Foundations of statistical mechanics and the status of the Born rule in de Broglie-Bohm pilot-wave theory

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We compare and contrast two distinct approaches to understanding the Born rule in de Broglie-Bohm pilot-wave theory, one based on dynamical relaxation over time (advocated by this author and collaborators) and the other based on typicality of initial conditions (advocated by the 'Bohmian mechanics' school). It is argued that the latter approach is inherently circular and physically misguided. The typicality approach has engendered a deep-seated confusion between contingent and law-like features, leading to misleading claims not only about the Born rule but also about the nature of the wave function. By artificially restricting the theory to equilibrium, the typicality approach has led to further misunderstandings concerning the status of the uncertainty principle, the role of quantum measurement theory, and the kinematics of the theory (including the status of Galilean and Lorentz invariance). The restriction to equilibrium has also made an erroneously-constructed stochastic model of particle creation appear more plausible than it actually is. To avoid needless controversy, we advocate a modest 'empirical approach' to the foundations of statistical mechanics. We argue that the existence or otherwise of quantum nonequilibrium in our world is an empirical question to be settled by experiment.

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1 Introduction

The pilot-wave theory of de Broglie (1928) and Bohm (1952a,b) is a deterministic theory of motion for individual systems. In the version first given by de Broglie, a system with configuration q and wave function $\psi(q,t)$ has an actual trajectory $q(t)$ determined by de Broglie's equation of motion

$$\frac{dq}{dt} = \frac{j}{|\psi|^2} \ , \tag{1}$$

where ψ obeys the usual Schrödinger equation (units $\hbar = 1$)

$$i\frac{\partial \psi}{\partial t} = \hat{H}\psi \tag{2}$$

(with a Hamiltonian operator $\hat{H}$) and j is a current satisfying the continuity equation

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot j = 0 \tag{3}$$

(with ∂_q the gradient operator in configuration space).[1] Equation (3) is a straightforward consequence of (2), and using (1) it may be rewritten as

$$\frac{\partial |\psi|^2}{\partial t} + \partial_q \cdot \left(|\psi|^2 \dot{q} \right) = 0 \tag{4}$$

(where $\dot{q} = j/|\psi|^2$ is the configuration-space velocity field).

Thus, for example, for a single low-energy spinless particle of mass m we find a current

$$\mathbf{j} = |\psi|^2 \frac{\nabla S}{m} \tag{5}$$

(where S is the phase of $\psi = |\psi| \exp(iS)$) and (1) reads

$$\frac{d\mathbf{x}}{dt} = \frac{\nabla S}{m} \ . \tag{6}$$

Given an initial wave function $\psi(q,0)$, (2) determines $\psi(q,t)$ at all times and so the right-hand side of (1) is also determined at all times. Given an initial configuration $q(0)$, (1) then determines the trajectory $q(t)$. Thus, for example, in a two-slit experiment with a single particle, if the incident wave function is known then (6) determines the trajectory $\mathbf{x}(t)$ for any initial position $\mathbf{x}(0)$.

Mathematically, for a given wave function, the law of motion (1) defines a trajectory $q(t)$ for each initial configuration $q(0)$. In practice, however, we do not know the value of $q(0)$ within the initial packet. For an ensemble of systems with the same $\psi(q,0)$, the value of $q(0)$ will generally vary from one system to another. We may then consider an initial distribution $\rho(q,0)$ of values of $q(0)$

[1] This construction applies to any system with a Hamiltonian $\hat{H}$ given by a differential operator on configuration space (Struyve and Valentini 2009).

over the ensemble. As the trajectories $q(t)$ evolve, so will the distribution $\rho(q,t)$. By construction $\rho(q,t)$ will obey the continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_q \cdot (\rho \dot{q}) = 0 \ . \tag{7}$$

In principle there is no reason why we could not consider an arbitrary initial distribution $\rho(q,0)$. De Broglie's equation (1) determines the time evolution of a trajectory $q(t)$ for any initial $q(0)$, and over the ensemble the continuity equation (7) determines the time evolution of a density $\rho(q,t)$ for any initial $\rho(q,0)$. There is certainly no reason of principle why $\rho(q,0)$ should be equal to $|\psi(q,0)|^2$.

As an extreme example, an ensemble of one-particle systems could have the initial distribution $\rho(\mathbf{x},0) = \delta^3(\mathbf{x}-\mathbf{x}_0)$, with every particle beginning at the same point $\mathbf{x}_0$. As the distribution evolves, it will remain a delta-function concentrated on the evolved point $\mathbf{x}(t)$. Every particle in the ensemble would follow the same trajectory. If such an ensemble were fired at a screen with two slits, every particle would land at the same final point $\mathbf{x}_f$ on the backstop and there would be no interference pattern (indeed no pattern at all), in gross violation of quantum mechanics.

If instead it so happens that $\rho(q,0) = |\psi(q,0)|^2$, then since ρ and $|\psi|^2$ obey identical continuity equations ((7) and (4) respectively) it follows that

$$\rho(q,t) = |\psi(q,t)|^2 \tag{8}$$

for all t. This is the usual Born rule. In conventional quantum mechanics (8) is taken to be an axiom or law of nature. Whereas in pilot-wave theory (8) is a special state of 'quantum equilibrium': if it happens to hold at one time it will hold at all times (for an ensemble of isolated systems). Thus, for example, if such an ensemble of particles is fired at a screen with two slits, the incoming equilibrium ensemble will evolve into an equilibrium ensemble at the backstop, and hence the usual interference pattern $\rho = |\psi|^2$ will be trivially obtained. More generally, as first shown in detail by Bohm (1952b), for systems and apparatus initially in quantum equilibrium, the distribution of outcomes of quantum measurements will agree with the conventional quantum formalism.

But in principle the theory allows for 'quantum nonequilibrium' ($\rho \neq |\psi|^2$). How then can pilot-wave theory explain the success of the Born rule (8), which has been confirmed to high accuracy in every laboratory experiment? Most workers in the field (past and present) simply take it as a postulate. Thus, for example, according to Bell (1987, p. 112) '[i]t is *assumed* that the particles are so delivered initially by the source', while according to Holland (1993, p. 67) the Born rule is one of the 'basic postulates'. This is unsatisfactory. There is after all a basic conceptual distinction between equations of motion and initial conditions. The former are regarded as immutable laws (they could not be otherwise), whereas the latter are contingencies (there is no reason of principle why they could not be otherwise). Once the laws are known they are the same for all systems, whereas for a given system the initial conditions must be determined empirically. Thus Newton, for example, wrote down laws that explain the motion of the moon, but he made no attempt to explain the current position and velocity of the moon: the latter are arbitrary or contingent initial conditions to be determined empirically, which may then be inserted into the laws of motion to determine the position and velocity at other times. In pilot-wave theory, if we consider only ensembles restricted by the additional postulate (8), then this is closely analogous to considering Newtonian mechanics only for ensembles restricted to thermal equilibrium (with a uniform distribution on the energy surface in phase space). In both theories there is a much wider nonequilibrium physics, which is lost if we simply adopt initial equilibrium as a postulate.

Most workers in the field seem unperturbed by this and continue to treat (8) as a postulate. Others are convinced that some further explanation is required and that the question – if pilot-wave theory is true, why do we always observe the Born rule? – requires a more satisfying answer.

There are currently two main approaches to understanding the Born rule in pilot-wave theory, which we briefly summarise here.

The first approach, associated primarily with this author and collaborators, proposes that the Born rule we observe today should be explained by a process of 'quantum relaxation' (analogous to thermal relaxation), whereby initial nonequilibrium distributions $\rho \neq |\psi|^2$ evolve towards equilibrium on a coarse-grained level, $\bar{\rho} \to \overline{|\psi|^2}$ (in terms of coarse-grained densities $\bar{\rho}$ and $\overline{|\psi|^2}$, much as in Gibbs' classical account of thermal relaxation for the coarse-grained density on phase space). This process may be understood in terms of a 'subquantum' coarse-graining H-theorem on configuration space, analogous to the classical coarse-graining H-theorem on phase space (Valentini 1991a,b). Extensive numerical simulations, carried out with wave functions that are superpositions of different energy states, have confirmed the general expectation that initial densities $\rho(q, 0)$ lacking in fine-grained microstructure rapidly become highly filamentary on small scales and indeed approach the equilibrium density $|\psi|^2$ on a coarse-grained level (Valentini 1992, 2001; Valentini and Westman 2005; Towler, Russell and Valentini 2012; Colin 2012; Abraham, Colin and Valentini 2014). This may be quantified by a decrease of the coarse-grained H-function

$$\bar{H}(t) = \int dq\ \bar{\rho} \ln \left(\bar{\rho} / \overline{|\psi|^2} \right) \ , \tag{9}$$

which reaches its minimum $\bar{H} = 0$ if and only if $\bar{\rho} = \overline{|\psi|^2}$, and which is found to decay approximately exponentially with time (Valentini and Westman 2005; Towler, Russell and Valentini 2012; Abraham, Colin and Valentini 2014). Similar studies and simulations have been carried out for field theory in an expanding universe, for which relaxation is found to be suppressed at very long cosmological wavelengths (Valentini 2007, 2008a, 2010a; Colin and Valentini 2013, 2015, 2016). This opens the door to possible empirical evidence for quantum nonequilibrium in the cosmic microwave background (Valentini 2010a; Colin and Valentini 2015; Vitenti, Peter and Valentini 2019) – as well as in relic particles left over today from the very early universe (Valentini 2001, 2007; Underwood and Valentini 2015, 2016). It has further been shown that if nonequilibrium systems were discovered today, their physics would be radically different from the physics we currently know, involving practical superluminal signalling, violations of the uncertainty principle, and a general breakdown of standard quantum constraints (such as expectation additivity and the indistinguishability of non-orthogonal quantum states) (Valentini 1991a,b, 1992, 2002a, 2004, 2009; Pearle and Valentini 2006). On this view, quantum theory is merely a special 'equilibrium' case of a much wider nonequilibrium physics, which may have existed in the early universe and which could still exist in some exotic systems today.

The second approach, associated primarily with Dürr, Goldstein, and Zanghì, as well as with Tumulka and other collaborators, proposes that the Born rule we observe today should be explained in terms of the 'typicality' of configurations $q_{\mathrm{univ}}(0)$ for the whole universe at the initial time $t = 0$ (Dürr, Goldstein and Zanghì 1992; Dürr and Teufel 2009; Goldstein 2017; Tumulka 2018). In this approach, if $\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)$ is the initial universal wave function then $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)|^2$ is assumed to be the natural measure on the set of possible initial universal configurations $q_{\mathrm{univ}}(0)$. It may then be shown that the Born rule (8) is almost always obtained for ensembles of sub-systems prepared with wave function ψ – where 'almost always' is defined with respect to the measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)|^2$. This is regarded as an explanation for the empirical success of the Born rule (8). On this view there is no realistic chance of ever observing quantum nonequilibrium, which is intrinsically unlikely (as defined with respect to $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)|^2$). The Born rule is in effect regarded as an intrinsic part of the theory, though instead of postulating the probability distribution (8) for sub-systems this approach postulates the typicality measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)|^2$ for the whole universe at $t = 0$. If this is correct, quantum nonequilibrium will never be observed and de Broglie-Bohm theory will never be experimentally distinguishable from conventional quantum theory.

The typicality approach has given rise to a distinctive physical perspective on pilot-wave theory – concerning for example the status of the uncertainty principle and of Lorentz invariance, among other important topics. These views may be classified under the heading of the 'Bohmian mechanics school', where the term 'Bohmian mechanics' was first introduced by Dürr, Goldstein and Zanghì (1992) to denote the dynamical theory defined by equations (1) and (2).

It should however be noted that, historically speaking, the dynamics defined by (1) and (2) was first proposed by de Broglie at the 1927 Solvay conference (for a many-body system with a pilot wave in configuration space) (Bacciagaluppi and Valentini 2009). De Broglie called his new form of dynamics 'pilot-wave theory'. The theory was revived by Bohm in 1952, though rewritten in a second-order form with a law of motion for acceleration that includes a 'quantum potential' Q. Bohm's version of the dynamics is physically distinct from de Broglie's: in principle it allows for non-standard initial momenta $p \neq \partial_q S$ (Bohm 1952a, pp. 170, 179; Colin and Valentini 2014).[2] Thus there are impor-

[2]In Bohm's dynamics there arises the additional question of why we observe today an

tant physical differences between de Broglie's dynamics and Bohm's dynamics. The terminology 'Bohmian mechanics', as applied to de Broglie's equations (1) and (2), is therefore misleading: it does not give due credit to de Broglie and it misrepresents the views of Bohm. Our concern in this paper is with the status of the Born rule in de Broglie's dynamics which, following de Broglie's own usage (as well as Bell's), we refer to as 'pilot-wave theory'.

Writings by the Bohmian mechanics school generally fail to recognise the significance and priority of de Broglie's work. For example, in rather glib sections entitled 'History', both Goldstein (2017) and Tumulka (2018) portray de Broglie as having simply proposed or considered the guidance equation at the 1927 Solvay conference. But in fact, as early as 1923 de Broglie had postulated the single-particle guidance equation – as a new law of motion, expressing a unification of the principles of Maupertuis and Fermat – and in that same year de Broglie used his theory to predict electron interference (four years before it was observed by Davisson and Germer). Furthermore, it was de Broglie's early research into his new form of dynamics (with particles guided by waves) that led Schrödinger to the wave equation in 1926.[3]

The distinctive approach of the Bohmian mechanics school has been reiterated and developed in a number of papers and reviews. A textbook has also been published (Dürr and Teufel 2009). For the sake of perspective it is worth remarking that writings by members of this school generally focus on their own interpretation. There are other approaches to de Broglie-Bohm theory, not only that taken by this author and collaborators but also others that lie outside the scope of this paper.[4] The Bohmian mechanics school has been particularly influential among philosophers of physics. The entry 'Bohmian mechanics' in *The Stanford Encyclopedia of Philosophy* is written by a leading member of the school (Goldstein 2017) (regularly updated by the same author since 2001). It is noteworthy that such an extensive reference encyclopedia does not contain an entry on de Broglie-Bohm theory generally; only this one particular school is represented, suggesting a skewed perception of the field among philosophers. One of the aims of this paper is to redress this imbalance in the philosophy of physics literature.

We shall compare and contrast the two approaches outlined above, in particular regarding the status of the Born rule and related physical questions. As we shall discuss, in our view the typicality approach is essentially circular (Valentini 1996, 2001). With respect to a different initial measure (such as $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^4$), we will almost always obtain initial violations of the Born rule (such as $\rho \propto |\psi|^4$). While it may be said that nonequilibrium is 'untypical' (has zero measure) with respect to the univeral Born-rule measure, it may

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'extended quantum equilibrium' in phase space, with momenta satisfying $p = \partial_q S$ as well as configurations distributed according to (8). Colin and Valentini (2014) show that extended nonequilibrium does not relax and is unstable, and argue that as a result Bohm's dynamics is physically untenable.

[3]For an extensive historical analysis of de Broglie's remarkable work in the period 1923–27, see Bacciagaluppi and Valentini (2009, chapter 2).

[4]See, for example, the books by Holland (1993) and by Bohm and Hiley (1993).

equally be said that nonequilibrium is 'typical' (has unit measure) with respect to a non-Born-rule measure. In effect, in the typicality approach the Born rule is taken as an axiom, albeit at the level of the universe as a whole. This is misleading, not least because a postulate about initial conditions can have no fundamental status in a theory of dynamics.

As we shall see, the typicality approach has engendered a basic confusion between contingent and law-like features. This has led to misleading claims not only about the Born rule but also about the nature of the wave function (or pilot wave). The artificial restriction to equilibrium has led to further misunderstandings concerning the status of the uncertainty principle, the role of quantum measurement theory, and the kinematics of the theory (including the status of Galilean and Lorentz invariance). The restriction to equilibrium has also made an erroneously-constructed stochastic model of particle creation seem more plausible than it actually is.

By considering how hidden variables can account for the Born rule (8), workers in quantum foundations find themselves confronted by issues in the foundations of statistical mechanics – a subject which is no less fractious and controversial than quantum foundations itself. We begin by outlining our own views on the subject (Section 2), summarise some of the key results in quantum relaxation and how these apply to cosmology (Sections 3 and 4), and then provide a critique of the typicality approach (Section 5) and of related viewpoints (Sections 6 and 7), ending with some concluding remarks (Section 8).

2 Empirical approach to statistical mechanics

Pilot-wave dynamics is a deterministic theory of motion. As in classical physics, there is a clear conceptual distinction between the laws of motion on the one hand and initial conditions on the other. The initial conditions (for the wave function ψ and for the configuration q) are in principle arbitrary – which is to say, perhaps more properly, that they are contingent. Whatever the actual initial conditions were, there is no known reason of principle why they could not have been different. In order to find out what the initial conditions actually were, we do not appeal to laws or principles but to simple empiricism: we carry out observations today and on that basis we try (using our knowledge of the dynamical laws) to deduce the initial conditions. This is as true for ensembles as it is for single systems. In pilot-wave theory, for an ensemble of systems with the same initial wave function $\psi(q, 0)$, the initial distribution $\rho(q, 0)$ of actual initial configurations $q(0)$ could in principle be anything. To find out what $\rho(q, 0)$ was in an actual case we must resort to empirical observation.

This raises the subtle question of what it might mean for pilot-wave dynamics to 'explain' the Born rule (8). If the distribution ρ is purely empirical, there might then seem to be no question of 'explaining' the particular distribution (8): it is not something one explains, it is something one finds empirically.

The matter is further complicated by the time-reversal invariance of pilot-wave dynamics. If all initial conditions are in principle possible, then any distribution $\rho(q, t_0)$ today is in principle possible, since it could have evolved from some appropriate $\rho(q, 0)$ (which can in principle be calculated from $\rho(q, t_0)$ by time-reversal of the equations of motion). Again, there might seem to be no question of 'explaining' (8): it is simply a brute 'matter of fact' established by observation.

In our view there is in fact considerable scope for explaining the presently-observed Born rule (8), in the sense that it can be explained in terms of past conditions (with the aid of dynamical laws) – where the past conditions are, however, ultimately empirical and not fixed by any fundamental laws or principles. On this view, the present is explained in terms of the past, while the past is itself something we establish empirically. This of course leaves open the possibility of further explanation by peering even further into the past, and in our universe this chain of causal explanations eventually leads us back to the big bang. To understand the origins of the Born rule, then, we are led to consider conditions in the very early universe. Specifically: what initial conditions (at or close to the big bang) could have given rise to the all-pervasive distribution (8) which we see today?

In the context of statistical mechanics, it might be objected that to explain the state (8) seen today we must not merely deduce which past conditions (for example, which particular initial states $\rho(q, 0) \neq |\psi(q, 0)|^2$) could have evolved into (8) today, we must instead show that 'all' or 'most' past conditions evolve into (8) today. For otherwise, it might be said, we have simply replaced one unexplained empirical fact (conditions today) with another unexplained empirical fact (conditions in the past), so that in a sense we are not really making progress. In our view this objection is misguided and has roots in some unfortunate misunderstandings in the early history of statistical mechanics.

First of all, it is perfectly reasonable to explain the present in terms of the past. This is standard practice across the physical sciences – from astrophysics to geology. As a simple example, suppose that today at time t_0 the moon is observed to have a certain position and momentum, so that it now occupies a particular location (q_0, p_0) in phase space. With the aid of Newton's laws, this fact today may be explained by the fact that the moon was at a location $(q(t), p(t))$ in phase space at some earlier time $t < t_0$. If t is very far in the past, pre-dating direct human observation, then in practice we would deduce that the moon must have been at $(q(t), p(t))$ at time t. That we have had to deduce the past from the present would not undermine our physical intuition that the moon may be said to be where it is now *because* it was in the deduced earlier state at an earlier time. This is normal scientific practice. On the other hand, one can imagine the philosophical objection being raised, that the past state is a mere deduction (or retrodiction) and not a *bona fide* 'explanation' for the observed state today. It might also be suggested that we would have a satisfactory explanation only if we could show that *all* – or in some sense 'most' – possible earlier states $(q(t), p(t))$ of the moon evolve into the moon being in the state (q_0, p_0) today. Needless to say, most physicists would disagree with this objection (not least because, from what we understand of lunar dynamics, such a suggestion has no chance of being correct). The objection seems unfounded. And yet similar objections are frequently heard in the context of statistical mechanics. Why?

In our view the trouble stems from a mistake in the early history of the subject. Boltzmann originally hoped to deduce the second law of thermodynamics from mechanics alone. As is well known, this ambitious project was fundamentally misguided. It is impossible to deduce any kind of necessary unidirectional evolution in time in a time-reversal invariant theory. For any initial molecular state that evolves towards thermal equilibrium, one can always construct a time-reversed initial state that evolves away from thermal equilibrium. Boltzmann's program was dogged by such 'reversibility objections', resulting in heated debate about the foundations of the subject. The debate rages even today.[5] It is now widely accepted that the laws of mechanics alone do not suffice: one must also assume something about the initial conditions (such as an absence of fine-grained microstructure, or an absence of correlations among molecular velocities). Debates then continue about the status of the assumption about the initial conditions, with many authors attempting to justify the assumption on the basis of some fundamental principle or other. Running like a thread through these debates is the expectation that, in order for the program to succeed, it must be shown either that the required initial conditions are 'almost always' satisfied or that they are required by some principle (where such attempts invariably lead to further controversy). In our view, this expectation is misguided and reflects the historical error in Boltzmann's original program. There was never any reason to expect all allowed initial conditions to give rise to thermal relaxation, and subsequent attempts to show that 'most' initial conditions will do so, or that the required initial conditions are consequences of some fundamental principle or other, in effect propagate the original error (albeit in a reduced or weaker form).

We advocate a more modest – and in our view more reasonable – 'empirical' approach to statistical mechanics (Valentini 1996, 2001). On this view the observed thermodynamical behaviour is an empirical fact which must be explained (with the aid of dynamical laws) in terms of past conditions, where the latter are themselves also empirical. The past conditions do not need to be 'almost always' or 'typically' true, nor do they need to be true by virtue of some deep principle or other: they simply need to explain or be consistent with what is observed. Just as in the case of the moon, where we try to deduce – or if necessary guess – its state in the past given its state today, in the case of a box of gas that is evolving towards thermal equilibrium we try to deduce or guess the required character of the initial (microscopic) state. For a gas, of course, there will be a set or class of microstates yielding the observed behaviour. Unlike for the moon, we make no attempt to deduce the exact initial micro-state. And with so many variables involved, it is convenient to apply statistical methods. The essential aim and method of statistical mechanics is then this. First, to find a class of initial conditions that yields the observed behaviour. And second,

[5]For an even-handed and scholarly review see Uffink (2007).

to understand the evolution of those initial conditions towards equilibrium in terms of a general mechanism – without having to solve the exact equations of motion for the huge number of variables involved.

In the case of pilot-wave theory we wish to explain the observed validity of the Born rule (8) for laboratory systems – to within a certain experimental accuracy. For example, if we prepare a large number N of hydrogen atoms in the ground state with wave function $\psi_{100}(\mathbf{x})$, and if we measure the electron position $\mathbf{x}$ (relative to the nucleus) for each atom, then for large N we find an empirical distribution $\rho(\mathbf{x})$ of the schematic form

$$\rho = |\psi_{100}|^2 \pm \epsilon \ , \tag{10}$$

where ϵ characterises the accuracy to which the Born rule has been confirmed (where this will depend on the accuracy of the position measurements as well as on the value of N). How can this be explained in terms of past conditions?

The first thing to note is that, when we encounter a hydrogen atom in the laboratory, the atom has not been floating freely in a vacuum for billions of years prior to us experimenting with it. The atom has a past history during which it has interacted with other things. That past history traces back ultimately to the formation of the earth, the solar system, our galaxy, and ultimately merges with the history of the universe as a whole, which as we know began with a hot and violent phase called the big bang. In fact, every system we have access to in the laboratory has a long and violent astrophysical history. Therefore, when attempting to explain the Born rule today by conditions in the past, we should make use of our knowledge of that history. In other words, when we attempt to deduce what earlier conditions are required to explain what we observe in the laboratory today, we should take into account what we already know about the history of the systems in question.

The second thing to note is that the Born rule for a sub-system such as an atom is a simple consequence of the Born rule applied to a larger system from which the atom may have been extracted. If we consider an ensemble of many-body systems, all with the same wave function $\Psi(q, t_0)$ and with an ensemble distribution $P(q, t_0) = |\Psi(q, t_0)|^2$ of configurations q at some time t_0, then it is readily shown that if an ensemble of sub-systems with configurations x are extracted from the parent ensemble and prepared with an effective (or reduced) wave function $\psi(x, t)$ at $t > t_0$, then the distribution of extracted configurations x will be $\rho(x, t) = |\psi(x, t)|^2$ (Valentini 1991a).[6] In other words, equilibrium for a many-body system implies equilibrium for extracted sub-systems (a property which is sometimes called 'nesting'). This means that we can explain the Born rule for extracted sub-systems such as atoms if we are able to explain the Born rule for larger parent systems.

Wherever we look, in fact, we find the Born rule – not only in the laboratory today but also further afield. For example, the relative intensities of atomic spectral lines emitted from the outskirts of distant quasars agree with the Born rule (as applied to atomic transitions). The observed cosmological

[6] A similar result was obtained by Dürr, Goldstein and Zanghì (1992).

helium abundance agrees with calculations based on the Born rule (as applied to nuclear reactions in the early hot universe). Perhaps the ultimate test of the Born rule is currently taking place in satellite observations of the small temperature and polarization anisotropies in the cosmic microwave background, which were caused by classical inhomogeneities which existed at the time of photon decoupling (around 400,000 years after the big bang), which in turn grew from classical inhomogeneities which existed in the very early universe, and which according to inflationary cosmology were in turn formed from quantum vacuum fluctuations in an 'inflaton' scalar field. Ultimately, on our current understanding, the 'primordial power spectrum' (the spectrum of very early classical inhomogeneities) was generated by a Born-rule spectrum of primordial quantum field fluctuations.

To explain the success of the Born rule, then, we can consider the earliest possible conditions in the history of our universe. Clearly, the initial conditions must be such as to evolve into or imply the Born rule at relevant later times. What initial conditions should we assume?

One possibility, of course, is to simply assume that the universe began in a state of quantum equilibrium. Below we argue that this is, in effect, the assumption made by Dürr, Goldstein and Zanghì (1992). Unlike in the thermal case, we have not observed relaxation to the Born rule actually taking place over time. All we see is the equilibrium Born rule. Since the equations of motion preserve the Born rule over time, a simple way to explain our observation of the Born rule now is to assume that the Born rule was true at the beginning.

But initial equilibrium is only one possibility among uncountably many. Given the known violent history of our universe and of everything in it, and given the results for quantum relaxation summarised in the next section, there clearly exists a large class of initial nonequilibrium states that will evolve to yield the Born rule today to an excellent approximation (in particular at the short wavelengths relevant to local physics). As we shall see, the said initial nonequilibrium distributions may be broadly characterised as having no fine-grained microstructure (with respect to some coarse-graining length) while the initial wave functions are superpositions of at least a few energy eigenstates (in order to guarantee a sufficiently complex de Broglie velocity field). A coarse-graining H-theorem provides a general mechanism in terms of which we can understand how equilibrium is approached, without having to solve the exact equations of motion for the system. By itself, of course, the H-theorem does not prove that equilibrium is actually reached. The rate and extent of relaxation depend on the system and on its initial wave function, as shown by extensive numerical simulations. The Born rule today may then be understood to have arisen dynamically, by a process of relaxation from an earlier nonequilibrium state for which the Born rule was not valid.

Once again, in a time-reversal invariant dynamics it cannot be true that all initial nonequilibrium states relax to equilibrium. But it does not need to be true: non-relaxing initial conditions (with fine-grained microstructure, or with very simple wave functions with trivial velocity fields) are ruled out empirically, not by theoretical fiat.

As we shall see in Section 4, modern developments in theoretical and observational cosmology make it possible to test the Born rule for quantum fields at very early times. Thus the existence of initial equilibrium or nonequilibrium is an empirical question – not only in principle but also in practice.

3 Overview of quantum relaxation

In this section we provide a brief overview of quantum relaxation. Further details may be found in the cited papers.

3.1 Coarse-graining H-theorem

For a nonequilibrium ensemble of isolated systems with configurations q, it follows from (4) and (7) that the ratio $f = \rho / |\psi|^2$ is preserved along trajectories: $df/dt = 0$. (This is the analogue of Liouville's theorem, $d\rho_{\mathrm{cl}}/dt = 0$, for a classical phase-space density ρ_{cl}.) The exact H-function $H(t) = \int dq\ \rho \ln(\rho/|\psi|^2)$ is then constant, $dH/dt = 0$, and there is no fine-grained relaxation. However, if we average the densities ρ and $|\psi|^2$ over small coarse-graining cells of volume δV,[7] we may assume the absence of fine-grained microstructure at $t = 0$,[8]

$$\bar{\rho}(q,0) = \rho(q,0) \ , \quad \overline{|\psi(q,0)|^2} = |\psi(q,0)|^2 \ , \tag{11}$$

and consider the time evolution of the coarse-grained H-function (9). Defining $\tilde{f} \equiv \bar{\rho}/\overline{|\psi|^2}$, straightforward manipulations show that

$$\bar{H}_0 - \bar{H} = \int dq\ |\psi|^2 \left(f \ln(f/\tilde{f}) - f + \tilde{f} \right) \tag{12}$$

(where the subscript 0 denotes a quantity at $t = 0$ and the absence of a subscript denotes a quantity at a general time t). Use of the inequality $x \ln(x/y) - x + y \geq 0$ – for all real and non-negative x, y, with equality if and only if $x = y$ – then implies the coarse-graining H-theorem (Valentini 1991a, 1992)[9]

$$\bar{H}(t) \leq \bar{H}(0) \ . \tag{13}$$

From (12) it also follows that (13) becomes a strict inequality, $\bar{H}(t) < \bar{H}(0)$, when $f \neq \tilde{f}$. Since $|\psi|^2$ remains smooth this occurs when $\rho \neq \bar{\rho}$, that is, when ρ develops fine-grained structure – which it generally will for non-trivial velocity fields that vary over the coarse-graining cells.

The quantity $\bar{H}$ is equal to minus the relative entropy of $\bar{\rho}$ with respect to $\overline{|\psi|^2}$. As already noted, $\bar{H}$ is bounded from below by zero and the minimum

[7]For systems with N degrees of freedom, $q = (q_1, q_2, ..., q_N)$ and $\delta V = (\delta q)^N$.

[8]This will hold to arbitrary accuracy as $\delta V \to 0$ if ρ_0 and $|\psi_0|^2$ are smooth functions.

[9]As is well known for the classical case, the result (13) is time-symmetric, with $t = 0$ a local maximum of $\bar{H}(t)$.

$\bar{H} = 0$ is attained if and only if $\bar{\rho} = \overline{|\psi|^2}$. Thus a decrease of $\bar{H}$ quantifies relaxation to the Born rule.[10]

The result (13) formalises an intuitive understanding of relaxation in terms of a mixing of two 'fluids' with densities ρ and $|\psi|^2$ in configuration space. These obey the same continuity equation and are therefore 'stirred' by the same velocity field $\dot{q}$. For a sufficiently complicated flow, ρ and $|\psi|^2$ tend to become indistinguishable on a coarse-grained level (Valentini 1991a). This is similar to the classical stirring of two fluids that was famously discussed by Gibbs (1902), where his fluids were analogous to the classical phase-space densities ρ_{cl} (for a general ensemble) and $\rho_{\mathrm{eq}} = \mathrm{const.}$ (for an equilibrium ensemble) on the energy surface, and where the mixing of ρ_{cl} and ρ_{eq} may be quantified by a decrease of the classical H-function $\bar{H}_{\mathrm{cl}} = \int dq\ \bar{\rho}_{\mathrm{cl}} \ln\left(\bar{\rho}_{\mathrm{cl}}/\overline{\rho_{\mathrm{eq}}}\right)$. In both cases the nonequilibrium density develops fine-grained structure while the equilibrium density remains smooth. The increase of the 'subquantum entropy' $\bar{S} = -\bar{H}$ may be associated with the mixing of ρ and $|\psi|^2$ in configuration space, just as the increase of the Gibbs entropy $\bar{S}_{\mathrm{Gibbs}} = -\bar{H}_{\mathrm{cl}}$ may be associated with the mixing of ρ_{cl} and ρ_{eq} in phase space.

Note that we use the word 'mixing' informally in the simple sense of 'stirring' (as in Gibbs' original analogy). Mathematical mixing is defined by an infinite-time limit.[11] Its physical relevance is therefore questionable, and in any case it might not apply to realistic systems even in that limit (Uffink 2007). In our view the above process – whereby ρ develops fine-grained structure while $|\psi|^2$ does not – is the essential physical mechanism that drives quantum relaxation over realistic timescales. Though as noted, the actual rate and extent of relaxation will depend on the system.

An assumption about initial conditions is of course necessary to explain relaxation in a time-reversal invariant theory. We offer no 'principle' to justify the initial conditions (11). In the spirit of our empirical approach, they are justified only by the extent to which they help us explain observations. We take (11) to be matters of fact about our world, while acknowledging that in principle they could be false. Their truth or falsity is ultimately a matter for experiment.

It is also worth remarking that our approach (like its classical Gibbsean counterpart) does not rely on any particular interpretation of probability theory. The density ρ might represent a subjective probability for a single system, a distribution over a theoretical ensemble, or the distribution of a real ensemble of existing systems, according to taste or requirement.

Note also that, for a finite real ensemble of N systems, the actual density ρ will be a sum of delta-functions, which can approach a smooth function only in the large-N limit (Valentini 1992, pp. 18, 36). To obtain a density with no fine-grained structure on a coarse-graining scale δV, we must of course take the

[10] The quantity $\bar{H}$ is also equal to the well-known (in mathematical statistics) Kullback-Leibler divergence $D_{\mathrm{KL}}(\bar{\rho} \parallel \overline{|\psi|^2})$, which measures how $\bar{\rho}$ differs from $\overline{|\psi|^2}$.

[11] Formally, a dynamical system (with measure μ on a set Γ subject to measure-preserving transformations T_t) is 'mixing' if and only if $\lim_{t\to\infty} \mu(T_t A \cap B) = \mu(A)\mu(B)$ for all relevant subsets A and B of Γ.

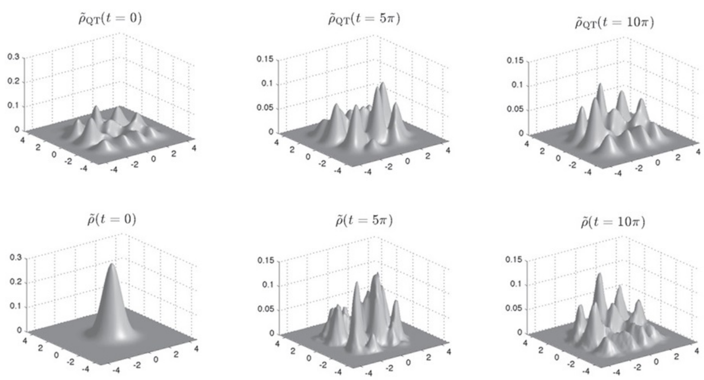

Figure 1: Illustration of numerical quantum relaxation for an oscillator in a superposition of $M = 25$ modes (Abraham, Colin and Valentini 2014). The wave function has period 2π. The top row shows the evolving coarse-grained density $\tilde{\rho}_{\rm QT}$ as predicted by quantum theory, while the bottom row shows the coarse-grained relaxation of an initial nonequilibrium density $\rho(x, y, 0) = (1/\pi)e^{-(x^2+y^2)}$ (where tildes denote a smoothed coarse-graining with overlapping cells). After five periods the coarse-grained densities are almost indistinguishable. (Note the different vertical scale at $t = 0$.)

appropriate large-N limit *before* considering small δV.[12]

3.2 Numerical simulations

Extensive numerical simulations demonstrate that quantum relaxation takes place efficiently for wave functions ψ that are superpositions of multiple energy eigenstates (Valentini and Westman 2005; Towler, Russell and Valentini 2012; Colin 2012; Abraham, Colin and Valentini 2014). An example is shown in Figure 1, for a two-dimensional oscillator in a superposition of $M = 25$ modes and with an initial Gaussian nonequilibrium density ρ_0. The top row displays the time evolution of the (coarse-grained) equilibrium density $|\psi|^2$, while the bottom row displays relaxation of the actual (coarse-grained) density ρ. Comparable results are obtained for superpositions with as little as $M = 4$ modes, and also for a two-dimensional box.

In all of these simulations $\bar{H}$ is found to decay approximately exponentially with time: $\bar{H}(t) \approx \bar{H}_0 e^{-t/\tau}$ for some constant τ whose value depends on the initial wave function as well as on the coarse-graining length (Towler, Russell and Valentini 2012). An example is shown in Figure 2.

[12]This resolves a concern raised by Norsen (2018, p. 16) that the condition (11) on ρ_0 cannot be satisfied for realistic finite ensembles. The same concern arises, of course, in the classical Gibbsean approach and has the same resolution.

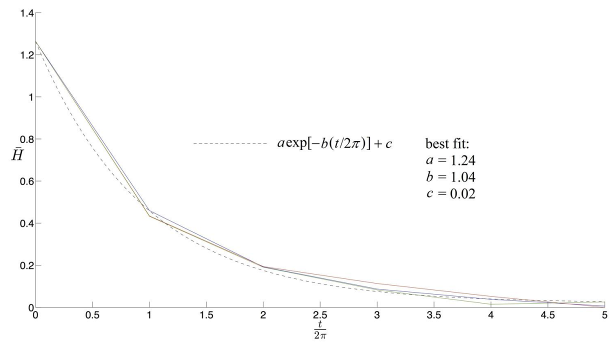

Figure 2: Approximate exponential decay of $\bar{H}(t)$ for the same simulation displayed in Figure 1 (Abraham, Colin and Valentini 2014). The error in $\bar{H}$ is estimated by running three separate simulations with different numerical grids (the solid curves). Fitting to an exponential (dashed curve) yields a best-fit residue $c = 0.02$ that is comparable to the late-time error, indicating no discernible late-time residue in $\bar{H}$. We find a decay timescale $\tau = 2\pi/b \simeq 6$ (units $\hbar = m = \omega = 1$).

Some simulations show a small but discernible non-zero 'residue' in $\bar{H}$ at large times (unlike the case displayed in Figure 2), indicating that equilibrium is not reached exactly (Abraham, Colin and Valentini 2014). For these cases, the trajectories tend to show some degree of confinement (not fully exploring the support of $|\psi|^2$). Numerical evidence shows that this is less likely to happen for larger M. Because all laboratory systems have a long and violent astrophysical history, during which the relevant value of M will have been very large, we may expect that in the remote past they will have reached equilibrium on a very small coarse-graining scale.

3.3 The early universe

This motivates us to consider quantum relaxation in the early universe. This may be discussed for a free massless scalar field ϕ on flat expanding space with spacetime metric

$$d\tau^2 = dt^2 - a^2 d\mathbf{x}^2 \tag{14}$$

and a scale factor $a(t) \propto t^{1/2}$ (corresponding to a radiation-dominated expansion), where t is standard cosmological time and physical wavelengths $\lambda_{\text{phys}}(t)$ are proportional to $a(t)$. The Fourier components $\phi_{\mathbf{k}}(t)$ may be written as

$$\phi_{\mathbf{k}} = \frac{\sqrt{V}}{(2\pi)^{3/2}} \left(q_{\mathbf{k}1} + i q_{\mathbf{k}2} \right) \ , \tag{15}$$

where $q_{\mathbf{k}1}$, $q_{\mathbf{k}2}$ are real and V is a normalisation volume. The field Hamiltonian then takes the form $\hat{H} = \sum_{\mathbf{k}r} \hat{H}_{\mathbf{k}r}$, where $\hat{H}_{\mathbf{k}r}$ $(r = 1, 2)$ coincides with the

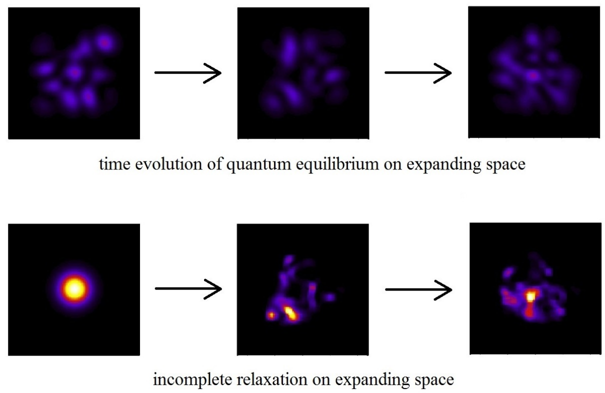

Figure 3: Incomplete quantum relaxation for a super-Hubble field mode $\phi_{\mathbf{k}}$ on expanding space over a time interval (t_i, t_f) (Colin and Valentini 2013). The final nonequilibrium width is noticeably smaller than the final equilibrium width.

Hamiltonian of a harmonic oscillator of mass $m = a^3$ and angular frequency $\omega = k/a$ (Valentini 2007, 2008a, 2010a). Thus a single (unentangled) field mode $\mathbf{k}$ in the early universe is mathematically equivalent to a two-dimensional oscillator with a time-dependent mass. This system is in turn equivalent to an ordinary oscillator (with constant mass and angular frequency) but with t replaced by a 'retarded time' $t_{\text{ret}} = t_{\text{ret}}(t, k)$ that depends on k (Colin and Valentini 2013). It is found that quantum relaxation depends crucially on how λ_{phys} compares with the Hubble radius $H^{-1} = a/\dot{a}$. For short wavelengths $\lambda_{\text{phys}} << H^{-1}$ we find $t_{\text{ret}}(t, k) \to t$ and we recover physics on static flat space, with the same rapid relaxation illustrated in Figure 1 for the oscillator. Whereas for long wavelengths $\lambda_{\text{phys}} \gtrsim H^{-1}$ we find $t_{\text{ret}}(t, k) < t$ and relaxation is retarded or suppressed (Valentini 2008a; Colin and Valentini 2013).

The suppression of quantum relaxation at long (super-Hubble) wavelengths is illustrated in Figure 3, where we show the time evolution of a nonequilibrium field mode with $\lambda_{\text{phys}} = 10H^{-1}$ (at initial time t_i). Over the given time interval (t_i, t_f) relaxation proceeds but is incomplete: in particular, the final nonequilibrium width (or variance) is smaller than the final equilibrium width. In contrast, in Figure 4 we show relaxation with the same initial conditions and over the same time interval (t_i, t_f) but with no spatial expansion: relaxation now takes place essentially completely and the final widths match closely.

Cosmologically speaking, the reduced width of the final distribution in Figure 3 is of particular interest. For a mode with wavenumber k we may write

$$\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle = \left\langle |\phi_{\mathbf{k}}|^2 \right\rangle_{\text{QT}} \xi(k) \ , \tag{16}$$

where $\langle ... \rangle$ and $\langle ... \rangle_{\text{QT}}$ denote respective nonequilibrium and equilibrium expectation values. The function $\xi(k)$ quantifies the degree of primordial quantum

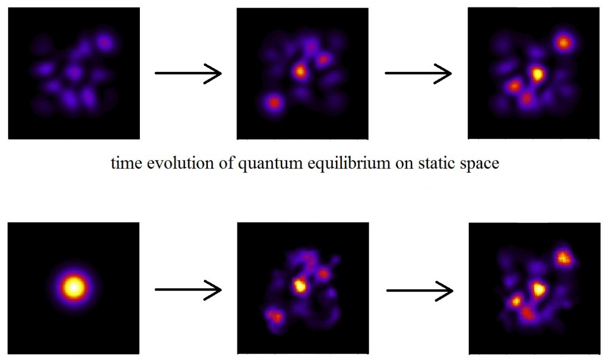

Figure 4: Essentially complete quantum relaxation for a field mode $\phi_{\mathbf{k}}$ on static space over the same time interval (t_i, t_f) as in Figure 3 and with the same initial conditions (Colin and Valentini 2013). The final nonequilibrium width closely matches the final equilibrium width.

nonequilibrium as a function of k. For the case shown in Figure 3, at the final time we clearly have $\xi(k) < 1$ (corresponding to a 'power deficit'). As a general trend we expect $\xi(k)$ to be smaller for smaller k, where longer wavelengths imply less relaxation. This has been broadly confirmed by running extensive simulations for varying values of k and plotting the function $\xi = \xi(k)$. The resulting curves show oscillations of magnitude $\lesssim 10\%$. As a first approximation we may ignore these, in which case we find a good fit to the function

$$\xi(k) = \tan^{-1}(c_1 \frac{k}{\pi} + c_2) - \frac{\pi}{2} + c_3 \ , \tag{17}$$

where the parameters c_1, c_2 and c_3 depend on the initial state and on the time interval (Colin and Valentini 2015). This function tends to a maximum $\xi \to c_3$ for $k \to \infty$, and decreases smoothly for smaller k. Simulations for a range of different initial nonequilibria – all assumed to have a narrower-than-quantum initial width[13] – show a good fit to the curve (17) (ignoring the oscillations) (Colin and Valentini 2016). Thus, with some mild assumptions about the initial state, the 'deficit function' (17) is a robust approximate prediction of the cosmological quantum relaxation scenario (for a free scalar field at the end of a radiation-dominated era).

[13]Heuristically, it seems natural to assume initial conditions with a subquantum statistical spread (so the initial state contains less noise than a conventional quantum state), but in principle this assumption could of course be incorrect.

4 Testing the primordial Born rule with cosmological data

We now outline how the Born rule may be tested at very early times with cosmological data.

The temperature of the cosmic microwave background (CMB) today is slightly anisotropic. It is customary to write a spherical harmonic expansion

$$\frac{\Delta T(\hat{\mathbf{n}})}{\bar{T}} = \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\hat{\mathbf{n}}) \tag{18}$$

for the measured anisotropy $\Delta T(\hat{\mathbf{n}}) \equiv T(\hat{\mathbf{n}}) - \bar{T}$, where the unit vector $\hat{\mathbf{n}}$ labels points on the sky and $\bar{T}$ is the mean temperature. As noted the CMB was formed around 400,000 years after the big bang, and its small anisotropies reflect small inhomogeneities of the universe at that time. Thus the coefficients a_{lm} are generated by the Fourier-space 'primordial curvature perturbation' $\mathcal{R}_{\mathbf{k}}$ according to the formula (Lyth and Riotto 1999)

$$a_{lm} = \frac{i^l}{2\pi^2} \int d^3\mathbf{k}\ \mathcal{T}(k,l) \mathcal{R}_{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) \ , \tag{19}$$

where the 'transfer function' $\mathcal{T}(k,l)$ encodes the relevant astrophysics.

Cosmologists generally assume that the measured function $\Delta T(\hat{\mathbf{n}})$ is a single realisation of a random variable with a probability distribution $P[\Delta T(\hat{\mathbf{n}})]$ associated with a 'theoretical ensemble' (which may be interpreted according to taste). It is usual to assume 'statistical isotropy', which means that P is invariant under a rotation $\hat{\mathbf{n}} \to \hat{\mathbf{n}}'$ (that is, $P[\Delta T(\hat{\mathbf{n}}')] = P[\Delta T(\hat{\mathbf{n}})]$). This implies that the ensemble average $\langle \Delta T(\hat{\mathbf{n}}_1) \Delta T(\hat{\mathbf{n}}_2) \rangle$ depends only on the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, which in turn implies (Hajian and Souradeep 2005, appendix B)

$$\langle a^*_{l'm'} a_{lm} \rangle = \delta_{ll'} \delta_{mm'} C_l \ , \tag{20}$$

where

$$C_l \equiv \left\langle |a_{lm}|^2 \right\rangle \tag{21}$$

(the 'angular power spectrum') is independent of m. Thus, while for fixed l there are $2l+1$ different quantities $|a_{lm}|^2$, statistical isotropy implies that they each have the same ensemble mean C_l.

For our one observed sky we may define a measured mean statistic

$$C_l^{\mathrm{sky}} \equiv \frac{1}{2l+1} \sum_{m=-l}^{+l} |a_{lm}|^2 \ .$$

This obviously satisfies $\left\langle C_l^{\mathrm{sky}} \right\rangle = C_l$. Thus C_l^{sky} (measured for one sky) is an unbiased estimate of C_l for the ensemble. Furthermore, assuming a Gaussian distribution it may be shown that $C_l^{\rm sky}$ has a 'cosmic variance'

$$\frac{\Delta C_l^{\rm sky}}{C_l} = \sqrt{\frac{2}{2l+1}} \ . \tag{22}$$

Thus for large l we expect to find $C_l^{\rm sky} \approx C_l$, whereas for small l the accuracy is limited.

It is also usually assumed that the theoretical ensemble for $\mathcal{R}$ is statistically homogeneous (that is, the probability distribution $P[\mathcal{R}(\mathbf{x})]$ is invariant under spatial translations). This implies that $\langle \mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{x}')\rangle$ depends only on $\mathbf{x}-\mathbf{x}'$, which implies

$$\langle \mathcal{R}^*_{\mathbf{k}'}\mathcal{R}_{\mathbf{k}}\rangle = \delta_{\mathbf{k}\mathbf{k}'}\left\langle |\mathcal{R}_{\mathbf{k}}|^2\right\rangle \ . \tag{23}$$

From (19) and (23) it follows that

$$C_l = \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{k}\ \mathcal{T}^2(k,l)\mathcal{P}_{\mathcal{R}}(k) \ , \tag{24}$$

where

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{4\pi k^3}{V}\left\langle |\mathcal{R}_{\mathbf{k}}|^2\right\rangle \tag{25}$$

is the 'primordial power spectrum' for $\mathcal{R}_{\mathbf{k}}$. Thus measurements for a single sky constrain the spectrum $\mathcal{P}_{\mathcal{R}}(k)$ – which is a property of the theoretical ensemble.

Note that in the discussion so far the quantities a_{lm}, C_l and $\mathcal{R}_{\mathbf{k}}$ are treated classically.[14]

What is the origin of the spectrum (25)? According to inflationary cosmology, during a very early period of approximately exponential expansion a perturbation $\phi_{\mathbf{k}}$ of the 'inflaton field' generates a curvature perturbation $\mathcal{R}_{\mathbf{k}} \propto \phi_{\mathbf{k}}$ (once the physical wavelength of the mode exceeds the Hubble radius) (Liddle and Lyth 2000). The quantum-theoretical variance $\left\langle |\phi_{\mathbf{k}}|^2\right\rangle_{\rm QT}$ is calculated from quantum field theory assuming the Born rule for an appropriate vacuum state. From this we readily obtain the corresponding variance $\left\langle |\mathcal{R}_{\mathbf{k}}|^2\right\rangle_{\rm QT}$ and hence a quantum-theoretical prediction $\mathcal{P}_{\mathcal{R}}^{\rm QT}(k)$ for the spectrum of $\mathcal{R}_{\mathbf{k}}$ (approximately flat with a slight tilt). If instead we have a nonequilibrium variance (16) for $\phi_{\mathbf{k}}$, the predicted spectrum for $\mathcal{R}_{\mathbf{k}}$ is corrected by the factor $\xi(k)$:

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}}^{\rm QT}(k)\xi(k) \ . \tag{26}$$

Inserting this into (24) yields a corrected angular power spectrum C_l. From measurements of the CMB we may then set observational bounds on $\xi(k)$ – that is, on corrections to the Born rule in the very early universe (Valentini 2010a).

To obtain a prediction for $\xi(k)$, we may for example assume that quantum relaxation took place during a pre-inflationary era. It is not uncommon for

[14]Goldstein, Struyve and Tumulka (2015) give a rather confused account of the relation between primordial perturbations and CMB anisotropies, missing in particular the crucial role played by statistical isotropy.

cosmologists to assume that such an era was radiation-dominated.[15] In this case, at the end of pre-inflation we expect the nonequilibrium variance to be corrected by a deficit function of the approximate form (17). It may be shown that relaxation does not take place during inflation itself (Valentini 2010a). If we make the simplifying assumption that the spectrum is unaffected by the transition from pre-inflation to inflation, we obtain a prediction for a corrected primordial power spectrum (26) with $\xi(k)$ of the form (17) (with three unknown parameters). Note, however, that if the lengthscale c_1 is too large the dip in the spectrum will be essentially unobservable (even if it exists).

CMB data from the *Planck* satellite show hints of a power deficit at large scales (small k and low l) (Aghanim *et al.* 2016). Because of the large cosmic variance in this region, it is however difficult to draw firm conclusions. Extensive data analysis shows that the predicted deficit (17) fits the data more or less as well as the standard 'power-law' model (where the evaluated significance takes into acount the larger number of parameters) (Vitenti, Peter and Valentini 2019). This is a modest success, in the sense that with three extra parameters the significance could have been worse. But evidence from this fitting alone neither supports nor rules out the prediction (17). If one is inclined to invoke Ockham's razor in favour of the simplest cosmological model, then the observed low-power anomaly may reasonably be regarded as a statistical fluctuation. To obtain evidence for or against our model we must include more detailed predictions, such as oscillations around the curve (17) (Colin and Valentini 2015; Kandhadai and Valentini 2019) or possible violations of statistical isotropy (Valentini 2015).

5 Critique of 'typicality' as an explanation for the Born rule

In this section we provide a critical assessment of the typicality approach to understanding the Born rule.

5.1 Typicality, probability, and intrinsic likelihood

As we noted in the Introduction, the Bohmian mechanics school attempts to explain the success of the Born rule today by appealing to a notion of 'typicality' for the initial configuration $q_{\text{univ}}(0)$ of the universe. In this approach, if $\Psi_{\text{univ}}(q_{\text{univ}}, 0)$ is the initial universal wave function then $|\Psi_{\text{univ}}(q_{\text{univ}}, 0)|^2$ is assumed to be the 'natural measure' on the set of initial universal configurations $q_{\text{univ}}(0)$.

In our view the typicality approach amounts to assuming, without justification, that the universe as a whole began in quantum equilibrium (Valentini 1996, 2001). The approach then seems circular. Defenders of the approach might attempt to avoid the charge of circularity by claiming that typicality and

[15] See, for example, Wang and Ng (2008).

probability are conceptually distinct. In our view, however, typicality is synonomous with probability. The Bohmian mechanics school employs the word 'typicality' when referring to probability for the whole universe and employs the word 'probability' when referring to probability for sub-systems. But in our view the two words mean the same thing.

Typicality and the Born rule

Dürr, Goldstein and Zanghì (1992) showed that, if we consider sub-systems within the universe, we will obtain the Born rule for 'almost all' initial configurations $q_{\mathrm{univ}}(0)$ – where 'almost all' is defined with respect to $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^2$. Thus it may be said that for sub-systems the quantum equilibrium distribution $\rho = |\psi|^2$ is 'typical', where the notion of 'typicality' is defined with respect to the measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^2$.

This result may be illustrated by a simple example. We consider a model universe at $t = 0$ containing a large number n of unentangled sub-systems all with the same initial wave function $\psi(q,0)$ and with initial configurations $q_1(0),\ q_2(0),\ ...,\ q_n(0)$ that generally vary from one sub-system to another. We may write

$$q_{\mathrm{univ}}(0) = (q_1(0),\ q_2(0),\ ...,\ q_n(0)) \tag{27}$$

and

$$\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0) = \psi(q_1,0)\psi(q_2,0)...\psi(q_n,0)\ . \tag{28}$$

For large n, a given initial configuration $q_{\mathrm{univ}}(0)$ determines an initial distribution $\rho(q,0)$ over the ensemble of sub-systems. Thus we have a schematic correspondence

$$q_{\mathrm{univ}}(0) \longleftrightarrow \rho(q,0) \qquad (n \to \infty)\ .$$

Note that the induced distribution $\rho(q,0)$ need not be equal to (or even close to) $|\psi(q,0)|^2$. On the other hand, if we adopt the measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^2$ on the set of possible initial configurations $q_{\mathrm{univ}}(0)$, then it is easy to see that, with respect to this measure, in the limit $n \to \infty$ almost all points $q_{\mathrm{univ}}(0)$ correspond to the Born-rule distribution $\rho(q,0) = |\psi(q,0)|^2$. This is because, with respect to the universal measure

$$|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^2 = |\psi(q_1,0)|^2|\psi(q_2,0)|^2...|\psi(q_n,0)|^2\ ,$$

in effect we have n independent and identically-distributed random variables $q_1,\ q_2,\ q_3,\ ...,\ q_n$, each with the same probability distribution $|\psi(q,0)|^2$. In the limit $n \to \infty$ we then necessarily find $\rho(q,0) = |\psi(q,0)|^2$.

It might then appear that nonequilibrium configurations – that is, configurations $q_{\mathrm{univ}}(0)$ corresponding to distributions $\rho(q,0) \neq |\psi(q,0)|^2$ – comprise a vanishingly small set (in the limit $n \to \infty$) and may then be regarded as intrinsically unlikely or 'untypical'. But this conclusion rests crucially on the choice of measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^2$. For example, if instead we choose a measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}},0)|^4$ (up to overall normalisation), then by the same argument almost all configurations $q_{\mathrm{univ}}(0)$ now correspond to a nonequilibrium distribution $\rho(q,0) \propto |\psi(q,0)|^4$ for sub-systems. More generally, if we choose a measure $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^p$ (for constant $p>0$), we will almost always obtain a nonequilibrium distribution $\rho(q,0)\propto|\psi(q,0)|^p$ for sub-systems. The typicality approach then seems circular: by assuming a universal Born-rule measure $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2$, one is simply assuming the Born rule at the initial time $t=0$ (Valentini 1996, 2001).

In our illustrative example $\Psi_{\text{univ}}(q_{\text{univ}},0)$ takes the simple form (28) and we only consider measurements at $t=0$. The original argument by Dürr, Goldstein and Zanghì (1992) is more general than this and includes a discussion of time ensembles of measurements. But the key objection remains: the Born rule is guaranteed to hold for sub-systems in the early universe only because the Born rule is assumed to hold for the whole universe at $t=0$.

It is important to emphasise that there is a qualitative difference between cases with large-but-finite n and the literal limit $n\to\infty$. For any finite n, however large, a set of points $q_{\text{univ}}=(q_1,\ q_2,\ ...,\ q_n)$ that has zero measure with respect to $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2$ will arguably have zero measure with respect to any reasonable density function on configuration space. Thus, for example, if a particle moving in two spatial dimensions has an initial wave function $\psi(x,y,0)$ then points and lines in the two-dimensional configuration space will have zero measure with respect to $|\psi(x,y,0)|^2$. Those same points and lines will also have zero Lebesgue measure (or zero area), and furthermore they will have zero measure with respect to any density proportional to $|\psi(x,y,0)|^p$ (with $p>0$). Much the same may be said for general configuration spaces. This means that, for finite n, if a set of points $q_{\text{univ}}(0)$ has zero measure with respect to $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2$ then that same set of points may reasonably be regarded as objectively small and hence physically negligible. But this objectivity vanishes when we take the limit $n\to\infty$ (the limit where the typicality argument is applied). For example, as we noted above, for $n\to\infty$ the nonequilibrium set

$$S_{\text{noneq}}=\left\{q_{\text{univ}}(0)\mid\rho(q,0)\propto|\psi(q,0)|^4\right\}\tag{29}$$

(the set of initial points $q_{\text{univ}}(0)$ yielding a nonequilibrium sub-system density $\rho(q,0)\propto|\psi(q,0)|^4$) has zero measure with respect to $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2$ (that is, $\mu_{\text{eq}}[S_{\text{noneq}}]=0$ where $d\mu_{\text{eq}}=|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2dq_{\text{univ}}$). On the other hand, the same set S_{noneq} has unit measure with respect to $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^4$ (suitably normalised) (that is, $\mu_{\text{noneq}}[S_{\text{noneq}}]=1$ where $d\mu_{\text{noneq}}\propto|\Psi_{\text{univ}}(q_{\text{univ}},0)|^4dq_{\text{univ}}$). Thus, in the limit $n\to\infty$, there is *no* objective sense in which the nonequilibrium set S_{noneq} is 'small'.

Technically, the qualitative difference between cases with finite n and cases where the literal limit $n\to\infty$ is taken may be highlighted in terms of the notion of 'absolute continuity'. For finite n, the alternative measure $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^4$ is absolutely continuous with respect to the equilibrium measure $|\Psi_{\text{univ}}(q_{\text{univ}},0)|^2$. This simply means, by definition, that the alternative measure of a set S is equal to zero whenever the equilibrium measure of S is equal to zero ($\mu_{\text{eq}}[S]=0$ implies $\mu_{\text{noneq}}[S]=0$). But this will *not* hold for $n\to\infty$, where we can have both $\mu_{\text{eq}}[S_{\text{noneq}}]=0$ and $\mu_{\text{noneq}}[S_{\text{noneq}}]=1$.

In this context one should beware of statements appealing to absolute continuity. For example, Dürr and Teufel (2009, p. 222) write:

> ... any measure ... which is absolutely continuous with respect to the equivariant measure $[|\Psi_{\mathrm{univ}}|^2]$ ... defines the same sense of typicality.

Taken by itself this statement is correct and simply amounts to a statement of the definition of absolute continuity. But to avoid misunderstandings in this context, it is important to add (as Dürr and Teufel omit to) that absolute continuity fails in the relevant limit $n \to \infty$. In that limit, running the argument with an alternative typicality measure will *not* yield the Born rule for subsystems, because the alternative measure will not be absolutely continuous with respect to the Born-rule measure.

Intrinsic likelihood and unlikelihood

We have seen that at $t = 0$ nonequilibrium is 'untypical' (has zero measure) with respect to the equilibrium measure, and that equally nonequilibrium is 'typical' (has unit measure) with respect to a corresponding nonequilibrium measure. It might then be said that initial nonequilibrium is unlikely with respect to the equilibrium measure, and that it is *likely* with respect to a nonequilibrium measure.

With these clarifications in mind, there is clearly no sense in which quantum nonequilibrium is instrinsically unlikely – contrary to claims made by the Bohmian mechanics school, according to which the theory will always yield the Born rule. For example, in their founding paper Dürr *et al.* claim that

> ... in a universe governed by Bohmian mechanics it is in principle impossible to know more about the configuration of any subsystem than what is expressed by [the Born rule]. (Dürr, Goldstein and Zanghì 1992)

Similarly, Tumulka writes that

> ... in a universe governed by Bohmian mechanics, observers will see outcomes with exactly the probabilities specified by the usual rules of quantum mechanics ... . (Tumulka 2018)

But as we have noted this is true only in quantum equilibrium: for general nonequilibrium ensembles the Born rule is violated (Valentini 1991a,b) and it is perfectly possible to know more about a sub-system than is allowed by the Born rule or by the associated uncertainty principle (Valentini 2002a).

The claims made by the Bohmian mechanics school amount to an artificial and unjustified restriction of the theory to equilibrium only. This is most explicit in the presentation by Tumulka, who writes that it is

> ... a fundamental law of Bohmian mechanics, to demand that $[q]$ be *typical* with respect to $|\Psi|^2$. (Tumulka 2018)

But as we have emphasised, only the dynamical equations have the status of fundamental laws. In a theory of dynamics it is physically nonsensical to regard a restriction on the initial state as a 'fundamental law'.

Dürr and Teufel (2009, p. 224) go so far as to compare searching for nonequilibrium to waiting for a stone to jump up spontaneously in the air:

> ... one could sit in front of a stone and wait for the stone to jump into the air, because in a very atypical world, that could happen, now, tomorrow, maybe the day after tomorrow. (Dürr and Teufel 2009, p. 224)

But there is no scientific basis for the claim that quantum nonequilibrium is intrinsically unlikely. This claim stems, as we have seen, from a circular argument in which the Born-rule measure is taken to *define* 'typicality' for the initial conditions of the universe. In our view, in contrast, initial conditions are ultimately empirical.

Typicality and probability

If we replace the word 'typicality' by its synonym 'probability', it becomes apparent that the argument given by Dürr *et al.* simply assumes a universal Born-rule probability density $|\Psi_{\mathrm{univ}}|^2$, which implies the Born-rule probability density $|\psi|^2$ for sub-systems.[16]

Goldstein (2001) has, however, defended the view that typicality and probability are *not* synonymous. Goldstein argues that for a set S the precise value of the typicality measure $\mu(S)$ (say 1/2 or 3/4) is immaterial: the only thing that matters is whether $\mu(S)$ is very small or not. If $\mu(S)$ is very small, this is regarded as a sufficient explanation for why events in S do not occur. According to Goldstein, this concept of typicality plays an important role in scientific explanation. In our view such attempts to elevate the notion of typicality from a synonym for probability to a fundamentally new kind of explanatory principle raise more questions than they answer. It is claimed that what counts is only whether $\mu(S)$ is very small or not. How small is small enough? Questions remain as to precisely how typicality differs from probability. One may also ask why the notion of probability alone does not suffice. The nature of probability is already controversial: it seems misguided to introduce a conceptual variant which proves to be even more controversial.

Other responses to circularity

Dürr and Teufel (2009, pp. 220–222) make some attempt to respond to our charge of circularity.

The first response appeals to the equivariance of the Born-rule measure over time. This measure has the special property that its form as a function of Ψ is preserved by the dynamics (that is, it is 'equivariant'). Dürr and Teufel argue

[16] For measurements at a single time this result is a restatement of the 'nesting' property discussed in Section 2 (Valentini 1991a).

that this property singles out the Born-rule measure as the preferred measure of typicality:

> Would another measure ... say the one with density $|\Psi|^4$ ... not yield typicality for the empirical distribution $|\varphi|^4$ by the same argument? In fact, it would, but only at *exactly* that moment of time where the measure has the $|\Psi|^4$ density. Since the measure is not equivariant, its density will soon change to something completely different. ... The equivariant measure of typicality on the other hand is special ... . Typicality defined by this measure does not depend on time. (Dürr and Teufel 2009, pp. 220–221)

This defence is physically unconvincing. As we have seen, in their argument for the typicality of the Born rule for sub-systems the universal Born-rule measure is applied to the initial universal configuration $q_{\mathrm{univ}}(0)$ at only one time (the initial time $t = 0$). From this we may immediately derive the Born rule for sub-systems at that same initial time $t = 0$. The time evolution of the measure at $t > 0$ is irrelevant to the statistics of sub-systems at $t = 0$. Furthermore, as Dürr and Teufel themselves admit, if an alternative universal measure were applied at $t = 0$ then by the same derivation we would obtain an alternative (non-Born-rule) distribution for sub-systems at $t = 0$ – regardless of how the measure evolves at later times. Dürr and Teufel wish to argue in favour of Born-rule typicality for initial configurations at $t = 0$, but their argument appeals to properties of the time evolution of the measure at $t > 0$. Physically speaking, it is hard to understand how initial conditions at $t = 0$ can be dictated by, or influenced by, a convenient mathematical property of the evolution at $t > 0$.

In a similar vein, while commenting on the quantum relaxation approach advocated by this author, and in particular on the idea of primordial quantum nonequilibrium, Tumulka claims outright that an initial nonequilibrium measure would be intrinsically unnatural and puzzling because it would not be equivariant:

> ... the $|\Psi|^2$ distribution is special since it is equivariant ... if we found empirically (which we have not) that it was necessary to assume that $[q(0)]$ was $|\Psi|^4$-distributed, then it would be a big puzzle needing explanation why it was $|\Psi|^4$ of all distributions instead of the natural, equivariant $|\Psi|^2$. (Tumulka 2018)

The argument seems to be that initial equilibrium is natural and needs no explanation because it is equivariant. Whereas, because initial nonequilibrium is not equivariant, it is unnatural and so – if it were observed – would present a major puzzle. Again, physically it is difficult to see how initial conditions at $t = 0$ can be dictated by a convenient mathematical property of the evolution at later times.

A second response to the charge of circularity appeals to ease of proof:

> ... the equivariant measure is a highly valuable technical tool, because this is the measure which allows us to prove the theorem! At time t, let us say today when we do the experiment, any other measure would look so odd (it would depend on Ψ_t in such an intricate way) that we would have no chance of proving anything! And if it did not look odd today, then it would look terribly odd tomorrow! The equivariant measure always looks the same and what we prove today about the empirical distribution will hold forever. (Dürr and Teufel 2009, p. 222)

But we are concerned with objective physical facts about the initial configuration of our universe. In a scientific theory, the initial state of the universe is not determined by mathematical convenience or ease of proof of a certain theorem, but by empirical observation and measurement.

Typicality in classical and quantum statistical mechanics

In defence of the typicality approach it might be asserted that, in statistical mechanics generally, certain undesirable initial conditions are often ruled out on the grounds that they are exceptional (or untypical) with respect to a particular measure. Some authors do indeed justify the required restrictions on initial conditions along these lines. But it should be admitted that all initial conditions are allowed in principle. The actual realised initial conditions are ultimately an empirical matter to be constrained by experiment. Furthermore, the mere use of a different word does not entail the use of a genuinely different concept: in our view ruling out undesirable initial conditions as 'untypical' simply amounts to ruling them out as 'improbable'.

The use of typicality in classical statistical mechanics has also been defended by Goldstein (2001), who argues that to explain thermal relaxation it suffices to note that phase-space points corresponding to thermal equilibrium occupy an overwhelmingly larger volume than phase-space points corresponding to thermal nonequilibrium. According to this argument, if a system begins at an initial point corresponding to nonequilibrium, then it is overwhelmingly likely to evolve (and quickly) to a final point corresponding to equilibrium, merely by virtue of the much larger volume occupied by the latter points. It may then be said that thermal relaxation is 'typical' with respect to the phase-space volume measure. However, as noted by Uffink (2007, pp. 979–980), a specific system trajectory $(q(t), p(t))$ traces out a set of points S_0 of measure zero (with respect to phase-space volume), whereas the set of points S_1 never visited by the trajectory is of measure one. By definition the system remains in the zero-measure set S_0 for all time, and does *not* move into the set S_1 even though the latter has an overwhelmingly larger phase-space measure. It is then untenable to claim that a system is more likely to move into a set of points simply because that set has a larger phase-space volume. Uffink concludes – in our view correctly – that a *bona fide* explanation for relaxation must appeal to properties of the dynamics and not merely to a measure-theoretic counting of states.

In our view measure-theoretic arguments are misleading and by themselves give no indication of likelihood. We emphasise once again that initial conditions are ultimately a matter for experiment. As scientists we need to understand which initial conditions are consistent with present observations. To this end we must consider the dynamics, and also take into account our knowledge of past history (whether locally in the laboratory or at cosmological scales).

5.2 Probability and ensembles in cosmology

According to the Bohmian mechanics school it is meaningless to consider probabilities or ensembles for the whole universe:

> What physical significance can be assigned to a probability distribution on the initial configurations for the entire universe? What can be the relevance to physics of such an ensemble of universes? (Dürr, Goldstein and Zanghì 1992)
>
> ... since we only have access to one universe ... an ensemble of universes is meaningless for physics. (Dürr and Teufel 2009, p. 224)

This claim conflicts with common practice in both theoretical and observational cosmology. In recent decades hundreds of millions of dollars have been spent on satellite observations of the CMB (as well as on galaxy surveys) with the express aim of putting empirical constraints on the probability distribution for primordial cosmological perturbations. These observations do have a well-defined meaning, in spite of the above claim.

First of all, to suggest that there is only one universe is correct only as a trivial tautology. In practice cosmologists employ the word 'universe' to denote the totality of what we are currently able to observe. Given our current knowledge it is perfectly plausible that what we see is only one element of a very large and perhaps even infinite ensemble. In current observational cosmology, the data are well described by a 'standard' cosmological model according to which what we see is only a tiny patch within an infinite flat (expanding) space – a conclusion that is arguably the most conservative option at the present time. Further afield, contemporary theoretical cosmology includes 'eternal' inflationary models with an infinity of pocket universes, while string theories are widely believed to imply the existence of a 'multiverse'. Thus in theory there is no difficulty in imagining that the universe we see is merely one of a large ensemble, and this may well be the case as a matter of physical fact.

Secondly, the meaning of a probability distribution 'for the universe' is by no means as problematic as the Bohmian mechanics school portrays it. As we outlined in Section 4, practising cosmologists routinely test the predictions of such distributions via measurements of the CMB. As we explained, such measurements probe the primordial power spectrum (25) for a theoretical ensemble. By this means, whole classes of cosmological models have been ruled out by observation because they predict an incorrect spectrum.

This is not to say that one cannot or should not question the meaning of a theoretical ensemble. But we would argue that such foundational questions have no special connection with pilot-wave theory or cosmology; they arise in any practical application of probability theory or statistical inference, whether one is considering genetic populations on earth or the distribution of galaxies in space. Furthermore, the relevant mathematical properties of the assumed 'probability distribution over a theoretical ensemble' do not depend on any particular interpretation of probability theory. One could, for example, regard the distribution as expressing a subjective degree of belief, or as representing a really existing ensemble; it would make no difference to how the distribution is employed in mathematical practice. Indeed, if one prefers one may avoid the notion of a theoretical ensemble of 'universes' and instead consider a real ensemble of approximately independent sub-regions of a single universe. Given that the 'universe' we see may in any case be just such a sub-region, which approach one takes is immaterial.

In this context one should also beware of the claim (sometimes made by the Bohmian mechanics school[17]) that pilot-wave dynamics is in some special sense fundamentally a dynamics of the whole universe. If this were true, we would need a complete theory of cosmology to work with and apply pilot-wave theory. In a trivial sense, of course, there will always be small interactions between even very distant systems, and in all known theories of physics it could be said that fundamentally the theory is a theory of the whole universe. But this is no more true in pilot-wave theory than it is in classical mechanics, classical field theory, or general relativity. In principle one needs to consider the whole universe in all these theories; but in practice, the universe divides into approximately independent pieces, at least in the real situations occurring in our actual world. The situation in pilot-wave theory shows no essential difference from that of other physical theories.

In our view, the question of the meaning of probability and of ensembles in cosmology has its place as a valid and interesting philosophical question, but the emphasis placed on this question by the Bohmian mechanics school has proved to be misleading and (we would argue) a distraction. The question of the existence or otherwise of primordial quantum nonequilbrium is empirical. It will be answered by detailed work in theoretical and observational cosmology, not by foundational debates about the meaning of probability and related topics.

On a related note we comment on a recent paper by Norsen (2018), which attempts to combine quantum relaxation ideas with typicality arguments. In our view considerations of typicality add nothing substantial to a quantum relaxation scenario, and merely introduce a new word to denote the probability measure for a universal theoretical ensemble.[18] Furthermore if, as Norsen (p. 24) advocates, we also consider 'reasonably smooth' non-Born-rule typicality measures on the initial universal configuration, then while (as Norsen notes) our relaxation results suggest that the Born rule will still be obtained on a coarse-grained level at later times, the fact remains that such initial measures will

[17]See, for example, Goldstein and Zanghì (2013).

[18]Norsen (p. 15) follows the Bohmian mechanics school in making the misleading claim that '... there is simply no such thing as the probability distribution P for particle configurations of the universe as a whole, because there is just one universe'.

imply nonequilibrium for sub-systems in the early universe (with all the novel physical implications we have described). This then contradicts Norsen's claim (p. 22) that early nonequilibrium is intrinsically unlikely (a claim made, confusingly, by appealing to the initial Born-rule measure). To avoid such needless controversy, we should bear in mind that there simply is no intrinsic typicality (or probability) measure for initial conditions, and that in the end the existence or non-existence of early nonequilibrium can only be established by observation.

6 Contingency and the nature of the universal wave function

The typicality approach has led to misunderstandings not only of the Born rule in pilot-wave theory but also of the nature of the wave function (or pilot wave).

As we have emphasised, in a theory of dynamics the initial conditions are contingent and only the laws of motion are law-like. And yet, in the typicality approach the initial configuration $q_{\mathrm{univ}}(0)$ of the universe is restricted by the requirement that it be typical with respect to the measure $|\Psi_{\mathrm{univ}}(q_{\mathrm{univ}}, 0)|^2$. The latter measure is treated as if it had a law-like status (as explicitly claimed by Tumulka (2018)). In our view, in contrast, the initial probability distribution for the universe is a contingency which can be constrained only by cosmological observation.

This confusion between contingent and law-like entities has been taken to an extreme in claims made by the Bohmian mechanics school regarding the nature of the universal wave function Ψ_{univ}, specifically: (1) that Ψ_{univ} cannot be regarded as contingent, and (2) that Ψ_{univ} is not a physical object but a law-like entity ('nomological' rather than 'ontological').

The argument that Ψ_{univ} cannot be regarded as contingent is essentially this: the wave function for the whole universe

> is not controllable: it is what it is. (Goldstein 2010)

There are several problems with this argument. Firstly, the same could be said of the universal configuration q_{univ}, resulting in the remarkable conclusion that no properties of our universe can be regarded as contingent (not even the position of the moon). Secondly, and in a similar vein, one could just as well say that our universe has only one spacetime geometry, and indeed only one intergalactic magnetic field. Each of these objects 'is not controllable' and 'is what it is'. And yet, according to standard thinking in physics and cosmology, the detailed form of either object is not completely determined by physical laws: each has a strong element of contingency. Thirdly and finally, as noted in Section 5.2, in this context we ought to beware of statements that there is 'only one universe': in principle such statements are trivially and tautologically true, but in practice the universe studied by cosmologists may well be one element of a large (and possibly infinite) ensemble, where the object which we call Ψ_{univ} can vary contingently across the ensemble.

The argument that Ψ_{univ} is not a physical object but a law-like entity is based on three assertions: (a) that Ψ_{univ} cannot be regarded as contingent (claim (1) above, which we have argued to be unfounded), (b) that Ψ_{univ} is static, and (c) that Ψ_{univ} is uniquely determined by the laws of quantum gravity.[19] On these grounds it has been claimed that Ψ_{univ} is a law-like entity roughly analogous to a classical Hamiltonian (Dürr, Goldstein and Zanghì 1997; Goldstein and Zanghì 2013). Thus:

> ... the wave function is a component of physical law rather than of the reality described by the law. (Dürr, Goldstein and Zanghì 1997, p. 33)

But the arguments (a)–(c) do not bear scrutiny. We have already seen that argument (a) is spurious. Argument (b) is also questionable, based as it is on the time-independence of the Wheeler-DeWitt equation (the analogue of the Schrödinger equation) in canonical quantum gravity.[20] However, the physical meaning and consistency of the quantum-gravitational formalism remains in doubt, in particular because of the notorious 'problem of time' (the problem of explaining the emergence of apparent temporal evolution in some appropriate limit). Many workers have suggested that a physical time parameter is in effect hidden within the formalism and that, when correctly written as a function of physical degrees of freedom, the wave function is in fact time dependent.[21] As for argument (c), in canonical quantum gravity the solutions for Ψ_{univ} (satisfying the Wheeler-DeWitt equation as well as the other required constraints) are in fact highly *non*-unique (Rovelli 2004). In quantum cosmological models, for example, the solutions for Ψ_{univ} have the same kind of contingency that we are used to for quantum states in other areas of physics (Bojowald 2015).

It is worth emphasising that even if a consistent theory of quantum gravity did require Ψ_{univ} to be static, this still would not by any means establish that Ψ_{univ} is law-like. The key aspect of Ψ_{univ} that makes it count as a physical object is its contingency, in other words its under-determination by known physical laws. This implies that Ψ_{univ} contains a lot of independent and contingent structure – just like the electromagnetic field or the universal spacetime geometry – and so should be regarded as part of the physical state of the world (Valentini 1992, p. 17; Brown and Wallace 2005, p. 532; Valentini 2010b).

7 Further criticisms of 'Bohmian mechanics'

Pilot-wave theory is, in general, a nonequilibrium physics that violates the statistical predictions of quantum theory (Valentini 1991a,b, 1992). It can only

[19]Other authors express concern about regarding a field on configuration space as a physical object. In our view it is not unreasonable for configuration space to be the fundamental arena of a realistic physics, with physical objects propagating on it (cf. footnote 23).

[20]The Wheeler-DeWitt equation takes the schematic atemporal form $\hat{\mathcal{H}}\Psi = 0$, where $\hat{\mathcal{H}}$ is an appropriate operator for the Hamiltonian density (Rovelli 2004).

[21]See, for example, Roser and Valentini (2014) and the exhaustive review of the problem of time by Anderson (2017).

be properly understood from this general perspective. The Bohmian mechanics school has instead promoted the belief that pilot-wave theory is instrinsically a theory of equilibrium. We now consider the principal physical misunderstandings that have arisen from this mistaken belief.

7.1 'Absolute uncertainty'

The Bohmian mechanics school has asserted that typicality with respect to the Born-rule measure is 'the origin of absolute uncertainty'. On this view the uncertainty principle is an 'absolute' and 'irreducible' limitation on our knowledge:

> In a universe governed by Bohmian mechanics there are sharp, precise, and irreducible limitations on the possibility of obtaining knowledge ... absolute uncertainty arises as a necessity, emerging as a remarkably clean and simple consequence of the existence of trajectories. (Dürr, Goldstein and Zanghì 1992)
>
> ... in a Bohmian universe we have an *absolute uncertainty* ... the [Born rule] is a sharp expression of the inaccessibility in a Bohmian universe of micro-reality, of the unattainability of knowledge of the configuration of a system that transcends the limits set by its wave function ψ. (Goldstein 2010)
>
> In Bohmian mechanics ... there are sharp limitations to knowledge and control: inhabitants of a Bohmian universe cannot know the position of a particle more precisely than allowed by the $|\psi|^2$ distribution ... . Furthermore, they cannot measure the position at time t without disturbing the particle ... . (Tumulka 2018)

But as we pointed out in Section 5.1, the typicality argument in effect inserts the Born rule by hand at the initial time. Furthermore, as we noted in the Introduction, the uncertainty principle is not absolute or irreducible but merely a peculiarity of the state of quantum equilibrium. In general, the uncertainty principle would be violated if we had access to quantum nonequilibrium systems (Valentini 1991b, 2002a).

7.2 The status of quantum measurement theory

As also briefly noted in the Introduction, in the presence of quantum nonequilibrium key quantum constraints are violated: these include statistical locality, expectation additivity for quantum observables, and the indistinguishability of non-orthogonal quantum states (Valentini 1991a,b, 1992, 2002a, 2004; Pearle and Valentini 2006). The physics of nonequilibrium is radically different from the physics of equilibrium, the latter being merely a highly restricted special case of the former. It should then come as no surprise that the nonequilibrium theory of measurement differs radically from its equilibrium counterpart. As philosophers of physics are well aware, measurement is 'theory laden': we need some body of theory in order to know how to perform measurements correctly. As Einstein put it, in an often-quoted conversation with Heisenberg:

> It is the theory which decides what we can observe. (Heisenberg 1971, p. 63)

In the presence of quantum nonequilibrium systems, pilot-wave theory itself tells us how to perform correct measurements. It is found, for example, that if we possessed an ensemble of 'apparatus pointers' with an arbitrarily narrow nonequilibrium distribution (much narrower than the standard quantum width as defined by the initial wave function of the pointer), then it would be possible to use the apparatus to perform 'subquantum measurements': in particular, we would be able to measure the position and trajectory of a particle without disturbing its wave function (to arbitrary accuracy) (Valentini 2002a, Pearle and Valentini 2006).

From this perspective, the physics of quantum equilibrium is highly misleading – and so is the associated equilibrium theory of measurement (also known as 'quantum measurement theory'). In fact, the detailed dynamics of pilot-wave theory shows that the procedures known as 'quantum measurements' are generally not correct measurements. Instead, those procedures are merely special kinds of experiments which have been designed to respect a formal analogy with classical measurements (where the analogy is implemented by a mathematical correspondence between classical and quantum Hamiltonians) (Valentini 1992, 1996, 2010b).

To put the so-called quantum theory of 'measurement' in a proper perspective, we must consider the more general physics of quantum nonequilibrium and its associated theory of subquantum measurement. But because the Bohmian mechanics school believes that the theory is fundamentally grounded in equilibrium, they are led to believe that the equilibrium theory *is* the theory – and that the associated quantum theory of measurement has a fundamental status. Thus Dürr, Goldstein and Zanghì (1996, 2004) argue that the quantum theory of measurement arises as an account of what they call 'reproducible experiments' and reproducible 'measurement-like' experiments. Measurements that lie outside of the domain of the quantum formalism are not considered. Thus both quantum equilibrium and its associated theory of measurement are in effect regarded as fundamental features of pilot-wave theory. In our view this is deeply mistaken. The physics of equilibrium is a special case of a much wider physics in which new kinds of measurements are possible. If instead we artificially restrict ourselves to the equilibrium domain, the result is a distorted understanding of measurement and an overstatement of the significance of the conventional quantum formalism.

7.3 The misleading kinematics of quantum equilibrium

It is worth noting how the artificial restriction to quantum equilibrium makes the idea of fundamental Lorentz invariance (at the level of the underlying equations of motion) seem much more plausible than it really is. As we have remarked, in general nonequilibrium gives rise to instantaneous signaling between remote entangled systems (Valentini 1991b).[22] The reality in principle of superluminal communication between widely-separated experimenters strongly suggests the existence of an absolute simultaneity associated with a preferred slicing of spacetime (Valentini 2008b). And indeed most versions of pilot-wave dynamics (and in particular of quantum field theory) are defined with respect to a preferred frame with a preferred time parameter t – where effective Lorentz invariance emerges only at the statistical level of quantum equilibrium (Bohm, Hiley and Kaloyerou 1987, Valentini 1992, Bohm and Hiley 1993, Holland 1993). If instead the theory is always and everywhere artificially restricted to equilibrium, locality will always hold at the statistical level and practical nonlocal signalling will be impossible. It may then seem plausible to search for a version of pilot-wave theory in which the dynamics is fundamentally Lorentz invariant, since one will never be faced directly with the awkward question of what happens when practical superluminal signals are viewed from a Lorentz-boosted frame and appear to travel backwards in time (potentially generating causal paradoxes). Even so, despite several attempts, a fundamentally Lorentz-invariant pilot-wave theory remains elusive and problematic (Dürr *et al.* 1999; Tumulka 2007; Dürr *et al.* 2014).

The attachment to fundamental Lorentz invariance has in turn encouraged a misunderstanding of the role of Galilean invariance, which the Bohmian mechanics school mistakenly regards as a fundamental symmetry of the low-energy theory (Dürr, Goldstein and Zanghì 1992; Allori *et al.* 2008; Dürr and Teufel 2009; Goldstein 2017; Tumulka 2018). Pilot-wave theory is a first-order or 'Aristotelian' dynamics with a law of motion (1) for velocity (as first envisaged by de Broglie in 1923), in contrast with Newtonian theory which is a second-order dynamics with a law of motion for acceleration. Because of this fundamental difference, the natural kinematics of pilot-wave theory is also Aristotelian with a preferred state of rest (Valentini 1997). Galilean invariance may be shown to be a fictitious symmetry of the low-energy pilot-wave theory of particles – just as invariance under uniform acceleration is well known to be a fictitious symmetry of Newtonian mechanics. If instead one tries to insist on Galilean invariance being a physical symmetry of the low-energy theory, the result is a conceptually incoherent combination of an Aristotelian dynamics with a Galilean kinematics.

In response it might be claimed that Galilean invariance plays an important role in selecting the form of the low-energy guidance equation (Dürr, Goldstein and Zanghì 1992; Dürr and Teufel 2009; Goldstein 2017). But in fact the de Broglie velocity $v = j/\left|\psi\right|^2$ is generally determined by the quantum current j, which may be derived as a Noether current associated with a global phase symmetry $\psi \to \psi e^{i\theta}$ on configuration space (Struyve and Valentini 2009). The derivation takes place in one frame of reference, with no need to consider boosts. The relevant symmetry is in configuration space, not in space or spacetime.[23]

[22] Similar conclusions hold in all nonlocal and deterministic hidden-variables theories (Valentini 2002b).

[23] This reinforces our view that configuration space is the fundamental physical arena of

7.4 Particle creation and indeterminism

It is also worth noting how the artificial restriction to quantum equilibrium makes a fundamentally stochastic model of particle creation – developed by the Bohmian mechanics school – appear more plausible than it really is. For if one denies the general contingency of the Born rule for initial conditions, it may seem no great loss to introduce a fixed and non-contingent probability into the dynamics as well.

The stochastic model promoted by the Bohmian mechanics school was constructed as follows. Bell (1986; 1987, chapter 19) had already proposed a discrete model of fermion numbers evolving stochastically on a lattice and had suggested that taking the continuum limit might yield a deterministic theory. The Bohmian mechanics school studied the continuum limit of Bell's model and arrived at a theory of particle trajectories with stochastic jumps at events where the particle numbers change (Dürr *et al.* 2004, 2005). They named their approach 'Bell-type quantum field theory', and have attempted to apply it to bosons as well as to fermions. The fundamental probability rule for the jumps is chosen so as to preserve the Born rule.

Any interacting quantum field theory will contain a plethora of events where the particle numbers change (photon emission, electron-positron pair creation, and so on), and the Bohmian mechanics school has suggested that determinism must be abandoned to describe them:

> In Bell-type [quantum field theories], God does play dice. There are no hidden variables which would fully predetermine the time and destination of a jump. (Dürr *et al.* 2004, p. 3)
>
> The quantum equilibrium distribution, playing a central role in Bohmian mechanics, then more or less dictates that creation of a particle occurs in a stochastic manner ... . (Dürr *et al.* 2005, p. 2)

It would, however, be remarkable indeed if indeterminism were required to describe particle creation – when determinism suffices to describe all other quantum-mechanical processes. But in fact, indeterminism is not required. The Bohmian mechanics school obtained a stochastic continuum limit of Bell's model because they adopted an erroneous definition of fermion number F. In quantum field theory, F is conventionally defined as the number of particles minus the number of anti-particles.[24] As a particle physicist, this is what Bell would have meant by fermion number. Unfortunately, Dürr *et al.* mistakenly took Bell's 'fermion number' to mean the number of particles *plus* the number of anti-particles.

The correct continuum limit of Bell's model was taken by Colin (2003), who employed the standard definition of F. As a result Colin obtained a 'Dirac sea' theory of fermions – anticipated by Bohm and Hiley (1993, p. 276) – in which

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pilot-wave dynamics (cf. footnote 19).

[24]In particle physics, for historical reasons F is defined as the sum $F = L + B$ of lepton and baryon numbers – where L is the number of leptons minus the number of antileptons and similarly for B.

particle trajectories are determined by a pilot wave that obeys the many-body Dirac equation.[25] The resulting model is fully deterministic, as Bell suggested it would be. There are no fixed or fundamental stochastic elements, and the usual contingency of probabilities applies to all processes.

For completeness we note that, for bosons, it is straightforward to develop a deterministic pilot-wave field theory, in which the time evolution of a (for example scalar) field ϕ is determined by the Schrödinger wave functional $\Psi[\phi, t]$ (Holland 1993). In such a theory, again, the Born rule $P = |\Psi|^2$ is contingent and may be understood as arising from a process of dynamical relaxation (Valentini 2007). In contrast, the Bohmian mechanics school encounters difficulties defining particle trajectories and a Born-rule position-space density for single bosons, as briefly noted by Dürr *et al.* (2005, p. 13). Such problems recall the long history (in standard quantum theory) of controversial attempts to define a position-space 'wave function' for single photons and other bosons, attempts which invariably lead to negative probabilities and superluminal wave packet propagation. Without a solution to this – probably insoluble – problem, so-called 'Bell-type quantum field theory' remains undefined for bosons.[26]

7.5 The problem of falsifiability

Finally, the artificial restriction to quantum equilibrium has compromised the status of pilot-wave theory as a falsifiable scientific theory. For it is then impossible to measure the trajectory of a system without disturbing its wave function; hence it is impossible to test the de Broglie equation of motion, which associates a specific set of trajectories with each given wave function. There are alternative pilot-wave theories, with alternative velocity fields, which nevertheless preserve the Born distribution and which therefore imply the same empirical predictions for the equilibrium state (Deotto and Ghirardi 1998). The physics of equilibrium is insensitive to the details of the trajectories. Thus, if we have access to equilibrium only, the alternative theories can never be tested against de Broglie's original theory. Indeed, in equilibrium, pilot-wave theories are forever experimentally indistinguishable from conventional quantum theory. As Dürr *et al.* put it:

> For every conceivable experiment, whenever quantum mechanics makes an unambiguous prediction, Bohmian mechanics makes exactly the same prediction. Thus, the two cannot be tested against each other. (Dürr, Goldstein, Tumulka and Zanghì 2009)

It is of course logically and mathematically possible for the world to be governed by pilot-wave theory and to be always and everywhere in quantum

[25] The Dirac-sea model requires regularisation (for example a cutoff) (Colin 2003, Colin and Struyve 2007). The same is true of the models developed by Dürr *et al.* in the presence of interactions.

[26] Dürr *et al.* (2005, p. 13) cite two papers 'in preparation' (their refs. [18] and [28]) purporting to address this problem. To the author's knowledge, and unsurprisingly, neither paper was completed.

equilibrium. But from a scientific point of view such a theory is unfalsifiable and therefore unacceptable. This shortcoming is, however, not a feature of pilot-wave theory itself – which abounds in new and potentially-observable physics – but rather stems from a misunderstanding of the status of the Born rule in this theory.

8 Conclusion

The foundations of statistical mechanics are notoriously controversial, and overlap with difficult questions concerning the nature of probability, the justification for standard methods of statistical inference, and even with philosophical questions concerning the foundations of the scientific method. However, important as these questions are, in our view they are not especially relevant to either pilot-wave theory or cosmology but instead arise generally across the sciences. We claim that attempts to forge a special link between such questions and pilot-wave theory are at best a distraction and at worst deeply misleading.

Something comparable took place during the early development of atomic theory in the late nineteenth century. At that time theoretical physics was divided between what we might now call 'operationalists' (who saw the macroscopic laws of thermodynamics as paradigmatic for physics generally) and 'realists' (who thought those laws required a deeper explanation in terms of atoms and kinetic theory). Boltzmann in particular was especially passionate about the philosophical importance of atomism as a basis for explanation in physics, vis à vis the competing operationalist views of Mach and Ostwald.[27] In retrospect it seems unfortunate that Boltzmann became embroiled in foundational controversies concerning probability, time reversal, and so on – important and interesting questions which, in hindsight, proved to be a distraction from the main goal of demonstrating the reality of atoms. The eventual atomistic explanation for Brownian motion by Einstein in 1905 owed little to such foundational debates and more to technical developments in kinetic theory – and the foundational debates persist to this day, more than a century after the existence of atoms was firmly established.

Similarly, theoretical physics today is again divided between operationalists (who see quantum mechanics as an operational theory of macroscopic observations) and realists (who think there must be a reality behind the formalism). Among the various realist approaches, pilot-wave theory is the most closely analogous to kinetic theory. Once again, it seems unfortunate that the subject has become embroiled in foundational controversies in statistical mechanics and probability theory, when surely the main goal is to find out whether the trajectories posited by pilot-wave theory really exist or not. In our view, while those foundational controversies are important and interesting, in the context of pilot-wave theory they have proved to be a distraction from the even more

[27]See, for example, Boltzmann's selected writings in the collection *Theoretical Physics and Philosophical Problems* (Boltzmann 1974).

important question of whether pilot-wave theory itself is true or not. Furthermore, the viewpoint championed by the Bohmian mechanics school (and widely followed by philosophers of physics) has played a major role in obscuring the physics of the theory – which is fundamentally a nonequilibrium physics that violates quantum mechanics. We emphasise, once again, that the existence or non-existence of quantum nonequilibrium in our universe (past and present) is an empirical question that will be settled only by detailed theoretical and observational work.

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