Leibniz Equivalence, Newton Equivalence, and Substantivalism

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Abstract

Active diffeomorphisms map a differentiable manifold to itself. They transform manifold points and objects without changing the system of local coordinates used to represent those objects. What has been called *Leibniz Equivalence* is the assertion that, although active diffeomorphisms do change manifold objects, they do not change what is called the "physical situation" being modeled by those objects. This paper introduces the contrasting idea of *Newton Equivalence,* which asserts that the different values of manifold objects produced by active diffeomorphisms do model different physical situations. But due to the assumption of general covariance, these different physical situations are all equally possible. They represent physically different situations all of which could happen. This paper compares these two interpretations of active diffeomorphisms, and comments on their importance in the substantivalism debate.

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1 Introduction

As he was working toward the field equation of general relativity, Einstein devised a thought experiment which convinced him that a generally covariant field equation could have multiple solutions.[1] He imagined a special but plausible case in which the energy-momentum tensor source term would vanish in a region he called a *hole*. A transformation that was an identity everywhere except the hole would then modify the solution in the hole region without any change to the sources. Stachel noted that Einstein had at first referred to this transformation as what translates as a *coordinate* transformation, and later, perhaps in response to criticism, as a *point* transformation.[2] Stachel suggested that the former term referred to diffeomorphic changes of local coordinates, and the latter term referred to what he called active diffeomorphisms, those that transform manifold points without changing the local coordinate system.[3]

In spite of Einstein's hole argument, the final form of his field equation is generally covariant. Einstein resolved this dilemma by asserting that the multiple solutions must all represent the same physical reality.

A decades old but still influential paper by Earman and Norton[4] suggested that Einstein's hole argument should be generalized. The Earman-Norton generalization makes no use of the detail of Einstein's special energy-momentum tensor source. Instead, it elevates Einstein's *resolution* of his dilemma, his assertion that multiple solutions represent the same physical reality, to a general principle referred to as *Leibniz Equivalence:* "Diffeomorphic models represent the same physical situation."[5] It is clear from context that the term "diffeomorphic models" here refers to models related to each other by Stachel's active diffeomorphisms.

Whereas Einstein did not address the case of differential equations other than his own field equation, the Earman-Norton paper asserts that the principle of Leibniz Equivalence also applies to "...Newtonian spacetime theories with all, one, or none of gravitation and electrodynamics; and special and general relativity, with and without electrodynamics."[6] Einstein's resolution of his hole argument dilemma is thus generalized into an assertion about the effect of active diffeomorphisms in essentially all differential geometric physical models.

The task of the present paper is to suggest that Leibniz Equivalence is not the only possible interpretation of active diffeomorphisms. We suggest an alternate

[1] See Chapter 5 of Torretti (1996) and Chapter 5 of Stachel (2002).

[2] Torretti (1996), Section 5.6, page 164; Stachel (1986).

[3] Using Einstein's form of his thought experiment, Johns (2019) outlines a mathematical proof that active diffeomorphisms can produce multiple solutions of the field equation of general relativity with the same energy-momentum source. However, it is also suggested that some of these solutions can be rejected as spurious, leading in some cases to a unique result.

[4] Earman and Norton (1987). Examples of its influence include recent papers by Muller (1995) completing, and by Schulman (2016) and Weatherall (2018) refuting, the "hole argument." These papers refer to the Earman and Norton (1987) generalization rather than to Einstein's version of it. See also the current encyclopedia article, Norton (2015).

[5] Earman and Norton, *op. cit.,* page 522.

[6] Earman and Norton, *op. cit.,* page 516.

interpretation called *Newton Equivalence* that is more in keeping with current practice in theoretical physics. In this interpretation, the changes of manifold objects produced by an active diffeomorphism do change the physical situation. But, due to general covariance, the new physical situation produced by an active diffeomorphism, while different, is equally possible.

Since the dispute here is between two alternate interpretations of active diffeomorphisms, it is essential to state clearly what the term means. To establish notation and avoid misunderstandings, Sections 2 and 3 review the underlying differential geometric definitions. Section 4 then summarizes the points of agreement between the Leibniz and Newton interpretations, the basic facts that are to be differently interpreted.

Section 5 provides two illustrative examples of the use of active diffeomorphisms in physics. Sections 6 and 7 use those examples to argue for and against the Newton and Leibniz interpretations.

Section 8 discusses Earman and Norton's argument that substantivalism fails because it requires denial of Leibniz Equivalence. It is shown that Newton Equivalence, which does deny Leibniz Equivalence, escapes the unfortunate consequences proposed by these authors. This section also shows that the attempt to derive indeterminism from the denial of Leibniz Equivalence fails to generalize the Einstein hole argument.

Section 9 summarizes the paper's conclusion that Newton Equivalence is a straightforward and correct interpretation of active diffeomorphisms, and is consistent with current practices in theoretical and experimental physics. Also, while choice of the Leibniz interpretation binds one to a rejection of substantivalism, choice of the Newton interpretation frees one from that binding and allows one to remain agnostic on this important issue.[7]

2 Passive Diffeomorphisms

Modern differential geometry makes a distinction between *manifold objects* and the *coordinate objects* that represent them in various systems of local coordinates.[8] Thus, given a differentiable manifold $\mathcal{M}$ of dimension m, a point $\mathbf{x} \in \mathcal{M}$ is a manifold object represented in different systems of local coordinates (here denoted as unprimed or primed local coordinates) by

$$\begin{aligned} x &= (x^1, \ldots, x^m) = \psi(\mathbf{x}) \\ x' &= (x'^1, \ldots, x'^m) = \psi'(\mathbf{x}) \end{aligned} \tag{2.1}$$

where ψ and ψ' are different homeomorphic mappings from $\mathcal{M}$ to $\mathbb{R}^m$. The definition of a smooth differentiable manifold is that both the relation eqn (2.2)

[7] With the current situation in theoretical physics (for example, the unknown nature of dark energy, the lack of a satisfactory quantum theory of gravity), it seems important not to restrict the models that theorists may try.

[8] To make this important distinction clear, throughout this paper manifold points and objects will be written in **bold roman type**, while coordinate objects will use *non-bold italic.*

between any two systems of local coordinates,

$$x' = \psi' \circ \psi^{-1}(x) \tag{2.2}$$

and its inverse, must be continuously differentiable to arbitrary order.[9]

Functions $\mathbf{f} : \mathcal{M} \to \mathbb{R}$ mapping manifold points $\mathbf{x}$ to real numbers are manifold objects that have local coordinate representations $f = \mathbf{f}\circ\psi^{-1}$ and $f' = \mathbf{f}\circ\psi'^{-1}$ related by

$$f(x) = \mathbf{f}(\mathbf{x}) = f'(x') \tag{2.3}$$

Smooth functions are defined as those for which $f(x)$ in some arbitrary system of local coordinates (and therefore in all such systems) is continuously differentiable to arbitrary order.

Tangent vector fields $\mathbf{V}(\mathbf{x})$ are manifold objects that map smooth functions to real numbers. These manifold objects are represented in unprimed and primed local coordinates, respectively, by

$$V(x) = \sum_{i=1}^{m} V^i(x)\,\partial/\partial x^i \quad \text{and} \quad V'(x') = \sum_{i=1}^{m} V'^i(x')\,\partial/\partial x'^i \tag{2.4}$$

The functions $V^i(x)$ and $V'^i(x')$ are called the components of $\mathbf{V}(\mathbf{x})$ in the two systems. The relation between manifold and coordinate objects is defined as

$$\sum_{i=1}^{m} V^i(x)\frac{\partial f(x)}{\partial x^i} = V(x)f(x) = \mathbf{V}(\mathbf{x})\,\mathbf{f}(\mathbf{x})$$
$$= V'(x')f'(x') = \sum_{i=1}^{m} V'^i(x')\frac{\partial f'(x')}{\partial x'^i} \tag{2.5}$$

Note that, by nearly universal custom, the operation of tangent vectors on functions is written in operator form with no parentheses around the function. The components of a vector field in the unprimed and primed systems are related by

$$V'^i(x') = \sum_{j=1}^{m} \frac{\partial x'^i}{\partial x^j} V^j(x) \tag{2.6}$$

Second rank covariant tensor fields are manifold objects $\mathbf{g}(\mathbf{x})$ that bilinearly map an ordered pair of tangent vector fields $\mathbf{U}(\mathbf{x})$ and $\mathbf{V}(\mathbf{x})$ to real numbers, denoted $\mathbf{g}(\mathbf{x})\{\mathbf{U}(\mathbf{x}), \mathbf{V}(\mathbf{x})\}$. These manifold objects also have local coordinate rep-

[9] The transformation from x to x' is a diffeomorphic change of local coordinates. We will refer to this transformation as a *passive diffeomorphism,* since it does not change the manifold object $\mathbf{x}$. The term *gauge transformation* is also sometimes used in the literature. This paper takes "passive diffeomorphism," "gauge transformation," and "diffeomorphic change of local coordinates" to refer to the same transformation, eqn (2.2).

resentations

$$\sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij}(x)U^{i}(x)V^{j}(x) = \mathbf{g}(\mathbf{x})\{\mathbf{U}(\mathbf{x}),\mathbf{V}(\mathbf{x})\}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} g'_{ij}(x')U'^{i}(x')V'^{j}(x') \quad (2.7)$$

The components of $\mathbf{g}(\mathbf{x})$ in unprimed and primed systems are related by

$$g'_{ij}(x') = \sum_{k=1}^{m}\sum_{l=1}^{m} g_{kl}(x)\frac{\partial x^{k}}{\partial x'^{i}}\frac{\partial x^{l}}{\partial x'^{j}} \quad (2.8)$$

Riemannian manifolds $(\mathcal{M},\mathbf{g})$ use such a second rank covariant tensor field to define a metric.[10] The invariant inner product of two vector fields is a manifold object defined as

$$\Big\langle \mathbf{U}(\mathbf{x}),\mathbf{V}(\mathbf{x})\Big\rangle = \mathbf{g}(\mathbf{x})\{\mathbf{U}(\mathbf{x}),\mathbf{V}(\mathbf{x})\} \quad (2.9)$$

Other forms and tensors are defined similarly. The important point is that passive diffeomorphisms do not change manifold objects, but only change the local coordinates and components used to represent them. For this reason, differential geometric models in theoretical physics universally assign reality only to manifold objects. Since they do not change these manifold objects, passive diffeomorphisms therefore do not change the physical reality being modeled. A magnetic field is not identified with its components $B^{i}(x)$ or $B'^{i}(x')$. It is modeled by a manifold object $\mathbf{B}(\mathbf{x})$ that does not change when the local coordinate system is changed.

3 Active Diffeomorphisms

Active diffeomorphisms are the opposite of passive diffeomorphisms. Passive diffeomorphisms change the system of local coordinates, but do not change the manifold objects. Active diffeomorphisms change the manifold objects but do not change the system of local coordinates being used to represent those manifold objects.

Active diffeomorphisms are a special case of a more general differential geometric construction, the differentiable mapping $\phi : \mathcal{M} \to \mathcal{N}$ from manifold $\mathcal{M}$ to manifold $\mathcal{N}$. In general, the two manifolds may have different dimensions, and the mapping need not have an inverse.

An active diffeomorphism is defined to be a differentiable, invertible mapping from $\mathcal{M}$ to itself rather than to some other manifold $\mathcal{N}$, with the target

[10]The term Riemannian manifold is assumed to include semi-Riemannian manifolds.

manifold therefore having the same dimension and the same system of local coordinates.[11]

The mapping is $\phi : \mathcal{M} \to \mathcal{M}$ with manifold points $\mathbf{x}$ and $\tilde{\mathbf{x}}$ before and after the mapping related by $\tilde{\mathbf{x}} = \phi(\mathbf{x})$. In terms of local coordinates $\tilde{x} = \psi(\tilde{\mathbf{x}})$ and $x = \psi(\mathbf{x})$, the active diffeomorphism is[12]

$$\tilde{x} = \psi \circ \phi \circ \psi^{-1}(x) \tag{3.1}$$

The mapping is defined to be diffeomorphic; both eqn (3.1) and its inverse are continuously differentiable to arbitrary order.

Note that the transformed local coordinate values $\tilde{x} = \left(\tilde{x}^1, \ldots, \tilde{x}^m\right) = \psi(\tilde{\mathbf{x}})$ differ from the original local coordinate values $x = (x^1, \ldots, x^m) = \psi(\mathbf{x})$. This is not because of a change of the local coordinate *system* but because the represented manifold object $\mathbf{x}$ has been transformed to a new manifold object $\tilde{\mathbf{x}} = \phi(\mathbf{x})$. The unchanged local coordinate *system* is defined by the same homeomorphic mapping ψ before and after an active diffeomorphism.

If manifold object $\mathbf{f}(\mathbf{x})$ is a smooth function defined on $\mathcal{M}$, then there is an actively transformed smooth manifold object $\tilde{\mathbf{f}} = \mathbf{f} \circ \phi^{-1}$ with

$$\tilde{\mathbf{f}}(\tilde{\mathbf{x}}) = \mathbf{f}(\mathbf{x}) \tag{3.2}$$

called the *push-forward* of $\mathbf{f}$ by ϕ and denoted $\tilde{\mathbf{f}} = \phi_* \mathbf{f}$. The original function $\mathbf{f}$ is also called the *pull-back* of $\tilde{\mathbf{f}}$ by ϕ, and is denoted $\mathbf{f} = \phi^* \tilde{\mathbf{f}}$. In terms of local coordinates, the relation between these functions is given by $\tilde{f}(\tilde{x}) = f(x)$.

Tangent vectors can also be pushed forward or pulled back.[13] The push-forward $\tilde{\mathbf{V}} = \phi_* \mathbf{V}$, or equivalently the pull-back $\mathbf{V} = \phi^* \tilde{\mathbf{V}}$, is defined by

$$\tilde{\mathbf{V}}(\tilde{\mathbf{x}})\, \tilde{\mathbf{f}}(\tilde{\mathbf{x}}) = \mathbf{V}(\mathbf{x})\, \mathbf{f}(\mathbf{x}) \tag{3.3}$$

In terms of local coordinates, $\tilde{V}(\tilde{x})\tilde{f}(\tilde{x}) = V(x) f(x)$. The component transformation, here written as a push-forward, is

$$\tilde{V}^i(\tilde{x}) = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j} V^j(x) \tag{3.4}$$

In Riemannian manifolds the metric tensor also can be equivalently pushed forward $\tilde{\mathbf{g}} = \phi_* \mathbf{g}$ or pulled back $\mathbf{g} = \phi^* \tilde{\mathbf{g}}$. The definition is

$$\tilde{\mathbf{g}}(\tilde{\mathbf{x}})\{\tilde{\mathbf{V}}(\tilde{\mathbf{x}}), \tilde{\mathbf{W}}(\tilde{\mathbf{x}})\} = \mathbf{g}(\mathbf{x})\{\mathbf{V}(\mathbf{x}), \mathbf{W}(\mathbf{x})\} \tag{3.5}$$

[11] This paper uses the term *active diffeomorphism,* from Stachel (1986). Einstein evidently called them what translates as *point transformations.* Differential geometric texts not aimed specifically at the general relativity community do not even discuss active diffeomorphisms, limiting themselves to the general case with $\phi : \mathcal{M} \to \mathcal{N}$ transforming between different manifolds. Several texts for general relativists refer to active diffeomorphisms as simply *diffeomorphisms* with no modifier. We will always precede the word diffeomorphism with a modifier, passive or active, to avoid confusion as to which one is intended.

[12] Objects after transformation by an active diffeomorphism will be denoted by a tilde over them, for example $\tilde{\mathbf{x}}$ and $\tilde{x}$.

[13] Unlike the general case where the mapping ϕ possibly had no inverse and hence some relations were ill defined, active diffeomorphisms have inverses and hence both pull-back and push-forward are always well defined.

for any general pair of tangent vectors. The component relation, here expressed as a push-forward, is

$$\tilde{g}_{ij}(\tilde{x}) = \sum_{k=1}^{m}\sum_{l=1}^{m} g_{kl}(x)\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j} \tag{3.6}$$

In pre-general-relativistic differential geometry, a Riemannian metric is a fixed part of the definition of a Riemannian manifold, denoted $(\mathcal{M}, \mathbf{g})$. But the metric $\phi_* \mathbf{g}$ pushed forward by an active diffeomorphism could possibly be different from that fixed metric of the manifold. Thus, in pre-general-relativistic physics, in order to preserve general covariance it is necessary to restrict active diffeomorphisms to those that do not change the Riemannian metric,[14] those with $\phi_* \mathbf{g} = \mathbf{g}$. Such active diffeomorphisms are called *isometric.*[15] In general relativity, however, the metric is unknown until the Einstein field equation is solved. So there is no need to restrict active diffeomorphisms in this way. In general relativity, the active diffeomorphism is of the form $\phi : (\mathcal{M}, \mathbf{g}) \to (\mathcal{M}, \phi_* \mathbf{g})$ with no isometric restriction on ϕ.

3.1 Generation of Active Diffeomorphisms

Some of the machinery of Lie Group theory can be borrowed to generate active diffeomorphisms from given tangent vector fields. A useful class of active diffeomorphisms can be constructed by considering the mapping ϕ_τ along a given tangent vector field $V(x)$.[16]

Given a chosen starting point $\mathbf{x} \in \mathcal{M}$, let a smooth mapping $(0, \tau_1) \to \mathcal{M}$ define a curve $\mathbf{x}(\tau)$ starting from $\mathbf{x}(0) = \mathbf{x}$. In terms of local coordinates, this is $x(\tau) = \psi(\mathbf{x}(\tau))$. Differentiating this curve with respect to τ gives what is sometimes called a "velocity" tangent vector, whose components are $\dot{x}^i(\tau) = dx^i(\tau)/d\tau$. Given a general tangent vector field $V(x)$, a curve whose velocity matches that tangent vector for every $\tau \in (0, \tau_1)$ is defined by the set of differential equations

$$\dot{x}^i(\tau) = V^i(x(\tau)) \qquad \text{where} \qquad i = 1, \ldots, m \tag{3.7}$$

whose solution $x(\tau)$ can be described as an integral curve or "field line" of $V(x)$ passing through $x(0) = x$. The corresponding field line in the manifold is then $\mathbf{x}(\tau) = \psi^{-1}(x(\tau))$.

Since the tangent vector field is assumed to be defined at all points of $\mathcal{M}$, we can consider the family of all such field-line curves beginning at every point $\mathbf{x} \in \mathcal{M}$. Consider an active diffeomorphic mapping $\phi_\tau : \mathcal{M} \to \mathcal{M}$ which *simultaneously* carries each $\mathbf{x}$ in $\mathcal{M}$ into an $\mathbf{x}(\tau)$ along the particular field line starting

[14]See Lee (1997), page 24. For example, in three-dimensional models with a fixed Euclidean metric, the only permissible active diffeomorphisms are rigid translations and rotations. A four-dimensional model with a fixed Minkowski metric permits only rigid translations, rotations, and boosts.

[15]In terms of local coordinates and components, an isometric active diffeomorphism has $\tilde{g}_{ij}(\tilde{x}) = g_{ij}(\tilde{x})$.

[16]Section 39 of Arnold (1978), pages 68-70 and Chapter 9 of Lee (2013), and pages 27-32 and 250-251 of O'Neill (1983).

at that $\mathbf{x}$. When $\tau = 0$, this mapping is the identity mapping $\phi_0 = I$. When $\tau > 0$, mapping ϕ_τ will move each point $\mathbf{x}$ of $\mathcal{M}$ along the appropriate field line to a new point $\tilde{\mathbf{x}} = \mathbf{x}(\tau) = \phi_\tau(\mathbf{x})$. Expressing the same mapping in local coordinates, each point $x = \psi(\mathbf{x})$ is moved by active diffeomorphism $\theta_\tau = \psi \circ \phi_\tau \circ \psi^{-1}$ into a new point $\tilde{x} = x(\tau) = \theta_\tau(x)$.

If $V(x)$ is a so-called Killing Vector Field,[17] then by definition the active diffeomorphism ϕ_τ is isometric. Generation of more general active diffeomorphisms with $\tilde{\mathbf{g}} = \phi_* \mathbf{g} \neq \mathbf{g}$ requires that $V(x)$ not be a Killing Vector Field.

3.2 Associated Passive Diffeomorphisms

An active diffeomorphism transforms manifold point $\mathbf{x}$ with local coordinate values $x = \psi(\mathbf{x})$ to a new manifold point $\tilde{\mathbf{x}} = \phi(\mathbf{x})$ with new local coordinate values $\tilde{x} = \psi(\tilde{\mathbf{x}})$. For each active diffeomorphism there is an *associated* passive diffeomorphism. If the active diffeomorphism is applied first, and then the associated passive diffeomorphism is applied, the end result is that the local coordinates are returned to their original numerical values. The associated passive diffeomorphism does not change the manifold point $\tilde{\mathbf{x}}$, but undoes the change of local coordinate values produced by the active diffeomorphism. It follows from eqn (2.2) and eqn (3.1) that the required associated passive diffeomorphism is $\tilde{x}'' = \psi'' \circ \psi^{-1}(\tilde{x})$ where $\psi'' = \psi \circ \phi^{-1}$. When it is used as described above, the final result is $\tilde{x}'' = x$.

It may seem that after this sequence nothing has changed. The local coordinates are back to their original values. Examination of the transformation rules for functions, tangent vectors, and general tensors shows that after the above sequence, the final components of these coordinate objects are also returned to their original numerical values. However, the correct reading is not that *nothing* has changed but that *everything* has changed, both the manifold objects and the system of local coordinates used to represent them. Due to general covariance, everything has changed in a consistent manner.

4 Uncontested Points

Newton equivalence and Leibniz equivalence are two different interpretations of active diffeomorphisms and their use in physics. But the two interpretations agree on the following points.

1. Coordinate objects change when the local coordinate system is changed by a passive diffeomorphism (a diffeomorphic change of local coordinates); manifold objects do not. Therefore, differential geometric models in theoretical physics universally assign reality only to manifold objects.

2. Passive diffeomorphisms do not change manifold objects. Therefore passive diffeomorphisms do not change the physical reality being modeled.

[17] Lee (2013), page 345.

3. Active diffeomorphisms change manifold objects. Whether this change is a true change of the physical situation being modeled is a point of contention between the two interpretations. But both agree that the manifold objects are modified by active diffeomorphisms.

4. Let a generally covariant model of an experiment consist of the set of manifold objects $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and the outcome of the experiment consist of the set of manifold objects $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_k)$. If an active diffeomorphism is performed on the model, the result will be a model consisting of transformed manifold objects $\tilde{\mathbf{A}} = (\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_n)$ with the transformed outcome manifold objects $\tilde{\mathbf{B}} = (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_k)$. If $\mathbf{A}$ produces $\mathbf{B}$, then $\tilde{\mathbf{A}}$ produces $\tilde{\mathbf{B}}$. Colloquially speaking, due to general covariance nothing is "broken" by the active diffeomorphism and the model will still "work" as before the active diffeomorphism was done.

5. As described in Section 3.2, any active diffeomorphism can be followed by an associated passive diffeomorphism that returns the local coordinates and components to their original numerical values.

5 Two Examples of Active Diffeomorphisms

Discussion of abstract issues like the role of active diffeomorphisms in differential geometry is often aided by concrete examples of what the abstractions actually produce. Sections 6 and 7 will make reference to the following two examples showing the effect of active diffeomorphisms on simple physical models.

5.1 Example 1

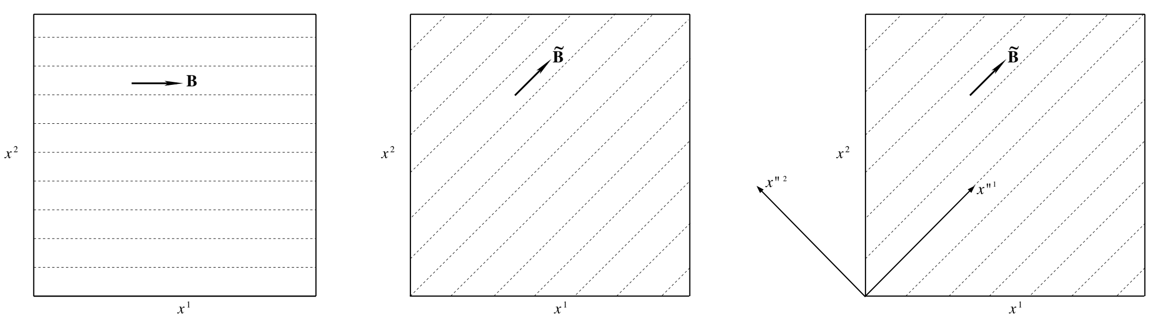

Figure 5.1: a, b, and c.

Figure 5.1a shows a model of an experiment containing a uniform magnetic field $\mathbf{B}(\mathbf{x})$ in a three-dimensional Euclidean space,[18] viewed from a system of local coordinates in which the magnetic field has the components $(B^1, 0, 0)$. The

[18] Or, equivalently, Examples 1 and 2, and the example in Section 7.5, show time slices of limited, approximately Minkowskian, regions of spacetime. See Section 10.3 of Rindler (2006).

field lines are shown as dashed lines, and a typical field value is shown as an arrow. An active diffeomorphism is generated using the methods of Section 3.1 using a tangent vector field $V(x)$ with components $(0, -x^2, x^1, 0)$. This vector field generates an orthogonal rotation by angle τ. This active diffeomorphism is assumed to be the identity everywhere except in the apparatus. Applying this active diffeomorphism with $\tau = \pi/4$ produces Figure 5.1b. The local coordinate system (possibly defined by the edges of the table) is unchanged, but the field lines are now at a 45° angle. A typical actively transformed field value $\tilde{\mathbf{B}}$ is shown. Figure 5.1c is the same as Figure 5.1b, except that the associated passive diffeomorphism has now been added, giving the local coordinate axis lines shown as x''^1 and x''^2. The actively transformed field lines are in the direction of the x''^1 axis, as described in Section 3.2, and $\tilde{B}''^i = B^i$ for $i = 1, 2, 3$.

5.2 Example 2

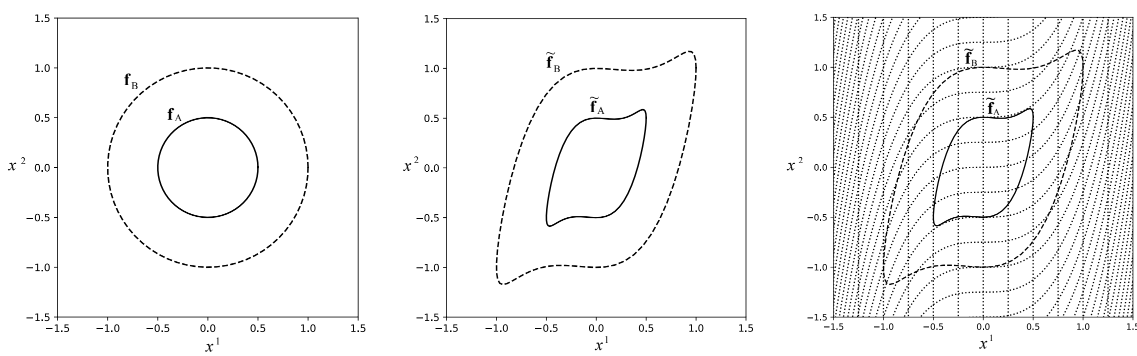

Figure 5.2: a, b, and c.

Figure 5.2a shows a model in which a mass distribution, as viewed initially on a hyperplane at time t_A, has a circular cylindrical boundary $\mathbf{f}_\mathrm{A}$. It later, as viewed on a hyperplane at time t_B, evolves into a circular cylindrical distribution with larger boundary $\mathbf{f}_\mathrm{B}$. An active diffeomorphism is generated using the methods of Section 3.1 using the tangent vector field $V(x)$ with components $(0, 0, (x^1)^3, 0)$. Applying this active diffeomorphism with $\tau = 1$ to the model gives Figure 5.2b in which the mass boundaries are distorted. But, as noted in point 4 of Section 4, due to general covariance the actively transformed initial boundary $\tilde{\mathbf{f}}_\mathrm{A}$ will still evolve into the actively transformed final boundary $\tilde{\mathbf{f}}_\mathrm{B}$. Note that the distortion in Figure 5.2b is due to distortion of the manifold object itself; the local coordinate system is unchanged. Figure 5.2c shows the grid lines of the associated reference system x''. In this distorted reference system, the actively transformed mass distributions would have the same local coordinate description as the original distributions had in Figure 5.2a.

6 Newton Equivalence

Newton Equivalence holds that active diffeomorphisms that change manifold objects necessarily change the physical reality being modeled by those objects. However, due to general covariance, the different physical situations reached by active diffeomorphisms are all equally possible. They model different but equally possible physical situations.

Thus, in Example 1 above, Figure 5.1b with its rotated $\tilde{\mathbf{B}}$ field models a physically modified experiment. This accords with the usual practice in experimental physics. If an experimenter finds that, with no change in the system of local coordinates, the magnetic field is now pointing in a different direction, his immediate conclusion is that something physical has changed. That the experiment still behaves correctly is taken as evidence for invariance under rotations. The fact that a rotation of reference system to match the changed magnetic field, as in Figure 5.1c, would return the local coordinates of that field to their original values would be taken as further evidence of rotational symmetry, and would not change the conclusion about Figure 5.1b.

In Example 2 above, an astrophysicist observes the mass distribution in Figure 5.2a to evolve into a larger one of the same circular shape. Her generally covariant model of this phenomenon can be transformed by an active diffeomorphism into a model that predicts the evolution of a distorted mass distribution into a distorted larger one, as in Figure 5.2b. However, in the model or in actual observation of the phenomena, the astrophysicist would never consider the two figures to represent the same physical reality. Figure 5.2b represents a different physical situation, but one that could happen given the validity of the generally covariant model.

6.1 Symmetry and Identity

An argument for Newton Equivalence is its agreement with the treatment of symmetry in physics. Active diffeomorphisms are a kind of symmetry operation allowed by generally covariant physical models.[19] The actively transformed version of a successful model is another successful model, something that could also happen. But the transformed model is not identical to the original one; it models a different experiment that is also possible, related to the original one by the symmetry but not identical to it.

In theoretical and experimental physics, great care is taken to distinguish *symmetry* from *identity*. This distinction is particularly emphasized in discussion of discrete symmetries like parity and time reversal. Parity symmetry means that the parity transformed experiment is one that is also possible, *i.e.,* is not prohibited by the laws of physics. But it does not mean that original and transformed experiments are identical, are physically the same. They are different. Parity

[19] A generally covariant model is one constructed using only manifold objects. Also, equalities are only between manifold objects of the same tensorial character: scalar equals scalar, vector equals vector, second rank covariant tensor equals second rank covariant tensor, etc. Thus an active diffeomorphism transforms a generally covariant model into another generally covariant model.

symmetry only means that the parity transformed experiment is one that can happen without violating the laws of nature.[20]

The distinction between symmetry and identity is even more pointed in time reversal invariance. Consider an experiment in which a particle fissions into two daughter particles. Time reversal symmetry means that the inverse process, two particles coalescing into one, is something that could possibly happen. The laws of physics do not prohibit it. But it does not mean that this inverse process is physically the same as the original one. It is different, and the experiment to demonstrate the time reversed process could be very difficult to perform.

Newton Equivalence echos this strong distinction between symmetry and identity found in theoretical physics. In a generally covariant model, an active diffeomorphism is a symmetry operation that changes the physical situation being modeled. The transformed model is then equally possible, something not prohibited by the generally diffeomorphic laws of nature.

6.2 Observer and Apparatus

Another argument for Newton Equivalence is its agreement with the general usage of coordinate objects and manifold objects in theoretical physics. The application of differential geometry to physics is based on a distinction between observer and apparatus. Local coordinates are the numbers resulting from the observation of an apparatus. The possibility of observation of an experiment from a viewpoint outside that experiment is fundamental to this distinction. Since they modify manifold objects, active diffeomorphisms do change the *values* of the data collected from an experiment, but they do not transform the *system* used by the observer to extract that data.

For example, let the magnetic field in Figure 5.1 be from an electromagnet sitting on a laboratory table, with a system of local coordinates fixed to the table. If that electromagnet apparatus is rotated by 45 degrees relative to the laboratory table, the result is a physical change, as in Figure 5.1b. The *system* of local coordinates is still fixed to the table, but the *values* of the local coordinates have changed, due to the physical rotation of the apparatus.

In the Newton Equivalence interpretation, the apparatus is modeled by coordinate independent manifold objects and the observation is modeled by local coordinates. Seeing the local components of the magnetic field change from B to $\tilde{B}$ with no change in the system of local coordinates, the observer correctly concludes that a physical change has taken place. This use of active and passive diffeomorphisms in differential geometry echos the usual practice in physics.

6.3 Agnosticism

The name *Newton* Equivalence was chosen to suggest a tenable interpretation of active diffeomorphisms that is also consistent with current practice in physics. It

[20] For example, Section 15.11 of Bjorken and Drell (1965) says of the active parity transformation, "If we invert the measuring apparatus, that is, consider a new physical system ... the dynamics of the new system is the same as that of the original one, provided parity is conserved."

is not intended to imply that an adherent of this interpretation must necessarily take Newton's side in the Newton-Leibniz debates of the early eighteenth century.[21] As discussed in Section 8 below, use of the Newton Equivalence interpretation allows a researcher to remain agnostic about the substantivalist-relativist debate.

7 Leibniz Equivalence

Leibniz Equivalence states that, "Diffeomorphic models represent the same physical situation."[22] While an active diffeomorphism does change the manifold objects of a model, it is asserted not to change the "physical situation" being modeled by those manifold objects. Thus the physical situation is not identified with a particular set of manifold object values, but with an *equivalence class* of manifold object values, a given set of values together with all the others reached from it by application of any active diffeomorphism. Figure 5.1a and Figure 5.1b thus are asserted to show the same physical situation. They are not models of a different physical reality; rather they are just parts of the same physical reality.

Some support for Leibniz Equivalence comes from its resonance with Leibniz' objections to Newton's absolute space. If we imagine the experiment in Figure 5.1 to be mounted in a closed room sitting on a turntable, then if someone rotates the turntable in the night while the experimenter sleeps, we would go directly from Figure 5.1a to Figure 5.1c and the experimenter would not discern any changes to his experiment. In this case, the interior of the room is like the experimenter's Leibnizian universe and the rest of the Earth is like Newton's absolute space. But the analogy is flawed; there are discernible changes in the physical situation. For example, the workers who rotated the laboratory in the night would have observed the change, and would have carefully measured it to be 45°. Also, Figure 5.1b would still require additional justification, since in it only the apparatus is rotated, and the unchanged laboratory reference system detects this change.

Leibniz Equivalence also holds Figure 5.2a and Figure 5.2b to show the same physical situation. Here the difference is not one of a Leibnizian rigid translation of a whole universe. It is a distortion, of a sort that neither Newton nor Leibniz would be likely to have imagined. The Leibniz interpretation ignores the large variety of available active diffeomorphisms; active diffeomorphisms can be much more general than the simple displacements argued by Leibniz.

7.1 Symmetry and Identity

Leibniz Equivalence erases the difference between symmetry and identity, the concepts that theoretical physics seeks to keep distinct. The name of the Leibniz Equivalence principle uses the word "equivalence" but the principle itself asserts

[21] See *The Leibniz-Clarke Correspondence,* Alexander (1956).

[22] Earman and Norton, *op. cit.,* page 522. As noted in Section 1, their term "diffeomorphic models" is to be interpreted as models related by active diffeomorphisms.

identity, the "same" physical situation. As noted in Section 6.1 above, general covariance in differential geometry is a kind of symmetry principle; given general covariance, applying an active diffeomorphism to a successful model produces another model that is equally successful. In theoretical physics, such symmetry transformations are carefully distinguished from identity. Leibniz Equivalence removes this distinction; active diffeomorphisms merely move the model from one to another member of an equivalence class, with both members representing the identical reality.

Thus, referring to Section 6.1 above, Leibniz Equivalence would say that an electromagnet sitting on a laboratory table, and the same electromagnet after the active diffeomorphism symmetry operation of rotation by 45 degrees relative to the table, both represent the identical *physical situation.* This interpretation diverges from the standard meanings of symmetry and *physical situation* in physics.

7.2 Passive and Active

Leibniz Equivalence blurs the difference between passive and active diffeomorphisms. As noted in items 1 and 2 of Section 5, it is generally accepted that *passive* diffeomorphisms do not change the physical reality being modeled. This is because they do not change manifold objects, which are used exclusively to model real quantities. Leibniz Equivalence extends this behaviour to *active* diffeomorphisms, even though active diffeomorphisms do change manifold objects. It says that different manifold objects produced by active diffeomorphisms "represent" the same physical situation. But, unlike coordinate objects which do merely "represent" an underlying manifold object in some arbitrary local coordinate system, the manifold objects themselves do not "represent" some deeper reality; the manifold objects *are* the differential geometric model of that reality. Changing them changes the reality being modeled.

7.3 Observer and Apparatus

In Section 4 of their paper, Earman and Norton deny the theoretical physicist's apparatus-observer distinction discussed in our Section 6.2. They assert: *To complete the dilemma we need only note that spatio-temporal positions by themselves are not observable. Observables are a subset of the relations between the structures defined on the spacetime manifold. Thus we cannot observe that body b is centered at position x. What we do observe are such things as the coincidence of body b with the x mark on a ruler, which is itself another physical system. Thus observables are unchanged under* [active] *diffeomorphism. Therefore* [active] *diffeomorphic models are observationally indistinguishable.*[23]

If applied only to a particular case—Leibniz' rigid translation of the entire universe relative to Newton's absolute space—this assertion is a paraphrase of Leibniz' argument that the unobservability of a rigid translation of the entire

[23] Earman and Norton, *op. cit.,* page 522.

universe renders spatiotemporal position relative to Newton's absolute space meaningless. However, the application of the same argument to active diffeomorphisms is untrue in general.

Their concluding statement that "...observables are unchanged under [active] diffeomorphism. Therefore [active] diffeomorphic models are observationally indistinguishable"[24] is untrue for the general case of *localized* active diffeomorphisms.[25] *Localized* active diffeomorphisms can transform one part of the universe while leaving the rest of it unmodified, and therefore are observable from a viewpoint in the unmodified region.

To illustrate by a simple example, consider an apparatus sitting on a laboratory table. A localized active diffeomorphism now spatially translates the entire apparatus across the table to a new position, but does not transform the table or the rest of the universe. Newton Equivalence holds that this move across the table is a real, observable physical change but, due to spatial translation symmetry, does not upset the working of the apparatus. If any part of the apparatus initially had local coordinate x_o in an unchanged and fixed system of local coordinates (for example one defined by the table edges), after the move it will now have a different local coordinate $\tilde{x}_o$.

The Earman-Norton statement says that, since this move of the apparatus across the table is done by an active diffeomorphism, the two positions of the apparatus must be *observationally indistinguishable*. But, relative to the untransformed table, both the initial position of the apparatus, and its changed position after the localized active diffeomorphism, are plainly observable and distinguishable.

This example illustrates a general result. In the general case of localized active diffeomorphisms there will always be a fixed reference systems, anchored in the untransformed region, that will register changes, and thus render an active diffeomorphism observable. It is untrue in general that "...[active] diffeomorphic models are observationally indistinguishable" as asserted in the above quotation.

Another failure is that the above quotation's application of Leibnizian relativity to active diffeomorphisms ignores complex distortions such as that in Figure 5.2 that cannot be reduced to Leibnizian arguments about translation with an unchanged relation of internal structures.

In summary, the Earman-Norton assertion quoted at the beginning of this subsection is an unjustified extrapolation, from Leibnizian relativism in the particular case of the Newton-Leibniz debates, to all active diffeomorphisms. It ignores the variety of available active diffeomorphisms, and is untrue for the general case of localized active diffeomorphisms.

[24] By "diffeomorphic models," Earman and Norton mean models related by an active diffeomorphism.

[25] The term *localized active diffeomorphism* will be used here to denote active diffeomorphisms whose domain is the whole manifold, but which act as the identity transformation outside of a specified region of spacetime. Einstein's active diffeomorphisms that are the identity except in a "hole" are an example.

7.4 History

The principal argument for Leibniz Equivalence appears to be historical, that it generalizes Einstein's resolution of his hole argument dilemma. Einstein maintained that the different metric solutions to his field equation represent the same physical reality. As expressed by Hawking and Ellis, "... the model for spacetime is not just one pair $(\mathcal{M}, \mathbf{g})$ but a whole equivalence class of all pairs $(\mathcal{M}', \mathbf{g}')$ which are equivalent to $(\mathcal{M}, \mathbf{g})$."[26] Some general relativity texts also echo this assertion, but, notably, others do not,[27] preferring to move directly from the field equation to its practical applications to cosmology.

It has been suggested by Johns (2019) that there may be a less drastic escape from Einstein's dilemma. The local coordinates used to write Einstein's field equation do not have any physical meaning until *after* a metric solution is found that defines their relation to physical quantities. Therefore, it may be possible to reject as spurious a metric solution that gives a physical meaning to its local coordinates that violates a desired symmetry, for example spherical symmetry for the Schwarzschild solution. In this way, as for any differential equation, rejection of spurious solutions can lead to a unique solution.

Aside from the question of the necessity of Einstein's solution to his hole dilemma is the question of Earman and Norton's generalization of that solution to a principle covering virtually all uses of differential geometry in physics. Einstein treated a specific example of a specific theory. Generalization of Einstein's solution to the principle of Leibniz Equivalence is by its nature an unprovable assertion.

7.5 Extrapolation

Earman and Norton assert without further proof that "...[active] diffeomorphism is the counterpart of Leibniz' replacement of all bodies in space in such a way that their relative relations are preserved."[28] But active diffeomorphisms are far more general than the rigid translation of the entire universe used by Leibniz. As noted in Section 5.2, they can distort the transformed objects rather than simply moving them, thus upsetting their "relative relations." And, as discussed in Section 7.3, they can transform parts of the universe while remaining the identity in other parts, thus permitting reference systems that render the transformation observable. The extrapolation of Leibnizian relativism to all active diffeomorphisms is unjustified.

These extrapolations will appeal to a researcher approaching the subject of differential geometry from a Leibnizian perspective. A researcher viewing the world through a Leibnizian lens may simply take the principle of Leibniz Equivalence as a *definition* of the technical phrase *physical situation,* a phrase now referring to an equivalence class of observationally distinct manifold objects related to each other by active diffeomorphisms. Then the Leibniz Equivalence

[26] Hawking and Ellis (1973), page 56.

[27] For example, Misner et al (1973), Weinberg (1972), and Rindler (2006).

[28] Earman and Norton, *op. cit.,* page 521.

principle becomes a tautology. Also such a researcher may want to replace the physics term *unobservable* by the technical term *operationally indistinguishable* defined by the assertion quoted at the beginning of Section 7.3. As noted in Section 7.3, this redefinition would define the *observable* differences produced by localized active diffeomorphisms to be *observationally indistinguishable.* With these redefinitions, the principle of Leibniz Equivalence, together with the assertion quoted at the beginning of Section 7.3, become true by definition.

But these extrapolations would produce an interpretation of differential geometry that misunderstands the generality of active diffeomorphisms, and is significantly at variance with the usages in current theoretical and experimental physics.

8 Substantivalism

In Section 3 of their paper,[29] Earman and Norton state the "acid test" of substantivalism to be the denial of the Leibnizian relativist proposition that the whole universe displaced "three feet East" is the same universe. The Earman-Norton arguments against substantivalism rest on their assertion that a substantivalist, in denying this Leibnizian relativism, must also deny their Leibniz Equivalence principle. Since Leibniz Equivalence is an extrapolation of Leibnizian relativism, this assertion is true. But this does *not* imply the converse, that one who denies Leibniz Equivalence must therefore deny Leibnizian relativism and take Newton's side in the Newton-Leibniz debates.[30] Instead, Newton Equivalence denies only Earman and Norton's *extrapolation,* from Leibnizian relativism for the universe, to their principle of Leibniz Equivalence for all active diffeomorphisms regardless of scale and type. An adherent to Newton Equivalence may be, but need not be, a substantivalist.

We now consider the two arguments that Earman-Norton make against substantivalism. Each of them suggests an undesirable consequence resulting from the denial of Leibniz Equivalence.

In their Section 4, titled "The Verificationist Dilemma,"[31] Earman and Norton argue that, since a substantivalist must deny Leibniz Equivalence, he or she is committed to accept that the *distinct states* produced by active diffeomorphisms are physically different, which runs counter to their argument, quoted at the beginning of our Section 7.3, that such differences are *observationally indistinguishable.* Newtonian Equivalence does indeed consider the *distinct states, i.e.,* the different values of manifold objects produced by active diffeomorphisms, to be observably different. But, as discussed in Section 7.3, the Earman-Norton claim that the differences produced by any active diffeomorphism must be unobservable is based on a misunderstanding of the generality of available active

[29]Earman and Norton, *op. cit.,* page 521.

[30]Leibniz Equivalence (LE) implies Leibnizian relativism (LR). Therefore substantivalism (not LR) implies (not LE). But the converse, that (LR) implies (LE), is not provable. It is an extrapolation that Newton Equivalence denies. Therefore (not LE) does not imply substantivalism (not LR).

[31]Earman and Norton, *op. cit.,* page 522.

diffeomorphisms, and is untrue in general. Thus Newton Equivalence escapes the dilemma.

In their Section 5, titled "The Indeterminism Dilemma,"[32] Earman and Norton argue that the substantivalist denial of Leibniz Equivalence commits one to a "...radical local indeterminism."[33] But their argument fails because it misunderstands Einstein's proof of indeterminism.

Einstein's proof requires a special experiment in which the energy-momentum tensor source term $T_{\mu\nu}(x)$ in his field equation vanishes in a *hole region.* Then an active diffeomorphism is carefully tailored to be the identity except in *that same hole region* where $T_{\mu\nu}(x)$ vanishes. This matched active diffeomorphism then changes the solution without changing the source,[34] and can therefore be used to prove indeterminacy of solution for that particular thought experiment. The *exact match* of active diffeomorphism to source hole is essential to the argument. Without it, the active diffeomorphism also modifies the source term and Einstein's proof fails.[35] The problem exposed by Einstein's hole argument is not multiple solutions *per se,* but multiple solutions *all of which have the same source;* specification of a source does not specify a unique solution produced by that source.[36]

But Earman-Norton do not assume a match between a source and what they call an active "hole diffeomorphism." Thus their unmatched active diffeomorphisms change both the solution *and the source.* This is, of course, just what one would expect in a well-defined theory. If the source changes, we have a different experiment, and would therefore expect a different solution. What Earman-Norton call "radical local indeterminism" is merely the correct action of a symmetry principle. General active diffeomorphisms produce models of possible new experiments with new sources and therefore new solutions.

Thus the Earman-Newton treatment of indeterminism does not generalize Einstein's version of the hole argument. Without an exact match of their active "hole diffeomorphisms" to an energy-momentum source, the Earman-Norton argument fails. With that match, it simply replicates Einstein's version.

The Newtonian denial of Leibniz Equivalence avoids the unfortunate consequences argued by Earman-Norton. Also, unlike a researcher who adopts Leibniz Equivalence and therefore must deny substantivalism and accept Leibnizian relativism, a researcher adopting Newton Equivalence is not thereby committed to either side in the substantivalist-relativist debate.

[32]Earman and Norton, *op. cit.,* page 522.

[33]Earman and Norton, *op. cit.,* page 524.

[34]Outside the hole, the active diffeomorphism is the identity and hence does not change the source there. Inside the hole, the source is identically zero and hence is not transformed, since zero tensors transform to zero tensors regardless of the active diffeomorphism applied.

[35]See Section 4 of Johns (2019).

[36]See, for example, Einstein's 1913 letter to Ludwig Hopf, quoted on page 163 of Torretti (1996), in which he says, "It is easily proved that a theory with generally covariant equations cannot exist if we demand that the field be mathematically *completely determined by matter.*" (Italics mine.)

9 Conclusion

Leibniz Equivalence and Newton Equivalence are two different interpretations of the set of agreed upon facts and practices listed in Section 4.

Newton Equivalence interprets active diffeomorphisms in a manner consistent with current theoretical and experimental physics. Active diffeomorphisms are taken to be symmetry operations that change manifold objects and therefore change the physical situation being modeled. But this changed physical situation is one that is equally possible, that does not violate the generally covariant laws of nature. Newton Equivalence also permits a researcher to remain agnostic about the relativist-substantivalist debate, thus avoiding restriction of the choices available to a theorist facing current problems in theoretical physics such as a quantum theory of gravity.

Leibniz Equivalence is based on an unjustified extrapolation of Leibnizian relativism, applying it to all active diffeomorphisms, including localized active diffeomorphisms that are non-identity only in limited regions, and those that may distort the manifold rather than just translate it. This extrapolation requires redefinition of the terms "physical situation" and "observationally indistinguishable," giving them meanings inconsistent with the usual practice in differential geometry and physics. Also, a researcher adopting Leibniz Equivalence must deny substantivalism, thus restricting the choices available to theorists.

These considerations would seem to favor Newton Equivalence. It is a straightforward and unencumbered interpretation of active diffeomorphisms, one that is consistent with the rules of differential geometry and with current practice in theoretical and experimental physics.

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