $\mathfrak{A}^c$ reverse the Lie product. They flip the generating relationship between observables and symmetries.

This algebraic story has an elegant geometric dual: the natural Lie product on $\mathfrak{A}$ corresponds to an orientation structure on its *state space*, $\mathcal{S}(\mathfrak{A})$, the collection

[19] If we start with $\mathfrak{A}_{SA}$ viewed as a real Jordan algebra, there will typically be infinitely many Lie products compatible with the given Jordan product. Every such compatible Lie product has a unique opposite. Alfsen and Shultz (2003, Thm. 6.15) prove that Jordan-compatible Lie products are in 1-1 correspondence with Jordan-compatible C^*-products on the complexification $\mathfrak{A}_{SA} + i\mathfrak{A}_{SA}$. The same reasoning applies to the opposite choice of complex unit, $\mathfrak{A}_{SA} - i\mathfrak{A}_{SA}$.

of all states on $\mathfrak{A}$. The state space is a compact convex set, with extremal points representing pure states. It has a natural order structure inherited from $\mathfrak{A}$ and its exposed faces form a lattice whose orthogonality relations mirror the spectral information encoded in the lattice of projection operators in $\mathfrak{A}$. Kadison (1951) proves that all of the spectral information encoded in the real Jordan algebra $(\mathfrak{A}_{SA}, \bullet)$ is captured by the convex geometry of $\mathcal{S}(\mathfrak{A})$.

Alfsen et al. (1980) prove that the full structure of $\mathfrak{A}$ can be recovered by equipping $\mathcal{S}(\mathfrak{A})$ with an orientation structure that determines a 1-1 pairing between observables, viewed as $\mathbb{R}$-valued affine functions on $\mathcal{S}(\mathfrak{A})$, and 1-parameter groups of symmetries of $\mathcal{S}(\mathfrak{A})$. For a 2-level quantum system, this orientation structure is easily visualized. In this case the state space, $\mathcal{S}(M_2)$, is isomorphic to a Euclidean 3-ball whose boundary points represent pure states and whose interior points represent mixed states. Each observable A determines a bounded affine function attaining maximum and minimum values on some pair of antipodal points. The non-self-adjoint operators iA and $-iA$ generate infinitesimal rotations of the 3-ball around the diameter connecting these antipodal points. There are two possible choices of orientation: iA can generate clockwise and $-iA$ counterclockwise rotations, or vice versa.

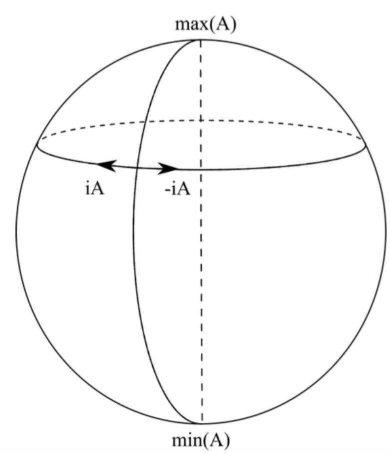

Figure 3: $\mathcal{S}(M_2)$ with clockwise orientation.

This basic idea forms the basis for the general case. For an arbitrary C^*-algebra, every minimal exposed face of $\mathcal{S}(\mathfrak{A})$ is either isomorphic to a Euclidean 3-ball or a line segment, the former if the face is generated by distinct pure states whose GNS representations are quasiequivalent and the latter if the GNS representations are inequivalent. An orientation for $\mathcal{S}(\mathfrak{A})$ is then given by a suitably continuous choice of orientation, clockwise or counterclockwise, for each facial 3-ball. Unlike a total manifold orientation where there are only two choices, there are in general, infinitely many orientation structures of $\mathcal{S}(\mathfrak{A})$ that are in 1-1 correspondence with Jordan-compatible Lie products on $\mathfrak{A}_{SA}$ (Alfsen and Shultz, 2001, Thm. 5.73). Every such orientation, however, has a unique opposite. Down the line, this geometric interpretation of the Lie product will give us valuable insight into the CPT theorem. Symmetries that reverse the Lie product correspond to orientation-reversing reflections of the theory's state space.

3.3 Tomita-Takesaki Modular Theory

We have gone from no suspects to an entire slew of them. Every C^*-algebra used by a model of AQFT has a canonical conjugate-antiautomorphism as well as a family of related mappings connecting the algebra to its opposite, conjugate, and conjugate-opposite algebra. We do not expect every algebraic symmetry to be a physical symmetry, however, since they will not necessarily preserve superselection structure in the given set of physically significant GNS representations.[20]

There is an additional constraint. In many, possibly all, models of AQFT satisfying the assumptions of §2.1, vacuum states are CPT-invariant. This entails that the algebraic CPT automorphism, θ, is implementable in vacuum representations, i.e., $\Theta\pi_\omega(A)\Theta = \pi_\omega \circ \theta(A)$ for all $A \in \pi_\omega(\mathfrak{A})$.[21] In general, the canonical involution will not be unitarily or antiunitarily implementable in any physically significant representations, but under certain technical conditions that are guaranteed to hold for every local algebra in vacuum representations (via the Reeh-Schlieder theorem), the involution can be split into two pieces, one of which is antiunitary. Furthermore

[20]Since each of these mappings induces a bijection on pure states, they must act as a permutation on the unitary equivalence classes of irreducible representations of $\mathfrak{A}$. So they permute sectors in a broad sense. The difficulty is that there is no guarantee that all of these sectors are physical sectors. They need not satisfy the DHR/BF selection criteria (or any other physically significant selection criteria). As emphasized in §2.3, physical symmetries must induce a bijection on pure states in the set of physically possible states, $\{\phi\}$, which is a significantly stronger condition. See Baker and Halvorson (2013) for further discussion of these issues.

[21]Must vacuum states be CPT-invariant? Borchers and Yngvason (2000, Thm. 2.1) prove that the family of vacuum representations is CPT-invariant, but this leaves open the possibility that CPT-symmetry could be spontaneously broken in vacuum states, and Θ permutes disjoint vacuum representations. The chain of argument sketched in §3.5, however, appears to rule this scenario out. The interplay between CPT symmetry and spontaneous symmetry breaking is a subtle issue that requires further study.

this antiunitary piece implements the canonical anti-isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^{op}$. This is our third important observation and the subject of *Tomita-Takesaki modular theory.*

In its most general mathematical setting, modular theory studies the action of a von Neumann algebra, $\mathfrak{M}$, on a Hilbert space, $\mathcal{H}$, with a cyclic, separating vector, Φ. The former means that $\overline{\mathfrak{M}\Phi} = \mathcal{H}$ (where the overline denotes closure in the Hilbert space norm topology), and the latter means that $A\Phi = B\Phi$ entails $A = B$. As a result, we can use Φ to translate between algebraic structure on $\mathfrak{M}$ and geometric properties of $\mathcal{H}$. In general, the canonical algebraic involution does not give rise to an isometry of the Hilbert space structure, but can always be represented as a reflection with an additional "twist."

Using Φ, we can define the (generally unbounded) antilinear operator,

$$S_0 A\Phi = A^*\Phi \ , \tag{5}$$

for all $A \in \mathfrak{M}$. This can be extended to a closed, antilinear operator, S, defined on a dense subset of $\mathcal{H}$. Any such operator has a unique polar decomposition into a partial isometry and a positive, self-adjoint (generally unbounded) operator called the modulus. In the present case, the polar decomposition of S is given by:

$$S = J\Delta^{1/2} \ , \tag{6}$$

with partial isometry J and modulus $(S^*S)^{1/2} = \Delta^{1/2}$. It can be shown that $J = J^* = J^{-1}$, and thus J, called the *modular conjugation*, is antiunitary, self-adjoint, and involutive. The positive, self-adjoint operator Δ is called the *modular operator*. The Hilbert space action of the algebraic involution can thus be broken up into a reflection, J, with an additional twist, $\Delta^{1/2}$.[22]

Together, the operators Δ, J have a rich structure that forms the basis of Tomita-Takesaki modular theory. Its central theorem establishes the existence of a canonical group of automorphisms of $\mathfrak{M}$ and a canonical anti-isomorphism between $\mathfrak{M}$ and its commutant $\mathfrak{M}'$:

[22]The Tomita operator, S, implementing the canonical involution is isometric iff $S = J$ and Δ is the identity operator. This is the case iff Φ is a tracial state. Since local algebras in AQFT are generically type III von Neumann algebras and thus lack tracial states, their canonical involutions will not be antiunitarily implementable in any physically significant representation. Global algebras are generically type I, but physical states are typically not separating for these, and so the conditions for applying modular theory do not apply.

Theorem (Tomita-Takesaki)**.** *If $\mathfrak{M}$ is a von Neumann algebra acting on a separable Hilbert space, $\mathcal{H}$, with a cyclic and separating vector, $\Phi \in \mathcal{H}$, then*

$$\begin{aligned} (i)\ & J\Phi = \Phi = \Delta\Phi\ , \\ (ii)\ & \Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}, \forall t \in \mathbb{R}, \\ (iii)\ & J\mathfrak{M}J = \mathfrak{M}' \end{aligned}$$

where Δ, J are the associated modular invariants.[23]

Since Δ is positive, Δ^{it} is unitary, and (ii) defines a strongly continuous 1-parameter automorphism group of $\mathfrak{M}$ — the *modular automorphism group*. By (iii), the adjoint action of the modular conjugation generates an anti-isomorphism and an equivalent conjugate-isomorphism between $\mathfrak{M}$ and $\mathfrak{M}'$. This allows us to canonically identify $\mathfrak{M}'$ with $\mathfrak{M}^{op}$ and $\mathfrak{M}^c$:

Lemma 2. *Let $\mathfrak{M}$ be a von Neumann algebra with a cyclic and separating vector, $\Phi \in \mathcal{H}$, and let Δ, J be the associated modular objects. J defines natural *-isomorphisms $\psi^{op} : \mathfrak{M}' \to \mathfrak{M}^{op}$ and $\psi^c : \mathfrak{M}' \to \mathfrak{M}^c$.*

Thus within the setting of modular theory, we find that the reflection symmetry relating $\mathfrak{M}$ and $\mathfrak{M}^{op}$ (equivalently $\mathfrak{M}^c$) is always antiunitarily implemented by J, with the commutant $\mathfrak{M}'$ identified with $\mathfrak{M}^{op}$ (equivalently $\mathfrak{M}^c$).

In AQFT, the Reeh-Schlieder theorem guarantees that these conditions hold for local algebras in vacuum representations:

Theorem (Reeh-Schlieder)**.** *If a model of AQFT satisfies the Haag-Kastler axioms and weak additivity, then any vacuum state, ω, is cyclic and separating for every local von Neumann algebra, $\mathfrak{R}_\omega(O)$, where O' is a proper subset of Minkowsi spacetime.*[24]

[23]In the case where S is a bounded operator an elementary proof can be given. See Blackadar (2006, Thm. III.4.3.2.). The unbounded case is highly non-trivial. See Takesaki (2000, Ch. VI-VII) and Kadison and Ringrose (1997, Ch. 9.2) for different versions of the full proof.

[24]This is the first instance of an analytic continuation argument in the proof of the CPT theorem. Here is the main idea behind the proof: let Ω be the vector representing ω in the GNS Hilbert space $\mathcal{H}_\omega$. For some region O, suppose that an arbitrary vector $\Psi \in \mathcal{H}_\omega$ is orthogonal to $\mathfrak{R}_\omega(O)\Omega$. Ψ will also be orthogonal to $U(a)\mathfrak{R}_\omega(\tilde{O})U(-a)\Omega$ where $\tilde{O}$ is any subregion strictly contained in O and $U(a)$ are sufficiently small translations. The spectrum condition entails that the vector-valued function $U(a)\Omega$ has analytic extension to the forward tube $T(V^+) := \{z \in \mathbb{C}^4 | \text{Im } z \in \overline{V^+}\}$, where V^+ is the forward lightcone. This fact is used to show that the function $\langle \Omega, U(a)\mathfrak{R}_\omega(\tilde{O})U(-a)\Omega \rangle$ is the boundary value of a holomorphic function on the

So every local algebra in a vacuum representation has a canonical antiunitary conjugation, J, the reflection portion of the Tomita operator, S, implementing the algebraic involution. Lemma 2 then entails that the reflection symmetry linking $\mathfrak{R}_\omega(O)$ to $\mathfrak{R}_\omega(O)^{op}$ (equivalently $\mathfrak{R}_\omega(O)^c$) is always antiunitarily implemented with $\mathfrak{R}_\omega(O)'$ identified with $\mathfrak{R}_\omega(O)^{op}$ (equivalently $\mathfrak{R}_\omega(O)^c$). As we will go on to see, this has important geometric ramifications since commuting algebras are associated with spacelike separated regions.

3.4 Time Reversal

The local modular conjugation operators are prime suspects for representing time reversal transformations. In AQFT, the dynamics are encoded in the representation of the translation subgroup whose existence is posited by the covariance axiom. Relative to a given Lorentz frame, we can write the time evolution of an arbitrary observable as

$$\alpha_t(X) = e^{itP_0} X e^{-itP_0} \ , \tag{7}$$

where P_0 is the global Hamiltonian.

Choosing an arbitrary $t = 0$ allows us to identify time evolved observables in two distinct temporal directions, t and $-t$. As Roberts (2017) emphasizes, in any quantum theory, a time reversal transformation should reverse the temporal ordering of observables (in the Heisenberg picture) while preserving the length of temporal intervals, thus mapping observables at time t to observables at time $-t$. It should also be an involution. If a theory is invariant under a symmetry implementing a time reversal transformation (possibly along with other transformations), this symmetry will be represented by a unitary or antiunitary operator, T. In addition, it should either commute or anticommute with P_0 so that the form of the dynamical laws is unaffected by the reversal. Putting these four constraints together, we have

$$T\alpha_t(X)T = Te^{itP_0} X e^{-itP_0} T = e^{-itP_0} TXT e^{itP0} = \alpha_{-t}(TXT) \ . \tag{8}$$

forward tube that vanishes in some neighborhood of the origin and therefore vanishes everywhere. Consequently Ψ is orthogonal to $U(a)\mathfrak{R}_\omega(\tilde{O})U(-a)\Omega$ for all translations. Weak additivity then entails that $\Psi = 0$, and so Ω is cyclic for $\mathfrak{R}_\omega(O)$. Microcausality entails the separating property. (For a full proof, see Horuzhy 1990, Thm. 1.3.1.) Note that the theorem generalizes to any state *analytic for the energy.* This requirement entails that the field strength cannot grow too large as a function of the energy and is guaranteed to hold for any states satisfying the DHR/BF selection criteria. We will not need this generalization in the ensuing discussion.

The spectrum condition entails that the spectrum of P_0 is positive in all Lorentz frames. As a consequence, Roberts shows that the only way to consistently implement (8) with $P_0 \neq 0$ is for T to be antiunitary. The key idea is simple, but illuminating. Since the generators of the translations are unique, it follows from (8) that $TitP_0T = -itP_0$. If T is unitary, linearity entails that $Tit = itT$ and hence that $TP_0T = -P_0$. (In this case, T anticommutes with P_0.) Since unitary operators preserve inner products, if $P_0 \neq 0$ the spectrum of P_0 cannot be bounded from below, violating the spectrum condition. So T must be antiunitary, in which case antilinearity entails that $Tit = -itT$, and T commutes with P_0.

Roberts's argument sheds considerable light on why time reversal symmetries in quantum mechanics must be antiunitary. Our discussion of the dual Lie-Jordan product in §3.2 yields additional insight. Any time reversal symmetry worth the name must send $\alpha_t \mapsto \alpha_{-t}$. There are only two ways to to do this. The first is a unitary transformation sending $P_0 \mapsto -P_0$. The second is an antiunitary transformation that reverses the Lie product and thus the generating relationship between P_0 and α_t.[25] The first route is blocked by the spectrum condition, leaving the second as the only viable way to implement time reversal symmetry in QFT. Given the constraints linking the Lie product, C^*-product, and complex unit, antiunitary time reversal can be viewed either as a conjugate-isomorphism sending $i \mapsto -i$ with fixed C^*-product (thus sending $e^{it} \mapsto e^{-it}$) or an anti-isomorphism sending $AB \mapsto BA$ with fixed complex unit (thus sending $e^{it}(\cdot)e^{-it} \mapsto e^{-it}(\cdot)e^{it}$). Since $\mathfrak{A}^c$ is naturally isomorphic to $\mathfrak{A}^{op}$, the viewpoints are completely equivalent. Physically speaking, both are ways of reversing the generating relationship between observables and 1-parameter groups of symmetries.

But more is required. In order to be a symmetry of AQFT, a time reversal transformation should also preserve subsystem localization information; T should be a symmetry of the net of observable algebras, not just the global algebra. In the vacuum representation, if $O_1 \subset O_2$, and thus $\mathfrak{R}_\omega(O_1) \subset \mathfrak{R}_\omega(O_2)$, we require that $T\mathfrak{R}_\omega(O_1)T \subset T\mathfrak{R}_\omega(O_2)T$. Moreover, if a system is localized in a particular type of region (e.g., a lightcone, doublecone, spacelike wedge, spacelike cone), time reversal symmetry should preserve this localization. It should map like-regions onto like-regions.

By the main Tomita-Takesaki theorem, $J\mathfrak{R}_\omega(O)J = \mathfrak{R}_\omega(O)'$, therefore in order for J to represent a time reversal symmetry, $\mathfrak{R}_\omega(O)'$ must be the local algebra of a region with the same geometry as O. In general, there is no guarantee that $\mathfrak{R}_\omega(O)'$

[25] A symmetry preserves the Lie product iff it is unitarily represented and reverses the Lie product iff it is antiunitarily represented. See Alfsen and Shultz (2001, Thm. 4.27).

will be a local algebra at all, however microcausality entails that $\mathfrak{R}_\omega(O') \subset \mathfrak{R}_\omega(O)'$. If a stronger *duality* relation obtains, the local algebras are as large as possible consistent with microcausality, and $\mathfrak{R}_\omega(O') = \mathfrak{R}_\omega(O)'$. If O has the same geometry as O', then the modular conjugation meets this necessary requirement.

Are there regions like this — regions that are isometric to their spacelike complement and for which we expect duality to hold quite generally? The answer is yes. In Minkowski spacetime, the causal complement of a spacelike wedge is another spacelike wedge. In fact, spacelike wedges are essentially the only causally well-behaved regions with this property:

Lemma 3. *If O is an open, convex, causally complete proper subregion of Minkowski spacetime such that O and O' are isometric, then O is a spacelike wedge.*

Moreover, *wedge duality*, $\mathfrak{R}_\omega(W)' = \mathfrak{R}_\omega(W')$, is a sufficient condition (in conjunction with the Haag-Kastler axioms) for applying the tools of DHR/BF superselection theory. It is expected to hold (in the vacuum representation) with greater generality than other forms of duality.[26]

3.5 Wedge Reflection

The focus of our investigation has narrowed to modular conjugations associated with spacelike wedges in the vacuum representation. (Incidentally, this is where most presentations of the algebraic CPT theorem begin, obscuring the physical detective work that has gotten us this far.) If wedge duality holds, then $J_W \mathfrak{R}_\omega(W) J_W = \mathfrak{R}_\omega(W)' = \mathfrak{R}_\omega(W')$, where J_W is the modular conjugation associated with wedge W, and W' is the opposite wedge, the reflection of W across one spatial direction (perpendicular to the edge of W). Thus, because of the fact that J_W is a modular conjugation, mapping $\mathfrak{R}_\omega(W)$ onto its commutant, combined with duality and the unique geometry of W, J_W is a candidate for representing a spatial reflection. If it does so (and thus preserves subsystem localization), and if it commutes with the Hamiltonian, J_W will also represent a time reversal symmetry since it is an antiunitary involution.

[26]Wedge duality entails *essential duality*, a technical condition needed to prove that the DHR/BF category has sufficient structure to represent charges. Essential duality requires that the dual net, $\mathfrak{R}_\omega(O)^d := \mathfrak{R}_\omega(O')'$, satisfies microcausality. (It should be noted that as a restriction on the family of physical states, $\{\phi\}$, the DHR/BF selection criteria can be applied whether or not duality obtains.) Another widely discussed duality condition, *Haag duality*, requires that $\mathfrak{R}_\omega(D)' = \mathfrak{R}_\omega(D')$ for any doublecone D. It is equivalent to the absence of spontaneous symmetry breaking in the vacuum sector and is therefore of more limited interest, although it does hold in a number of important models, such as the free Bosonic field.

Under the same elementary conditions needed for the Reeh-Schlieder theorem, Borchers (1992) and Weisbrock (1992) establish a remarkable result settling the latter question:

Theorem (Borchers-Weisbrock)**.** *If a model of AQFT satisfies the Haag-Kastler axioms (except the spectrum condition) and weak additivity, then the spectrum condition holds iff*

$$\Delta_W^{it} U(a) \Delta_W^{-it} = U(\Lambda_W(t)a)$$
$$J_W U(a) J_W = U(ra)$$

where $U(a)$ is the unitary implementing an arbitrary translation in the vacuum representation, $\Lambda_W(t)$ is the unique 1-parameter group of W-preserving Lorentz boosts, and r is the P_1T reflection defined by $r(a_0, a_1, a_2, a_3) = (-a_0, -a_1, a_2, a_3)$, with a_1 being a spacelike translation perpendicular to the edge of the wedge.[27]

The modular objects Δ^{it} and J_W thus have the right commutation relations with the translations to be interpreted as wedge-preserving Lorentz boosts and a P_1T reflection. For 2-dimensional AQFTs, wedge duality is a direct corollary of the Borchers-Weisbrock theorem. In higher dimensions, however, counterexamples constructed by Yngvason (1994) show that things can go haywire in the directions along the edge of the wedge and the modular invariants may not map doublecones onto doublecones.

To ensure that the modular invariants act geometrically, additional analyticity assumptions are needed. Borchers (1995, 1996a, 1998, 2000) identifies two such

[27]Borchers originally proved the forward direction and Weisbrock the converse. The forward proof has since been greatly streamlined by Florig (1998). The key idea is as follows: spacelike translations along any direction in the characteristic 2-plane of W form a positively generated 1-parameter group $U(s)$, such that $U(s)\Omega = \Omega$ and $U(s)\mathfrak{R}_\omega(W)U(-s) \subset \mathfrak{R}_\omega(W)$ for $s \geq 0$ (a group of so-called *half-sided translations*). Using the fact that $U(s)$ is positively generated, along with the analytic properties of the modular automorphism group encoded in the KMS condition, Florig shows that the function

$$f(z) = \langle \Delta_W^{i\overline{z}} A'\Omega, U(e^{2\pi z})\Delta_W^{-iz} A\Omega \rangle$$

which is analytic in the interior of the complex strip $S(0, 1/2)$, can be extended to a holomorphic function which is bounded, and thus constant. (Here, A and A' are arbitrary elements of $\mathfrak{R}_\omega(W)$ and $\mathfrak{R}_\omega(W)'$ respectively.) This entails, in particular, that $\Delta_W^{it} U(e^{2\pi t}s)\Delta_W^{-it} = \Delta_W^{i0} U(e^{2\pi 0}s)\Delta_W^{-i0} = U(s)$ and $J_W U(-s) J_W = U(s)$. Extending these commutation relations to arbitrary translation vectors, a, is then a straightforward calculation exploiting the algebraic and analytic properties of Δ_W^{it} and J_W.

conditions that are equivalent to wedge duality and the Bisognano-Wichmann property. Let D be a doublecone contained within a spacelike wedge, W, and let $K(D) \subset W$ denote the cylindrical set obtained by translating D in some direction parallel to the edge of the wedge. Wedge duality requires that there be enough elements $A \in \mathfrak{R}_\omega(K(D))$ such that $U(\Lambda_W(t))A\Omega$ has bounded analytic continuation into the strip $S(-1/2, 0)$. For any such element, it can be proven that there exists an operator, $\hat{A}$, affiliated with $\mathfrak{R}_\omega(K(rD)) \subset \mathfrak{R}_\omega(W')$, such that

$$U(\Lambda_W(-i/2))A\Omega = \hat{A}\Omega \ . \tag{9}$$

The Bisognano-Wichmann property requires that in addition, there is a large enough set of such analytic elements closed under involution. This motivates the following:

> **B-Analyticity** *The set of $A \in \mathfrak{R}_\omega(K(D))$ such that $U(\Lambda_W(t))A\Omega$ has bounded analytic continuation into the strip $S(-1/2, 0)$ is *-strong dense in $\mathfrak{R}_\omega(K(D))$.*

> **B-Reality** *The set of $A \in \mathfrak{R}_\omega(K(D))$ such that both $U(\Lambda_W(t))A\Omega$ and $U(\Lambda_W(t))A^*\Omega$ can be analytically continued, with $\hat{A}^* = \widehat{A^*}$, is *-strong dense in $\mathfrak{R}_\omega(K(D))$.*

Theorem (Borchers)**.** *If a model of AQFT satisfies the Haag-Kastler axioms and the wedge intersection property, then*

(i) wedge duality holds in the vacuum representation iff B-analyticity holds,

(ii) the Bisognano-Wichmann property holds iff wedge duality and B-reality hold.[28]

[28]The wedge intersection property is only needed for (ii). Borchers's proof is very technical, but we can gain some understanding of it by focusing on the significance of the cylindrical sets $K(D)$ and equation (9). Note that for any $K(D) \subset W$, $K(D) = (W + a) \cap (W' + b)$ where a, b, are spacelike translations in the characteristic 2-plane of W. We consider two algebras $\mathfrak{R}_\omega(K(D)) = \mathfrak{R}_\omega(W + a) \cap \mathfrak{R}_\omega(W' + b)$ and $\widehat{\mathfrak{R}}_\omega(K(D)) = \mathfrak{R}_\omega(W + a) \cap \mathfrak{R}_\omega(W + b)'$. It follows that, $\mathfrak{R}_\omega(K(D)) \subseteq \widehat{\mathfrak{R}}_\omega(K(D))$.

The Borchers-Weisbrock theorem entails that Δ^{it} and $U(\Lambda_W(t))$ commute, and since both leave Ω invariant, they differ by a 1-parameter group $F_W(t)$. If $A \in \mathfrak{R}_\omega(K(D))$ is an analytic element, then at the lower boundary of $S(-1/2, 0)$,

$$\begin{aligned} U(\Lambda_W(-i/2))A\Omega &= F_W(-i/2)\Delta_W^{1/2}A\Omega \\ &= F_W(-i/2)J_W A^* J_W\Omega \\ &= J_W F_W(-i/2)A^* J_W\Omega \ , \end{aligned}$$

Setting $V(\Lambda_W(t)) := \Delta_W^{it}$, this theorem allows us to define a unitary representation of the Poincaré group, called the *minimal representation*, which acts covariantly on the observable net and satisfies the spectrum condition.[29] Because of the analytic properties of these unitaries, $\Delta_W^{1/2} = V(\Lambda_W(-i/2))$ is an element of the complex Lorentz group and thus $J_W = \Delta_W^{1/2} S_W$ is the product of a complex Lorentz transformation and the canonical involution on the wedge algebra implemented by S_W. This allows us to show that the minimal representation contains additional reflection symmetries. In particular, we can define a "PT" operator,

$$\Theta := J_W V(R_W(\pi)) \ , \tag{10}$$

where $V(R_W(\pi))$ implements a spatial rotation by π in the plane along the edge of the wedge.[30] The scare quotes are included to emphasize that Θ may do more than just represent a PT reflection. Indeed, in the next section we will see that it must also conjugate charge, making it a CPT operator.

But before we can proceed, one final wrinkle must be ironed out. As a consequence of the Haag-Kastler axioms and weak additivity, we have discovered that J_W has the right algebraic properties to represent a $\mathrm{P}_1\mathrm{T}$ reflection. Adding

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where the second line follows from the definition of the modular invariants, $S = J\Delta^{1/2}$, and the last line by the Borchers-Weisbrock theorem. In general, $F_W(-i/2)A^*$ is not a bounded operator, but it is affiliated with $\mathfrak{R}_\omega(K(D)) \subset \mathfrak{R}_\omega(W+a) \subset \mathfrak{R}_\omega(W)$. Thus by the Tomita-Takesaki theorem and the Borchers-Weisbrock theorem, $J_W F_W(-i/2)A^* J_W$ is affiliated with $\widehat{\mathfrak{R}}_\omega(K(rD)) \subset \mathfrak{R}_\omega(W-a)' \subset \mathfrak{R}_\omega(W)'$. Wedge duality holds iff $\widehat{\mathfrak{R}}_\omega(K(rD)) = \mathfrak{R}_\omega(K(rD))$. Using Lorentz invariance, Borchers shows that this is the case iff the set of analytic elements is *-strong dense. The Bisognano-Wichmann property holds iff $F_W(t)$ is trivial. If so, then we have $\widehat{A^*}\Omega = J_W A J_W \Omega = (J_W A^* J_W)^*\Omega = \hat{A}^*\Omega$, and B-reality holds. The converse requires a detailed analytic continuation argument. See Borchers (2000, Thm. IV.2.2) for details.

[29] The key to defining the Lorentz group is to note that any Lorentz transformation is the product of boosts in three linearly independent spacelike directions, and that each such boost is part of the stabilizer subgroup of some wedge. The translations are a bit trickier. Consider two wedges $W + a \subset W$, where a is a lightlike translation in the characteristic 2-plane of W. Using the Borchers-Weisbrock theorem, it can be shown that

$$\lim_{t\to\infty} \Delta_{W+a}^{it}\Delta_W^{-it} = \lim_{t\to\infty} V(\Lambda_{W+a}(t))V(\Lambda_W(-t))$$

converges strongly and therefore defines a unitary operator $V(a)$ acting like a lightlike translation in the a-direction. The remaining translations can then be constructed as products of lightlike translations. See Borchers (2000, §IV.4) for the complete construction.

[30] In order to prove this it must be shown that the product J_W maps doublecones onto doublecones and that $J_W V(R_W(\pi))$ does not depend on the choice of a particular wedge W. This hinges on the analytic properties of the modular invariants, Poincaré covariance, and the special geometry of wedges. See Borchers (2000, §IV.3) for details.

B-analyticity and *B-reality* ensures that it does in fact have such a geometric interpretation as part of an antiunitary representation of the Poincaré group which includes an operator, Θ, defined by equation (10), representing a PT reflection. The difficulty is that this antiunitary representation may not be an extension of the original unitary representation posited by the covariance and vacuum axioms. In this case we have two distinct representations, $U(\Lambda, a)$ and $V(\Lambda, a)$, encoding potentially different physics. The physics described by the minimal representation $V(\Lambda, a)$ must be Θ-invariant, but there is no similar guarantee for $U(\Lambda, a)$.

Streater (1967) and Oksak and Todorov (1968) exploit this gap to construct counterexamples to the CPT and spin-statistics theorems. All of these examples employ fields transforming under infinite-dimensional representations of the cover of the Lorentz group, so-called *infinite spin representations*. As a result, such QFTs violate the split property and are thermodynamically ill-behaved. Exactly how physically pathological they are remains to be fully investigated, but Brunetti et al. (1993) prove that if the split property obtains, even in its weaker distal form, $U(\Lambda, a)$ is the unique covariant representation of the (cover of the) Poincaré group acting on the vacuum representation:

Theorem (Brunetti-Guido-Longo)**.** *If a model of AQFT satisfies the Haag-Kastler axioms and the distal split property, then there can only be one covariant representation of the Poincaré group or its covering group in the vacuum representation.*[31]

So if the minimal representations exists, then the distal split property ensures that $U(\Lambda, a) = V(\Lambda, a)$ and the relevant physics is Θ-invariant. It also ensures that Θ is unique up to unitary equivalence.

3.6 Charge Conjugation

The hardest part of the CPT theorem is to understand why charge conjugation is connected to a spatiotemporal symmetry like PT. The answer lies in how the PT transformation constructed above is implemented. In effect we are performing a spatiotemporal reflection by flipping the Lie product, by changing how quantities

[31] Here is the central idea: the Doplicher-Roberts reconstruction theorem (Doplicher and Roberts, 1990) shows that if the distal split property holds, the gauge group G is compact and commutes with any representation of the Poincaré group. If $U(\Lambda, a)$ and $V(\Lambda, a)$ are two different representations of the Poincaré group, then the adjoint action of $U(\Lambda, a)V(\Lambda^{-1}, -a)$ is an internal symmetry, and thus an element of G. This defines an action of the Poincaré group in G. Since G is compact and the Poincaré group has no non-trivial finite dimensional representations, the action must be trivial, and $U(\Lambda, a) = V(\Lambda, a)$.

and symmetries are linked at a fundamental level. The Lie product not only defines how spatiotemporal symmetries are tied to quantities like mass and spin, it also defines how internal symmetries are tied to gauge charges. Flipping the Lie product, while preserving the charge localization structure, maps each charge to its conjugate. This is exactly what J_W does.

Recall from §2 that in AQFT information about global gauge symmetries is encoded in the structure of the category of localized transportable morphisms of the quasilocal algebra. Conjugate charges ϱ and $\bar{\varrho}$ have the defining property that $\omega \circ \varrho \circ \bar{\varrho}$ contains a component in the vacuum sector. If a model of the Haag-Kastler axioms satisfies additivity and the distal split property, the statistical dimension of each charge sector is finite, and each charge has a unique conjugate up to unitary equivalence (some charges may be self-conjugate). At the heart of the algebraic CPT theorem, Guido and Longo (1992) establish the following:

Theorem (Guido-Longo)**.** *If a model of AQFT satisfies the Haag-Kastler axioms, additivity, the distal split property, the Bisognano-Wichmann property, and the DHR/BF selection criterion, then for any charge morphism, ϱ, localized in a doublecone/spacelike cone*

$$\bar{\varrho} = j_W \circ \varrho \circ j_W,$$

where $j_W(A) = J_W A J_W$ (for all $A \in \mathfrak{A}$) is the morphism defined by the adjoint action of the vacuum wedge modular conjugation, J_W.

Rotational invariance then entails that $\bar{\varrho} = \theta \circ \varrho \circ \theta$.

In order to understand the Guido-Longo theorem, there are two questions that must be answered — why is $j_W \circ \varrho \circ j_W$ a suitably localized transportable morphism, and why is it conjugate to ϱ? The answer to the first question is relatively straightforward. Since J_W implements a $\mathrm{P_1T}$ reflection in the vacuum representation, it induces an algebraic $\mathrm{P_1T}$ reflection on the defining net of local C^*-algebras, $\{\mathfrak{A}(O)\}$, common to all sectors of the theory. Consequently $J_W \mathfrak{A}(rO) J_W = \mathfrak{A}(O)$, where r is a $\mathrm{P_1T}$ reflection around the edge of the wedge, W. Since $J_W^2 = I$, it follows that $j_W \circ \varrho \circ j_W$ is a nontrivial morphism on $\mathfrak{A}(rO)$ and the identity on $\mathfrak{A}(rO')$. Therefore $j_W \circ \varrho \circ j_W$ is localized in rO, a region with the same geometry at O. Since ϱ is transportable and J_W acts geometrically, $j_W \circ \varrho \circ j_W$ is similarly transportable.

The answer to the second question is less obvious and comes from a deep connection between conjugacy and modular inclusions. Let $\mathfrak{M}$ be an infinite factor (i.e., an infinite von Neumann algebra with a trivial center, $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}I$) acting on a separable Hilbert space with a cyclic, separating vector Φ. (Eventually $\mathfrak{M}$

will be identified with $\mathfrak{R}_\omega(W)$.) It follows from the Tomita-Takesaki theorem that $\mathfrak{M}'$ is also an infinite factor. Let ϱ be an irreducible morphism of $\mathfrak{M}$ (i.e., $\varrho(\mathfrak{M})' \cap \mathfrak{M} = \mathbb{C}$), and assume that Φ is also cyclic and separating for $\varrho(\mathfrak{M})$. In this setting, Longo (1984) establishes the existence of a canonical endomorphism, $\gamma_\varrho : \mathfrak{M} \to \varrho(\mathfrak{M})$, defined by,

$$\gamma_\varrho(A) := J_\varrho J A J J_\varrho \ , \tag{11}$$

where J and J_ϱ are the modular conjugations of $\mathfrak{M}$ and $\varrho(\mathfrak{M})$ with respect to Φ. This canonical endomorphism in turn defines a *conditional expectation* relating the statistical properties of $\mathfrak{M}$ and the subalgebra $\varrho(\mathfrak{M})$. We can loosely think of a conditional expectation as a device for optimally estimating the statistics of measurements in $\mathfrak{M}$ given information about measurements in the subalgebra, $\varrho(\mathfrak{M})$. Accardi and Checchini (1982) describe it as characterizing the "statistical location" of $\varrho(\mathfrak{M})$ within $\mathfrak{M}$ relative to a given state.

If $\bar{\varrho}$ is conjugate to ϱ, then by definition there exist isometries $V, W \in \mathfrak{M}$ such that $\varrho \circ \bar{\varrho}(A)V = VA$ and $\bar{\varrho} \circ \varrho(A)W = WA$ for all $A \in \mathfrak{M}$. Longo (1990) proves that the equation,

$$\varepsilon_{\varrho\bar{\varrho}}(A) := \varrho(V)^* \varrho \circ \bar{\varrho}(A) \varrho(V) \ , \tag{12}$$

also defines a conditional expectation relating $\mathfrak{M}$ and $\varrho(\mathfrak{M})$. A seminal theorem by Takesaki (1970) establishes that there can only be one such conditional expectation, and thus $\gamma_\varrho = \varrho \circ \bar{\varrho}$ up to unitary equivalence. Combining these ingredients, we find that the conjugate morphism must have the form (up to unitary equivalence),

$$\bar{\varrho} = \varrho^{-1} \circ \gamma_\varrho \ , \tag{13}$$

where ϱ^{-1} is the inverse of ϱ.

If we choose a unitary operator implementing the morphism, $\varrho(\cdot) = U(\cdot)U^*$, then the inverse, ϱ^{-1}, is the result of simply reversing the order of multiplication, $\varrho^{-1}(\cdot) = U^*(\cdot)U$.[32] The conjugate morphism, $\bar{\varrho}$, is revealed to be something slightly more complex, the result of reversing the order of multiplication combined with the canonical endomorphism, γ_ϱ, relating the statistical structure of $\mathfrak{M}$ and $\varrho(\mathfrak{M})$. Repeated iteration of conjugate morphisms generates a sequence of nested subalgebras:

$$\mathfrak{M} \supset \varrho(\mathfrak{M}) \supset \bar{\varrho} \circ \varrho(\mathfrak{M}) \supset \varrho \circ \bar{\varrho} \circ \varrho(\mathfrak{M}) \supset \ldots \tag{14}$$

[32] In the DHR/BF picture where ϱ is interpreted as a localized charge creating morphism, ϱ will always be unitarily implementable within its localization region, however the corresponding unitaries do not give rise to a unitary equivalence between charge sectors.

A proper charge conjugation symmetry fulfilling condition (e) from §2.3 must do more than invert ϱ while preserving the type of spacetime region ϱ is localized in. It must also preserve statistical information about this infinite sequence of inclusions. The fact that the modular conjugation operator does so is linked to its role in defining the canonical endomorphism, γ_ϱ. Relative to the previous choice of unitaries, the modular conjugation of $\varrho(\mathfrak{M})$ can be written as $J_\varrho = \varrho(J) = UJU^*$. Therefore $\bar{\varrho}(\cdot) = JU^*J(\cdot)JUJ$.

Returning now to physics, consider a transportable DHR/BF charge morphism, ϱ, localized in a doublecone or spacelike cone, $O \subset W$. Wedge duality entails that ϱ generates a transportable morphism, $\varrho_W : \mathfrak{R}_\omega(W) \to \mathfrak{R}_\omega(W)$, localized in W. The wedge algebra $\mathfrak{R}_\omega(W)$ is an infinite factor, and by the Reeh-Schlieder theorem, the vacuum state is cyclic and separating for both $\mathfrak{R}_\omega(W)$ and $\varrho_W(\mathfrak{R}_\omega(W))$. For simplicity we assume that ϱ_W is irreducible. (Nothing turns on this simplification. See Guido and Longo 1992 for the general case.) This places us in the general mathematical setting discussed above.

Using equation (13), we can define a W-localized conjugate morphism, $\bar{\varrho}_W = \varrho_W^{-1} \circ \gamma_{\varrho_W}$. Choosing a unitary implementing ϱ_W, a straightforward calculation shows that

$$\varrho_W(\cdot) = \begin{cases} U(\cdot)U^* & \text{on } W \\ \text{id} & \text{on } W' \end{cases} \qquad \bar{\varrho}_W(\cdot) = \begin{cases} J_W U^* J_W(\cdot) J_W U J_W & \text{on } W \\ \text{id} & \text{on } W', \end{cases} \tag{16}$$

while $j_W \circ \varrho_W \circ j_W$ is localized in the opposite wedge,

$$j_W \circ \varrho_W \circ j_W(\cdot) = \begin{cases} \text{id} & \text{on } W \\ J_W U J_W(\cdot) J_W U^* J_W & \text{on } W'. \end{cases} \tag{17}$$

It follows that $J_W U J_W \bar{\varrho}_W J_W U^* J_W = j_W \circ \varrho_W \circ j_W$, and since $J_W U J_W$ is unitary, $j_W \circ \varrho_W \circ j_W$ and $\bar{\varrho}_W$ are thus unitarily equivalent.

This is the situation for every $W \supset O$. If it is possible to choose a consistent family of conjugates, $\{\bar{\varrho}_{W_i}\}$, such that

$$\bar{\varrho}_{W_1}|_{W_1 \cap W_2} = \bar{\varrho}_{W_2}|_{W_1 \cap W_2} \tag{18}$$

for every $W_1, W_2 \supset O$, then this consistent family will define a DHR/BF charge morphism localized in O conjugate to the original ϱ. (The flexibility to choose different $\bar{\varrho}_W$ comes from the fact that charge morphisms are only defined up to unitary equivalence.) Guido and Longo (1992) prove that it is in fact possible to choose such a consistent family as a consequence of the distal split property, Poincaré covariance, and the geometric action of J_W.

Thus we find that all of the special properties of J_W are essential for explaining why it conjugates charge. Because it represents a $\mathrm{P_1T}$ reflection, it preserves the regions that DHR/BF charges are localized in, and thus maps objects to objects in the relevant category of localized transportable morphisms. Because it is an antiunitary operator, it reverses the Lie bracket and thus the order of C^*-multiplication, inverting the morphism. But because it is also a modular conjugation for the spacelike wedge W, it defines the canonical endomorphism γ_{ϱ_W}, characterizing the statistical location of $\varrho_W(\mathfrak{R}_\omega(W)) \subset \mathfrak{R}_\omega(W)$. It therefore maps W-localized morphisms onto their conjugates given by the formula $\bar{\varrho}_W = \varrho_W^{-1} \circ \gamma_{\varrho_W}$. In DHR/BF representations of a Poincaré covariant QFT satisfying the distal split property, each DHR/BF localized morphism is generated by a consistent family of such wedge-localized morphisms, and therefore in addition to representing a $\mathrm{P_1T}$ reflection, J_W conjugates charge.

The role that the Lie product plays in characterizing conjugate charges can be further illuminated by considering the more traditional view of charges as conserved quantities associated with internal gauge symmetries. One of the greatest insights of the DHR/BF analysis, is that the algebraic description of charge structure outlined in §2.2 is physically equivalent to this traditional picture. If the observable algebras are generated by field operators as in Lagrangian and Wightman QFT, the observable net corresponds to the gauge-invariant portion of the underlying field algebra. In this context, we can characterize superselection sectors using irreducible representations of the relevant gauge group G. Superselected charges are defined using the Casimir invariants of the conserved currents generated by these gauge transformations. The field operators act on a single, underlying Hilbert space which splits into a direct sum of G-invariant subspaces, $H = \bigoplus H_\sigma$. These subspaces are in 1-1 correspondence with the superselection sectors in DHR/BF theory. Restricting the action of G to H_σ yields a direct sum of irreducible representations of G with the same character, σ. These subspaces are also $\mathfrak{A}$-invariant. Restricting $\mathfrak{A}$ to H_σ yields a direct sum of quasiequivalent, irreducible representations of $\mathfrak{A}$ satisfying the DHR/BF selection criteria. Conversely, the reconstruction theorem proven by Doplicher and Roberts (1990) shows that given the category of DHR/BF representations, one can naturally reconstruct a unique minimal algebra of field operators and gauge group, G.

In the field algebra picture, we find that the action of J_W takes a given representation of G to its complex conjugate representation, which models the conjugate charge sector. We do not have to look far to see why. A representation (π, H) of G can be specified by a set of generating fields, T^a, lying in the (weak closure) of the field algebra, that satisfy the commutation relations $[T^a, T^b] = i f^{abc} T^c$ (where f^{abc}

are the group structure constants for G). The complex conjugate representation is given by $-(T^a)^*$. Within these relations we immediately recognize the ever-present Lie product. The same structure which encodes how observables generate symmetries also encodes how unobservable field operators generate internal gauge transformations. Flipping this structure yields the complex conjugate representation. It is because the same Lie structure is employed in describing both internal gauge symmetries and external spacetime symmetries that we find a connection between them.

3.7 Summary

The algebraic proof of the CPT theorem shows how it is always possible, in a broad class of thermodynamically well-behaved models of AQFT, to systematically reverse the generating relationship between symmetries and observables while preserving the dynamics, spectra, and localization properties of physical systems. As a consequence of the Reeh-Schlieder theorem, for any local algebra in the vacuum sector of a generic model of AQFT, the canonical involution can be broken into two pieces, one of which is the antiunitary modular conjugation operator, J. The modular conjugation maps the relevant local algebra onto its commutant, reversing the Lie product in the process. Commuting algebras are associated with spacelike separated regions, hinting at a possible geometric interpretation, and since J is antiunitary, it is a candidate for a physical symmetry.

For spacelike wedges, the associated modular conjugation, J_W, is in fact always a physical symmetry. Because of the spectrum condition, any generalized time reversal symmetry must be represented by an antiunitary involution that reverses the Lie product, commutes with the dynamics, and acts uniformly on spacetime. Because J_W is a modular conjugation operator, $J_W \mathfrak{R}_\omega(W) J_W = \mathfrak{R}_\omega(W)'$, and if wedge duality holds, $\mathfrak{R}_\omega(W)' = \mathfrak{R}_\omega(W')$, suggesting that J_W represents a $\mathrm{P}_1\mathrm{T}$ reflection. Proving that J_W commutes with the dynamics, that wedge duality holds, and that J_W represents a uniform geometric reflection requires a detailed technical argument exploiting analyticity properties derived from covariance, the spectrum condition, and the distal split property, as well as auxiliary assumptions B-analyticity and B-reality.

The Lie product also appears in the characterization of wedge-localized charge morphisms, related by the formula $\bar{\varrho} = \varrho^{-1} \circ \gamma_\varrho$. The Lie product encodes the relational distinction between ϱ and its inverse ϱ^{-1}, and since J_W reverses the Lie product, it flips this distinction. Moreover since J_W is a modular conjugation, it preserves the statistical properties of subalgebras related by charge morphisms (encoded by the conditional expectation associated with γ_ϱ). Because J_W acts uniformly on spacetime, it preserves all types of localization regions, and since the theory is Poincaré covariant, morphisms localized in doublecones and spacelike cones can be constructed from families of compatible wedge-localized morphisms. Consequently, J_W also conjugates DHR/BF charges.

The connection between the Lie product and state space orientation discussed in §3.2 gives us further geometric insight into the theorem. Since Θ reverses the Lie product, it reflects the corresponding state space orientation. It is not a product of three separate reflection symmetries, C, P, and T, but rather a single, global reflection of state space. This conclusion is reinforced by proofs of the CPT theorem in Wightman QFT (Streater and Wightman, 1989) and rigorous Lagrangian proofs (Greaves and Thomas, 2014). In both cases, a CPT operator is proven to exist without first decomposing it into separate C, P, and T reflections as is commonly done in textbook presentations. The algebraic framework gives us a clearer picture of the geometric origins of this operator as a systematic reversal of the generating relationship between observables and state space symmetries.

4 Philosophical Consequences

4.1 Bain's Skeptical Challenge

The story just outlined is an example of what Bain (2016) calls a *structural explanation*, insofar as it appeals to "mathematical constraints on a theory's state space that are independent of the specification of the theory's dynamics" (p. 155). It should be emphasized, however, that the explanation is not a purely mathematical one. Though they do take the form of mathematical conditions, the assumptions in §2.1 represent important physical constraints imposed on any causal, Lorentz-invariant, thermodynamically well-behaved QFT. By connecting each stage of the proof back to the Haag-Kastler axioms, and by linking CPT reflection to a systematic reversal of the generating relationship between observables and symmetries, the aim has been to illuminate how the steps of the algebraic proof trace out important physical dependency relations present in any such QFT.[33] The upshot is

[33] For our purposes it can be left open exactly what sorts of things these dependency relations are. They might be nomological or meta-nomological relations, counterfactual relations, grounding or constitutive relations, or a mixture thereof. (It seems unlikely that they are causal relations, however, making the CPT theorem an important prima facie example of non-causal explanation.) At this level, the story will depend not only on further analysis of the relevant physics, but on metaphysical debates about laws, modality, and fundamentality.

that the only consistent way to realize these constraints is for the theory's state space to be CPT-invariant, a fact which has physical consequences for its particle spectrum as well as for scattering and decay processes. A structural explanation is only as good as our understanding of how the mathematics represents the physics, both at the level of a proof's inputs and outputs, as well as its logical structure.

While Bain is broadly sympathetic to this style of explanation, he is skeptical that current versions of the CPT theorem actually give us explanations, even provisional ones. Citing the current inability of AQFT to rigorously model local gauge theories, he argues that the algebraic CPT theorem does not explain why the actual laws of nature are CPT-invariant: "the systems of interest; those that make contact with empirical tests, lie outside the subclass of systems for which the CPT and spin-statistics theorems provide structural explanations" (Bain, 2016, p. 157). These systems, the Yang-Mills theories that comprise the standard model, can be described using techniques from perturbative Lagrangian and S-matrix formulations of QFT, but proofs of the CPT theorem in these frameworks differ significantly from those in AQFT on Bain's telling. For example, they disagree about whether or not Lorentz invariance is necessary to prove the CPT theorem, as well as about whether CPT invariance entails the spin-statistics connection or vice versa. Consequently Bain thinks "it will be hard to make a case for a common underlying mathematical structure," shared across frameworks, that a structuralist explanation of CPT invariance can appeal to (p. 156).

I agree that right now the algebraic CPT theorem is only a potential explanation for the CPT symmetry found in nature; however, I think that Bain's conclusion is overly pessimistic. In my review of Bain's book (Swanson, 2018), I argue that Bain's presentation of the algebraic CPT theorem misinterprets the physical content of modular covariance and obscures the role of Lorentz invariance, the spectrum condition, and the split property in the physical argument at the center of the theorem. In doing so, it overemphasizes differences and underestimates commonalities between the algebraic CPT theorem and proofs couched in other frameworks. Upon closer inspection, there is a set of core of assumptions which appear (in slightly different forms) in virtually all known versions of the CPT theorem: restricted Lorentz invariance, energy positivity, causality, finite spin, and analyticity.

Our detective work in §3 reinforces this critique. Unlike some versions of the algebraic CPT theorem, the proof outlined here explicitly displays this common logical form. The covariance axiom, spectrum condition, and microcausality axiom enforce Lorentz invariance, energy positivity, and causality, respectively, while the split property ensures that there are no fields transforming under infinite spin representations. Together, the covariance axiom, spectrum condition, and Tomita-Takesaki modular theory entail important analyticity properties (sufficient to establish the CPT theorem in 2-dimensional models), while B-analyticity and B-reality supply additional constraints needed in higher dimensions. Moreover, our analysis locates the seed of CPT reflection symmetry in a systematic reversal of the algebraic Lie product, a structure found not only in AQFT, but in every formulation of QFT that represents observables using Hilbert space operators including Lagrangian, S-matrix, and Wightman QFT.[34]

While there are significant structural commonalities between proofs in different frameworks, there are also non-trivial mathematical differences. Whether these stem from a deep disagreement about the fundamental character of QFT, as Bain contends, or if they are simply the product of different modeling techniques and goals, remains an open question. Regardless, a strong case can be made that the algebraic approach offers us some of the best explanatory insight into the CPT theorem at present. Unlike Lagrangian and Wightman QFT which start with assumptions about gauge-dependent field operators, the Haag-Kastler axioms characterize constraints on gauge-independent physical quantities, making their physical interpretation and justification more direct. Moreover, there are models of CPT-invariant AQFTs that violate the Wightman axioms, that are not the quantization of any known classical Lagrangian, and which do not satisfy the assumptions of Haag-Ruelle scattering theory, a mathematically rigorous analogue of the standard S-matrix picture (Summers, 2012; Lechner, 2015). While the Wightman axioms are known to fail for QFTs with local gauge symmetry (Strocchi, 2013, ch. 6), the jury is still out on the Haag-Kastler axioms. All of this suggests that AQFT has a wider scope than other frameworks, providing a better characterization of what all relativistic QFTs have in common.

Of course, we do not understand the entire story yet. B-analyticity and B-reality are bootstrap assumptions. Their physical interpretation and justification is a major question mark at the heart of the algebraic proof. The role of analytic continuation arguments must be better understood, even in cases like the Reeh-Schlieder and Borchers-Weisbrock theorems where the starting assumptions are physically well-motivated. Whether or not additivity and the wedge intersection property can be eliminated or physically motivated remains to be seen.

In addition, there are significant limitations to the scope of existing algebraic

[34]In some frameworks, including perturbative AQFT (Brunetti et al., 2009), the local algebras are not C^*-algebras but more general types of *-algebras. Nonetheless, such algebras contain a canonical involution operation, and thus we might hope to find a suitable generalization of the ideas sketched in §3.1-2.

proofs. It is currently not known how to extend the DHR/BF picture to describe charges associated with local gauge symmetries, and because of the infrared problem, it cannot be applied to theories with massless particles yet. The argument also crucially relies on analyticity properties associated with symmetries of a continuous spacetime manifold, as well as on microcausality holding at arbitrarily short length-scales. Consequently, it is not clear how to generalize the algebraic argument to cover effective QFTs.[35] The version of the proof outlined here makes the further assumption that spacetime is flat.

Despite these limitations, there are reasons for optimism on several fronts. Recent model-dependent results due to Morinelli (2018) suggest that the Bisognano-Wichmann property is logically weaker than the split property, breathing life into Haag's conjecture that it can be proven from existing, physically justified axioms. The Bisognano-Wichmann property can be formulated in any spacetime with well-defined wedge regions, and it has been used to prove the CPT theorem in curved spacetimes with a large group of global isometries (Buchholz et al., 2000). More recently, the generally covariant AQFT program initiated by Brunetti, Fredenhagen, and Verch has made substantial progress towards an axiomatic version of AQFT in arbitrary curved spacetimes (Brunetti et al., 2015).[36] Combined with developments in perturbative AQFT, this program has also started to provide a provisional picture of what effective QFTs with local gauge symmetry might look like in the algebraic framework (Fredenhagen and Rejzner, 2013, 2016). Strocchi (2013) tentatively identifies the physical significance of local gauge invariance with the holding of Gauss-type conservation laws for the associated charge densities,

[35]Although the split property and weak additivity can be replaced by distal versions that hold at some sufficiently long length-scale without affecting the details of the proof, it is much more difficult to envision relaxing microcausality and covariance.

[36]Fewster (2016) applies this framework to prove a general curved spacetime version of the the spin-statistics theorem, the CPT theorem's close cousin. The proof shows that any QFT on curved spacetime that can be related to a QFT on flat spacetime by certain deformations must obey the spin-statistics connection if the flat spacetime QFT does. This suggests that flat spacetime versions of the CPT and spin-statistics theorems might continue to carry important explanatory insight in a curved spacetime context. It also could help diffuse a potentially serious objection to the logic of the algebraic proof sketched in §3. In an arbitrary curved spacetime, it is no longer true that the causal complement of a wedge is always a wedge, and this is a necessary condition in order for J_W to represent a P_1CT reflection. Borchers has conjectured that W' is a wedge only if the background spacetime is conformally equivalent to either Minkowski or deSitter spacetime (see Hollands and Rheren 2012). If Fewster's strategy works for the CPT theorem too, then although we would not expect the CPT operator to be represented by J_W in general curved spacetime models, it could be defined by deformations of J_W from a corresponding flat spacetime model.

and Kijowski and Rudolph (2003) apply the DHR picture to study the superselection structure of lattice quantum chromodynamics. Meanwhile, by studying certain equivalence classes of superselection sectors called *charge classes*, Buchholz and Roberts (2014) have started to clarify the complex superselection structure of massless theories.[37] Together, this work suggests that key ideas from the DHR/BF picture will apply to theories like the standard model.

So even though we must wait for further developments in constructive AQFT to determine whether the algebraic CPT theorem gives us the right story about our own universe, we have good reason to believe that central ideas from the proof will be part of the eventual explanation. In the interim, even as a provisional explanation, it can shed light on a number of interpretive debates.

4.2 Greaves's Lagrangian Approach

In her agenda-setting study of the Lagrangian CPT theorem, Greaves (2010) provides a different structural explanation for the origins of CPT symmetry. (Arntzenius 2011 offers a similar story.) Although it has since been superseded by the more nuanced account developed by Greaves and Thomas (2014), the original, simpler version is worth investigating on its own terms first.

The explanation has two main components. The first is a theory of antimatter that conceptually links time reversal and charge conjugation. Motivated by Feynman's famous view of antiparticles as regular particles traveling backwards in time, Greaves argues that particles should be represented by oriented worldlines. Regular particles are co-aligned with the direction of time, while antiparticles are anti-aligned. Exactly what does the orientation work is left open. It could be a 4-momentum as in Feynman's picture. It could be a wavevector as in the standard Lagrangian picture (Wallace, 2009). It could be something else entirely. The upshot is that any time reversal symmetry will transform particles into antiparticles and vice versa. This collapses the CPT puzzle into a PT puzzle.

The second component is a PT theorem for classical field theories in Minkowski

[37]As in the massive case, charge morphisms are localized in spacelike cones. Unlike the BF picture, however, where the direction of the cone is arbitrary, in massless theories the direction of the cone determines an additional superselected global observable associated with the asymptotic boundary conditions of soft photon clouds. Sectors are labeled by the value of the total charge along with this asymptotic flux parameter. Although we currently lack tools to define a tensor product on the relevant category of representations, Buchholz and Roberts show that considering equivalence classes of sectors with the same global charge, many of the tools from the DHR/BF picture can be imported into this new setting.

spacetime. The theorem applies to Lorentz-invariant theories satisfying two main assumptions: (i) the fields transform as tensors under spacetime diffeomorphisms, and (ii) the dynamical laws are partial differential equations with polynomial interaction terms. In this situation, Greaves proves that it is impossible for the laws to encode just a temporal orientation or just a spatial orientation.

Theorem (Greaves). *Any polynomial combination of tensor fields that is invariant under connected Lorentz transformations is also invariant under PT symmetry.*[38]

If these are the only tools the laws have at their disposal, they cannot break PT symmetry. Accordingly, any reasonable field theory, quantum or classical, has to be PT-invariant. If the theory has antiparticles, the first part of the story entails that PT invariance is just the same thing as CPT invariance. The CPT theorem is therefore an essentially relativistic result.

Although it is an appealing story on many levels, Greaves's explanation faces several challenges. The most immediate worry is that assumptions (i) and (ii) in the classical PT theorem are not physically well-motivated. The first assumption, (i), rules out spinor-valued fields, which are used to describe matter with half-integer spin, an essential ingredient in any theory like the standard model with a wide assortment of fermions in its particle zoo. Spinors can be used to construct PT-pseudotensors that are not PT-invariant. For example, the common bilinear currents $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\frac{i}{2}[\gamma^\mu, \gamma^\nu]\psi$ change sign under PT transformations. (Here ψ and $\bar{\psi}$ are conjugate Dirac spinors and γ^μ are Dirac spin matrices.) Prima facie, classical spinor-valued field theories need not be PT-invariant. This conclusion is confirmed by Greaves and Thomas (2014, §8), who prove that there is no direct analogue of Greaves's PT theorem for classical spinor fields. The fact that quantum spinor fields are constrained by the CPT theorem cannot simply be explained by appealing to the classical PT theorem.

[38]See Greaves (2010, fn. 12) for a proof. The same basic argument underwrites the classical PT theorem (Thm. 5.6) in Greaves and Thomas (2014). A temporal orientation can be defined by specifying a privileged timelike vector-field, t^a, however t^a will not be invariant under all connected Lorentz transformations. (It defines more than just a temporal orientation.) This problem can be avoided by choosing an equivalence class of coaligned timelike vector fields, $[t^a]$, but this sort of object is not suitable to appear in a partial differential equation so the laws cannot make use of it. Similar reasoning rules out the ability to encode just a spatial orientation and can be generalized to any polynomial combination of tensor-fields. The situation is markedly different in Galilean spacetime, where it is possible to directly encode a temporal orientation using a special 1-form field. In this setting, there is no analogue of the PT theorem to be found. The difference boils down to what kinds of orientation structures can be encoded by polynomial combinations of tensor fields on a given background spacetime.

The second assumption (ii), rules out field theories with non-polynomial interactions, a rich area of active study in mathematical physics. Notable examples include Sine-Gordon models, Liouville field theory, and Weinberg's chiral Lagrangian for π-mesons, all of which are expected to be CPT-invariant. Greaves and Thomas (2014) suggest that insofar as non-polynomial interactions terms can be approximated by power series expansions, we can import techniques from the polynomial case and extend the explanation. This may not always be possible, and it is difficult to ascertain at this stage what limitations in scope this puts on the proposed explanation. Regardless, the algebraic story sketched in §3 directly circumvents both of these challenges. It applies to both spinor and tensor theories and is not restricted to polynomial interactions. In principle, it has the potential to encompass models in which *primitive causality* fails and the dynamical laws cannot be expressed in the form of differential equations at all.

Perhaps an even greater advantage, the algebraic story incorporates the DHR/BF analysis of charge structure, a more powerful, unified picture of antimatter than the Feynman-inspired view advocated by Greaves. The latter requires a coherent notion of particle worldline, and if particles must be emergent structures in QFT as suggested by numerous no-go results (e.g., Halvorson and Clifton 2002), the characterization of antimatter will be similarly emergent. (Even if a QFT has emergent particles, quantum effects may render the notion of a particle worldline unintelligible in many contexts.) In contrast, the algebraic picture picture draws a more fundamental distinction between matter and antimatter which is known to apply to rigorous models of low-dimensional interacting theories like 2-dimensional Yukawa theory that lack a particle interpretation (Baker and Halvorson, 2010). The CPT theorem is generally viewed as a deep, foundational result. It would be odd if it turned out merely to describe high-level, emergent phenomena.

In addition, the connection between charge structure and worldline orientation lacks a clear explanation on Greaves's view. Even if time-reversal entails that we relabel particles and antiparticles, it does not say anything about charge conjugation as such. An electron will have -1 electric charge and a co-aligned worldline. If we reverse the direction of time, its worldline is now anti-aligned so we redescribe it as a positron. But intuitively, it should be a positron with charge -1! If the 4-momentum or wavevector is responsible for orienting the worldline, there is no obvious link to charge structure. If somehow the charge itself is responsible, then we need a story about how it does this orientation work. Unless this explanatory gap can be bridged, the view turns out to rely on a primitive CT symmetry built into the laws from the very start.

The algebraic CPT theorem, on the other hand, characterizes antimatter solely in terms of charge structure and provides an explanation for when and why particle/antiparticle partners must have the same mass, spin, and lifetime. From this perspective, the validity of the CPT theorem justifies why Feynman's interpretation is possible in the first place. It is because a theory is CPT-invariant that we can interpret a forward moving positron with charge $+1$ as a backwards moving (opposite handed) electron with charge -1. Moreover, it explains why time-reversal and charge-conjugation are so closely linked. The spectrum condition entails that the only way to reverse the direction of time is to reverse the Lie product. But since the Lie product is also responsible for encoding the relationship between conjugate charges, antiunitary time reversal will also conjugate charge.[39]

Thirdly, if time reversal and charge conjugation are linked by definition, as in Greaves's view, it is metaphysically impossible for there to be particle/antiparticle pairs with different masses and spins. But this is arguably a coherent possibility. Tureanu (2013) reviews a number of CPT violating QFTs with these features. Although there is ongoing debate about whether these theories can satisfy Lorentz invariance and locality, the examples prove that particle/antiparticle mass splitting is a metaphysical possibility, even if the models are not well-behaved relativistic QFTs. Unlike Greaves's picture, the algebraic view does not build this restriction into its definition of antimatter. The possibility of creation/annihilation events only requires that partners have conjugate charge, not that they are otherwise identical. One might object on externalist grounds that antiparticles in our world must be anti-aligned particles, even if in other worlds they are realized by different sorts of entities. Alternatively, it could be argued that this identification is a physical rather than metaphysical necessity. Either way, the algebraic picture is revealed to have greater unifying power as an explanation, describing charges and antimatter in a broader class of theories with a common structure.

In more recent work, Greaves and Thomas (2014) give a mathematically expanded interpretation of the Lagrangian CPT theorem drawing upon a general result dubbed *strong reflection invariance*. (A strong reflection is defined as a PT

[39] There is an important caveat here. Although it is always possible to conjugate charge by reversing the Lie product using the CPT operator, in theories where charges and parity are treated symmetrically by the laws, it is also possible to define unitary C and P operators, U_C and U_P , that preserve the Lie product. In this case one can combine an antiunitary CPT reflection with unitary C and P transformations to produce a net antiunitary time reversal represented by $V_T = U_C U_P \Theta$. This allows for theories like quantum electrodynamics which are invariant under C, P, and T symmetries independently. In other theories it is possible to define a unitary CP operator, allowing for theories invariant under T, CP, and CPT transformations. See Bogolubov et al. (1975, Ch. 12.4) and Mund (2001) for examples of such constructions.

operation on the field symbols combined with a reversal of the order of products, reducing to just a PT transformation in the classical case.) In this new work, the Feynman picture of antimatter no longer plays a prominent role. Instead, Greaves and Thomas characterize antimatter by a splitting of the space of field configurations into complex subspaces similar to the standard Lagrangian picture defended by Wallace (2009). The restriction to polynomial interactions persists in their concept of a classical *formal field theory*, defined as a complex affine subspace of the set of polynomial combinations of field symbols and their derivatives. Perhaps most significantly, by assuming a classical version of the spin-statistics connection they expand the story to include spinor fields. They use this to prove a more general version of the classical CPT theorem that will be discussed in the next section.

There are striking parallels between a strong reflection and a global reversal of the Lie product. As we have seen in §3.2, reversing the order of the C^*-product is one way of reversing the Lie product in AQFT. Thus at the heart of the new Greaves-Thomas theorems we can find a signature of the algebraic proof's guiding idea. Given the common structural core discussed in §4.1, it is perhaps unsurprising to discover additional commonalities between Lagrangian and algebraic proofs. This important connection deserves further study.

At the same time, the algebraic proof continues to enjoy certain explanatory advantages over these new Lagrangian proofs. The picture of antimatter advocated by Greaves and Thomas is subject to many of the same criticisms that applied to the Feynman view. In interacting QFTs, our ability to decompose the space of field configurations into positive and negative frequency subspaces is an idealization. In the limit of no interactions, quantum electrodynamics admits an interpretation in terms of plane waves with opposite frequency. Conventionally, positive frequency solutions are labeled as particle states and negative frequency solutions are labeled as antiparticle states. This division into positive and negative frequency states requires a choice of complex structure on the theory's state space, forging a link between frequency sign and translations in opposite temporal directions. Conventionally, positive frequency solutions have a wavevector co-aligned with the direction of time and negative frequency solutions have an anti-aligned wavevector. Reversing the direction of time switches which wavevectors are co-aligned and anti-aligned, but the view is silent on the connection between this wavevector-induced orientation and charge structure. Identifying symmetric positive and negative frequency solutions with field configurations carrying conjugate charge requires an additional, unmotivated ansatz. Thus the new picture has the same explanatory gap as the old one. It postulates a form of CT symmetry at the outset by requiring any symmetry transformation that reverses the wavevector orientation to also conjugate charge. This eliminates the possibility of particle/antiparticle partners with different masses and spins by fiat. Moreover, it is an emergent rather than fundamental characterization of antimatter that only applies asymptotically in the free-field limit.

In models of AQFT that have a mass gap and are asymptotically complete (and thus have a limiting particle interpretation), one can show how matter and antimatter states are conventionally linked to opposite frequency wave solutions in the no-interaction limit (Mund, 2001). The DHR/BF picture can thereby help explain the origins of the emergent Lagrangian picture in certain well-understood examples. In principle, nothing prevents Lagrangian QFT from making use of tools from AQFT to provide a more fundamental characterization of antimatter. Existing Lagrangian proofs simply have not done so. Since a convergent structural explanation of CPT symmetry within different frameworks would greatly improve our understanding of the theorem, this marks another important question for further study.

4.3 Classical or Quantum?

One of the surprising corollaries of Greaves's Lagrangian account is that apart from the existence of antimatter, the CPT theorem does not crucially rely on quantum mechanical assumptions. Most of the heavy lifting is done by the PT theorem for tensor fields, which applies to both classical and quantum field theories. In contrast, the algebraic proof of the CPT theorem employs foundational assumptions from quantum mechanics as well as relativity. Covariance and microcausality appear to be primarily relativistic constraints, while the spectrum condition and modular theory are quantum mechanical in origin. This suggests, contra Greaves, that the explanation of CPT symmetry requires ingredients from both theories. To what extent is the theorem classical or quantum?

Intriguingly, Greaves and Thomas (2014) give a proof of a classical CPT theorem which they claim has the same logical structure as their version of the full quantum CPT theorem. True, they are both instances of the same schema:

Theorem (Greaves-Thomas). *If a classical/quantum formal field theory is (a) supercommutative and (b) invariant under a representation of the (cover of the) connected Lorentz group, then the theory is invariant under CPT reflections iff it is $-hermitian.*[40]

[40] For proof, see Greaves and Thomas (2014, Thm. 9.6).

Supercommutativity is a version of the standard spin-statistics connection for formal field theories, while \$-hermiticity requires invariance under a certain involution mapping. The problem is that different notions of \$-hermiticity are required to prove the classical and quantum versions. In the quantum case, the relevant notion is defined with respect to the canonical involution (providing another point of convergence with the algebraic proof). In the classical case, it is defined with respect to the charge conjugation involution. Thus once it is unpacked, the classical theorem asserts that if a classical field theory obeys the spin-statistics connection and is invariant under (a cover of) the Lorentz group, then it is CPT-invariant iff it is C-invariant. The quantum CPT theorem permits C-violating theories which are nonetheless still CPT-invariant, like the weak sector of the standard model. Thus there is interesting interaction between spatiotemporal and charge structure in the quantum theorem that is entirely absent in the classical version. Moreover, while the theorem covers classical spinor theories, it does so by assuming the spin-statistics connection. In relativistic quantum theories this can be motivated by the spin-statistics theorem, but in classical theories it is unclear if this sort of physical motivation can be given (Bain, 2016, §4.2). The Greaves-Thomas classical CPT theorem is therefore markedly different from its quantum counterpart, despite initial appearances to the contrary.[41]

Our investigation of the algebraic proof in §3 casts further doubt on the existence of a classical theorem with similar structure, although the state of play is more complicated than it first appears. Naively, we might try to model algebraic classical field theories by starting with a net of Poincaré-covariant commutative C^*-algebras. But in an abelian algebra, there is no natural Lie product and the modular structure becomes trivial.[42] To compound these difficulties, Borchers (1996b, Thm. IV.6.2) shows that no net of abelian C^*-algebras can satisfy the spectrum condition. Given the central role played by both modular theory and

[41] Flato and Raczka (1977) construct an example of a classical scalar theory with $\lambda\varphi^5$ self-interaction that is not C-invariant. Since this is a polynomial field theory, combined with Greaves and Thomas's classical CPT theorem, this provides a prima facie example of a CPT-violating classical field theory. Flato and Raczka draw a different conclusion: whether we describe this example as CPT-violating and T-invariant or as T-violating and CPT-invariant depends on a conventional choice of phase for the T operator. This shows that even in the classical case, the status of reflection symmetries is a subtle problem requiring further investigation.

[42] The modular automorphism group acts as the identity iff the generating state is tracial, but in an abelian von Neumann algebra, every state is tracial. As a result there is no meaningful analogue of the Bisognano-Wichmann property. Furthermore, every abelian von Neumann algebra is maximal, $\mathfrak{M} = \mathfrak{M}'$. Thus $J\mathfrak{M}J = \mathfrak{M}$, and so the modular conjugation operator cannot carry the geometric significance of a CPT operator.

the spectrum condition in the algebraic CPT theorem, these observations apparently preclude a classical analogue of the proof. But this skeptical conclusion is too quick. In classical field theories, physical quantities are linked to symmetries by Noether's theorem just as in QFT. Since this generating relationship is not captured by the structure of abelian C^*-algebras, they are not the right mathematical tools to model the full structure of classical field theories. This view is adopted by the deformation quantization program, which models classical theories using dual Lie-Jordan algebras just like quantum theories.[43] The principle difference between classical and quantum Lie-Jordan algebras is the associativity or non-associativity of the underlying Jordan product. Weinstein (1997) develops the basic tools of modular theory within the setting of classical associative Lie-Jordan algebras.

Whether or not such algebras can be used to evade Borchers's no-go result remains to be seen. Even if there is a classical analogue of the spectrum condition and modular theory, though, it is not at all clear that these will entail the analyticity properties required to establish CPT invariance. The link between observables and symmetries is more tightly constrained in the quantum case than in the classical case. As Zalamea (2018) shows, since the classical Jordan product is associative, the spectral properties of classical observables are completely independent of their role as generators. In contrast, the non-associative quantum Jordan product directly relates the spectral properties of quantum observables to the state space symmetries that they generate.[44] In quantum theories, constraints on symmetries entail constraints on observable spectra and vice versa. This interplay between constraints is entirely absent in classical theories. We should expect the joint consequences of covariance, microcausality, the spectrum condition, and modular theory to diverge significantly in the classical and quantum cases.

A satisfactory resolution of this problem must await the development of an algebraic formalism for classical field theories based on Lie-Jordan algebras. For

[43] See Landsman (1998) for an overview.

[44] Zalamea compares the symplectic manifold formulation of classical mechanics and the Kähler manifold formulation of quantum mechanics in which quantum mechanical state space is viewed as a Kähler manifold, (M, ω, g, J), i.e., a symplectic manifold equipped with a compatible Riemannian metric, g, and almost-complex structure, J. In both the classical and quantum case, observables can be defined as continuous $\mathbb{R}$-valued functions that preserve all of the geometric structure of the relevant state space. In classical mechanics, the Jordan product is just the product of functions $f \bullet g = fg$. In quantum mechanics, the Jordan product is given by $f \bullet g = fg + g(v_f, v_g)$, where $g(v_f, v_g)$ is the metric-induced inner product of the Hamiltonian vector fields v_f and v_g generated by f and g. On this picture, the uncertainty of a quantum observable, $\Delta f = g(v_f, v_f)$, in a given state, ϕ, is a measure of how much the symmetry generated by f changes ϕ.

initial work in this direction, see Brunetti et al. (2019). To make better contact with the algebraic picture of antimatter we also need a classical analogue of the DHR/BF analysis of charge structure, which is nowhere in sight. These are important areas for future research. Although the structural disanalogies between classical and quantum Lie-Jordan algebras give us significant reason for doubt, all we can definitively say at this stage is that a convincing case for a classical theorem with the same scope and physical motivation as the quantum CPT theorem has not been made.

5 Conclusion: Greaves's Puzzle

Greaves (2010) frames the challenge of explaining CPT invariance in the form of a puzzle. If the laws of nature violate C, P, or T symmetry it is because they somehow define a privileged direction of time, spatial handedness, or charge sign. If the laws are CPT-invariant, this means that they cannot define one such orientation structure independently of the other kinds. The puzzle is that these orientation structures seem to be "paradigm cases of distinct existences; it's odd to find such necessary connections between them" (2010, p. 38). Indeed, in relativistic spacetime one can show that spatial orientation and temporal orientation are mathematically independent; a choice of one does not fix the other. (This follows from the existence of isometries that preserve spatial orientation while reversing temporal orientation and vice versa.) While their relationship to charge orientation is less clear cut, charge superselection structure arises from internal gauge symmetries associated with the particular forces that the charges couple to, and these internal symmetries have no obvious connection to spacetime structure.

Our investigation of the algebraic CPT theorem has revealed a hidden connection. The mathematical structure that AQFT uses to distinguish between the forward dynamics and its time-reversal is the very same structure that it uses to distinguish between translations in different spatial directions as well as between charges and their conjugates. CPT symmetry is not an ad hoc combination of different reflections at all, but rather a single reflection of state space that reverses the generating relationship between observables and symmetries. By tracing this idea through the steps of the algebraic CPT theorem we have come to a better physical understanding of the argument. The proof reveals non-trivial constraints on how a model of AQFT can deploy the Lie product in conjunction with the Haag-Kastler axioms and auxiliary physical assumptions. These constraints manifest themselves in complex analyticity properties derived from the interplay between the spectrum condition, microcausality, and covariance.[45] As we have seen though, additional analyticity assumptions are input by hand. It is unclear if they can be derived from existing assumptions or must be independently motivated, so at this stage the algebraic CPT theorem only gives us a partial answer to Greaves's puzzle.

By framing the algebraic proof in this fashion, though, we have made more direct contact with the Haag-Kastler axioms than other presentations of the theorem (e.g., Bain 2016, Ch. 1.2.4) and revealed important structural commonalities that it shares with proofs in Lagrangian, S-matrix, and Wightman QFT. This goes a considerable way towards responding to Bain's skeptical challenge. Although our analysis also highlights shortcomings of Greaves's Lagrangian approach, the parallels between the algebraic proof and the recent Greaves-Thomas proof hint at additional structural commonalities that warrant further investigation.

While the scope of Greaves's original geometric explanation of CPT symmetry appears to be too narrow, its central insight is promising. Even if different types of orientation structures are logically independent, there may be constraints on how laws of a particular type can encode these structures. Greaves locates these constraints in the definability properties of Lorentz-invariant polynomial combinations of tensor fields, but the story might go differently. In addition to temporal, spatial, and charge orientation, there are other orientation structures that the laws can employ. And in the presence of these other orientation structures, temporal, spatial, and charge orientation might be less independent than it first seems. For example, although temporal and spatial orientations are logically independent in Minkowski spacetime, if the laws somehow make use of a total orientation of the underlying spacetime manifold, the choice of a temporal orientation naturally defines a spatial orientation and vice versa.[46]

The laws of QFT do not apparently use a total spacetime orientation, but they do use a Lie product, a natural orientation structure on state space. This provides the basis for an intriguing conjecture:

Conjecture. *In a causal, Lorentz-invariant, thermodynamically well-behaved QFT, the choice of a state space orientation naturally defines a temporal orientation up to a choice of spatial orientation and charge sign.*

This has the same flavor as Greaves's original PT theorem, but with significantly

[45]These constraints are either absent or substantially weakened in Galilean QFT where the CPT theorem is known to fail (Lévy-Leblond, 1967). The fact that a theory employs a single Lie product to describe both internal and external symmetries is insufficient to answer Greaves's puzzle on its own.

[46]See Wald (1984, p. 60, 429-434).

different mathematical and physical content. A proof of this conjecture stands to shed further light on the origins of CPT symmetry and will be the subject of future work.[47]

6 Appendix: Proofs of Lemmas 1-3

Proof of Lemma 1. All four algebras have the same involution structure and self-adjoint subspace (which includes the identity element). To prove (i), define the isomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}^{cop}$ as the linear mapping whose restriction to $\mathfrak{A}_{SA}$ is the identity and that sends $i \mapsto i^c$ and $AB \mapsto (AB)^{op}$. It is well-defined because the product of two self-adjoint elements is self adjoint iff they commute (and thus iff $AB = (AB)^{op}$), and iA is not self-adjoint for any self-adjoint element A. Using the fact from the main text, every element $A \in \mathfrak{A}$ can be uniquely written as $A = H + iK$ with $H, K \in \mathfrak{A}_{SA}$. Consequently $\varphi(A) = H + i^c K = H - iK = A^*$, so the map is a bijection. Moreover, $\varphi(I) = I$ and $\varphi(A^*) = H - i^c K = H + iK = A^{**} = \varphi(A)^*$, so it preserves the identity element and involution structure. Finally, it is multiplicative, since $\varphi(AB) = (AB)^* = B^*A^* = (A^*B^*)^{op} = (\varphi(A)\varphi(B))^{op}$, and is thus a *-isomorphism. Naturality follows from the fact that for any *-homomorphism, π, $\pi(A^*) = \pi(A)^*$, and therefore $\pi \circ \varphi(A) = \pi(A^*) = \pi(A)^* = \varphi \circ \pi(A)$.

The proof of (ii) follows the same reasoning, with $\tilde{\varphi} : \mathfrak{A}^{op} \to \mathfrak{A}^c$ the analogously defined natural *-isomorphism. To prove (iii) define the anti-isomorphism $\varphi^{op} : \mathfrak{A} \to \mathfrak{A}^{op}$ as the linear mapping whose restriction to $\mathfrak{A}_{SA}$ is the identity and that sends $i \mapsto i$ and $AB \mapsto (BA)^{op}$. $\tilde{\varphi} \circ \varphi^{op} : \mathfrak{A} \to \mathfrak{A}^c$ then defines an anti-isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^c$. Similarly, to prove (iv) define the conjugate-isomorphism $\varphi^c : \mathfrak{A} \to \mathfrak{A}^c$ as the conjugate-linear mapping whose restriction to $\mathfrak{A}_{SA}$ is the identity and that sends $i \mapsto -i^c$ and $AB \mapsto AB$. $\tilde{\varphi}^{-1} \circ \varphi^c : \mathfrak{A} \to \mathfrak{A}^{op}$ then defines a conjugate-isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^{op}$. Naturality follows from the naturalness of $\tilde{\varphi}$ and the fact that $\varphi^{op}(A) = \varphi^c(A) = A$, and so both mappings commute with *-homomorphisms. □

Note: Interestingly, $\mathfrak{A}$ is not necessarily isomorphic to $\mathfrak{A}^{op}$ or $\mathfrak{A}^c$. The first such examples are due to Connes (1975). It remains an open question which (if any) natural constraints might entail that an algebra is antiautomorphic (equiv. conjugate-automorphic) to itself.

Proof of Lemma 2: The mapping $j^* : \mathfrak{M} \to \mathfrak{M}'$ defined $j(A) := JA^*J$ is the re-

[47] "CPT, Spin-Statistics, and State Space Geometry" (in preparation).

quired anti-isomorphism, and the mapping $j : \mathfrak{M} \to \mathfrak{M}^c$ defined $j(A) := JAJ$ is the equivalent conjugate isomorphism. To check this, note that $j^*(AB) = J(AB)^*J = JB^*A^*J = JB^*JJA^*J = j^*(B)j^*(A)$ and $j^*(iI) = J(iI)^*J = J(-iI)J = iI$ since $iI \in \mathfrak{M} \cap \mathfrak{M}'$ and for any such central element $JAJ = A^*$. Thus j^* is an anti-isomorphism. It follows from a similar calculation that j preserves products but $j(iI) = -iI$, so j is a conjugate-isomorphism. (These morphisms are natural since J is uniquely fixed by $\mathfrak{M}$ and Φ.) Using Lemma 1, we can then define the natural isomorphisms $\psi^{op} := \varphi^{op} \circ (j^*)^{-1}$ and $\psi^c := \varphi^c \circ j^{-1}$. □

Proof of Lemma 3: Suppose some open, convex, causally complete region O is isometric to O' and is not a spacelike wedge. Thomas and Wichmann (1997, Thm. 3.2) prove that every closed, convex, causally, complete subset of Minkowski spacetime is the intersection of closed spacelike wedges, so $O \subset W$ for some wedge W and thus $W' \subset O'$. Spacelike wedges are maximal in the lattice of open, convex, causally complete subregions, so it follows that O' is a wedge. The causal complement of a wedge is a wedge and $(O')' = O$ (since O is causally complete), so O is a wedge, contradicting the initial assumption. □

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